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The $K(\pi, 1)$ conjecture for Artin groups of spherical type

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Abstract

In these notes, we introduce the 50-year-old $K(\pi, 1)$ conjecture alongside Coxeter and Artin groups. Roughly speaking, the conjecture states that the complement in $\mathbb{C}^n$ of a "symmetric" configuration of hyperplanes is a $K(\pi, 1)$ space. Our end goal is to present a proof of the conjecture in the so-called spherical case, where only a finite number of hyperplanes are removed, through methods from combinatorial topology. This proof draws inspiration from the original proof of the spherical case, which is a special case of a celebrated 1972 theorem by Pierre Deligne.

1. Introduction

The $K(\pi, 1)$ conjecture, dating back to the 1960s and 1970s and usually attributed to Brieskorn, Arnol'd, Pham, and Thom, states that the orbit configuration space of any Artin group is an Eilenberg-Maclane space (or classifying space, or $K(\pi, 1)$). The most famous special case is the space of configurations of n distinct points in $\mathbb{C}$, which was proved to be a classifying space for the braid group Br_n by Fox and Neuwirth [FN62]; see Figure 1.1.

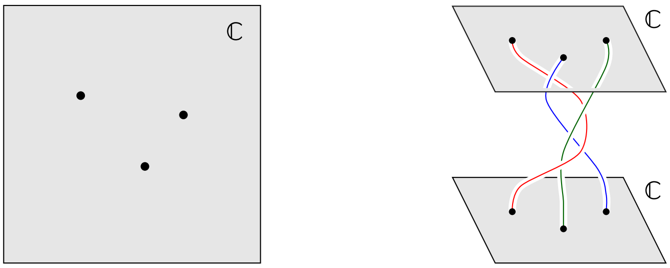

Figure 1.1. On the left, a configuration of 3 distinct points in $\mathbb{C}$. The space of all such configurations has a fundamental group given by the braid group on 3 strands: one closed loop is shown on the right, and its homotopy class forms a braid.

The present notes are based on a mini-course held by the author at WinterBraids 2023 in Tours. After providing the necessary foundations on Coxeter and Artin groups, our main goal is to describe a proof of the $K(\pi, 1)$ conjecture for Artin groups of spherical type (which include the braid group as a special case), using the modern language of discrete Morse theory. This proof is a bit different from the original proof by Deligne [Del72] (although the core idea is the same), and draws ingredients from a survey paper by Paris [Par14] and the author's master's thesis [Pao15].

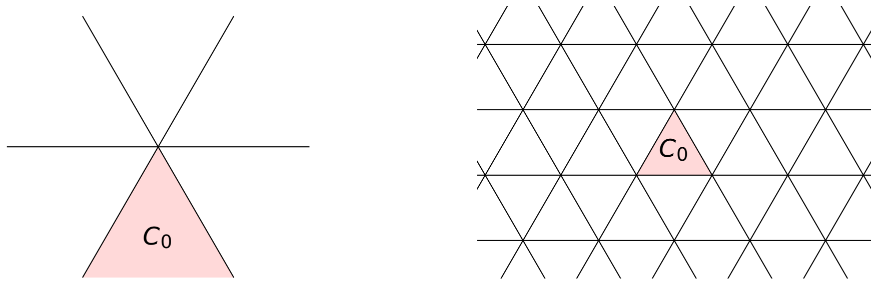

Figure 2.1. Arrangements of lines associated with two real reflection groups in $\mathbb{R}^2$: the symmetric group $\mathfrak{S}_3$ on the left (also known as type A_2), and the affine symmetric group of type $\tilde{A}_2$ on the right. A chamber C_0 is highlighted. The dihedral angles between the walls of C_0 are all $\pi/3$, giving rise to relations such as $aba = bab$.

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2. Coxeter groups

A Coxeter system is a pair (W,S) where S is a finite set and W is a group with a presentation of the following form:

$$W = \langle S \mid s^2 = 1 \;\; \forall s \in S, \text{ and } \underbrace{stst\cdots}_{m(s,t)\text{ terms}} = \underbrace{tsts\cdots}_{m(s,t)\text{ terms}} \;\; \forall s \neq t \text{ with } m(s,t) \neq \infty \,\rangle \tag{2.1}$$

for some fixed numbers $m(s,t) = m(t,s) \in \{2,3,4,\ldots,\infty\}$ for all $s \neq t$. The value $m(s,t) = \infty$ conventionally means no relation between s and t. The group W is called a Coxeter group. Whenever we talk about a Coxeter group W, we implicitly fix a Coxeter system (W,S).

Standard textbooks about Coxeter groups are [Bou68, Hum92, BB06]. The most important example of a Coxeter group is the symmetric group $\mathfrak{S}_n$ with its generating set $S = \{(1\;2),(2\;3),\ldots,(n-1\;\;n)\}$. If we call s_i the i-th generator, namely the transposition $(i\;\;i+1)$, then we can indeed observe that the following relations are satisfied:

- $s_i^2 = 1$;
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} = (i\;\;i+2)$;
- $s_i s_j = s_j s_i = (i\;\;i+1)(j\;\;j+1)$ whenever $|i-j| \geq 2$.

These relations turn out to present exactly the symmetric group $\mathfrak{S}_n$, and not some larger group. For instance, the symmetric group $\mathfrak{S}_3$ has the following Coxeter presentation:

$$\mathfrak{S}_3 = \langle a,b \mid a^2 = b^2 = 1,\; aba = bab \rangle,$$

where $a = s_1 = (1\;2)$ and $b = s_2 = (2\;3)$.

The next fundamental example of Coxeter groups is given by *real reflection groups*, namely the discrete groups of Euclidean isometries of $\mathbb{R}^n$ generated by orthogonal reflections. Finite reflection groups must fix at least one point of $\mathbb{R}^n$; they are also called *spherical* because they are groups of isometries of a sphere S^{n-1} centered at a fixed point. Infinite reflection groups are also called *affine*. The collection of all fixed hyperplanes of reflections in a reflection group W is called the *hyperplane arrangement* associated with W. See Figure 2.1.

Note that the symmetric group $\mathfrak{S}_n$ is a reflection group: it acts on $\mathbb{R}^n$ by permuting the coordinates, and transpositions act as orthogonal reflections with respect to the diagonal hyperplanes $\{x_i = x_j\}$. Since all these hyperplanes contain the line spanned by $(1,\dots,1)$, one often restricts this representation to the codimension-one subspace orthogonal to $(1,\dots,1)$. For $n = 3$, one obtains the arrangement in $\mathbb{R}^2$ depicted on the left of Figure 2.1.

Theorem 2.1 ([Cox34, Wit41])**.** *Every real reflection group is a Coxeter group.*

Sketch of proof. Given a real reflection group W acting on $\mathbb{R}^n$, one can construct a Coxeter presentation as follows. Fix any *chamber* C_0, i.e., a connected component of the complement of the hyperplane arrangement associated with W (see Figure 2.1). The codimension-one faces of C_0 span reflection hyperplanes $H_1,\dots,H_k$, called the *walls* of the chamber C_0. Let $S = \{s_1,\dots,s_k\} \subseteq W$ be the set of orthogonal reflections with respect to the hyperplanes $H_1,\dots,H_k$. Since W is discrete, any two distinct hyperplanes, H_i and H_j, must either intersect at a dihedral angle of the form $\pi/m(s_i,s_j)$ for some integer $m(s_i,s_j) \geq 2$, or be parallel (in which case we set $m(s_i,s_j) = \infty$). The set S with the numbers $m(s_i,s_j)$ yields a presentation of W of the form (2.1). See also [Hum92, Section 1.9] for more details. □

It turns out that the class of spherical real reflection groups exactly coincides with the class of finite Coxeter groups.

Theorem 2.2 ([Cox35, Wit41])**.** *Every finite Coxeter group is a (spherical) real reflection group.*

From now on, we say that a Coxeter group is spherical if it is finite, or equivalently if it is a spherical real reflection group. We say that a Coxeter group is affine if it is an affine real reflection group.

It turns out that every Coxeter group can be represented as a reflection group in a weak sense: the generators (and all their conjugates) act as linear reflections, but not necessarily as orthogonal reflections. By linear reflection we mean a linear automorphism of a real vector space having order 2 and fixing a hyperplane pointwise.

Theorem 2.3 ([Tit61])**.** *Any Coxeter group W has a faithful representation (called the contragradient representation) $\rho\colon W \hookrightarrow \mathrm{GL}(V)$ with $V = \mathbb{R}^{|S|}$, satisfying the following properties.*

(i) For every element $r \in W$ that is conjugate to an element of S, we have that $\rho(r)$ is a linear reflection fixing some hyperplane $H_r \subset V$.

(ii) The elements of S act as reflections fixing the walls of a chamber C_0 of the arrangement $\mathcal{A} = \{H_r\}$ (C_0 is often called the fundamental chamber).

(iii) Consider the union of all closed chambers in the orbit of $\overline{C_0}$:

$$I = \bigcup_{u \in W} \rho(u)(\overline{C_0}).$$

Then $I \subseteq V$ is a convex cone, called the Tits cone.

(iv) W acts simply transitively on the set of chambers contained in the Tits cone. In other words, for $u \in W$, we have that $\rho(u)(C_0) = C_0$ if and only if $u = 1$.

(v) The Tits cone I coincides with the entire space V if and only if W is spherical; it is an open half-space, with the origin added, if and only if W is affine; it does not contain any affine line if W is not spherical nor affine.

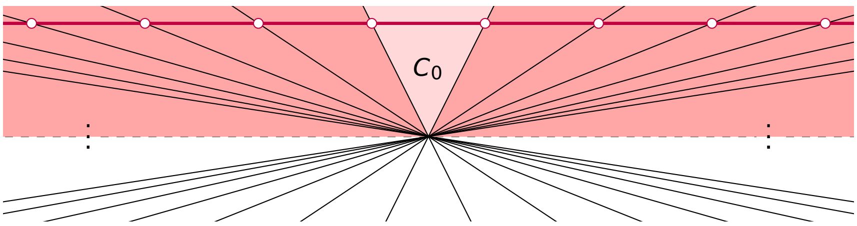

Figure 2.2. The contragradient representation of the affine Coxeter group of type $\tilde{A}_1$, defined as $W = \langle a, b \mid a^2 = b^2 = 1\rangle$, also known as the infinite dihedral group. Here $m(a,b) = \infty$, so there is no relation between a and b. The chamber C_0 is highlighted in light red, and the rest of the Tits cone I is in dark red. The action restricts to any horizontal affine line (such as the purple one), making W an affine real reflection group acting on $\mathbb{R}$.

If W is an affine Coxeter group, its contragradient representation restricts to any affine hyperplane H fully contained in the Tits cone I. The linear reflections with respect to the hyperplanes H_r restrict to affine orthogonal reflections in H. This makes W an affine real reflection group, as defined earlier. An example is shown in Figure 2.2.

Denote by $l(u)$ the minimal length of an expression of an element $u \in W$ as a product of generators. If W is spherical, then the length function is bounded and there is a unique element of maximal length.

Theorem 2.4. *Let W be a spherical Coxeter group. Then there is exactly one element $\delta \in W$, called the longest element, whose length $l(\delta)$ is maximized. It satisfies $l(u) + l(u^{-1}\delta) = l(\delta)$ for every $u \in W$.*

Sketch of proof. By simple transitivity of the action of W on the chambers, there exists a unique element $\delta \in W$ such that $\delta(C_0) = -C_0$, where $-C_0$ is the chamber opposite to C_0. The chamber $-C_0$ belongs to the Tits cone because W is spherical, and is the unique chamber of maximal distance from C_0 (where the distance between two chambers is defined as the number of reflection hyperplanes separating them). This can be shown to imply that δ is the unique element of maximal length. □

For every $T \subseteq S$, the elements of T generate a subgroup $W_T \subseteq W$ called a *standard parabolic subgroup*. We end this section with two important properties of parabolic subgroups and their cosets.

Theorem 2.5. *Let $T \subseteq S$.*

(i) The pair (W_T, T) is a Coxeter system. More precisely, a Coxeter presentation for the subgroup W_T can be naturally obtained from (2.1) *by removing all generators except those in T and all relations except those involving elements of T.*

(ii) Every (left) coset of W_T in W has a unique element of smallest length u_0, called its minimal coset representative. It satisfies $l(u_0u) = l(u_0) + l(u)$ for every $u \in W_T$.

3. Artin groups

To every Coxeter group W presented as in (2.1), there is an associated Artin group defined as follows:

$$G_W = \langle S \mid \underbrace{stst\cdots}_{m(s,t)\text{ terms}} = \underbrace{tsts\cdots}_{m(s,t)\text{ terms}} \quad \forall s \neq t \text{ with } m(s,t) \neq \infty \rangle. \tag{3.1}$$

While there is no standard reference about Artin groups, a recommended read is a survey by Paris [Par14], which develops many of the topics covered in these notes in greater detail.

Two basic yet important examples of Artin groups are the free group F_n (obtained by setting all $m(s,t) = \infty$) and the free abelian group $\mathbb{Z}^n$ (obtained by setting all $m(s,t) = 2$). However, the example that started the whole theory of Artin groups is the *braid group*, introduced by Artin in 1925 [Art25]. The braid group Br_n is the Artin group associated with the symmetric group $\mathfrak{S}_n$. For instance, the braid group Br_3 has the following presentation:

$$\mathrm{Br}_3 = \langle a, b \mid aba = bab \rangle.$$

Elements of Br_n can be interpreted as isotopy classes of braids on n strands, as shown in Figure 1.1. Artin groups associated with spherical (resp., affine) Coxeter groups are called of spherical type (resp., affine type). For instance, the braid group Br_n and the free abelian group $\mathbb{Z}^n$ are Artin groups of spherical type; the free group F_2 is an Artin group of affine type, as it is associated with the affine Coxeter group of type $\tilde{A}_1$ (introduced in Figure 2.2).

The contragradient representation of a Coxeter group W (Theorem 2.3) gives rise to the *orbit configuration space* of W:

$$Y_W = \left((I + iV) \setminus \bigcup_r H_r \otimes_{\mathbb{R}} \mathbb{C} \right) / W.$$

Here, $I + iV$ is the subset of the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ consisting of all vectors with real part belonging to the Tits cone I; for each reflection $r \in W$, the set $H_r \otimes \mathbb{C} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ is the complex hyperplane with the same (real) equation as H_r; the Coxeter group W acts diagonally on the real and the imaginary parts of vectors in $V \otimes_{\mathbb{R}} \mathbb{C}$, and we are taking the quotient by this action. Since the (complexified) reflection hyperplanes are removed, the action of W is a covering space action.

For $W = \mathfrak{S}_n$, the orbit configuration space is

$$Y_{\mathfrak{S}_n} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \ \forall\, i \neq j\} / \mathfrak{S}_n.$$

This is exactly the space of configurations of n distinct points in $\mathbb{C}$; note that the quotient by the $\mathfrak{S}_n$ action has the effect of forgetting the order of the n points. As illustrated in Figure 1.1, the fundamental group of $Y_{\mathfrak{S}_n}$ is the braid group Br_n, and this is, in fact, true more generally if we replace $\mathfrak{S}_n$ with an arbitrary Coxeter group.

Theorem 3.1 ([Bri71, VdL83])**.** *For every Coxeter group W, we have an isomorphism*

$$\pi_1(Y_W) \cong G_W.$$

Fox and Neuwirth proved that $Y_{\mathfrak{S}_n}$ is a classifying space for the braid group Br_n [FN62]. This led to a formulation of the following "$K(\pi, 1)$ conjecture," which is one of the most important open problems about Artin groups.

Conjecture 3.2. *For every Coxeter group W, we have $Y_W \simeq K(G_W, 1)$.*

The $K(\pi, 1)$ conjecture is known for instance in the following cases: spherical [Del72], affine [PS21], two dimensional and FC-type [CD95]. See also [Bri73, Oko79, CMS10, Hen85, DPS22, HH23, Hua23, Hua24].

The orbit configuration space Y_W has the homotopy type of a CW complex with a finite number of cells. This was proved by Salvetti [Sal87, Sal94], and the complex he constructed is known as the Salvetti complex.

Theorem 3.3 ([Sal94])**.** *The orbit configuration space Y_W is homotopy equivalent to a CW complex X_W, called the Salvetti complex, having one cell σ_T of dimension $|T|$ for every subset $T \subseteq S$ such that the standard parabolic subgroup $W_T \subseteq W$ is spherical (i.e., finite).*

The cells σ_T of the Salvetti complex can be described geometrically as follows. Given a spherical standard parabolic subgroup W_T, consider its representation as a spherical real reflection group acting on $\mathbb{R}^n$. Fix any point x in the fundamental chamber C_0, and denote by

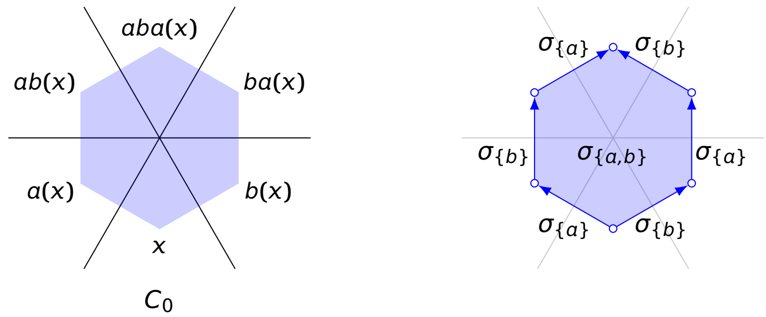

Figure 3.1. On the left, the polytope dual to the hyperplane arrangement associated with the symmetric group $\mathfrak{S}_3 = \langle a, b \mid a^2 = b^2 = 1,\ aba = bab \rangle$. On the right, the cell σ_T for $T = \{a, b\}$ when $m(a, b) = 3$. The arrows point away from the minimal coset representative, and describe how the 1-dimensional faces with the same label ($\sigma_{\{a\}}$ or $\sigma_{\{b\}}$) are identified. All six 0-dimensional faces are identified with $\sigma_\varnothing$.

$Q \subseteq \mathbb{R}^n$ the polytope obtained as the convex hull of the points $u(x)$ for $u \in W$. The polytope Q is dual to the hyperplane arrangement associated with W_T, as shown in Figure 3.1 on the left. Identify the cell σ_T with Q.

For every face $F \subset Q$, there exist a subset $R \subset T$ and a minimal coset representative $u_0 \in W$ for the coset $u_0 W_R \subset W$ such that F is the convex hull of the vertices $u_0 u(x)$ for $u \in W_R$. Then F can be canonically identified with the cell σ_R, as illustrated in Figure 3.1 on the right. This construction describes how σ_T is attached to the lower-dimensional cells σ_R for $R \subset T$ in the Salvetti complex X_W.

Note that, for every standard parabolic subgroup W_T, the Salvetti complex X_{W_T} is naturally a subcomplex of the Salvetti complex X_T. Geometrically, this corresponds to an inclusion of Y_{W_T} as a subspace of Y_W in a neighborhood of a point fixed by W_T (see e.g. [LP21, Section 6]).

4. Artin monoids

While Artin groups are fairly complicated to understand, the closely related *Artin monoids* are more approachable. For every Coxeter system (W, S), the associated Artin monoid G_W^+ is defined through a monoid presentation identical to the Artin group presentation (3.1):

$$G_W^+ = \langle S \mid \underbrace{stst\cdots}_{m(s,t)\text{ terms}} = \underbrace{tsts\cdots}_{m(s,t)\text{ terms}} \quad \forall\, s \neq t \rangle_+$$

(the "+" on the right-hand side denotes a monoid presentation). There is a natural monoid map $G_W^+ \to G_W$ sending each generator of the Artin monoid G_W^+ to its copy inside the Artin group G_W. It is not obvious, but true, that an Artin monoid injects into its Artin group.

Theorem 4.1 ([BS72, Del72, Par02])**.** *For every Coxeter group W, the natural map $G_W^+ \to G_W$ is injective.*

For example, the element $aba = bab \in \mathrm{Br}_3$ also belongs to the Artin monoid Br_3^+, whereas the element ab^{-1} does not, as it cannot be written as a product of the generators (without using their inverses).

Since the defining relations are homogeneous, every word (over the alphabet S) representing a given monoid element $\alpha \in G_W^+$ has the same length, and we denote it by $l(\alpha)$. In addition, every element u of the Coxeter group W can be lifted to an Artin monoid element by taking any expression of minimal length $u = s_{i_1} s_{i_2} \cdots s_{i_k}$ and reading it as a word in the Artin monoid G_W^+. Call $\iota\colon W \to G_W^+$ the resulting function, which is a section of the natural projection $\pi\colon G_W^+ \to W$.

Theorem 4.2 ([Tit69]). *The section $\iota\colon W \to G_W^+$ is well-defined, i.e., it does not depend on the chosen minimal-length expression.*

The section ι preserves the length. However, it is not a monoid homomorphism. The previous theorem essentially states that any two minimal-length expressions of an element $u \in W$ can be transformed into each other through a sequence of substitutions $stst\cdots \longleftrightarrow tsts\cdots$ (where both subwords have $m(s,t)$ factors). This is known as the *Matsumoto property* and it is closely related to the *exchange property*.

In the Artin monoid, we have divisibility relations on the left and the right: we say that α left-divides β (resp. α right-divides β) if $\beta = \alpha\gamma$ (resp. $\beta = \gamma\alpha$) for some $\gamma \in G_W^+$; when this happens, we also write $\alpha \leq_{\mathrm{L}} \beta$ (resp. $\alpha \leq_{\mathrm{R}} \beta$).

Theorem 4.3 ([BS72]). *Let W be any Coxeter group.*

(i) Any subset $E \subseteq G_W^+$ has a (left or right) greatest common divisor.

(ii) A subset $T \subseteq S$ admits a (left or right) least common multiple in G_W^+ if and only if the standard parabolic subgroup $W_T \subseteq W$ is spherical. When this happens, the left and right least common multiples coincide and are denoted by Δ_T (the Garside element of the Artin group G_{W_T}).

The Garside element of an Artin group of spherical type enjoys many useful properties, some of which are presented in the following theorem.

Theorem 4.4 ([BS72]). *Let W be a spherical Coxeter group and $\Delta \in G_W^+$ the Garside element.*

(i) $\Delta = \iota(\delta)$, where δ is the longest element of the Coxeter group W.

(ii) Every element $\alpha \in G_W$ can be written in the form $\alpha = \Delta^k \beta$ for some $\beta \in G_W^+$ and $k \in \mathbb{Z}$. This expression is unique if we require k to be as large as possible.

5. The $K(\pi,1)$ conjecture for Artin groups of spherical type

In this section, we outline a proof of the $K(\pi,1)$ conjecture for Artin groups of spherical type. To do this, we need to show that the universal cover $\tilde{X}_W$ of the Salvetti complex X_W is contractible when W is a spherical Coxeter group.

The universal cover $\tilde{X}_W$ inherits a cellular structure from X_W; the cells of $\tilde{X}_W$ are indexed by pairs (α, T) for $\alpha \in G_W = \pi_1(X_W)$ and $T \subseteq S$ with W_T finite. The boundary of a cell $\sigma_{(\alpha,T)}$ of $\tilde{X}_W$ consists of all the cells $\sigma_{(\alpha\,\iota(u_0),R)}$ for all cosets $u_0 W_R \subseteq W_T$ (here u_0 is the minimal representative of the coset $u_0 W_R$). These cells are all distinct, and $\tilde{X}_W$ is a regular CW complex. See Figure 5.1.

The universal cover $\tilde{X}_W$ has a subcomplex $\tilde{X}_W^+$ consisting of all cells $\sigma_{(\alpha,T)}$ such that $\alpha \in G_W^+$. We call $\tilde{X}_W^+$ the positive subcomplex of $\tilde{X}_W$.

Lemma 5.1. *If W is spherical and $\tilde{X}_W^+$ is contractible, then $\tilde{X}_W$ is also contractible.*

Proof. Let $\Delta \in G_W^+$ be the Garside element of G_W. If $\sigma_{(\alpha,T)}$ is any cell of X_W, we can write $\alpha = \Delta^k \beta$ for some $k \in \mathbb{Z}$ and $\beta \in G_W^+$. Then $\sigma_{(\alpha,T)}$ belongs to the subcomplex $Z_k = \Delta^k \tilde{X}_W^+ \subseteq \tilde{X}_W$; here we are letting the elements $\Delta^k \in G_W$ act on $\tilde{X}_W$ as deck transformations.

We have therefore written $\tilde{X}_W$ as the union of a chain of subcomplexes $\ldots \supset Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \supset \ldots$, all homeomorphic to $\tilde{X}_W^+$ and thus contractible. The homotopy groups of $\tilde{X}_W$ must vanish because any homotopy class can be represented by a map $S^n \to Z_k$ for some k. By Whitehead's theorem, the entire universal cover $\tilde{X}_W$ is also contractible. □

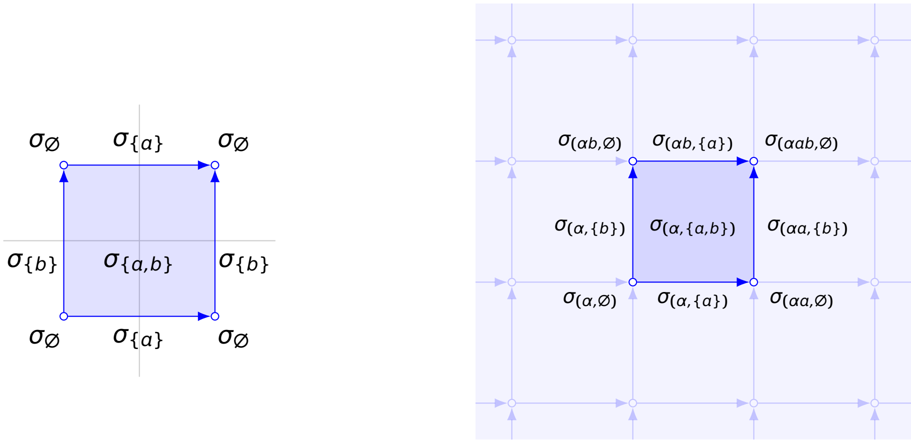

Figure 5.1. On the left, the Salvetti complex X_W for the Coxeter group $W = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1, ab = ba\rangle$. In this case, X_W is a torus $S^1 \times S^1$ with its minimal cell structure. On the right, the universal cover $\tilde{X}_W \cong \mathbb{R}^2$. For every $\alpha \in G_W = \mathbb{Z} \times \mathbb{Z}$, the universal cover has: one 2-cell $\sigma_{(\alpha,\{a,b\})}$; two 1-cells $\sigma_{(\alpha,\{a\})}$ and $\sigma_{(\alpha,\{b\})}$; and one 0-cell $\sigma_{(\alpha,\varnothing)}$. A generic 2-cell $\sigma_{(\alpha,\{a,b\})}$ is highlighted, together with all the cells in its boundary.

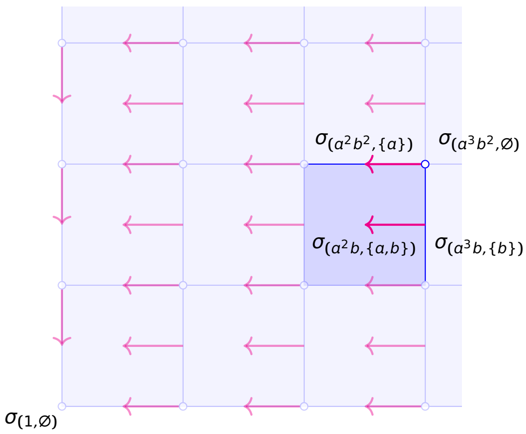

Figure 5.2. The positive subcomplex $\tilde{X}_W^+$ for the Coxeter group $\mathbb{Z}_2 \times \mathbb{Z}_2$, with arrows indicating a Morse matching (or *discrete Morse vector field*) showing that $\tilde{X}_W^+$ deformation retracts onto the 0-cell $\sigma_{(1,\varnothing)}$. The four cells in $\eta^{-1}(a^3b^2)$ are highlighted: the 0-cell $\sigma_{(a^3b^2,\varnothing)}$ is matched with the 1-cell $\sigma_{(a^2b^2,\{a\})}$, and the 1-cell $\sigma_{(a^3b,\{b\})}$ is matched with the 2-cell $\sigma_{(a^2b,\{a,b\})}$.

So it is enough to prove that the positive subcomplex $\tilde{X}_W^+$ is contractible, and this can be done for all Coxeter groups W (not necessarily finite). The idea is to construct a deformation retraction of $\tilde{X}_W^+$ onto the 0-cell $\sigma_{(1,\varnothing)}$ through a discrete Morse vector field [For98], as illustrated in Figure 5.2. We outline the main results we are going to use from Forman's *discrete Morse theory*, developed in [For98, For02, Cha00, Bat02]; we refer the reader to [Koz07, Chapter 11] for a textbook introduction to the topic.

A discrete Morse vector field is encoded through a matching on the cells of a CW complex Z, which for simplicity we assume to be regular. More precisely, let $\mathcal{G}(Z) = (\mathcal{F}(Z), \mathcal{E}(Z))$ be the directed graph where the set of vertices $\mathcal{F}(Z)$ consists of all cells of Z, and the set of edges $\mathcal{E}(Z)$ consists of all pairs $\sigma \to \tau$ where τ is a codimension-one face of σ. A *Morse matching* is a subset $\mathcal{M} \subseteq \mathcal{E}(Z)$ satisfying the following properties:

1. Every cell appears at most once in $\mathcal{M}$ (so $\mathcal{M}$ is a partial matching on the graph $\mathcal{G}(Z)$).

2. The graph $\mathcal{G}_{\mathcal{M}}(Z)$, obtained from $\mathcal{G}(Z)$ by reversing all the edges in $\mathcal{M}$, is acyclic.

3. For every $\sigma \in \mathcal{F}(Z)$, the set of cells reachable from σ in the graph $\mathcal{G}_{\mathcal{M}}(Z)$ is finite.

The edges that are reversed when passing from $\mathcal{G}(Z)$ to $\mathcal{G}_{\mathcal{M}}(Z)$ are precisely the arrows in Figure 5.2. Given a Morse matching $\mathcal{M}$ on Z, the cells of Z that do not appear in $\mathcal{M}$ are called *$\mathcal{M}$-critical*. The following is a special case of the main theorem of discrete Morse theory.

Theorem 5.2 ([For98, Bat02])**.** *Let $\mathcal{M}$ be a Morse matching on a regular CW complex Z. If the $\mathcal{M}$-critical cells form a subcomplex $Z_{\mathcal{M}}$ of Z, then Z deformation retracts onto $Z_{\mathcal{M}}$.*

The assumptions that Z is regular and that the $\mathcal{M}$-critical cells form a subcomplex can both be relaxed, but we do not need such generality in the present notes.

We shall make use of a standard tool to construct Morse matchings, often referred to as the Patchwork Theorem. For this, let $(P, \leq)$ be any poset, and suppose to have a map $\eta\colon \mathcal{F}(Z) \to P$ that respects the partial ordering, i.e., such that $\eta(\tau) \leq \eta(\sigma)$ for every directed edge $\sigma \to \tau$ in the graph $\mathcal{G}(Z)$. Denote by $Z_{\leq p}$ the subcomplex of Z consisting of all cells σ with $\eta(\sigma) \leq p$.

Theorem 5.3 ([Her05, Jon08])**.** *For every $p \in P$, assume that the following properties hold.*

1. *There is a partial matching $\mathcal{M}_p \subseteq \mathcal{E}(Z)$ only involving the cells in the fiber $\eta^{-1}(p)$.*

2. *The induced subgraph of $\mathcal{G}_{\mathcal{M}_p}(Z)$ on the vertex set $\eta^{-1}(p)$ is acyclic.*

3. *The subcomplex $Z_{\leq p}$ has a finite number of cells (i.e., it is compact).*

Then the union $\mathcal{M} = \bigcup_{p \in P} \mathcal{M}_p$ is a Morse matching on Z.

We are now ready to construct a Morse matching on the positive subcomplex X_W^+ for any Coxeter group W. Let $\eta\colon \mathcal{F}(\tilde{X}_W^+) \to G_W^+$ be the map that sends a cell $\sigma_{(\alpha,T)}$ to its "longest vertex" $\alpha\Delta_T$. To apply Theorem 5.3, we now show that η respects the partial ordering, where the Artin monoid G_W^+ is equipped with the partial order of left divisibility $\leq_{\mathrm{L}}$.

Lemma 5.4. *If $\sigma \to \tau$ is an edge in $\mathcal{G}(X_W^+)$, then $\eta(\tau) \leq_L \eta(\sigma)$.*

Proof. Let $\sigma = \sigma_{(\alpha,T)}$ and $\tau = \sigma_{(\alpha\,\iota(u_0),R)}$, where u_0 is the minimal representative of the coset $u_0 W_R \subset W_T$. We need to prove that $\alpha\,\iota(u_0)\Delta_R \leq_{\mathrm{L}} \alpha\Delta_T$, which is equivalent to $\iota(u_0)\Delta_R \leq_{\mathrm{L}} \Delta_T$. In the Coxeter group W_T, we have $l(u_0) + l(\delta_R) = l(u_0\delta_R)$ by Theorem 2.5. Since δ_T is the longest element of W_T, we can write $\delta_T = u_0\delta_R v$ with $l(u_0\delta_R) + l(v) = l(\delta_T)$ by Theorem 2.4. Applying ι, we obtain the factorization $\Delta_T = \iota(u_0)\Delta_R\,\iota(v)$, so in particular $\iota(u_0)\Delta_R \leq_{\mathrm{L}} \Delta_T$. □

Next, we construct perfect acyclic matchings $\mathcal{M}_\beta$ on all fibers $\eta^{-1}(\beta)$ for $\beta \neq 1$ by using the results on greatest common divisors and least common multiples in Artin monoids. The matching we construct is the one exemplified in Figure 5.2.

Lemma 5.5. *Let $\beta \in G_W^+ \setminus \{1\}$. Then there exists a matching $\mathcal{M}_\beta \subseteq \mathcal{E}(\tilde{X}_W^+)$, involving all the vertices in $\eta^{-1}(\beta)$, such that the induced subgraph of $\mathcal{G}_{\mathcal{M}_\beta}(\tilde{X}_W^+)$ on the vertex set $\eta^{-1}(\beta)$ is acyclic.*

Proof. Let $T_\beta = \{s \in S \mid s \leq_{\mathrm{R}} \beta\}$. By Theorem 4.3, we have that $\Delta_R \leq_{\mathrm{R}} \beta$ for every $R \subseteq T$. Therefore, the fiber $\eta^{-1}(\beta)$ is naturally in bijection with the powerset of T_β, and this bijection is order-preserving: $\sigma_{(\alpha_1,R_1)}$ is a face of $\sigma_{(\alpha_2,R_2)}$ if and only if $R_1 \subseteq R_2$. The induced subgraph of $\mathcal{G}(\tilde{X}_W^+)$ on the vertex set $\eta^{-1}(\beta)$ is then the 1-skeleton of a $|T_\beta|$-dimensional hypercube. To construct a matching $\mathcal{M}_\beta$, we can fix any $\bar{s} \in T_\beta$ (the set T_β is non-empty because $\beta \neq 1$) and pair $R \cup \{\bar{s}\}$ with R, for every $R \subseteq T_\beta \setminus \{\bar{s}\}$. In the induced subgraph of $\mathcal{G}_{\mathcal{M}_\beta}(\tilde{X}_W^+)$, edges correspond to either removing an element of $T_\beta \setminus \{\bar{s}\}$ or adding $\bar{s}$; such subgraph has no cycles. □

We can finally put all the ingredients together.

Theorem 5.6. *For every Coxeter group W, the positive subcomplex $\tilde{X}_W^+$ is contractible.*

Proof. Let $Z = \tilde{X}_W^+$ and recall the order-preserving map $\eta\colon \mathcal{F}(Z) \to G_W^+$ defined earlier. For every $\beta \in G_W^+ \setminus \{1\}$, the subcomplex $Z_{\leq_{\mathrm{L}}\beta}$ has a finite number of cells because each fiber is finite (as observed in the proof of Lemma 5.5) and there is a finite number of monoid elements that are $\leq_{\mathrm{L}} \beta$.

Consider now the matching $\mathcal{M} = \bigcup_{\beta \in G_W^+ \setminus \{1\}} \mathcal{M}_\beta$ on Z. Thanks to Lemma 5.4, Lemma 5.5 and the previous observation, we can apply Theorem 5.3 and conclude that $\mathcal{M}$ is a Morse matching. The only critical cell is the 0-cell $(1, \varnothing)$, so Z is contractible by Theorem 5.2. □

Corollary 5.7. *The $K(\pi,1)$ conjecture holds for all Artin groups of spherical type.*

Proof. Apply Lemma 5.1 and Theorem 5.6 to conclude that the universal cover $\tilde{X}_W$ is contractible. □

In Artin monoids of non-spherical type, there is no common multiple of all standard generators, and thus, no Garside element. As a result, the proof of Lemma 5.1 is not applicable, and the proof strategy outlined in the present notes is unlikely to be generalizable. Alternative Garside structures, using a different generating set, have proven to be a fruitful avenue for studying certain Artin groups of non-spherical type. This has led, in particular, to the solution of the word problem and the $K(\pi,1)$ conjecture in the affine case [MS17, PS21].

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