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EXTENDABILITY OF C. R. FUNCTIONS : A MICROLOCAL VERSION OF BOCHNER'S TUBE THEOREM

by M. S. BAOUENDI

We present some recent results obtained jointly with F. Treves. Details and complete proofs can be found in [1].

Let m and n be two positive integers, we shall denote by $t = (t_1,\dots,t_m)$ the variable in $\mathbb{R}^m$ and by $x = (x_1,\dots,x_n)$ the variable in $\mathbb{R}^n$. Let U be an open connected set in $\mathbb{R}^m$ and $\phi = (\phi_1,\dots,\phi_n)$ a Lipschitz continuous mapping $U \to \mathbb{R}^n$. We consider the associated complex vector fields in $U \times \mathbb{R}^n$

(1) $$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{n} \frac{\partial \phi_k(t)}{\partial t_j} \frac{\partial}{\partial x_k} \quad , \; j = 1,\dots,m.$$

We have

(2) $$\begin{cases} L_j z_k = 0 & 1 \leqslant j \leqslant m, \quad 1 \leqslant k \leqslant n \\ z_k(t,x) = x_k + i\,\phi_k(t). \end{cases}$$

We denote by $z = z(t,x)$ the mapping $U \times \mathbb{R}^n \longrightarrow \mathbb{C}^n$ defined by $z = (z_1,\dots,z_n)$.

<u>Definition 1</u> : Assume ϕ to be real analytic and let $t^o \in U$ and $x^o \in \mathbb{R}^n$. The system $\mathbb{L} = (L_1,\dots,L_m)$ defined by (1) is said to be analytic hypoelliptic at (t^o,x^o) if and only if any distribution u in some open neighborhood ω of (t^o,x^o), such that $L_j u$ is analytic for $j = 1,\dots,m$, is itself analytic in a possibly smaller open neighborhood ω' of (t^o,x^o).

Before giving a necessary and sufficient condition for the system $\mathbb{L}$ to be analytic hypoelliptic at (t^o,x^o) we state some simple reductions and remarks.

<u>Remarks</u> :

1. In order to prove the analytic hypoellipticity of $\mathbb{L}$ it suffices to prove the analyticity of the solutions of the homogeneous equations

(3) $$L_j h = 0 \qquad 1 \leqslant j \leqslant m .$$

Indeed if $L_j u = f_j$ is analytic, we can solve $L_j v = f_j$ with an analytic solution v.

Since $L_j(u-v) = 0$ it suffices to show the analyticity of $u - v$.

2. We can restrict ourselves to the study of the C^1 solutions of (3). Indeed it can be easily proved [2] that any distribution solution of (3) near (t^o,x^o) is of the form

$$h = \Delta_x^q h'$$

where h' is of class C^1 and also solution of (3).

3. In order to prove the analytic hypoellipticity of $\mathbb{L}$ at (t^o,x^o) it suffices to show that if h is a C^1 solution of (3) near (t^o,x^o) then the function

$$h_o(x) = h(t^o,x) \tag{4}$$

is analytic at x^o. This can be easily seen using Remarks 1, 2 and the fact that the local Cauchy problem $L_j v = 0$, $1 \leqslant j \leqslant m$, with Cauchy datum at $t = t_o$, has a solution in the class of analytic functions and uniqueness holds in the class C^1 functions.

C.R. Functions

Let V be an open set of $\mathbb{R}^n$. We denote

$$\Omega = U \times V.$$

We consider the "tuboïd" of $\mathbb{C}^n$

$$z(\Omega) = V + i\phi(U).$$

Définition 2 : A function u defined on the set $z(\Omega)$ is said to be Lipschitz continuous if its pull-back via z, $\tilde{u} = u \circ z$ is Lipschitz continuous on $\Omega = U \times V$. Moreover u is said to be a C.R. function if $\tilde{u}$ satisfies (3) in $U \times V$.

Observe that the push via z of L_j, $1 \leqslant j \leqslant m$ is given by

$$\sum_{k=1}^{n} (L_j z_k)\frac{\partial}{\partial z_k} + (L_j \bar{z}_k)\frac{\partial}{\partial \bar{z}_k} = -2i \sum_{k=1}^{n} \frac{\partial \phi_k}{\partial t_j}\frac{\partial}{\partial \bar{z}_k} .$$

Therefore if $\phi(U)$ is an immersed submanifold of $\mathbb{R}^n$, a function u is a C.R. function according to Definition 2 if and only if it satisfies the usual induced Cauchy-Riemann equations on $z(\Omega)$.

If f is a holomorphic function in an open neighborhood of $z(\Omega)$ in $\mathbb{C}^n$, clearly its restriction to $z(\Omega)$ is a C.R. function. We are interested here in the following local extendability question : Let $(t^o,x^o) \in \Omega$ and u a C.R. function on $z(\Omega)$ when does u extend holomorphically to a neighborhood of $z(t^o,x^o)$?

We have the following :

<u>Proposition 1</u> : Let u be a C.R. function defined on $z(\Omega)$ and $(t^o,x^o) \in \Omega$. The function u extends holomorphically to a neighborhood of $z(t^o,x^o)$ if and only if the function

$$x \longmapsto \tilde{u}(t^o,x) = u(z(t^o,x))$$

is analytic at x^o.

When ϕ is analytic the analytic hypoellipticity of the system $\mathbb{L}$ defined by (1) and the local holomorphic extendability are therefore equivalent (Prop. 1 and Remark 3).

<u>Theorem 1</u> : Assume ϕ to be analytic. The following conditions are equivalent :

(i) The system $\mathbb{L} = (L_1,\dots,L_m)$ defined by (1) is analytic hypoelliptic at (t^o,x^o).

(ii) Any C.R. function defined on a neighborhood of $z(t^o,x^o)$ in $z(\Omega)$ extends holomorphically to a full neighborhood of $z(t^o,x^o)$ in $\mathbb{C}^n$.

(iii) For every $\xi \in \mathbb{R}^n \setminus 0$, t^o is not a local extremum of the function $t \mapsto \phi(t).\xi$.

Theorem 1 follows essentially from the following microlocal result.

<u>Theorem 2</u> : Assume ϕ to be analytic and let $\xi^o \in \mathbb{R}^n \setminus 0$. The following conditions are equivalent :

(a) For every distribution h defined in some neighborhood of (t^o,x^o) and satisfying (3) (x^o,ξ^o) is not in the analytic wave-front set of h_o (defined by (4)).

(b) t^o is not a local minimum of the function $t \mapsto \phi(t).\xi^o$.

We can assume that (t^o,x^o) is the origin of $\mathbb{R}^m \times \mathbb{R}^n$ and that $\phi(0) = 0$. In order to prove that (a) implies (b) it suffices to observe that if $\phi(t).\xi^o \geq 0$ for all $t \in U$, the function

$$h(t,x) = (x.\xi^o + i\,\phi(t).\xi^o)^{3/2},$$

with the principal determination of $\zeta^{3/2}$ for $\zeta \in \mathbb{C}$ Im $\zeta \geq 0$, satisfies (3) and

$(0, \xi^o)$ is in the analytic wave-front set of $h_o(x) = (x.\xi^o)^{3/2}$.

The proof of (b) $\Rightarrow$ (a) is an easy corollary of the following more general result :

Theorem 3 : Assume ϕ to be Lipschitz continuous in $U(0 \in U)$ and let V be the open ball of $\mathbb{R}^n$ centered at the origin of radius $r > 0$. Let $\xi^o \in \mathbb{R}^n \setminus 0$ and assume there are $t^* \in U \setminus 0$ and a Lipschitz curve γ in U with 0 and t^* as its end-points satisfying :

(5) $$- \phi(t^*).\xi^o > 0,$$

(6) $$\sup_{t \in \gamma} |\phi(t)| < r,$$

(7) $$|\phi(t^*)|^2 \sup_{t \in \gamma} \phi(t).\xi^o < [r^2 - \sup_{t \in \gamma}|\phi(t)|^2] [- \phi(t^*).\xi^o].$$

Then if h is any Lipschitz continuous solution of (3) in $\Omega = U \times V$, $(0,\xi^o)$ is not in the analytic wave-front set of $h_o(x) = h(0,x)$.

Idea of the proof of Theorem 3

Let $\varepsilon > 0$ and $K > 0$ be determined later. Let $g \in C_o^\infty(V)$, $g(x) \equiv 1$ for $|x| \leqslant (1 - \varepsilon)r$. Consider the integral

(8) $$I(x,\xi) = \int_{\mathbb{R}^n} \int_\gamma e^{i(x-y-i\phi(t)).\xi - K(x-y-i\phi(t))^2|\xi|} L[g(y)h(t,y)]dtdy .$$

We have used the notation $z^2 = \sum_{j=1}^n z_j^2$, and

$$Lf(t,y)dt = \sum_{j=1}^m L_j f(t,y)dt_j$$

which is a one form on U depending on y.

Integrating (8) by parts with respect to t and y and using (2) we obtain

(9) $$I(x,\xi) = I_*(x,\xi) - I_o(x.\xi)$$

with

$$I_*(x,\xi) = \int_{\mathbb{R}^n} e^{i(x-y-i\phi(t^*)).\xi - K(x-y-i\phi(t^*))^2|\xi|} g(y)h(t^*,y)dy$$

$$I_o(x,\xi) = \int_{\mathbb{R}^n} e^{i(x-y).\xi - K(x-y)^2|\xi|} g(y)h_o(y)dy .$$

In order to show that $(0,\xi^o)$ is not in the analytic wave front set of h_o, it suffices to show that the estimate

$$|I_o(x,\xi)| \leqslant C\, e^{-|\xi|/C} \tag{10}$$

with $C > 0$, holds for (x,ξ) in a conic neighborhood of $(0,\xi^o)$ (see Sjöstrand [3]). Assumptions (5), (6), (7) and (3) allow us to find $\varepsilon > 0$ and $K > 0$ so that estimates of the form (10) hold for $I(x,\xi)$ and $I_*(x,\xi)$; thus the desired estimate (10) follows from (9).

Other remarks

4. The microlocal results of this paper can yield holomorphic extendability of C.R. functions not only to full neighborhood of a point in $z(\Omega)$ in $\mathbb{C}^n$, but also to open sets of $\mathbb{C}^n$ whose boundary contains part of $z(\Omega)$.

5. It should be mentioned that other extendability results generalizing Bochner's tube theorem appeared in the literature : H. Lewy, Hörmander, Komatsu, Hill, Kazlow (see [1] for references).

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