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Let $l > 0$ and $m > 0$ be integers. Let $P(D)$ be a linear operator in $\mathbb{R}^n$. Let P_m be its principal part. We say that the Cauchy problem

(1) $\quad P(D)u = f,\ u - g = O(x_1^{\ l})$

is uniquely solvable in the class of analytic functions if to each f analytic in $\mathbb{R}^n$ and each g analytic in a neighbourhood of $x_1 = 0$ there is an unique function u analytic in R^n such that (1) is true. We show the following theorem [5].

Theorem 1. The problem (1) is uniquely solvable in the class of analytic functions if and only if $m = l$ and P_m is hyperbolic in the $(1,0,\ldots,0)$ direction.

In the proof we use

Theorem 2. Let $P(D)$ be a linear operator with constant coefficients such that P_m is not hyperbolic in the $(1,0,\ldots,0)$ direction. Then there is a v such that v is analytic in $x_1 > 0$, $P(D)v = 0$ in $x_1 > 0$ and v is not bounded near $x = 0$.

The proof of Theorem 2 makes use of

Theorem 3. Let $P(D)$ be a linear operator in $\mathbb{C}^n$ of the form

$$P(D) = D_1^l D_2^{m-l} + \sum_{\substack{|\alpha|=m \\ \alpha_1 = l}} a_\alpha D^\alpha + \sum_{|\alpha|<m} a_\alpha D^\alpha$$

with $0 \leq l < m$.

Then there is a function v holomorphic when $z_1 \notin (-\infty,0]$ such that

$$P(D)v = 0,\ v(z_1,0) = z_1^{-1},\ z_1 \notin (-\infty,0].$$

Hadamard's example with $u = n^{-1}\sin nx_2 \sinh nx_1$ shows that the Cauchy problem for the Laplace equation is not uniquely solvable in C^∞. The function $u = (1 - x_1 + ix_2)^{-1}$ shows that this is also the case in the smaller class of analytic functions.

Theorem 2 is a generalization of this example to general operators.

We like to remark that the "if" part of Theorem 1 is due to J.-M. Bony and P. Schapira [1].

As another application of Theorem 2 we prove

Theorem 4. Let $P(D)$ be an operator with constant coefficients in $\mathbb{R}^n$. Let ω and Ω be open convex sets in $\mathbb{R}^n$ such that $\omega \subset \Omega$. Then the following two conditions are equivalent.

a) Let u be analytic in ω and assume that $P(D)u$ can be continued analytically to Ω . Then u can be continued to a function analytic in Ω .

b) Every hyperplane intersecting Ω but not ω has a normal hyperbolic with respect to P_m .

Proof. If follows from [1, Thēoreme 4.2, p. 88-89] that b) implies a). Here we notice that the set of hyperbolic directions is open when the coefficients are constant. See [3, Lemma 5.5.1, p. 133].

Assume that there is a hyperplane H with non-hyperbolic normal with respect to P_m such that $H \cap \Omega \neq \emptyset$ and $H \cap \omega = \emptyset$. We rotate and translate the coordinate system such that $H = \{x; x_1 = 0\}$, $\omega \subset \{x; x_1 > 0\}$, $0 \in \Omega$. Then we choose u from Theorem 2 and get a u analytic in ω and fulfilling $P(D)u = 0$ there. But u cannot be continued analytically to Ω . The theorem is proved.

A local version of Theorem 3 for operators with holomorphic coefficients in $\mathbb{C}^n$ can be found in [4, Theorem 4.1]. We may also notice that a refinement of the technique in [4] has been used to prove an existence theorem for the non-characteristic Cauchy problem when data are singular. See J. Persson [6]. A similar but much more complicated technique has been used on the same problem by Y. Hamada, J. Leray and C. Wagschal [2].

References.

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