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François Trèves

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INTEGRAL REPRESENTATION OF SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS, II

par

François TRÈVES

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0. - INTRODUCTION.

The present article studies a first-order linear partial differential operator L, with C^∞ coefficients, nondegenerate, which satisfies the solvability condition (P) (of [1], [2]; see (1.3)). If U is a sufficiently small open set (in the manifold where L is defined and has the above properties), we exhibit an integral operator G, akin to Fourier integral operators (his not quite one), such that

(1) L G f = f in U,

for any $f \in C_c^\infty(U)$. This generalizes part of the results proved in [5] when the coefficients are analytic. However, unlike in the analytic case, I have not succeeded in constructing a right-inverse G to L which maps $C_c^\infty(U)$ into $C^\infty(U)$. Rather, given any positive integer m, one can choose the neighborhood U and adapt the construction so as to insure that G maps $C_c^\infty(U)$ into $C^m(U)$. Thus, except in certain particular cases (see end of Section 1), the problem of the C^∞ solvability of the equation Lu = f is still open. Nevertheless the extension of the partial result from analytic to non analytic coefficients requires some modifications which are not quite straight forward, and seem to warrant publication.

Notation and definitions are those of [5], to which we refer.

1. STATEMENT OF THE RESULT AND PRELIMINARY REDUCTION.

Let Ω be an open subset of $\mathbb{R}^N$, containing the origin, where the coefficients of L are defined and smooth (i.e. C^∞). We suppose that the coordinates in $\mathbb{R}^N$, which we systematically denote by $(x^1, \dots, x^n, t)$, setting $n = N-1$, are such that

$$(1.1)\quad L = \zeta(x,t)\{\frac{\partial}{\partial t} + i\sum_{j=1}^{n} b^j(x,t)\frac{\partial}{\partial x^j} + c(x,t)\}$$

where ζ does not vanish at any point of Ω, and the b^j are real valued (ζ, the b^j and c are C^∞ functions in Ω). To put L in the form (1.1) is always possible, provided that Ω is small enough (thanks to the hypothesis that L is nondegenerate). We deal with open neighborhoods of the origin in $\mathbb{R}^{n+1}$, of the form

$$(1.2)\quad U(T) = \{(x,t)\ ;\ x \in U_0,\ |t| < T\},$$

where U_0 is an open neighborhood of 0 in $\mathbb{R}^n$. We assume that $\overline{U(T)} \subset \Omega$ and is compact. We make the hypothesis that Condition (P) is satisfied in Ω, in the following strong sense :

$$(1.3)\quad \vec{b}(x,t) = |\vec{b}(x,t)|\vec{v}(x), \qquad (x,t) \in \Omega,$$

where $\vec{b} = (b^1, \dots, b^n)$. Of course, $\vec{v}$ is a unit vector, unambiguously defined for a given x if there is some $t \in \mathbb{R}$ such that $(x,t) \in \Omega$ and $|\vec{b}(x,t)| \neq 0$.

THEOREM 1.1. - Under Hypothesis (1.3), to every non negative integer m there is a number $T_m > 0$ and a continuous linear operator

$$(1.4)\quad G_m : C_c^\infty(U(T_m)) \to C^m(U(T_m)) \to C^m(U(T_m))$$

such that $LG_m = I$, identity of $C_c^\infty(U(T_m))$.

As it is stated, Th. 1.1 follows essentially from the results of [2], [3]. The novelty in the present approach lies in the construction, and the resulting integral expression of the operators G_m.

We may of course assume that the multiplicative factor ζ is identically equal to one, and write

$$(1.5)\quad L = L_0 + c(x,t),$$

with

$$(1.6)\quad L_0 = \frac{\partial}{\partial t} + i\,\vec{b}(x,t)\cdot\frac{\partial}{\partial x}$$

(stands for the gradient operator).

We begin by applying Th. 2.1 of [5] (in a slightly more precise form). Let us introduce the function in $U = U(T)$,

$$(1.7)\quad \rho(x,t) = \int_{-T}^{t} |\vec{b}(x,s)|ds.$$

We say that a C^∞ function f in U is ρ-flat if, given any integer $M \geq 0$ and any linear partial differential operator P with C^∞ coefficients in U, $\rho^{-M}Pf$ is a continuous function in U. We denote by $\varepsilon_{\rho\text{-flat}}(U)$ the space of those functions, by $\mathcal{D}_{\rho\text{-flat}}(U)$ the subspace made up by those with compact support.

LEMME 1.1. - If $T > 0$ is small enough, there is a continuous linear map E : $C_c^\infty(U(T)) \to C^\infty(U(T))$ such that $R = LE - I$ maps $C_c^\infty(U(T))$ into $\varepsilon_{\rho\text{-flat}}(U(T))$.

Note that the open set U_0 is left unchanged ; that this can be is easily seen on the proofs of Theorems 1 of [4], and of 2.1 of [5], to which we refer.

Remark 1.1. - The proof of Th. 2.1. [5] yields an operator E which resembles a Fourier integral operator (with respect to the variables x) with complex phase, but which is not really one. As it is written in [5] it does not act on arbitrary distributions, only on functions whose Fourier transform have a certain rate of decay at infinity. It can be extended to functions with a finite degree of regularity (and with compact support), which it transforms into functions with a smaller degree of regularity. The same is then true for the "error term" R which furthermore introduces a certain degree of ρ-flatness, related to the degree of regularity of the functions on which it acts.

Lemma 1.1 reduces the construction of the operator G of Th. 1.1 to that of an operator

(1.8) $\mathcal{K}_m : \mathcal{D}_{\rho\text{-flat}}(U(T_m)) \to C^m(U(T_m))$,

such that

(1.9) $L\mathcal{K}_m = I$, identity of $\mathcal{D}_{\rho\text{-flat}}(U(T_m))$.

We take then

(1.10) $G_m f = Ef - \mathcal{K}_m(\psi Rf)$,

where $\psi \in C_c^\infty(U(T_m))$ is equal to one in a relatively compact open subneighborhood $U'_0 \times]-T'_m , T'_m[$ of $U(T_m)$.

In the forthcoming argument of foremost importance will be the behaviour of the vector $\vec{b}(x,t)$ (cf. (1.3)) near the "critical set"

(1.11) $\mathcal{N}_0 = \{x \in U_0 \; ; \; \forall t \in [-T,T], \; b(x,t) = 0\}$.

As a matter of fact, by pushing further the analysis begun here, one could construct the operator G_m of Th. 1.1 in such a way that, whatever $f \in C_c^\infty(U(T_m))$, $G_m f$ is a C^∞ function of (x,t) for $t \notin \mathcal{N}_0$. We have decided not to include in the present article the proof of this property, so as not to increase excessively its length. Note that the property in question has the implication that $G_m f \in C^\infty(U(T_m))$ whatever $f \in C_c^\infty(U(T_m))$ whenever <u>the critical set $\mathcal{N}_0$ is empty</u>. Actually one can prove that G_m maps $C_c^\infty(U(T_m))$ into $C^\infty(U(T_m))$ if there is a factorization of b of the kind

$$(1.12) \qquad \vec{b}(x,t) = \lambda(x,t)\vec{w}(x),$$

where $\lambda \in C^\infty(U(T))$, $\lambda \geq 0$, and $\vec{w} \in C^\infty(U_0; \mathbb{R}^n)$. A non trivial case in which (1.12) holds is that where we have:

$$(1.13) \qquad \forall\, x \in \mathcal{N}_0,\ \exists t,\ |t| < T,\ \partial_x \vec{b}(x,t) \neq 0$$

(notice that $\partial_x \vec{b}(x,t)$ is an n x n matrix ; (1.13) does not require it to be invertible, merely not to vanish).

II. <u>THE FUNCTION</u> r <u>AND THE PARAMETRIZATION OF THE ORBITS OF</u> $\vec{v}$

Every point $x_0 \in U_0 \backslash \mathcal{N}_0$ has an open neighborhood $\mathcal{O}_0$ such that $|\vec{b}(x,t_0)| \neq 0$ for all $x \in \mathcal{O}_0$ and some fixed t_0, $|t_0| < T$. This implies that $\vec{v}(x) = \vec{b}(x,t_0)/|\vec{b}(x,t_0)|$ is C^∞ in $\mathcal{O}$. Furthermore, if $\mathcal{O}_0$ is small enough, we have $|v^j(x)| \geq 1/2n$ in $\mathcal{O}_0$, for some index j, $1 \leq j \leq n$. This implies that $|\vec{b}(x,t)| = b^j(x,t)/v^j(x)$ is C^∞ in $\mathcal{O} \times]-T,T[$. Thus:

PROPOSITION 2.1. - <u>The mappings</u> $x \to \vec{v}(x)$, of $U_0 \backslash \mathcal{N}_0$ <u>into</u> $\mathbb{R}^n$, <u>and</u> $(x,t) \to |\vec{b}(x,t)|$, <u>of</u> $(U_0 \backslash \mathcal{N}_0) \times]-T,T[$ <u>into</u> $\mathbb{R}$, <u>are</u> C^∞.

We shall consider the <u>orbits</u> of $\vec{v}$ in $U_0 \backslash \mathcal{N}_0$, that is the <u>maximal</u> <u>connected</u> integral curves of $\vec{v}$ in $U_0 \backslash \mathcal{N}_0$ according to Prop. 2.1 they are smooth curves. With such an orbit Γ we associate the cylinder

$$\Sigma = \Gamma \times]-T,T[\ .$$

If $x \in U_0 \backslash \mathcal{N}_0$ lies on Γ we often write Γ_x and Σ_x. We are going to select a convenient parametrization of the curves Γ, which, among other properties, will have the effect that the parametric distance between two points x and x_0 of Γ will grow to infinity whenever x nears the boundary of $U_0 \backslash \mathcal{N}_0$ while x_0 stays away from it. This is achieved by means of a strictly positive C^∞ function r(x) in $U_0 \backslash \mathcal{N}_0$ which tends to zero at the boundary, $\mathcal{N}_0 \cup \complement U_0$, at a suitably fast rate. Actually it is convenient to assume that $\vec{b}(x,t)$ vanishes identically for $x \notin U_0$, which can always be achieved after multiplication of $\vec{b}(x,t)$ by a cut-off function $g(x) \in C_c^\infty(\mathbb{R}^n)$, g = 0 in $\complement U_0$, g > 0 in U_0, g = 1 in an arbitrary relatively compact open subset U'_0 of U_0 (containing the origin). This of course modifies L outside U'_0, but the latter is as arbitrary as U_0 was (U_0 was solely submitted to the requirement that $\overline{U} \subset (\Omega)$.

Then we may introduce the function (in $\mathbb{R}^n$):

(2.1) $\quad r^*(x) = \sup_{|t|\le T} |\vec{b}(x,t)|,$

which is uniformly Lipschitz continuous ; $r^*(x) > 0$ if and only if $x \in U_0 \setminus \mathcal{N}_0$. With this function we may associate a Whitney partition of unity in $U_0 \setminus \mathcal{N}_0$ in the manner of [3], Appendix. Its elements ζ_j ($j = 1, 2, \ldots$) are non negative and have the following important properties:

(2.2) there is an integer $\nu \ge 1$ such that $\bigcap_{j\in J} \operatorname{supp} \zeta_j = \emptyset$ if Card $J > \nu$;

(2.3) to every $\alpha \in \mathbb{Z}_+^n$ there is $C_\alpha > 0$ such that (in $U_0 \setminus \mathcal{N}_0$)

$$\sum_{j=1}^{n} |\partial_x^\alpha \zeta_j| \le C_\alpha (r^*)^{-|\alpha|};$$

(2.4) $$\sup_{\zeta_j(x)\ne 0} r^*(x) \le 2 \inf_{\zeta_j(x)\ne 0} r^*(x), \forall j=1, 2, \ldots$$

For each $j = 1, 2, \ldots$, we select arbitrarily a point x_j in supp ζ_j and set

(2.5) $\quad r(x) = \sum_{j=1}^{\infty} r^*(x_j)\, \zeta_j(x),\ x \in U_0 \setminus \mathcal{N}_0.$

Clearly $r \in C^\infty(U_0 \setminus \mathcal{N}_0)$ and by virtue of (2.3) and (2.4),

(2.6) $\quad \frac{1}{2} r^* \le r \le 2r^*,$

(2.7) $\quad |\partial_x^\alpha r| \le 2\nu C_\alpha (r^*)^{1-|\alpha|} < 2^{|\alpha|} \nu C_\alpha r^{1-|\alpha|}$

If $x_0 \in U_0 \setminus \mathcal{N}_0$ there is t_0, $|t_0| \le T$, such that $|\vec{b}(x_0,t_0)| = \sup_{|t|\le T} |\vec{b}(x,t)|$. In a sufficiently small neighborhood $\mathcal{O}_0$ of x_0 we may write $\vec{v}(x) = \vec{b}(x,t_0)/|\vec{b}(x,t_0)|$. By differentiating with respect to x and putting $x=x_0$ in the result we obtain at once

(2.8) $\quad \alpha \in \mathbb{Z}_+^n$, $\exists\, C_\alpha > 0$ such that, in $U_0 \setminus \mathcal{N}_0$

$$|\partial_x^\alpha \vec{v}| \le C_\alpha (r^*)^{-|\alpha|} \le 2^{|\alpha|} C_\alpha r^{-|\alpha|}.$$

If we combine (2.7) and (2.8) and increase C_α we obtain

(2.9) $\quad |\partial_x^\alpha [r(x)\vec{v}(x)]| \le C_\alpha r(x)^{1-|\alpha|},\ x \in U_0 \setminus \mathcal{N}_0.$

In particular, $r\vec{v}$ is uniformly Lipschitz continuous in $U_0 \setminus \mathcal{N}_0$. Therefore the solution $x = x(\chi, x_0)$ of the problem

(2.10) $\quad \frac{dx}{d\chi} = r(x)\vec{v}(x),\ x|_{\chi=0} = x_0 \in U_0 \setminus \mathcal{N}_0$

exists for all $\chi \in \mathbb{R}$ and defines a local diffeomorphism of $\mathbb{R}$ onto the orbit Γ_{x0}. Of course it need not be a global diffeomorphism, as we see when Γ_{x0} is "closed", i.e., when $x(\chi, x_0)$ is periodic with respect to χ. We have (as we see by the Picard iteration method):

(2.11) $\quad |x-x_0| \le r(x)\ |e^{C|\chi|}-1|.$

Thus, if we keep x_0 fixed and let x go to the boundary of $U_0\backslash\mathcal{N}_0$, which imply that $r(x) \to 0$ we must have $|\mathcal{X}| \to +\infty$.

The "inversion" of the local diffeomorphism $\mathcal{X} \to x(\mathcal{X}, x_0)$ defines $\mathcal{X} = (x, x_0)$ as a C^∞ function on the universal covering of Γ_{x_0}; it is the solution of the ordinary differential equation

(2.12) $$r(x) \sum_{j=1}^{n} v^j(x) \frac{\partial \mathcal{X}}{\partial x^j} = 1,$$

with "initial" condition

(2.13) $$\mathcal{X}|_{x=x_0} = 0.$$

We recall the standard relation

(2.14) $$\mathcal{X}(x,x_0) + \mathcal{X}(x_0,x_1) = \mathcal{X}(x,x_1),$$

if x, x_0, x_1 belong to the same orbit Γ; (2.14) implies

(2.15) $$\mathcal{X}(x_0,x) = - \mathcal{X}(x,x_0).$$

By vitue of (2.12) we have the right to use the notation

(2.16) $$\frac{\partial}{\partial \mathcal{X}} = r(x) \sum_{j=1}^{n} v^j(x) \frac{\partial}{\partial x^j},$$

and we shall do so in the future. Note, for future reference, that (2.7) implies:

(2.17) $$\left|\frac{\partial r}{\partial \mathcal{X}}\right| \leq C\ r$$

on each orbit Γ of $\vec{v}$; the constant C is independent of the orbit.

III. APPROXIMATE PHASE AND AMPLITUDE FUNCTIONS

We are going to define an operator $\mathcal{K}$ acting on elements f of $\mathcal{D}_{\rho\text{-flat}}(U(T))$ by a formula

(3.1) $$\mathcal{K}f(x,t) = \iint K(x,t,x',t')\ f(x',t')d\mathcal{X}'\ dt',$$

where K(x,t,x',t') is a kernel in $\Sigma_x \times \Sigma_x$ (i.e., is defined only for x, x' belonging to one and the same orbit, Γ_x), and $d\mathcal{X}'$ is the measure on Γ_x defined by $\mathcal{X}' = \mathcal{X}(x,x')$. The operator $\mathcal{K}$, after one last modification, will become the songht operator $\mathcal{K}_m$ of (1.8)-(1.9). We begin to describe the kernel . In doing so the model we have in mind is a special <u>fundamental kernel</u> of the operator $\frac{\partial}{\partial t} + i \frac{\partial}{\partial x}$ in $\mathbb{R}^2$, specifically

(3.2) $$\frac{1}{2\pi i} \frac{e^{-z^2}}{z}, \quad z = x-x'-i(t-t').$$

The reason for the exponential factor e^{-z^2} is the same as in [5] ; it lies in the need to handle on a uniform manner the various kinds of orbits of Γ, periodic, almost-periodic, flowing to the boundary of $U_0\backslash\mathcal{N}_0$, which might occur.

For a general operator L restricted to the two-leaf Σ_x the fonction z of (3.2) should be replaced by "the" solution of the Cauchy problem

(3.3) $\quad L_0 Z = 0, \qquad Z|_{t=t'} = \chi(x,x'),$

except, of course, that for arbitrary C^∞ coefficients, we shall not be able to solve (3.3) exactly. This will generate an error whose "absorption" will force us to modify $\mathcal{H}$ in order to get $\mathcal{H}_m$.

Before pursuing the description of the kernel K in (3.1) we state the lemma about Problem (3.3) which we shall need later. Let us set:

(3.4) $\quad Z = \chi(x,x') + \varphi(x,t,t').$

The function φ must be (approximately) a solution of

(3.5) $\quad \frac{\partial\varphi}{\partial t} + i \sum_{j=1}^{n} b^j(x,t)\frac{\partial\varphi}{\partial x^j} + i\,\frac{|\vec{b}(x,t)|}{r(x)} = 0,$

(3.6) $\quad \varphi|_{t=t'}$.

In (3.5)-(3.6) $\chi(x,x')$, hence x' and the orbit Γ_x have disappeared. We may try to solve (approximately) this Cauchy problem for x ranging freely in $U_0 \backslash \mathcal{N}_0$.

LEMMA 3.1. - Let $\vec{b}(x,t)$ satisfy the hypotheses of Th. 1.1. There is a C^∞ function $\varphi(x,t,t')$ in the region

(3.7) $\quad x \in U_0\backslash\mathcal{N}_0,\ |t| < T,\ |t'| < T,$

satisfying , for these (x,t,t'), the equations

(3.8) $\quad \frac{\partial\varphi}{\partial t} + i\vec{b}(x,t).\frac{\partial\varphi}{\partial x} + i\,\frac{|\vec{b}(x,t)|}{r(x)} = H(x,t,t'),$

(3.9) $\quad \varphi|_{t=t'} = 0,$

with H having the following properties :

(3.10) to every α in $\mathbb{Z}_+^n$, ℓ, ℓ', M in $\mathbb{Z}_+$ there is a positive constant $C = C(\alpha, \ell, \ell', M) > 0$ such that, for all (x,t,t') in the set (3.7),

(3.11) $\quad |\partial_x^\alpha \partial_t^\ell \partial_t^{\ell'} H(x,t,t')| \le C\, r(x)^{-|\alpha|-\ell-\ell'} \left| \int_{t'}^{t} \frac{|\vec{b}(x,s)|}{r(x)}\, ds \right|^M .$

Furthermore, to every α, ℓ, ℓ', there is $C_{\alpha,\ell,\ell'} > 0$ such that, in the set (3.7),

(3.12) $$|\partial_x^\alpha \partial_t^\ell \partial_{t'}^{\ell'} \varphi(x,t,t')| \begin{cases} \le C_{\alpha,\ell,\ell'}\, r(x)^{-|\alpha|-\ell-\ell'-1} & \text{if } \ell+\ell' > 0 \\ \le C_{\alpha,0,0}|t-t'|\, r(x)^{-|\alpha|} & \text{if } \ell+\ell' = 0 \end{cases}$$

The proof of Lemma 3.1, which is rather technical, has been postponed to the Appendix. But we draw right now some of its consequences :

Corollary 3.1. - For some C > 0 and all (x,t,t') in (3.7)

(3.13) $\quad |\varphi(x,t,t') + i \int_{t'}^{t} \frac{|\vec{b}(x,s)|}{r(x)}\, ds| \le C\, |t-t'| \left| \int_{t'}^{t} \frac{|\vec{b}(x,s)|}{r(x)}\, ds \right| .$

Proof. - Integrate (3.8) from t' to t, taking (3.9) into account, then apply (3.11) and the fact, deriving from (3.12), that $|\partial_x \varphi| \leq \text{const.}\ |t-t'|/r(x)$.

Corollary 3.2. - In (3.7) (and with the notation (2.16))

$$(3.14)\quad \left|\frac{\partial\varphi}{\partial\chi}\right| \leq \text{const.}\ |t-t'|.$$

Immediate consequence of (3.12).

Corollary 3.3. - To every α in $\mathbb{Z}_+^n$, ℓ, ℓ' in $\mathbb{Z}_+$ such that $\ell+\ell' > 0$ there are constants $C_\alpha > 0$, $C_{\alpha,\ell,\ell'} > 0$ such that, in (3.7),

$$(3.15)\quad \left|\partial_x^\alpha\left(\frac{\partial\varphi}{\partial\chi}\right)\right| \leq C_\alpha\ |t-t'|\ r(x)^{-|\alpha|},$$

$$(3.16)\quad \left|\partial_x^\alpha \partial_t^\ell \partial_{t'}^{\ell'}\left(\frac{\partial\varphi}{\partial\chi}\right)\right| \leq C_{\alpha,\ell} r(x)^{-|\alpha|-\ell-\ell'-1}.$$

Combine (2.9) with (3.12).

Due to the fact that, in general, the operator L under study is not equal to its leading part, L_0, we need also an amplitude function (in dealing with $\frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + c(x,t)$ the kernel to use is not (3.2) : we must multiply (3.2) by a function $k = k(x,t,x',t')$ such that $\frac{\partial k}{\partial t} + i\frac{\partial k}{\partial x} + ck = 0$, $k(x',t',x',t') = 1$). Here we shall solve approximately

$$(3.17)\quad [L_0 + c(x,t)]k = 0, \qquad k|_{t=t'} = 1 .$$

Because of our choice of the condition at t=t' we may take $k = k(x,t,t')$ independent of x' (and of the orbit Γ_x). Actually we apply directly Th. 1 of [4] and state:

LEMMA 3.2. - There is a C^∞ function $k(x,t,t')$ in the region

$$(3.18)\quad x \in U_0, \quad |t| < T, \quad |t'| < T',$$

satisfying there

$$(3.19)\quad \frac{\partial k}{\partial t} + i\,\vec{b}(x,t).\frac{\partial k}{\partial x} + c(x,t)k = h(x,t,t'),$$

$$(3.20)\quad k|_{t=t'} = 1,$$

where $h(x,t,t')$ is $\left|\int_{t'}^{t} |\vec{b}(x,s)|ds\right|$ - flat (cf. p. 1-03).

IV. SOLUTION ON THE INDIVIDUAL LEAVES

We proceed with the definition of the operator $\mathcal{K}$ in (3.1). The next step mimicks what can be done with the kernel (3.2), namely that one may write (setting $Z = x-x'-i(t-t')$),

$$(4.1)\quad \frac{1}{2\pi}\iint_{\mathbb{R}^2} \frac{e^{-z^2}}{z} f(x',t')dx'\,dt' = \frac{1}{2\pi}\int_{-\infty}^{t}\int_{\theta=-\infty}^{0}\int_{\mathbb{R}} e^{i\theta z - z^2} f(x',t')dx'\,d\theta\,dt'$$

$$- \frac{1}{2\pi} \int_{t}^{+\infty} \int_{\theta=0}^{+\infty} \int_{\mathbb{R}} e^{i\theta Z - Z^2} f(x',t')dx'\, d\theta\, dt'.$$

Thus we take

(4.2) $\mathcal{K} = \mathcal{K}_+ - \mathcal{K}_-$,

(4.3) $$\begin{cases} \mathcal{K}_+ f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{t} \int_{\theta=-\infty}^{0} \mathcal{F}(x,t;\theta,t')d\theta\, dt', \\ \mathcal{K}_- f(x,t) = \frac{1}{2\pi} \int_{t}^{+\infty} \int_{0}^{+\infty} \mathcal{F}(x,t;\theta,t')d\theta\, dt', \end{cases}$$

where we use the notation

(4.4) $$\mathcal{F} = \int_{-\infty}^{+\infty} e^{i\theta Z - Z^2} k(x,t,t') f(x',t')d\mathcal{X}',$$

with Z given by (3.4) (in (3.4) we take φ as in Lemma 3.1) and k as in Lemma 3.2. By (3.13) $|\mathrm{Im}\,\varphi| = |\mathrm{Im}\, Z| \leq |t-t'| + C|t-t'|^2$, hence the integral (4.4) converges.

We have, according to (3.4), (3.9) and (3.20):

$$\mathcal{F}(x,t;\theta,t) = \int_{-\infty}^{+\infty} e^{i\theta\mathcal{X}' - \mathcal{X}'^2} f(x',t)d\mathcal{X}',$$

and thus:

$$L\mathcal{K}f = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{i\theta\mathcal{X}' - \mathcal{X}'^2} f(x',t)d\mathcal{X}'d\theta + Sf,$$

where

(4.5) $S = S_+ - S_-$,

(4.6) $$\begin{cases} S_+ f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{t} \int_{-\infty}^{0} L\mathcal{F}(x,t;\theta,t')d\theta dt', \\ S_- f(x,t) = \frac{1}{2\pi} \int_{t}^{+\infty} \int_{0}^{+\infty} L\mathcal{F}(x,t;\theta,t')d\theta dt'. \end{cases}$$

But

$$\frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{i\theta\mathcal{X}' - \mathcal{X}'^2} f(x',t')d\mathcal{X}'d\theta = e^{-\mathcal{X}'^2} f(x',t)\Big|_{\mathcal{X}'=0} = f(x,t),$$

and thus, with the notation (4.5)-(4.6),

(4.7) $L\mathcal{K}f = f + Sf.$

We have, in connection with (4.6),

(4.8) $$L\mathcal{F} = \int_{-\infty}^{+\infty} e^{i\theta Z - Z^2} \sigma f(x',t')d\mathcal{X}',$$

where $\sigma = \sigma(x,t,x',t',\theta)$ is defined by taking (3.8) and (3.19) into account:

(4.9) $\sigma = (i\theta - 2Z)H + h.$

The last step in the proof consists of showing that, if the length T of the time interval is small enough, S defines a bounded linear operator in a suitable Banach space of fuctions, with a norm so small that I + S can be inverted -and that

$\mathfrak{K}_m = \mathfrak{K}(I+S)^{-1}$ possesses the properties we are looking for.

V. EXISTENCE AND ACROSS - THE - BOARD. REGULARITY OF THE SOLUTIONS

In this section we explain how to prove the estimates about $\mathfrak{K}$, defined in (4.2)-(4.3), and S, defined in (4.5)-(4.6), which we need to complete the proof of Th. 1.2.

We shall need some notation. Let us set

(5.1) $\quad \mathcal{N} = \mathcal{N}_0 \times]-T,T[$,

and for $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, $(x,t) \in U \backslash \mathcal{N}$,

(5.2) $\quad N_{p,q}(f;x,t) = r(x)^{-p} \sum_{|\alpha|+\ell \leq q} r(x)^{|\alpha|+\ell} \, |\partial_x^\alpha \partial_t^\ell f(x,t)|$.

Furthermore we denote by $\mathcal{B}_{p,q}(U,r)$ the space of C^q functions f in $U\backslash\mathcal{N}$ such that

(5.3) $\quad N_{p,q}(f) = \sup_{U\backslash\mathcal{N}} N_{p,q}(f;x,t) < +\infty$,

equipped with the norm $N_{p,q}$ (which obviously turns it into a Banach space).

In the integral (4.4) x' must be regarded as a function of $x \in U_0 \backslash \mathcal{N}_0$ and of $\chi' \in \mathbb{R}$, the one defined by

(5.4) $\quad \frac{\partial x'}{\partial \chi'} = r(x')\,\vec{v}(x')$, $\quad x'\big|_{\chi'=0} = x$.

Observe that we can differentiate this initial value problem with respect to x and to χ'. In particular:

(5.5) $\quad \frac{\partial}{\partial \chi'}\left(\frac{\partial x'}{\partial x}\right) = [\partial_x(r\vec{v})](x')\left(\frac{\partial x'}{\partial x}\right)$, $\frac{\partial x'}{\partial x}\Big|_{\chi'=0} = I$.

By repeated differentiations, with respect to x and to χ', by applying (2.9) and using induction on the order of differentiations one easily obtains the following:

LEMMA 5.1. - <u>To every</u> $\alpha \in \mathbb{Z}_+^n$, $j \in \mathbb{Z}_+$ <u>there is a constant</u> $C_{\alpha,j} > 0$ <u>such that</u>

(5.6) $\quad |\partial_x^\alpha \partial_{\chi'}^j x'| \leq C_{\alpha,j}\, r(x')^{1-|\alpha|-j}$

<u>whatever</u> $x \in U_0 \backslash \mathcal{N}_0$ <u>and</u> $\chi' \in \mathbb{R}$.

On the other hand, by applying Gronwall's inequality to (2.17) we obtain at once

LEMMA 5.2. - <u>There is a constant</u> C_0 <u>independent of</u> x <u>and</u> of χ' <u>such that</u>

(5.7) $\quad e^{-C|\chi'|} \leq \frac{r(x')}{r(x)} \leq e^{C|\chi'|}$.

We recall that $x' = x'(x,\chi')$, $\chi' = \chi(x,x')$.

We may exploit Lemma 5.1 to get:

$$(5.8)\quad |\partial_x^\gamma[f(x',t')]| \leq C_\gamma \sum_{0\neq\delta\leq\gamma} r(x')^{-|\gamma-\delta|}\,|(\partial_x^\delta f)(x',t')|.$$

We begin by proving an estimate concerning S.

PROPOSTION 5.1. - Given any p, q in $\mathbb{Z}$, $q \geq 0$, there is $C(p,q)>0$ such that, for every (x,t) in $U\backslash\mathcal{X}$,

$$(5.9)\quad N_{p,q}(Sf;x,t) \leq C(p,q)T \sup_{(x',t')\in\Sigma_x} N_{p,q}(f;x',t').$$

Proof. - The error terms H and h vanish of infinite order when t=t', and so does therefore σ defined in (4.9). This implies that, whatever $\alpha \in \mathbb{Z}_+^n$ and $\ell \in \mathbb{Z}^+$,

$$(5.10)\quad \partial_x^\alpha\partial_t^\ell S_+f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{t}\int_{-\infty}^{0}\partial_x^\alpha\partial_t^\ell\, L\mathcal{F}(x,t\;;\;\theta,t')d\theta\, dt',$$

and likewise with S_- in place of S_+. We have

$$(5.11)\quad \partial_x^\alpha\partial_t^\ell L\mathcal{F} = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\int_{-\infty}^{+\infty} e^{i\theta Z-Z^2}\,\sigma_{\beta,\ell}\,\partial_x^{\alpha-\beta}[f(x',t')]\,d\mathcal{X}'\;.$$

We have written

$$(5.12)\quad \sigma_{\beta,\ell} = e^{-i\theta Z+Z^2}\,\partial_x^\beta\partial_t^\ell(e^{i\theta Z-Z^2}\sigma).$$

We apply (3.11), (3.13), (3.19) : given any M and ℓ in $\mathbb{Z}_+$, β in $\mathbb{Z}_+^n$, there is $C_{M;\beta,\ell} > 0$ such that (for $|\theta| \geq 1$),

$$(5.13)\quad |\sigma_{\beta,\ell}| \leq C_{M;\beta,\ell}|\theta|\left\{\frac{|\theta|+|\mathcal{X}'|}{r(x)}\right\}^{|\beta|+\ell}|\operatorname{Im}\varphi|^M.$$

Thus, for a suitable constant $C' > 0$, (using (5.8)) we derive from (5.11)

$$(5.14)\quad r(x)^{-p+|\alpha|+\ell}\,|\partial_x^\alpha\partial_t^\ell L\mathcal{F}| \leq C'\int_{-\infty}^{+\infty} e^{-\theta\operatorname{Im}\varphi-\mathcal{X}'^2}\,(|\theta|+|\mathcal{X}'|)^{|\alpha|+\ell}\,|\operatorname{Im}\varphi|^M$$

$$\times\left[\frac{r(x)}{r(x')}+1\right]^{|\alpha|-p} N_{p,|\alpha|}(f\;;\;x',t')\;d\mathcal{X}'.$$

By (3.13) we know that $\operatorname{Im}\varphi$ has the sign of $-(t-t')$, hence $\theta\operatorname{Im}\varphi = |\theta\operatorname{Im}\varphi|$ on the domains of integration in (4.3) and (4.6). By applying (5.7) we draw from (5.14):

$$(5.15)\quad r(x)^{-p+|\alpha|+\ell}|\partial_x^\alpha\partial_t^\ell L\mathcal{F}| \leq C''|\theta|^{|\alpha|+\ell+1-M}\int_{-\infty}^{+\infty} e^{-\mathcal{X}'^2+C|p-|\alpha||\,|\mathcal{X}'|}N_{p,|\alpha|}(f;x',t')d\mathcal{X}$$

We take $M \geq q+\ell+3$, $q \geq |\alpha|$. By virtue of (5.15), (5.11) yields easily (5.9).

COROLLARY 5.1. - If $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, S defines a bounded linear operator of $\mathcal{B}_{p,q}(U,r)$ into itself, with norm $\leq C(p,q)T$.

COROLLARY 5.2. - If $T < C(p,q)^{-1}$, $I + S$ is an automorphism of $\mathcal{B}_{p,q}(U,r)$.

PROPOSITION 5.2. - Given p in $\mathbb{Z}$, q in $\mathbb{Z}_+$, there is $C(p,q) > 0$ such that, whatever (x,t) in $U\backslash\mathcal{N}$,

$$(5.16)\quad N_{p,q}(\mathcal{K}f;x,t) \leq C(p,q)T \sup_{(x',t')\in\Sigma_x} N_{p,q+2}(f;x',t').$$

Proof. - Since $\partial_t\mathcal{K}f = f + Sf - \vec{b}(x,t)\partial_x\mathcal{K}f$, by virtue of Prop. 5.1 we only need to establish estimates for the derivatives of $\mathcal{K}f$ with respect to x. We shall only do it when the order of differentiation does not exceed one. This will give a pretty good idea of what the argument in the general case would be ; the generalization is routine. We have therefore:

$$(5.17)\quad \partial_x\mathfrak{F} = \int_{-\infty}^{+\infty} e^{i\theta Z-Z^2}(k_1 f + kf_x \frac{\partial x'}{\partial x})d\chi'$$

where f stands for $f(x',t')$, f_x for $f_x(x',t')$ and

$$(5.18)\quad k_1 = (i\theta-2Z)Z_x k + k_x.$$

Recalling that $Z = \chi'+\varphi$, $Z_x = \varphi_x$, we see that

$$i\theta\int_{-\infty}^{+\infty} e^{i\theta Z-Z^2} Z_x kf\, d\chi' = -\int_{-\infty}^{+\infty} Z_x k\, e^{i\theta Z} \frac{\partial}{\partial\chi'}(e^{-Z^2} f)\, d\chi'$$

$$= \int_{-\infty}^{+\infty} e^{i\theta Z-Z^2}(2ZZ_x kf - Z_x kf_x \frac{\partial x'}{\partial\chi'})\, d\chi',$$

whence, by (5.17) and (5.18),

$$(5.19)\quad \partial_x\mathfrak{F} = \int_{-\infty}^{+\infty} e^{i\theta Z-Z^2}[k_x f + kf_x(\frac{\partial x'}{\partial x} - \varphi_x \frac{\partial x'}{\partial\chi'})]d\chi'.$$

After an integration by parts we have (for $j \in \mathbb{Z}_+$)

$$(5.20)\quad (-i\theta)^j\, \partial_x\mathfrak{F} = \int_{-\infty}^{+\infty} e^{i\theta Z}\, \partial_{\chi'}^j\{e^{-Z^2}[kf_x + kf_x(\frac{\partial x'}{\partial x} - \varphi_x \frac{\partial x'}{\partial\chi'})]\}d\chi'.$$

We apply (5.6) with j arbitrary and $\alpha=0$ or 1. We also use the fact that $\theta\,\mathrm{Im}\,\varphi \geq 0$ in the domains of integration (in (4.3)), and thus obtain

$$(5.21)\quad |\theta^j|\,|\partial_x\mathfrak{F}| \leq C'_j\int_{-\infty}^{+\infty} e^{-\chi'^2}\,|\chi'|^j \sum_{|\beta|\leq j} r(x')^{-j+|\beta|}[\,|\partial_x^\beta f(x',t')|$$

$$+ |\partial_x^\beta(\partial_x f)(x',t')|\}d\chi'.$$

We multiply both sides in (5.21) by $r(x)^{1-p}$, $p \in \mathbb{Z}$ and obtain:

$$(5.22)\quad r(x)^{-p+1}|\partial_x\mathfrak{F}|\,|\theta^j| \leq C''_j\int_{-\infty}^{+\infty} e^{-\chi'^2+C(|p|+1)|\chi'|}[N_{p,j}(f;x',t')+N_{p,j}(f_x;x',t')]\,d\chi$$

It would have been even easier to derive an estimate

$$(5.23)\quad r(x)^{-p}|\mathfrak{F}|\;|\theta^j| \le C_j'' \int_{-\infty}^{+\infty} e^{-\mathcal{X}'^2+C(|p|+1|)|\mathcal{X}'|} N_{p,j}(f;x',t')d\mathcal{X}'.$$

We always choose j=2 and integrate $|\mathfrak{F}|$ and $|\partial_x\mathfrak{F}|$ with respect to θ. We obtain thus in the cases q=0 and q=1, the inequality:

$$(5.24)\quad r(x)^{-p}\sum_{|\alpha|\le q} r(x)^{|\alpha|}\;|\partial_x^\alpha(\mathcal{K}f)(x,t)| \le C(p)T \sup_{(x',t')\in\Sigma_x} N_{p,q+2}(f;x',t').$$

As we said at the beginning this implies what we wanted.

COROLLARY 5.3. - If $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, $\mathcal{K}$ is a bounded linear operator $\mathcal{B}_{p,q+2}(U,r) \to \mathcal{B}_{p,q}(U,r)$ (with norm $\le C(p,q)T$).

Remark 5.1. - The two-derivative loss in the preceding result should not come as a surprise : it is due to the fact that we have used integration with respect to θ and "maximum normus" with respect to (x,t). No such loss would have occured, had we defined the normus as the supremum, over the collection of all leaves Σ, of the L^2 norm (with respect to $d\mathcal{X}'dt'$) on each individual leaf.

Finally, let $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$ be arbitrary and $TC(p,q) < 1$. We form the inverse $(I + S)^{-1}$ acting on $\mathcal{B}_{p,q+2}(U,r)$, and then the compose

$$(5.25)\quad \mathcal{K}(I+S)^{-1} : \mathcal{B}_{p,q+2}(U,r) \to \mathcal{B}_{p,q}(U,r)$$

We know we have

$$(5.26)\quad L\,\mathcal{K}(I+S)^{-1} = I \text{ (identity of } \mathcal{B}_{p,q+2}(U,r))$$

in $U\backslash\mathcal{N}$. But if $p \ge 0$ and if p and q are large enough we may achieve :

i) that (5.26) be true, not only in $U\backslash\mathcal{N}$, but everywhere in U ;

ii) that $\mathcal{B}_{p,q}(U,r) \subset C^m(\bar{U})$ ($m \in \mathbb{Z}_+$ given in advance).

The proof of Th. 1.1 is complete.

APPENDIX. PROOF OF LEMMA 3.1.

We need Prop. 2.1 with some added precision :

LEMMA A.1. - To every $\alpha \in \mathbb{Z}_+^n$, $\ell \in \mathbb{Z}_+$ there is $C_{\alpha,\ell} > 0$ such that

$$(A.1)\quad \Big|\partial_x^\alpha\partial_t^\ell|\vec{b}(x,t)|\Big| \le C_{\alpha,\ell}\{|\partial_t^\ell\vec{b}(x,t)| + r(x)\}\, r(x)^{-|\alpha|}$$

for all $x \in U_0\backslash\mathcal{N}_0$, $|t| \le T$.

Proof. - In any sufficiently small open subset $\mathcal{O}_0$ of $U_0\backslash\mathcal{N}_0$ one can find an index j, $1 \leq j \leq n$, such that $|v^j(x)| \geq 1/2n$. We have $|\vec{b}(x,t)| = b^j(x,t)/v^j(x)$, whence

$$\partial_x^\alpha \partial_t^\ell |\vec{b}(x,t)| = \partial_t^\ell b^j(x,t)\ \partial_x^\alpha[1/v^j(x)] + \sum_{0\neq\beta\leq\alpha} \binom{\alpha}{\beta}[\partial_x^\beta \partial_t^\ell b^j(x,t)]\ \partial_x^{\alpha-\beta}[1/v^j(x)],$$

and it suffices to apply (2.8), which implies

$$|\partial_x^\gamma[1/v^j(x)]| \leq \tilde{C}_\gamma\ r(x)^{-|\gamma|}.$$

Let us set

$$\text{(A.2)} \quad \omega(x,t) = |\vec{b}(x,t)|/r(x).$$

By (2.6) we recall that

$$\text{(A.3)} \quad |\vec{b}(x,t)| \leq 2r(x),\ x \in U_0\backslash\mathcal{N}_0,\ |t| \leq T.$$

COROLLARY A.1. - If $C_{\alpha,\ell} > 0$ is large enough we have in $U\backslash\mathcal{N}$

$$\text{(A.4)} \quad |\partial_x^\alpha \omega(x,t)| \leq C_{\alpha,0}\ r(x)^{-|\alpha|},$$

$$\text{(A.5)} \quad |\partial_x^\alpha \partial_t^\ell \omega(x,t)| \leq C_{\alpha,\ell}\ r(x)^{-|\alpha|-1} \text{ if } \ell > 0.$$

Remark A.1. - Thanks to Hyp. (P), i.e. (1.3), we can slightly improve the inequality (A.5) : the right-hand side can be taken equal to $C_{\alpha,1}\ r(x)^{-|\alpha|-1/2}$ when $\ell=1$.

We may now form an almost analytic extension of $\omega(x,t)$, with respect to the x variables, in a suitable neighborhood of $U_0\backslash\mathcal{N}_0$ in $\mathbb{C}^n$. We write :

$$\text{(A.7)} \quad \tilde{\omega}(z,t) = \sum_{\alpha\in\mathbb{Z}_+^n} \frac{(iy)^\alpha}{\alpha!}\ \partial_x^\alpha \omega(x,t)\ \zeta_\alpha(x,y).$$

The ζ_α are cut-off functions, chosen according to the following scheme. Select arbitrarily $\zeta \in C^\infty(\mathbb{R}^n)$, $\zeta(s) = 0$ if $s > 1$, $\zeta(s) = 1$ if $s < 1/2$, and take

$$\text{(A.8)} \quad \zeta_\alpha(x,y) = \zeta(s), \quad s = |y|^2/[\varepsilon_\alpha r(x)]^2.$$

The ε_α are > 0 and converge to zero, as $|\alpha| \to +\infty$, so fast that the series (A.7) converges, as a C^∞ function of (x,y,t), in the set

$$\text{(A.9)} \quad x \in U_0\backslash\mathcal{N}_0,\ y \in \mathbb{R}^n,\ |t| < T.$$

THat such a sequence $\varepsilon = (\varepsilon_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ indeed exists follows from Coroll. A.1 and from the properties (2.7) of r. (The proof of the existence of the sequence ε is standard and goes back to DuBois-Raymond, it essentially consists in proving that, if we have a sequence of sequences $\sigma^k = (\sigma_\alpha^k)_{\alpha \in \mathbb{Z}_+^n}$, $k = 1,2,\ldots$, of positive numbers, there is another such sequence, $\varepsilon = (\varepsilon_\alpha)_{\alpha \in \mathbb{Z}_+^n}$, such that $\sum_\alpha \sigma_\alpha^k \varepsilon_\alpha \leq C^{(k)} < +\infty$ for every k). By the same token we obtain :

LEMMA A.2. - If the sequence ε is well chosen, to every pair α,β in $\mathbb{Z}^n_+$ and to every ℓ in $\mathbb{Z}^n_+$, there are constants $C_{\alpha,\beta,\ell} > 0$ such that, in the set (A.9),

$$\text{(A.10)}\quad |\partial_x^\alpha \partial_y^\beta \partial_t^\ell \tilde{\omega}(z,t)| \le \begin{cases} C_{\alpha,\beta,0}\, r(x)^{-|\alpha+\beta|} & \text{if } \ell = 0, \\ C_{\alpha,\beta,\ell}\, r(x)^{-|\alpha+\beta|-1} & \text{if } \ell > 0. \end{cases}$$

On the other hand, if we use the fact that, for any $M > 0$ and a suitably selected $C_M > 0$,

$$\text{(A.12)}\quad |\zeta'(s)| \le C_M\, |s|^M,\ s \in \mathbb{R},$$

we also obtain -via Cor. A.1, (2.7) and (A.7) :

LEMMA A.3. - If the sequence ε is well chosen, to every pair α,β in $\mathbb{Z}^n_+$, ℓ, M in $\mathbb{Z}^n_+$, there is $C_{\alpha,\beta,\ell;M} > 0$ such that, in the region (A.9),

$$\text{(A.13)}\quad |\partial_x^\alpha \partial_y^\beta \partial_t^\ell \frac{\partial\omega}{\partial z}(z,t)| \le \begin{matrix} C_{\alpha,\beta,0;M}\, r(x)^{-|\alpha+\beta|-1} \left[\frac{|y|}{r(x)}\right]^M & \text{if } \ell = 0 \\ C_{\alpha,\beta,\ell;M}\, r(x)^{-|\alpha+\beta|-2} \left[\frac{|y|}{r(x)}\right]^M & \text{if } \ell > 0. \end{matrix}$$

Needless to say,

$$\text{(A.14)}\quad \tilde{\omega}(z,t) = \omega(x,t) \text{ if } z=x \in U_0 \setminus \mathcal{N}_0.$$

Next we apply Th. 1 of [ ]. We recall that

$$L_0 = \frac{\partial}{\partial t} + i \sum_{j=1}^{n} b^j(x,t) \frac{\partial}{\partial x^j}\ .$$

For the sake of brevity we shall write :

$$\text{(A.15)}\quad \rho^* = \rho(x,t,t') = \left|\int_{t'}^{t} |\vec{b}(x,s)|\,ds\right|$$

LEMMA A.4. - There is a C^∞ function $z(x,t,t')$ in (an open neighborhood of) the closed set

$$\text{(A.16)}\quad (x,t) \in \overline{U}, \qquad |t'| \le T,$$

such that

$$\text{(A.17)}\quad L_0 z \text{ is } \rho^*\text{-flat in (A.16)}, \qquad z\Big|_{t=t'} = x.$$

Moreover :

$$\text{(A.18)}\quad \left|z-x+i\int_{t'}^{t} \vec{b}(x,s)\,ds\right| \le \text{const.}\,|t-t'|\left|\int_{t'}^{t} |\vec{b}(x,s)|\,ds\right|.$$

We may now set

$$\text{(A.19)}\quad \varphi(x,t,t') = \int_{t'}^{t} \tilde{\omega}(z(x,t,s),s)\,ds.$$

We have, using tne fact that $z(x,t,t') = x$, and (A.14),

$$(A.20)\quad L_0\varphi = \omega(x,t) + \int_{t'}^{t} [\frac{\partial\tilde{\omega}}{\partial\bar{z}} L_0 z] ds.$$

In the integrals, at the right in (A.20), z stands for $z(x,t,s)$. The function H in Lemma 3.1 will be equal to $H_1 + H_2$, with

$$(A.21)\quad H_1 = \int_{t'}^{t} \frac{\partial\tilde{\omega}}{\partial z}(z,s)\, L_0 z\, ds,$$

$$(A.22)\quad H_2 = \int_{t'}^{t} \frac{\partial\tilde{\omega}}{\partial z}(z,s)\, L_0 \bar{z}\, ds.$$

If we keep in mind the fact that $L_0 z$ is ρ^*-flat we conclude easily (via Lemma A.2) that H_1 is also ρ^*-flat. Concerning H_2 we first note that

$$L_0\bar{z} = (L_0-\overline{L_0})\bar{z} + \overline{L_0 z} = 2i\, \vec{b}(x,t)\partial_x\bar{z} + \overline{L_0 z},$$

hence

$$(A.23)\quad L_0\bar{z} \equiv 2i\, \vec{b}(x,t)\, [1+O(t-t')]$$

$$\underline{\text{mod}}\ \rho^*\text{-}\underline{\text{flat functions}}.$$

We then apply Lemma A.3, and the obvious consequence of (A.15),

$$(A.24)\quad |\mathrm{Im}\, z| \leq C\, \rho^*.$$

It is a straight forward matter to derive that

$$(A.25)\quad |\partial_x^\alpha\partial_t^\ell\partial_{t'}^{\ell'} H_2| \leq C(\alpha,\ell,\ell';M)\; r(x)^{-|\alpha|-\ell-\ell'}\, [\frac{\rho^*}{r(x)}]^M .$$

This, of course, implies (3.10). The inequality (3.12) when $\ell+\ell'=0$ follows at once from Lemma A.2 and from the definition (A.16). On the other hand,

$$(A.26)\quad \frac{\partial\varphi}{\partial t} = -\,i\, b(x,t)\frac{\partial\varphi}{\partial x} - i\omega(x,t) + H(x,t,t'),$$

$$(A.27)\quad \frac{\partial\varphi}{\partial t'} = -\,\tilde{\omega}(z(x,t,t'),t').$$

If we apply (A.10), and (3.10), we easily obtain the inequality (3.12) when $\ell+\ell' > 0$.

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François TREVES
Department of Mathematics
Rutgers University
NEW BRUNSWICK, N.J. 08903
U.S.A.