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Winter Braids Lecture Notes

Peter Feller

**Positive braids and concordance**

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# Positive braids and concordance

PETER FELLER

**Abstract**

These lecture notes discuss notions of positivity for braids and links, their relation to complex plane curves, and how they provide strong restrictions on concordance properties of knots; e.g., sliceness. The notes are based on a three-hour mini-course delivered at Winter Braids XII.

## Introduction

These notes are the lecture notes for a three-part mini-course titled “Positive braids and concordance” held at Winter Braids XII in February 2023 in Tours.

### Structure of the lecture notes

A proverb asserts that a talk should contain at most one proof and at least one joke, but they should be different. While the jokes have not found their way into these notes, the structure of the mini-course—in particular the insistence on having one main argument per lecture—has remained.

The selection of the three arguments is such that, hopefully, even experts in the field will discover something novel. At least, as far as I know, the arguments cannot be found in this exact form in the literature. Here is a summary of the three arguments and the slight novelties, if there are any. The non-expert may safely skip this on the first reading.

1. Argument from Lecture I: Rudolphian variant of the Yamada-Vogel algorithm.

   Upshot: Preserves the strength of the Yamada-Vogel algorithm (efficiently turning a link diagram into a closed braid diagram while preserving the number of Seifert circles) with the added benefit of turning a positive diagram into a diagram of the closure of a strongly quasipositive braid.

2. Argument from Lecture II: Qualitative argument that links of singularities are closures of positive braids.

   Upshot: Rather than presenting the precise classification arguments for links of singularities, this qualitative argument aims to convince the reader that braid positivity naturally arises in the study of links of singularities.

   Bonus: We revisit the argument from Lecture I to see that the algorithm establishes Rudolph’s result that positive links are strongly quasipositive.

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*Keywords:* Braids, braid positive, quasipositive, strongly quasipositive, slice knots, slice-Bennequin-inequality, local Thom conjecture.

3. Argument from Lecture III: An elementary and short proof of the slice-Bennequin inequality using one input: the local Thom conjecture (smooth 4-genus of torus knots).

   Upshot: In the literature, the slice-Bennequin inequality may appear to be intricate (Rudolph's original argument mixes in complex curve considerations), or as a statement best understood in the context of 3-dimensional contact topology. Here, we (re)claim the slice-Bennequin inequality as a result within knot and braid theory: it relates the writhe of braids to the 4-genus of the links of their closure and its proof stays in the domain of geometric braids and their closure.

   Bonus: We also derive the local Thom conjecture from the Thom conjecture for $\mathbb{CP}^2$.

Additionally, the notes feature two exercises, one each in the first two lectures. The exercises are not meant to support a full exercise session, instead, the author considers them both helpful to have a better sense of the material that follows, and have one thing in mind to think about between the lectures.

## Acknowledgements

I am thankful to the organizers of Winter Braids XII, Paolo Bellingeri, Vincent Florens, and Thomas Gobet, for inviting me, for suggesting the topic of the mini-course, and their great patience as these notes were delayed due to personal issues. Further thanks go to the other lecturers, Federica Fanoni and Giovanni Paolini, for for the collaborative spirit they helped cultivate. Finally, I thank the anonymous referee for their careful review and insightful comments.

# 1. Lecture I: Braids, notions of positivity, links via braid closures, and links of singularities

In the first lecture we discuss the definition of the braid group, notions of positivity and the closure of braids, which form links. We further discuss a motivation for the study of links coming from the study of singularities of plane curves.

The main argument in Lecture I is a variant of the Yamada-Vogel algorithm [Yam87, Vog90], inspired by Rudolph's argument that positive links are strongly quasipositive [Rud99].

## 1.1. Four perspectives on the braid group

For a fixed integer $n \geq 1$, we denote by $B_n$ the braid group on $n$-strands and call its elements $n$-braids. We recall four different perspectives on braids and how they relate to one another. We will focus on the first two, but the other two lecture courses relate to the last two. We point to the excellent survey by Birman-Brendle [BB05] and book by Birman [Bir74] for more details.

### Artin's group presentation for the braid group

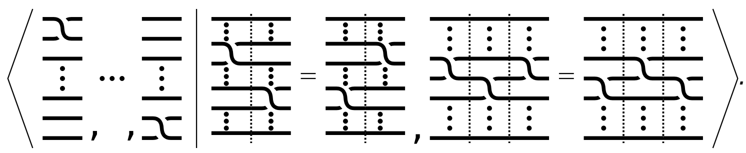
Artin first described the braid group using the following group presentation

$$B_n = \left\langle \sigma_1, \cdots, \sigma_{n-1} \,\middle|\, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \right\rangle,$$

now also known as the standard generators of the braid group with the standard braid relations [Art25].

### Geometric braids

Let $\mathbb{D}^2$ denote the complex unit disc and fix a set $N$ of $n$ points in the interior of $\mathbb{D}^2$. To justify the term 'braids', $n$-braids can be viewed as $n$ disjointly properly embedded intervals in the cylinder $\mathbb{D}^2 \times [0,1]$ that project regularly onto the second coordinate and have boundary in $N \times \{0,1\}$. Considering these embeddings modulo boundary-fixing ambient isotopy with the operation of stacking yields another model of the braid group.

$B_n =$ 

An isomorphism with the standard group presentation is provided by sending the geometric braid for which the $i$-strand crosses once positively over the $(i+1)$-strand, i.e. the geometric braids depicted above to the left, to $\sigma_i$. We use this isomorphism to identify these first two descriptions of the braid group $B_n$ throughout these notes.

### The braid groups as the mapping class group of the $n$-punctured disc

Another model for the braid group is given by the following mapping class group.

$$B_n = \mathrm{MCG}(\mathbb{D}_n) := \mathrm{Diff}^+(\mathbb{D}_n)/\mathrm{Diff}^+_0(\mathbb{D}_n) = \pi_0(\mathrm{Diff}^+(\mathbb{D}_n)).$$

Here, $\mathbb{D}_n = \mathbb{D}^2 \setminus N$ denotes the closed complex unit disc $\mathbb{D}^2$ from which a set $N$ of $n$ points that lie in the interior of $\mathbb{D}^2$ are removed, $\mathrm{Diff}^+$ denotes the set of self-diffeomorphisms that restrict to the identity on the boundary, and $\mathrm{Diff}^+_0$ denotes the subset of those such self-diffeomorphism that are isotopic to the identity via an isotopy that fixes the boundary.

Given a geometric braid, one obtains a mapping class by 'flowing the points $N \times \{0\}$ in $\mathbb{D}^2 \times \{0\}$ along the braid to $N \times \{1\}$ in $\mathbb{D}^2 \times \{1\}$', which can be formalized as follows. Pick a vector field $v$ on $\mathbb{D}^2 \times [0,1]$ that is tangent to the braid (given in the form of embedded intervals) and that projects to 1 in the tangent space $[0,1] = T[0,1] \subset T(\mathbb{D}^2 \times [0,1])$ of $[0,1] \subset \mathbb{D}^2 \times [0,1]$. Take $\phi\colon \mathbb{D}^2 \to \mathbb{D}^2$ to be the time one flow of $v$. Then the restriction of $\phi$ to $\mathbb{D}_n$ is a representative of the desired element of the mapping class group.

**The braid group as the fundamental group of a configuration space**

Finally, it is also common to consider the braid group as the fundamental group of the configuration of $n$ disjoint points in the plane.

$$B_n = \pi_1(\underbrace{\mathrm{Sym}^n(\mathbb{C}) \setminus \Delta}_{\{x=(x_1,\dots,x_n)\in\mathbb{C}^n\}\setminus\{x_i=x_j\}/S_n}).$$

Given a geometric braid $\beta \subset \mathbb{D}^2 \times [0,1]$, one finds a loop $[0,1] \to \mathrm{Sym}^n(\mathbb{C})\setminus\Delta$ by associating with $t \in [0,1]$ the subset $N_t \subset \mathbb{D}^2 \subset \mathbb{C}$ given by $N_t \times \{t\} = \mathbb{D}^2 \times \{t\} \pitchfork \beta \subset \mathbb{D}^2 \times [0,1]$. By construction, this is a loop based at $N \subset \mathbb{C}$.

## 1.2. Notions of positivity for braids

In each braid group $B_n$, we consider the following submonoids of $B_n$.

$$\begin{aligned} P_n := \langle \sigma_1, \cdots, \sigma_{n-1} \rangle_{SG} &\subseteq SQP_n := \left\langle \sigma_{i,j} \mid 1 \leq i < j \leq n \right\rangle_{SG} \\ &\subseteq QP_n := \left\langle \beta\sigma_i\beta^{-1} \mid 1 \leq i < j \leq n \right\rangle_{SG} \\ &\subseteq B_n, \end{aligned}$$

where $\sigma_{i,j} := (\sigma_i\sigma_{i+1}\dots\sigma_{j-2})\sigma_{j-1}(\sigma_i\sigma_{i+1}\dots\sigma_{j-2})^{-1}$. Note that $\sigma_{i,i+1} = \sigma_i$. Geometrically, $\sigma_{i,j}$ corresponds to the braid in which strand $i$ and $j$ cross once positively; see Figure 1.1.

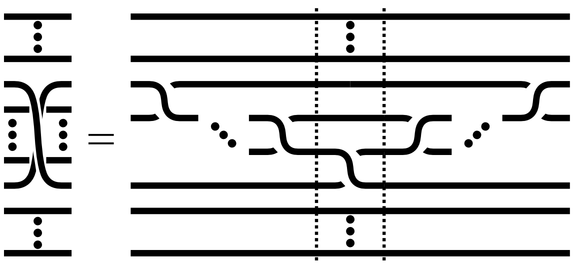
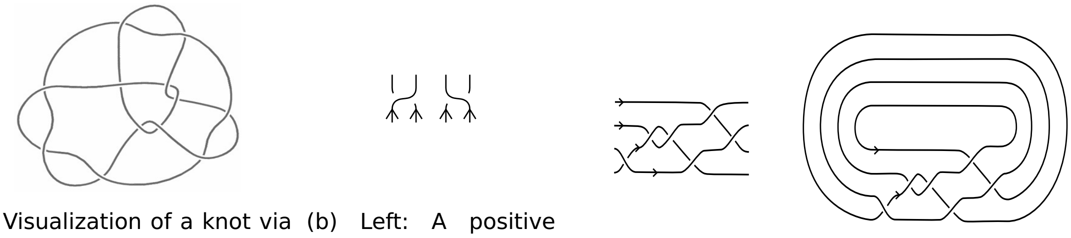


Figure 1.1. The strongly quasipositive standard generator $\sigma_{i,j} = (\sigma_i\sigma_{i+1}\dots\sigma_{j-2})\sigma_{j-1}(\sigma_i\sigma_{i+1}\dots\sigma_{j-2})^{-1}$.

The elements of $P_n$, $SQP_n$, and $QP_n$ are called *positive braids*, *strongly quasipositive braids*, and *quasipositive braids*. The study of quasipositive and strongly quasipositive braids and the knots and links that arise as their closure was initiated by Rudolph [Rud83a, Rud83c]. In spite of the somewhat convoluted name, quasipositivity and strong quasipositivity turn out to have surprising connections to smooth concordance theory (as discussed in the second lecture) and to the study of complex plane algebraic curves. While not the focus of our lecture, we point out that $P_n$ is the set of positive elements of the standard Garside structure on $B_n$, while $SQP_n$ is the set of positive elements of the dual Garside structure on $B_n$.

## 1.3. Knots and links

Non-empty smooth oriented 1-dimensional submanifolds of $S^3$ are called links. Connected links are called *knots*. Links are visually represented by link diagrams: generically immersed oriented 1-manifolds in the plane $\mathbb{R}^2$ with crossing information at each double point. See Figure 1.2a for a link diagram of a knot, which are called knot diagrams. Crossings are assigned a sign: + if they are positive, and − if they are negative, where positive and negative is defined in Figure 1.2b. Similarly, geometric braids can be represented by braid diagrams; in



(a) Visualization of a knot via a knot diagram. Orientation is suppressed.

(b) Left: A positive crossing. Right: A negative crossing.

(c) Left: A braid $\beta \in B_4$. Right: The closure $\mathrm{Cl}(\beta)$ of $\beta$.

fact, we have implicitly done so when introducing geometric braids and in Figure 1.1.

Braids are related to links via the so-called closure operation.

$$\mathrm{Cl}\colon \bigcup_{n\in\mathbb{N}} B_n \to \mathcal{L}\mathrm{inks}, \beta \mapsto \widehat{\beta},$$

where $\mathcal{L}$inks denotes the set of ambient isotopy classes of links and $\widehat{\beta}$ denotes the closure of a braid as illustrated (via braid and link diagrams) in Figure 1.2c. For this, we use the convention that geometric braids are oriented left to right.

The next two classical results establish surjectivity and describe the kernel of the closure operation.

**Theorem 1.1** (Alexander's Theorem [Ale23])**.** *The map* Cl *is surjective; i.e., every ambient isotopy class of a link arises as the closure of a braid.*

**Theorem 1.2** (Markov's Theorem [Mar36])**.** *For all braids $\alpha, \beta \in \bigcup_{n\in\mathbb{N}} B_n$,* $\mathrm{Cl}(\alpha)$ *and* $\mathrm{Cl}(\beta) \in \mathcal{L}$inks *are equal if and only if there exist a finite sequence of braids $\gamma_0, \ldots, \gamma_n$ such that $\alpha = \gamma_0$, $\beta = \gamma_n$ and for all $0 \le k < n$, $\gamma_{k+1}$ is obtained from $\gamma_k$ by conjugation (in $B_{n(k)}$, where $n(k)$ is such that $\gamma_k \in B_{n(k)}$) or by so-called stabilization or destabilization.*

Here, for $\beta \in B_n$, the positive (negative) *stabilization* of $\beta$ is the braid

$$\beta_{\mathrm{stab}}\iota(\beta)\sigma_n \in B_n \quad (\beta_{\mathrm{stab}}\iota(\beta)\sigma_n^{-1} \in B_n),$$

where $\iota$ denotes the inclusion group homomorphism of braid groups $B_n \to B_{n+1}$ defined by $\iota(\sigma_i) = \sigma_i$ for all $1 \le i \le n-1$; see Figure 1.3. A braid $\beta$ is said to be obtained from $\beta'$ by

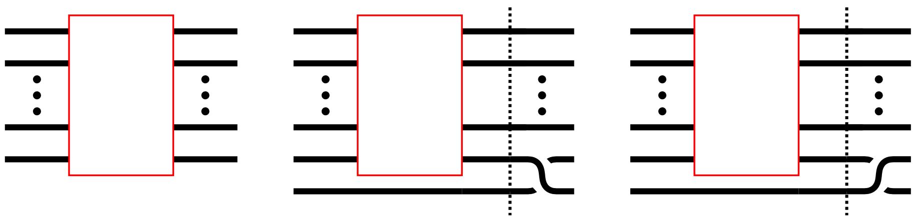


Figure 1.3. Left: a braid $\beta \in B_n$, all the braiding happens in the box (red).
Middle: the positive stabilization $\iota(\beta)\sigma_n \in B_{n+1}$ of $\beta$.
Right: the negative stabilization $\iota(\beta)\sigma_n^{-1} \in B_{n+1}$ of $\beta$.

(de-)stabilization if $\beta$ ($\beta'$) is the positive or negative stabilization of $\beta'$ ($\beta$).

We illustrate Theorem 1.2 with an example.

*Example* 1.3. Consider $\alpha = \sigma_1\sigma_1\sigma_1 \in B_2$ and $\beta = \sigma_1\sigma_2\sigma_1\sigma_2 \in B_3$. The closures of $\alpha$ and $\beta$ turn out to be isotopic. Here is a sequence of braids as guaranteed to exist by Theorem 1.2.

$$\gamma_0 = \sigma_1\sigma_1\sigma_1 \in B_2, \gamma_1 = \sigma_1\sigma_1\sigma_1\sigma_2 \in B_3, \gamma_2 = \sigma_1\sigma_1\sigma_2\sigma_1 = \sigma_1\sigma_2\sigma_1\sigma_2 \in B_3,$$

where $\gamma_1$ is obtained from $\gamma_0$ by a stabilization and $\gamma_2$ is obtained from $\gamma_1$ via conjugation by $\sigma_2 \in B_3$.

We invite the reader to draw the closures of these braids and verify that all resulting links are isotopic knots—the positive trefoil.

We provide a 'proof by example' for Theorem 1.1. The algorithm we explain by means of one example in fact yields more than the result that is claimed; see Remark 1.4. The algorithm is a reinterpretation of a proof by Rudolph [Rud99] as a variant of the Yamada-Vogel-algorithm [Yam87, Vog90].

*Proof of Theorem 1.1.* Let $L$ be any (isotopy class of a) link. Let $D$ be a diagram representing $L$. Without loss of generality, we can and do assume that the diagram is connected; i.e., the immersed 1-manifold obtained by forgetting the crossing information forms a connected subset of the plane. We first transform $D$ into a diagram of disjoint oriented circles and arcs as follows. We replace each crossing by an arc and merging the rest of the diagram into circles—called Seifert circles—as indicated in Figure 1.4. We will keep the diagram provided in Figure 1.4 as an illustration throughout the proof. By labeling each arc with the sign of its

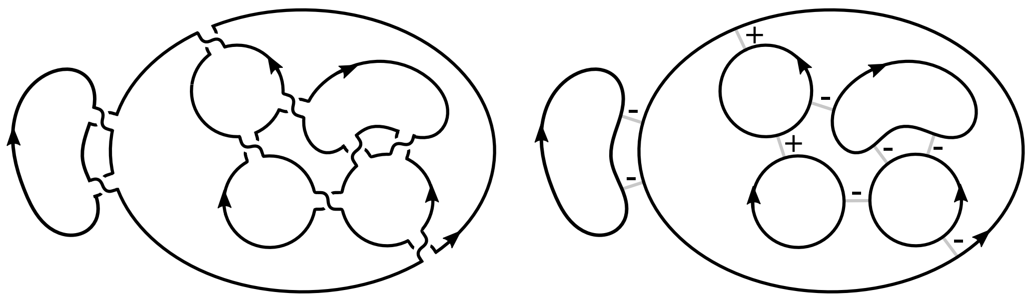


Figure 1.4. Left: A link diagram $D$ with two outermost Seifert circles.
Right: the result of replacing crossing with sign-labeled arcs (grey) and merging the rest of the diagram into Seifert circles.

corresponding crossing, this diagram retains the same information as the link diagram. We denote the result again by $D$. The point is that if the Seifert circles are maximally nested and all oriented in the same direction, then $D$ allows to read off a braid $\beta \in B_n$, where $n$ is the number of Seifert circles, such that $\mathrm{Cl}(\beta) = L$. Hence, we proceed to turn the diagram into one where the Seifert circles are nested and oriented in the same direction.

As a first step, we may change the diagram, without changing the link represented by it, to one that has only one outermost Seifert circle. For example, this can be achieved picking a point $p$ in the interior of a disc bounded by an innermost Seifert circle, then one-point compactifying $\mathbb{R}^2$ to get a diagram in $S^2 := \mathbb{R}^2 \cup \infty$, and considering the diagram in $\mathbb{R}^2$ that results via an identification of $S^2 \setminus \{p\}$ with $\mathbb{R}^2$. For example, in the case of the diagram in Figure 1.5, if we pick $p$ inside the left-most Seifert circle, the resulting diagram with one outermost Seifert circle is depicted in the top left of Figure 1.5.

We consider the outermost Seifert circle of $D$. and iteratively change the diagram to have more nested Seifert circles or more Seifert circles oriented in the same direction as the outermost circle. For this, we consider a *guiding arc*: an arc that starts and ends on two Seifert circles such that the interior of the guiding arc is disjoint of all arcs of $D$ and all Seifert circles and the two 'start and end' Seifert circles satisfy the following: they are either not nested and oriented the same as the outermost Seifert circle or nested and oriented oppositely such that the outer Seifert circle is oriented the same as the outermost Seifert circle; see Figure 1.5. We encourage the reader to check that, if no such guiding arc exists, then all Seifert circles are nested and oriented in the same direction.

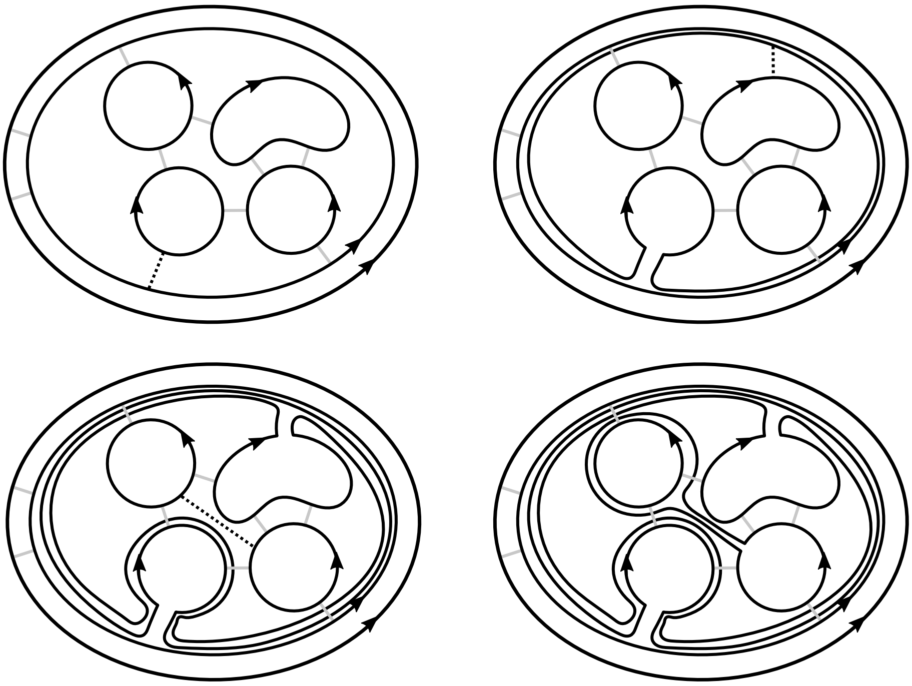


Figure 1.5. Top left: A diagram of a link with one outermost Seifert circle and a guiding arc (dotted) that starts and ends on two Seifert circles that are nested and oppositely oriented.
Top right: the result of changing the top left by modifying the inner of the two Seifert from circles touched by the guiding arc. And a next guiding arc.
Bottom left: Further step in nesting the Seifert circles and the next guiding arc.
Bottom right: Maximally nested Seifert circles all with the same orientation.
In all four diagrams the signs are suppressed.

The iteration goes as follows: modify one of the two given Seifert circles (if they are nested, we modify the inner one) by sliding it along the guiding arc over the other Seifert circle as indicated in Figure 1.5. We note that, in general, this transformation results in a diagram where arcs pass over Seifert circles. Translated back to a link diagram, this 'passing over' refers to the fact that both strands of the corresponding crossing pass over the part of the link diagram corresponding to the Seifert circles the arcs pass over; as indicated in Figure 1.6. We find a next guiding arc between Seifert circles as above, and iteratively modify Seifert circles by sliding one along that guiding arc, until no further guiding arc can be found. This process is illustrated for one diagram in Figure 1.5. We leave it to the reader to check that this process terminates (check that nesting or number of correctly oriented Seifert circles increases in each step) and that once no further guiding arcs exist, we have indeed a diagram where the Seifert circles are maximally nested and all oriented the same, as is the case in the bottom right of Figure 1.5.

At this point it is easy to read off a braid $\beta \in B_n$, where $n$ is the number of Seifert circles such that $\mathrm{Cl}(\beta) = L$. In fact, one reads of a word in $\sigma_{i,j}$ and their inverses. For the example at hand, as depicted on the right-hand side of Figure 1.6, we find

$$\beta = \sigma_{1,5}\sigma_{5,6}^{-1}\sigma_{5,6}^{-1}\sigma_{1,4}\sigma_{2,4}^{-1}\sigma_{2,5}^{-1}\sigma_{2,3}^{-1}\sigma_{2,3}^{-1}\sigma_{1,3}^{-1} \in B_6. \qquad \square$$

*Remark* 1.4. We note that throughout the above proof, the number of Seifert circles remains constant. In particular, the above proof recovers Yamada's result that, if a link $L$ is given by a link diagram with $n$ Seifert circles, then one can find an $n$-braid with closure $L$ [Yam87].

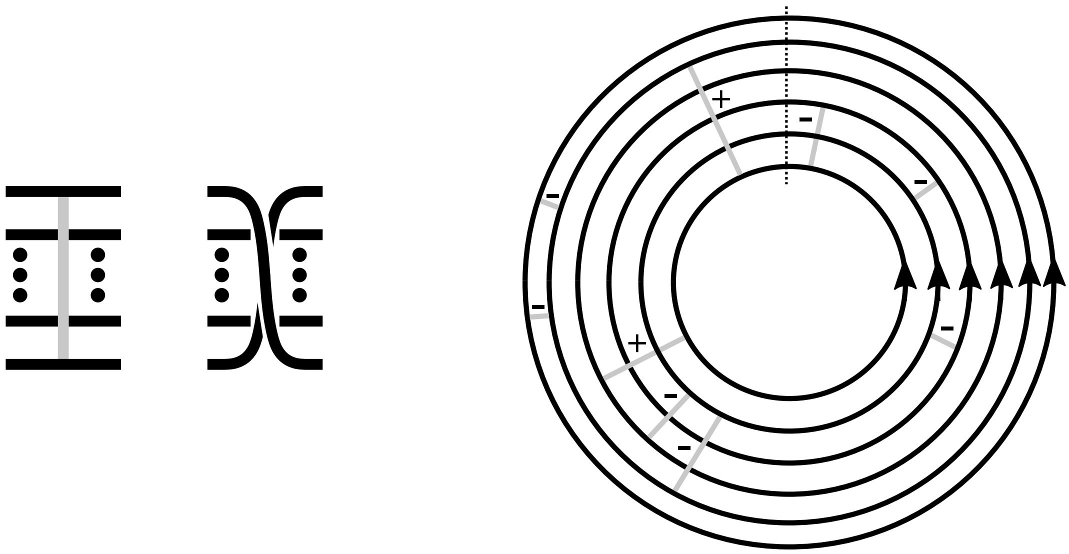


Figure 1.6. Left: Part of diagram with arcs crossing Seifert circles and the corresponding part of a link diagram. Signs and orientation suppressed.
Right: A diagram with maximally nested Seifert circles all with the same orientation. This is the result of isotoping diagram from the bottom right of Figure 1.5 to make the Seifert circles concentric. A line (dotted) cuts the diagram into a diagram for a geometric braid that is easily seen to be a product with factors $\sigma_{i,j}^{\pm 1}$, one for each arc (grey).

We also note that in the above proof, the number of generators $\sigma_{i,j}$ and their inverses in the word describing the braid $\beta$ is equal to the number of crossings in the original diagram. In fact, each positive crossing results in a generator $\sigma_{i,j}$, while each negative crossing results in a $\sigma_{i,j}^{-1}$. In particular, if the starting diagram has only positive crossings, then the resulting braid $\beta$ is strongly quasipositive.

Theorems 1.1 and 1.2 yield an approach to the study of links through braid theory. At this point the reader (an attendant of Winter Braids, hence naturally interested in braids but not necessarily links) may dare to wonder why one ought to study links. Within low-dimensional topology—the study of manifolds of dimension less than 5—are plentiful: 3-manifolds and 4-manifolds are intimately related to the study of knots and links; see also the interlude at the end of Lecture 2 for motivation regarding the study of 3-manifolds and 4-manifolds. In the next section, we provide motivation beyond low-dimensional topology: we consider singularity theory of complex plane curves, which turns out to be linked with the notions of positivity for braids introduced above, as we will see at the beginning of Lecture 2.

## 1.4. Links of Singularities

*Example* 1.5. We consider $f = y^2 - x^3$ and $g = y - x^3$. There is a qualitative difference between $f$ and $g$ (when considered as function germs defined in a neighborhood of $(0,0) \in \mathbb{C}^2$): $f$ is singular at $(0,0)$ while $g$ is not. This difference, and in fact many more properties, can be read off from the links $S^3 \cap V_f \subset S^3$ and $S^3 \cap V_f \subset S^3$, which are the positive trefoil, i.e. the closure of the 2-braid $\sigma_1^3$, and the unknot, i.e. the closure of the trivial 1-braid, respectively.

**Exercise 1.1.** *Visualize $S^3 \cap V_f$ and $S^3 \cap V_f$ by identifying these knots with knots in $\mathbb{R}^3 \cup \infty \cong S^3$. For this, consider the torus $\{(x,y) \in S^3 \mid |y|^p = |x|^q\}$ in $S^3 := \left\{v \in \mathbb{C}^2 \mid |v| = \epsilon\right\}$ for $p = 2$, $q = 3$ and $p = 1$, $q = 3$, respectively and visualize it in $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \infty \cong S^3$.*

For every $f \in \mathbb{C}[x,y]$ square-free polynomial with $f(0,0)=0$, we define (the isotopy class) of the link $L_f := S^3_\epsilon \cap V_f \subset S^3_\epsilon \cong S^3$, for small $\epsilon > 0$ as the *link of the singularity* of $f$. Here, $S^3_\epsilon := \left\{v \in \mathbb{C}^2 \mid |v| = \epsilon\right\}$ and 'small' means that, for $0 < \epsilon' \leq \epsilon$, $S^3_{\epsilon'}$ and $V_f$ intersect transversally. We end the first lecture with a remark describing how the link of a singularity depends on the properties of a singularity. We think of this as motivation for the philosophy that knot theory provides a tool to investigate the (topological) properties of singularities of complex plane curves—the zero-sets in $\mathbb{C}^2$ of one polynomial in $\mathbb{C}[x,y]$.

*Remark* 1.6*.* The link of singularity $L_f$ detects crucial information about $f$ (viewed as a germ of a function).

- $f$ is non-singular at $(0,0)$ if and only if $L_f$ is the unknot;
- $f$ is prime as a formal power series, i.e. an element of $\mathbb{C}[[x,y]]$ if and only if $L_f$ is a knot;
- the multiplicity of $f$ equals the braid index of $L_f$;
- $f$ and $g$ have the same topological type (i.e. there exists a homeomorphism $\phi\colon U_f \to U_g$ between open neighborhoods $U_f, U_g \subset \mathbb{C}^2$ of $(0,0)$ such that $g|_{U_g} \circ \phi = f|_{U_f}$) if and only if $L_f$ and $L_g$ are the same (isotopy class of a) link.

## 2. Lecture II: Notions of positivity, slice knots, and low-dimensional vs high-dimensional geometric topology

In this lecture, we first prove that links of singularities are positive braid links. We then put this result into broader context by discussing notions of positivity for links. Afterwards we turn to concordance, concretely we discuss the notion of a slice knot and how the existence of non-slice knots is related to the difference between low-dimensional and high-dimensional geometric topology.

### 2.1. Links of Singularities are positive braid links

The following result gives a first indication that notions of positivity of braids is related to the study of complex plane curves.

**Theorem 2.1** (Folklore)**.** *All links of singularities are positive braids links.*

A link $L$ is said to be a *positive braid link* if there exists a $\beta \in P_n$ for some $n$ such that $L = \widehat{\beta}$. Theorem 2.1 is in fact a consequence of a much more detailed description of the links of singularities. Rather than proving this more precise result, which is beautifully explained in the books by Brieskorn-Knörrer and Wall [BK86, Wal04], we provide a more qualitative argument, that focuses on why 'positivity' arises. (Spoiler: it is because holomorphic functions $g$ in one variable with $g(0) = 0$ wind positively around 0 in the target variable when the domain variable winds positively around 0.)

*Sketch of Theorem 2.1.* We use $f = y^2 - x^3$ for visualisations and as a motivating example.

To make explicit choice of coordinates simpler, rather than working with round spheres, we work with the boundary of bi-discs, which is readily seen to yield the same link (up to isotopy) see e.g. [BK86]. Furthermore, we arrange that $V_f$ intersects the boundary of the poly discs in the 'vertical sides' and the intersection is transversal. More precisely, we have the following:

- $S^3_\epsilon := \partial(\mathbb{D}_\epsilon \times \mathbb{D}_\epsilon)$, where $\mathbb{D}_\epsilon := \{z \in \mathbb{C} \mid |z| = \epsilon\}$, and
- $V_f \pitchfork S^3_\epsilon \subset S^1 \times \mathbb{D}_\epsilon \xrightarrow{\mathrm{pr}} \mathbb{D}_\epsilon$ is a regular map restricted to the smooth 1-manifold $V_f \pitchfork \subset S^1 \times \mathbb{D}_\epsilon$.

In the case of $f = y^2 - x^3$ and $\epsilon < 1$, we find

$$V_f \pitchfork S^3_\epsilon = \{(x,y) \in S^1_\epsilon \times \mathbb{D}_\epsilon \mid f(x,y) = 0\} = \{(x, \pm\epsilon^{\frac{2}{3}}\sqrt{x^3}) \mid x \in S^1_\epsilon\};$$

see Figure 2.1 for an illustration.

In general, the second point can be arranged by choosing $f$ to be of the following form. Let $n$ be the multiplicity—the smallest degree among the monomials of $f$—and, w.l.o.g., consider $f = x^n + \cdots$ with the coefficients of the other monomials small if needed. (This can easily be arranged by a linear coordinate change.) We assume that $f$ is of this form for the rest of the proof.

In this case, for sufficiently small $\epsilon$, we find

$$V_f \pitchfork S^3_\epsilon = \{(x,y) \in S^1_\epsilon \times \mathbb{D}_\epsilon \mid f(x,y) = 0\} = \{(x, r_1(x)), \dots, (x, r_n(x)) \mid x \in S^1_\epsilon\},$$

where $r_i$ are roots of $f(x,\cdot)$, meaning locally defined holomorphic functions on $\mathbb{C} \setminus \{0\}$ with $r_i(x) \to 0$ for $x \to 0$. In fact, the $r_i$ can be described as $r_i(x) = g_i(s)$, where $g_i$ are holomorphic function germs around 0 and $s = x^{\frac{1}{n}}$ by the so-called Puiseux theorem.

We identifying $S^3_\epsilon$ with $\mathbb{R}^3 \cup \infty$ such that $S^1_\epsilon \times \mathbb{D}_\epsilon$ is identified with a standard solid torus in $\mathbb{R}^3 \cup \infty$ in which we see braid closures. In fact, we choose the embedding such that projection $(x_1, y_1), (x_2, y_2) \in S^1_\epsilon \times \mathbb{D}_\epsilon$ project to the same point of the diagram plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ if and only if $x_1 = x_2$ and $y_1$ and $y_2$ have the same real part; see Figure 2.2. We show that under this identification, $V_f \pitchfork S^3_\epsilon$ is the closure of a positive braid.

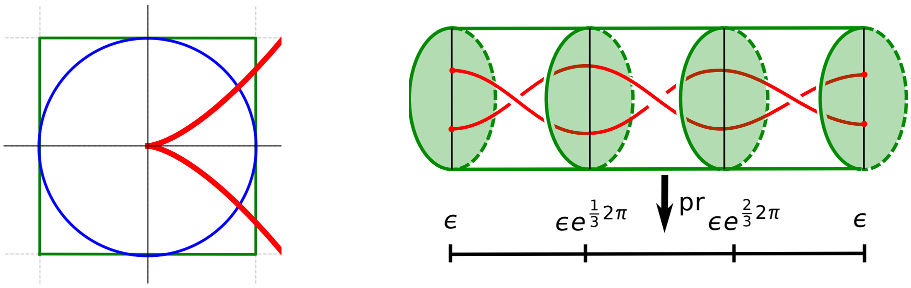



Figure 2.1. Left: The subsets $V_{y^2-x^3}$ (red), $S^3_\epsilon$ (once as $S^1_\epsilon \times \mathbb{D}^2_\epsilon \cup \mathbb{D}^2_\epsilon \times S^1_\epsilon$ (green) and once as the Euclidean sphere (blue)) of $\mathbb{C}^2$ illustrated by their intersection with $\mathbb{R}^2 \subset \mathbb{C}^2$.
Right: $S^1_\epsilon \times \mathbb{D}^2_\epsilon$ (green) and $S^1_\epsilon \times \mathbb{D}^2_\epsilon \cap V_{y^2-x^3}$ (red), the projection pr to $S^1_\epsilon$ (black) is realized as the vertical projection. $S^1_\epsilon$ is depicted as a line with identified endpoints.

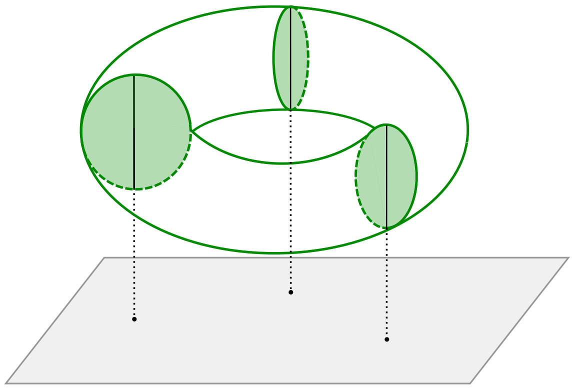


Figure 2.2. $S^1_\epsilon \times \mathbb{D}_\epsilon$ (green) in $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \infty \cong S^3_\epsilon$. The projection to the $\mathbb{R}^2 \times \{0\}$ (grey) is such that the preimages of a point corresponds to a line (black) in some $\{x\} \times \mathbb{D}^2_\epsilon$ with same real part in the second coordinate.

For this, for any choice of two different roots $r_i$ and $r_j$ (defined locally on some open subset), we consider their difference

$$h_{i,j}(x) = r_j(x) - r_i(x).$$

For example, in case of $f = y^2 - x^3$, we find that $h_{1,2}(x) = \epsilon^{\frac{2}{3}}\sqrt{x^3} - \epsilon^{\frac{2}{3}}\sqrt{x^3}$, where $\sqrt{x^3}$ denotes a choice of a local root of $x^3$. The following claim is the key observation needed to conclude the proof.

**Claim 2.2.** *If $x(t) = \epsilon e^{2\pi t}$ for $t \in (t_0 - \delta, t_0 + \delta)$ for sufficiently small $\epsilon, \delta > 0$, then $t \mapsto \arg\left(h_{i,j}(x(t))\right)$ is monotonically increasing.*

Assuming Claim 2.2, we conclude the proof as follows. Consider a double point of the diagram obtained by projecting $V_f \pitchfork S^3_\epsilon$ to $\mathbb{R}^2 \times \{0\}$, say $(x(t_0), r_i(x(t_0)), (x(t_0), r_j(x(t_0))$. We explicitly parametrize $V_f \pitchfork S^3_\epsilon$ in a neighborhood of the two points, finding the two parametrized intervals $(x(t), r_i(x(t)), (x(t_0), r_j(x(t))$ for $t \in (t_0 - \delta, t_0 + \delta)$. To determine the sign of the crossing in the diagram, we determine how the two points rotate relative to each other, that is, we consider the argument of $h_{i,j}(x(t))$ as $t$ varies. Therefore, by Claim 2.2, we have a positive crossing.

We end the proof with a short argument for why Claim 2.2 holds. Intuitively, up to taking roots, $h$ is a germ of a holomorphic function defined at 0 (since $r_i(x) \to 0$ for $x \to 0$) and as such rotates positively around the origin as $x$ turns positively around $S^1_\epsilon$. More precisely, as $h(x) = g_j(t) - g_i(t)$, where $t = x^{\frac{1}{n}}$, and we find $h(x) = h(x^{\frac{1}{n}})$ is a holomorphic function in the variable $x^{\frac{1}{n}}$ defined around 0. Hence, $h(x^{\frac{1}{n}}) = a(x^{\frac{1}{n}})^k + \ldots$ for some $a \in \mathbb{C}^*$ and $k$ a positive integer. At a small distance $\epsilon$ to the origin the term $a(x^{\frac{1}{n}})^k = ae^{kt\pi i/n}$ dominates, hence Claim 2.2 follows since strict monotonicity in $t$ is evident for $\arg(ae^{kt\pi i/n})$. □

### Rudolph's notions of positivity for links

After having seen that links of singularities are the closure of positive braids, we discuss ever more inclusive notions of positivity. Concretely, this concerns *positive links*—the links that arise from link diagrams with only positive crossings—and the notions of *strongly quasipositive* and *quasipositive links*—the links that arise as the closures of braids in $SQP_n$ and $QP_n$, respectively, for some $n$. We shorten the latter two to *sqp links* and to *qp links*, respectively. We first note following chain of (strict) inclusions.

(2.1) links of singularities $\subsetneq$ positive braid links $\subsetneq$ positive links $\subsetneq$ sqp links $\subsetneq$ qp links,

where the first inclusion is restating Theorem 2.1, and the second, fourth and fifth are immediate from the definitions. The third inclusion is discussed below; see Theorem 2.4.

We note that while qp links have an unappealing name, they are very much tied to the study of complex plane curves. Rudolph and Boileau-Orevkov established that qp links are exactly the links that arise when intersecting complex plane curves transversly with the standard 3-sphere in $\mathbb{C}^2$.

**Theorem 2.3** ([Rud83a, BO01])**.** *A link $L$ is quasipositive if and only if there exists a square-free polynomial $f \in \mathbb{C}[x, y]$ such that $V_f \pitchfork S^3 \subset S^3 \subset \mathbb{C}^2$ is isotopic to $L$.*

Here, $V_f \pitchfork S^3$ is understood to mean that all singular points of

$$V_f := \{(x, y) \mid f(x, y) = 0\}$$

are disjoint from $S^3$ and that the smooth surface given by the regular points of $V_f$ intersects $S^3$ transversally as smooth manifolds. A proof of this result is beyond the scope of this mini-course, but we mention it to show the strong ties between notions of positivity in knot theory and algebraic geometry of plane curves: not only are links of singularities—intersections of complex plane curves with sufficiently small 3-spheres—positive braid links, the larger class of sq links are exactly those links that arise when one considers the intersection between complex plane curve and arbitrary (round) 3-spheres. In contrast, links of singularities are the links that arise by intersecting algebraic curves with spheres of arbitrarily small radius centred at a singular point of the algeraic curve.

We conclude the second lecture by explaining the third inclusion in (2.1). In other words, we prove the following result of Rudolph.

**Theorem 2.4** ([Rud99])**.** *Positive links are strongly quasipositive.*

This in fact an immediate consequence of the proof of Alexander's theorem we provided in Lecture 2.

*Proof of Theorem 2.4.* Let $L$ be a positive link, given by a diagram $D$ with only positive crossings (which exists by the definition of being a positive link) and $n$ Seifert circles. Then, by Remark 1.4, there exists a $\beta \in SQP_n$ with $\mathrm{Cl}(\beta) = L$; in particular, $L$ is a strongly quasipositive link. □

## 2.2. Slice knots

The notion of a slice knot goes back to Fox. We denote by $B^4$ the standard closed 4-ball with boundary $S^3$.

**Definition 2.5.** A knot $K$ is said to be *slice* if there exists a smoothly embedded disc $\mathbb{D}^2 \cong D \subset B^4$ such that $\partial D = K \subset S^3 = \partial B^4$.

If there exists such a disc $D$ that is embedded as a topological neat submanifold (equivalently if $D$ is embedded topologically locally flatly), $K$ is said to be *topologically slice*.

The unknot and $K\#-K$ for any knot $K$ are examples of slice knots. The trefoil is an example of a knot that is not slice, which we will discuss below.

**Exercise 2.1.** *Let $K$ be any knot and denote by $-K$ the result of mirroring (apply an orientation reversing diffeomorphism of $S^3$) and orientation reversal.*

- *Given a diagram $D$ of $K$ construct a diagram for the* connected sum $K\#-K$ *formally defined as the result of the connected sum* $(S^3,K)$ *and* $(S^3,-K)$ *along* 3*-balls in* $S^3$ *that intersect $K$ and $-K$ in an interval.*

- *Show that $K\#-K$ is slice.*
*Extensive hint: Inspired by the description of $K\#-K$ above, identify $B^4$ with $\bigcup_{\theta\in[0,\pi]} B^3_\theta$ as a union of* 3*-balls $B^3 \times \{\theta\}$, where all the boundary* 2*-spheres are identified. Consider $K\#-K \subset B^3_0 \cup B^3_\pi \cong S^3 = \partial B^4$ where the intersection with $B^3_0$ and $B^3_\pi$ are as a set the same interval, say $I\times\{0\}$ and $I\times\{\pi\}$. Noting that the identification $B^3_0\cup B^3_\pi \cong S^3$ is orientation reversing on exactly one of the* 3*-balls, you find that $D := \bigcup_{\theta\in[0,\pi]} I\times\{\theta\}$ is a smoothly embedded disc witnessing the slice property of $K\#-K$.*

We make an attempt to place the study of slice knots with in geometric topology and hint at its relevance. None of what follows in the two short subsections below is needed logically for lecture 3.

### Why study slice knots?

The question of what knots are slice (and more generally which 1-manifolds in 3-manifolds are the boundary of what type of surfaces embedded in 4-manifolds) has grown to be a subfield of low-dimensional topology (aka concordance theory) with intimate connections to the study and classification of 4-manifolds. In particular, the difference between slice and topologically slice knots is intimately related to the stark difference between smooth and topological 4-manifolds that emerged as a focus of research starting with the seminal work by Donaldson [Don83] and Freedman [Fre82].

As a concrete connection with sliceness, an observation of Gompf (compare [GS99, p. 522]) is that topologically slice knots that are not slice can be used to build exotic $\mathbb{R}^4$s—smooth 4-manifolds that are homeomorphic but not diffeomorphic to $\mathbb{R}^4$. For context, such objects only exist in dimension 4, in fact, one has the following dichotomy between dimension 4 and all other dimensions:

- $d \neq 4$: if $R$ is a smooth $d$-dimensional manifold that is homeomorphic to $\mathbb{R}^d$, then $R$ is diffeomorphic to $\mathbb{R}^d$ [Sta62].

- $d = 4$: There exists smooth 4-dimensional manifolds $R_j$ indexed by an uncountable index set $J$ such that $R_j$ are not diffeomorphic to $\mathbb{R}^4$ and $R_j$ and $R_{j'}$ are not diffeomorphic for all $j \neq j' \in J$, but, for all $j \in J$, $R_j$ is homeomorphic to $\mathbb{R}^4$ [Tau87].

There are many algebraic topology tools that can be used to establish for that certain knots are not slice, arguably most famously the Fox-Milnor condition [FM66]. Often these obstructions work both in the smooth and topological category. In Lecture 3 we will focus on an obstruction that is specific to smooth sliceness and relates to notions of positivity as introduced in the last subsection. In particular, this obstruction is one way of establishing for concrete knots that they are topologically slice, but not smoothly slice; hence it is one possible route to understanding the existence of exotic $\mathbb{R}^4$s. Before we turn to this, here is a brief interlude that indicates why the problem of determining which knots are slice can be understood to lie at the heart of the mystery that is 4-manifold theory.

### Interlude: high-dim. vs low-dim. geometric topology

Smale's $h$-cobordism theorem [Sma60] started a revolution in geometric topology today known as surgery theory: a tool that brings problems about manifolds into the realm of being studied and fully resolved by methods of homotopy theory and algebraic topology. Crucially, these methods work for manifolds of 'high enough' dimension (where, depending on the application, high enough is usually 'at least 5' or 'at least 6'). It is this distinction of methods that has led to geometric topology having the subfields of low-dimensional and hight dimensional topology, and it is inapplicability of surgery methods in low-dimensions that makes the study of low-dimensional manifolds, most prominently 4-manifolds, a particularly fascinating subject. In what follows we go back to only concerning ourselves with the smooth setup. One key fact that can be understood to be at the core of why 'surgery works' is the following.

**Observation 2.1** (Key observation for surgery theory)**.** *Let $d \geq 5$ be an integer, $W$ a smooth $d$-dimensional manifold with boundary and $K \subset \partial W$ a smoothly embedded circle that is null-homotopic in $W$. Then there exists smoothly embedded disc $\mathbb{D}^2 \cong D \subset W$ such that $\partial D = K \subset \partial W$.*

The statement also holds for $d = 2$ (by the classification of surfaces) and for $d = 3$, where it is known as Dehn's lemma. The latter, proven by Papakyriakopoulos, is arguably one of the first key technical results in low-dimensional topology [Pap57]. The version stated (i.e. when $d \geq 5$), is a rather immediate consequence of generic transversality. Taking the vague motivation that this key observation is fundamental in the inner workings of surgery theory (buzzword: Whitney trick), we find an explanation why 4-manifolds behave fundamentally different. Namely, even the "local surgery problem" (the case when $W$ is the $d$-dimensional ball) is tricky in dimension 4: it corresponds exactly to asking which knots are slice.

*Example* 2.6. Consider $W = B^4$ the closed unit 4-ball in $R^4$ with boundary $\partial W = S^3$. All knots $K \subset S^3$ are null-homotopic in $W = B^4$, but not all knots are slice! For example the trefoil knot is not a slice knot.

## 3. The slice-Bennequin inequality, the local Thom conjecture, and the Thom conjecture for $\mathbb{CP}^2$

Given that not all knots are slice, one is led to define the slice genus, a quantity that measures how far a knot is from being slice.

**Definition 3.1.** Let $K$ be a knot. The non-negative integer

$$g_4(K) := \min\{\text{genus}(F) \mid \begin{array}{l} F \subset B^4 \text{ smooth oriented surfaces} \\ \text{such that } \partial F = K \subset S^3 = \partial B^4 \end{array} \}$$

is called the *slice genus* of $K$.

Before we turn to the main result of this lecture, the slice-Bennequin inequality, which allows one to bound the slice genus from below, we observe explicit constructions of surfaces that provide upper bounds for quasipositive knots (in particular for strongly quasipositve and positive braid knots), that as we will see below, turn out to be optimal.

**Observation 3.1.** *Let $K$ be a knot such that there exists a positive integer $n$ and $\beta \in QP_n$ such that $K = \mathrm{Cl}(\beta)$; in particular, $K$ is a quasipositive knot. Then, $g_4(K) \leq \frac{1-n+\mathrm{wr}(\beta)}{2}$.*

Here, $\mathrm{wr}(\beta)$ denotes the exponent sum of $\beta$ also known as the writhe. In case of $\beta \in QP_n$, i.e. $\beta = \Pi_{k=1}^{\ell} \omega_k \sigma_{i(k)} \omega_k^{-1}$ where the $\omega_k$ are any $n$-braid and $\sigma_{i(k)}$ are standard generators, we have $\mathrm{wr}(\beta) = l$.

*Proof of Observation 3.1.* This observation follows by noting that writing the braid $\beta$ as a product $\Pi_{k=1}^{\ell} \omega_k \sigma_{i(k)} \omega_k^{-1}$ guides the construction of a surface $F_K \subset B^4$ of genus $\frac{n-1+\mathrm{wr}(\beta)}{2}$ such that $\partial F_K = K$.

Concretely, we take $F_K$ to be the results of $n$-discs, one for each strand on the braid, and $l$ many half-twisted bands, one for each factor of the product. Both the discs and the half-twisted bands are taken to be in $S^3$, but, whenever needed to have an embedded rather than an immersed surface, their interior are slightly pushed into the interior of $B^4$ to avoid self-intersections. In case of $\beta$ being strongly quasipositive (in particular positive), i.e. when the factors $\omega_k \sigma_{i(k)} \omega_k^{-1}$ are of the form $\sigma_{i(k),j(k)}$, no pushing into $B^4$ is needed and the surface $F_K$ can be taken to be in $S^3$. The exact procedure is illustrated in Figure 3.1. For details of this construction of a so-called ribbon surface $F_K$ see e.g. [Rud83b].

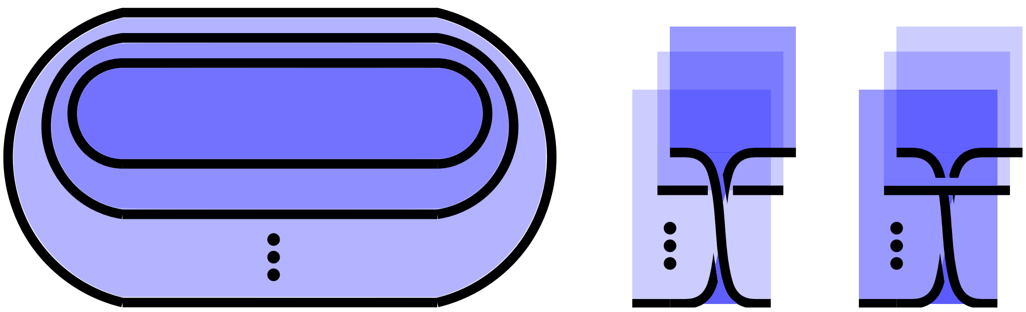


Figure 3.1. Left: The $n$ discs.
Middle: A band associated with a factor of the form $\sigma_{i(k),j(k)}$.
Right: A band associated with a factor of the form $\omega_k \sigma_{i(k)} \omega_k^{-1}$.

The Euler characteristic of $F_K$ equals the number of discs minus the number of bands; that is we have, $\chi(F_K) = n - \ell$. As the surface $F_K$ is connected and has one boundary component, genus$(F_K)$ equals $\frac{1-\chi(F_K)}{2} = \frac{1-n+\ell}{2}$ as desired. □

It turns out that the surfaces constructed in the proof of Observation 3.1 realize the slice genus; in other words, the inequality from Observation 3.1 is an equality. This follows from the theorem below, which constitutes the main result of the third lecture. Rudolph established it in a paper named *Quasipositivity as an obstruction to sliceness* [Rud93].

**Theorem 3.2** (Slice-Bennequin inequality [Rud93, KM93])**.** *Let $K$ be a knot and let $K = \mathrm{Cl}(\beta)$, where $n$ is a positive integer and $\beta \in B_n$. Then $g_4(K) \geq \frac{n-1+\mathrm{wr}(\beta)}{2}$.*

As an immediate corollary of the slice-Bennequin inequality and Observation 3.1, we determine the slice genus of quasipositive knots.

**Corollary 3.3.** *If a knot $K$ satisfies $K = \mathrm{Cl}(\beta)$ for some positive integer $n$ and $\beta \in QP_n$, then $g_4(K) = \frac{n-1+\mathrm{wr}(\beta)}{2}$.*

*Example* 3.4*.* We determine the slice genus of the trefoil and the so-called positive Whitehead double of it.

- Consider the trefoil $\mathrm{Cl}(\sigma_1^3)$, where $\sigma_1^3 \in B_2$. By Corollary 3.3, we have
$$g_4(\mathrm{Cl}(\sigma_1^3)) = \frac{1-2+3}{2} = 1.$$
- Consider the positive Whitehead double of the trefoil; that is, $\mathrm{Cl}(\beta)$ for
$$\beta = \sigma_{2,6}\sigma_6\sigma_{1,4}\sigma_{2,5}\sigma_{4,6}\sigma_{3,5}\sigma_{1,3}\sigma_6.$$
By Corollary 3.3, we find
$$g_4(\mathrm{Cl}(\beta)) = \frac{1-7+8}{2} = 1.$$

The second example is relevant because this knot is topologically slice. Topological sliceness follows from the fact that all Whitehead doubles are topologically slice since they all have Alexander polynomial 1 and a celebrated result of Freedman and Freedman and Quinn [Fre82, FQ90] states that knots with Alexander polynomial 1 are topologically slice. The details of this argument are beyond the scope of these lecture notes. However, the upshot is that $\mathrm{Cl}(\beta)$ is a strongly quasipositive knot that is topologically slice but not slice; hence, as discussed in the end of Lecture 2, this is a knot that can be used to construct an exotic $\mathbb{R}^4$.

## Torus knots

The proof of the slice-Bennequin inequality is surprisingly short, given one key input: the value of the slice genus for torus knots. Torus knots are prototypical examples of links of singularities. For positive coprime integers $p, q$, we define $T_{p,q} := V_{y^p - x^q} \cap S^3 \subset S^3$. By Theorem 2.1, we know that $T_{p,q}$ is a positive braid link. In fact, the positive braid
$$\beta_{p,q} = (\sigma_1\sigma_2\ldots\sigma_{p-1})^q$$
has closure $T_{p,q}$. (Indeed, following the proof of Theorem 2.1, one finds exactly the braid $\beta_{p,q}$; this was made explicit in the proof for the case $p = 2$ and $q = 3$ ; see Figure 2.1.)

The local Thom conjecture (also known as the Milnor conjecture) asserts that the torus knot $T_{p,q}$ has slice genus $(p-1)(q-1)/2$. As one of the first spectacular applications of Seiberg-Witten theory, Kronheimer and Mrowka proved the local Thom conjecture.

**Theorem 3.5** (Local Thom conjecture, [KM93])**.** *For all coprime positive integers $p, q$,*
$$g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

We note that this is exactly the statement of Corollary 3.3 for the braids $\beta = \beta_{p,q}$. From this perspective, the result might seem a bit arbitrary: why, after all, would we focus on these special braids and why would one conjecture that the smallest genus among surfaces in the 4-ball with boundary $T_{p,q}$ is $(p-1)(q-1)/2$. The reason is results from the study of complex plane curves, which was our motivation to study notions of (quasi)positivity all along. Concretely, it turns out that if $f$ is a square-free polynomial in $\mathbb{C}[x,y]$ such that $V_f \subset \mathbb{C}^2$ that intersects $S^3$ transversally and $V_f \cap S^3$ is (up to isotopy) $T_{p,q}$, then $\mathrm{genus}(V_f \cap B^4) = (p-1)(q-1)/2$. Hence, the local Thom conjecture asserts that among all smooth surfaces $F \subset B^4$ with boundary $T_{p,q}$, those that arise from intersecting $B^4$ with a smooth plane curve have smallest possible genus. We comment further on this (and the relation to the Thom Conjecture for $\mathbb{CP}^2$) in the last section of the third lecture. For now we assume Theorem 3.5 and use it to prove the slice-Bennequin inequality.

## The local Thom conjecture implies the slice-Bennequin inequality

The proof we provide is essentially Rudolph's proof but with care to only use knot theory and invoke the local Thom conjecture exactly as stated in Theorem 3.5.

*Proof of Theorem 3.2.* Let $K = \mathrm{Cl}(\beta)$ for some braid $\beta \in B_n$ for some $n$. We take a word $w$ in the generators $\sigma_i^{\pm}$ that expresses $\beta$, and we insert positive generators $\sigma_i$ into it until the resulting word expresses the braid $\beta_{n,q}$ for some positive integer $q$ that is coprime to $n$. We denote by $k$ the number of such insertions that were needed.

To be concrete, say $w$ has $q'$ positive and $l$ negative generators. By adding $l$ positive ones, one $\sigma_i$ after every $\sigma_i^{-1}$, after reducing the word by cancelling generators with their inverses next to them, we find a word consisting only of positive generators, $p'$ many to be exact. Adding to each of those positive $\sigma_i$ the word $\sigma_1\sigma_2\ldots\sigma_{i-1}$ in front and the word $\sigma_{i+1}\sigma_{i+2}\ldots\sigma_{n-1}$ after results in the braid $(\sigma_1\sigma_2\ldots\sigma_{n-1})^{q'}$. If $q'$ is coprime to $n$, we have just established that we can add $l+(n-2)q'$ positive generators to $w$ to reach a word representing the braid $\beta_{n,q'}$. In general, $q'$ and $n$ are not necessarily coprime. We pick $q \geq q'$ coprime to $n$ and add $(\sigma_1\sigma_2\ldots\sigma_{n-1})^{q-q'}$ at the end of the word. This shows there is a positive $q$ coprime to $n$ such that we can add $k := l+(n-2)q'+(q-q')(n-1)$ positive generators to $w$ resulting in a word that represents $\beta_{n,q} \in B_n$. Note also that $k = \mathrm{wr}(\beta_{n,q}) - \mathrm{wr}(\beta)$.

**Claim 3.6.** *If $\beta_1, \beta_2 \in B_n$ are such that $\mathrm{Cl}(\beta_1)$ and $\mathrm{Cl}(\beta_2)$ are knots and $\beta_2$ can be obtained from $\beta_1$ by adding $k$-many generators to a word representing $\beta_1$, then there exists a cobordism of genus $k/2$ between $\mathrm{Cl}(\beta_1)$ and $\mathrm{Cl}(\beta_2)$. Here, a* cobordism *between two knots is properly and smoothly embedded oriented surface $C \subset S^3 \times [0,1]$ with two boundary components, one lying in each of the two boundary components $S^3 \times \{0\}$ and $S^3 \times \{1\}$, that are the knots (seen in $S^3 \times \{0\} \cong S^3$ and $S^3 \times \{1\} \cong S^3$, respectively).*

Using the claim, we readily complete the proof. Let $F \subset B^4$ be any smooth oriented surface with boundary $K \subset \partial B^4$. Rescale $F$ by multiplication with $\frac{1}{2}$, which makes it lie inside $B^4_{\frac{1}{2}}$—the closed ball around the origin of radius $\frac{1}{2}$ in $B^4$. Denoting that surface (by abuse of notation) as $F$ consider the surface $F' = F \cup C$, where $C \subset B^4 \setminus \mathrm{int}(B^4_{\frac{1}{2}})$ is the cobordism from Claim 3.6 applied to $\beta$ and $\beta_{n,q}$ under an identification of $S^3 \times [0,1]$ with $B^4 \setminus \mathrm{int}(B^4_{\frac{1}{2}})$ such that the boundary of $C$ that is $K$ (up to isotopy) is exactly $\partial F$. The genus of $F'$ is $\mathrm{genus}(F') = \mathrm{genus}(F) + \mathrm{genus}(C)$; hence, using that $\mathrm{genus}(F') \geq (n-1)(q-1)/2$ by Theorem 3.5, we find

$$\begin{aligned}
\mathrm{genus}(F) &= \mathrm{genus}(F') - \mathrm{genus}(C)\\
&\geq \frac{(n-1)(q-1)}{2} - \frac{k}{2}\\
&= \frac{1-n+\mathrm{wr}(\beta_{n,q})}{2} - \frac{\mathrm{wr}(\beta_{n,q}) - \mathrm{wr}(\beta)}{2}\\
&= \frac{1-n+\mathrm{wr}(\beta)}{2}.
\end{aligned}$$

As $F \subset B^4$ was any smooth oriented surface with boundary $K \subset \partial B^4$, this shows $g_4(K) \geq \frac{1-n+\mathrm{wr}(\beta)}{2}$ as desired.

It remains to discuss Claim 3.6, which we sketch in this last paragraph to conclude the proof. The key observation is that adding a generator to a word that describes a braid $\beta$ amounts to performing a saddle move on $\mathrm{Cl}(\beta)$; see Figure 3.2. Two links related by a saddle

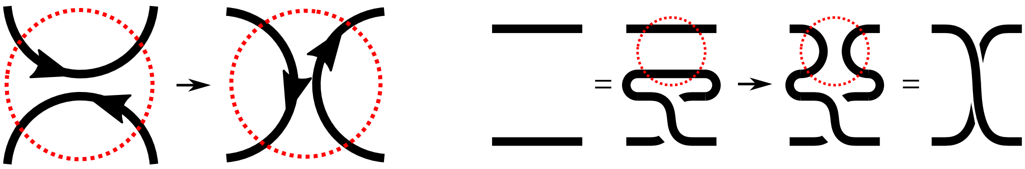


Figure 3.2. Left: A saddle move: local change (arrow) of a link given by changing it inside a 3-ball (red) that intersects the link in two arcs as indicated. Right: Realization of the effect of adding a generator into a braid word on the closure of the braid as isotopies (indicated by =) and one saddle move (arrow).

move can be realized as the two sides of the boundary of a cobordism given by one saddle; in particular, the cobordism has Euler characteristic $-1$. Iteratively doing $k$ saddle moves yields a cobordism $C$ given by $k$ saddles; in particular, $C$ has Euler characteristic $-k$. The cobordism $C$ is connected (as there are no maxima and minima, all of its components have boundary on both sides, and $\partial C$ consist of a knot on both sides). For a connected oriented surface $C$ with two boundary components, we have $\mathrm{genus}(C) = -\chi(C)/2 = k/2$; hence, $C$ is a cobordism of the form we claimed exists. □

We have established the slice-Bennequin inequality, a strong tool that can be used to illuminate the difference between topological and smooth sliceness, based on the local Thom conjecture. In turn the local Thom conjecture has been established by multiple methods. After the original Seiberg-Witten-invariant proof, Rasmussen found a proof using his $s$-invariant stemming from Khovanov homology [Ras10], in particular providing a proof not based on gauge-theoretic methods. Either of these proofs is beyond the scope of these notes. However, in the last section, we briefly connect the local Thom conjecture with the Thom conjecture for $\mathbb{CP}^2$.

## The Thom conjecture for $\mathbb{CP}^2$ implies the local Thom conjecture

The local Thom conjecture establishes that smooth complex plane curves intersected with $B^4$ realize the smallest genus among the smooth surfaces in $B^4$ with the same boundary. The Thom conjecture for $\mathbb{CP}^2$ establishes that smooth complexe curves in $\mathbb{CP}^2$ are genus minimizing in their second homology class. For context, recall that a smooth complex curve $C \subset \mathbb{CP}^2$ of degree $d$ (i.e. $C$ is defined by a square-free homogeneous polynomial of degree $d$ in three variables) has genus $(d-1)(d-2)/2$ (this is known as the genus-degree formula, see e.g. [Wal04, Chap.7]) and that it is in the homology class $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2;\mathbb{Z})$.

**Theorem 3.7** (Thom Conjecture for $\mathbb{CP}^2$, [KM94])**.** *Let $d$ be a positive integer. If $\Sigma \subset \mathbb{CP}^2$ is a smooth surface representing the homology class $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2;\mathbb{Z})$, then* $\mathrm{genus}(F) \geq \frac{(d-1)(d-2)}{2}$.

*Sketch of Theorem 3.5 using Theorem 3.7.* Without loss of generality, we only consider the case $T_{p,p-1}$ for $p \geq 2$. We invite the reader to check that the other cases follow; e.g., using Claim 3.6.

Let $F \subset B^4$ be a smooth surface with $\partial F = T_{p,p-1} \subset S^3$. We want to show that $\mathrm{genus}(F) \geq \frac{(p-2)(p-1)}{2}$.

We first observe that there exists a surface $F' \subset B^4$ with $\partial F' = T_{p,p} \subset S^3$ with the same genus as $F$. To see this, we note that there exists a cobordism $C$ consisting of $p-1$ saddles between the knot $T_{p,p-1}$ and the link with $p$ components $T_{p,p} := \mathrm{Cl}(\beta_{p,p})$. To see this, one can argue using Claim 3.6, or rather its analog for links rather than knots, and define $F' = F \cup C$ exactly as in the proof of Theorem 3.2. The resulting surface $F'$ is connected and has Euler characteristic $\chi(F') = \chi(F) - (p-1)$. Since $F$ has one boundary component but $F'$ has $p$, we find $\mathrm{genus}(F') = \mathrm{genus}(F)$.

Recall that $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1 = B^4 \cup N(\mathbb{CP}^1)$, where $N(\mathbb{CP}^1)$ denotes a closed neighborhood of $\mathbb{CP}^1$ that turns out to be diffeomorphic to the once twisted $\mathbb{D}^2$-bundle over $S^2$ (readily identified, seen as the Riemann sphere, with $\mathbb{CP}^1$), which we denote by $S^2 \tilde{\times} \mathbb{D}$. To be explicit, the latter is the total space of the bundle $S^2 \tilde{\times} \mathbb{D}^2 \xrightarrow{\pi} S^2 = \mathbb{CP}^1$ for which the boundary $\partial\left(S^2 \tilde{\times} \mathbb{D}\right) = S^2 \tilde{\times} S^1$ is diffeomorphic to $S^3$. We identify $S^3 = S^2 \tilde{\times} S^1$. The fibration $\pi_j \colon S^3 = S^2 \tilde{\times} S^1 \to S^2$ obtained by restricting $\pi$ to the boundary is the Hopf fibration.

Note that the preimage $\pi_j^{-1}(\{z_1, \ldots, z_p\})$ of the Hopf fibration of $p$ distinct points in $S^2$ is (up to isotopy) $T_{p,p}$. (Famously, the preimage of two points is the Hopf link $T_{2,2} = \mathrm{Cl}(\sigma_1^2)$.)

Fixing $z_1, \ldots, z_p$ distinct points and adapting our identification $S^3 = S^2 \tilde{\times} S^1$ by an isotopy as needed, we have that

$$\partial F' = T_{p,p} = \pi_j^{-1}\left(\{z_1, \ldots, z_p\}\right) = \partial\left(\pi^{-1}(\{z_1, \ldots, z_p\})\right),$$

where $\pi^{-1}(\{z_1, \ldots, z_p\}) \subset N(\mathbb{CP}^1)$ is a disjoint union of $p$ fibers, each of which is a disc since $\pi$ is a $\mathbb{D}^2$-bundle. With this setup, we define $\Sigma := F' \cup \pi^{-1}\left(\{z_1, \ldots, z_p\}\right) \subset B^4 \cup N(\mathbb{CP}^1) = \mathbb{CP}^2$. The surface $\Sigma$ arises from $F'$ by capping of each of its boundary components with a disc; hence, it is a connected orientable surface of genus $\mathrm{genus}(\Sigma) = \mathrm{genus}(F')$. The homology class represented by $\Sigma$ (after chosing the orientation correctly), is $p[\mathbb{CP}^1]$, which can for example be seen as follows. The transversal intersection $\Sigma \cap \mathbb{CP}^1 = \pi^{-1}\left(\{z_1, \ldots, z_p\}\right) \cap \mathbb{CP}^1 = \{z_1, \ldots, z_p\}$ consists of $p$ points all with the same orientation, namely the positive one. Hence, $\langle [\Sigma], [\mathbb{CP}^1] \rangle = p$, where $\langle \cdot, \cdot \rangle$ denotes the intersection form on $H_2(\mathbb{CP}^2; \mathbb{Z})$; thus, $[\Sigma] = p[\mathbb{CP}^1]$ since the generator $[\mathbb{CP}^1]$ of $H_2(\mathbb{CP}^2; \mathbb{Z})$ satisfies $\langle [\mathbb{CP}^1], [\mathbb{CP}^1] \rangle = 1$.

All in all we find, as desired, that

$$\mathrm{genus}(F) = \mathrm{genus}(F') = \mathrm{genus}(\Sigma) \overset{\text{Theorem 3.7}}{\geq} \frac{(p-2)(p-1)}{2}. \qquad \square$$

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Université de Neuchâtel, Rue Emile-Argand 11, 2000 Neuchâtel, Switzerland • peter.feller@unine.ch