Winter Braids Lecture Notes

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An invitation to big mapping class groups

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Abstract

The goal of these notes is to give an introduction to *big mapping class groups*, that is, mapping class groups of surfaces of infinite type. We first talk about infinite-type surfaces and their classification. We then describe some basic topological and algebraic properties of their mapping class groups.

1. Preface

In recent years, there has been a surge of interest in surfaces of infinite type (i.e. surfaces whose fundamental group is not finitely generated) and their mapping class groups (called *big* mapping class groups). The goal of these notes, which are a slighlty expanded version of a mini-course given at the Winter Braids XII school in Tours in 2023, is to give an introduction to this topic. Section 2 concentrates on basic definitions and examples, and it includes a discussion of the classification of surfaces of infinite type. In Section 3 we show how we can think of (big) mapping class groups as topological groups and what are some of their topological properties. We then talk about algebraic properties in Section 4. As the study of big mapping class groups is in great expansion, we necessarily have to make a (subjective) selection of topics that we present. In particular, we will not talk about the relations to Teichmüller theory or three-manifolds. Moreover, even if we discuss some geometric group theory aspects (coarse geometry and actions on graphs) in Section 3.1, it will be just a brief account. We also certainly do not claim to give a complete account of the known results in the topics we explore: we rather aim at presenting some theorems as examples of the type of questions and problems that arise when looking at big mapping class groups.

While we tried to make these notes mostly self-contained, some familiarity with mapping class groups of (finite-type) surfaces will certainly help, and we refer to the book by Farb and Margalit [FM12] for background and much more. The survey [AV20] by Aramayona and Vlamis has also been very useful to the author when preparing the mini-course and these notes, and we recommend it to the reader who wants to learn more about big mapping class groups.

Acknowledgements

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2. Definitions and examples

The objects we are interested in are surfaces and their mapping class groups. By *surface* we mean a manifold of (real) dimension two. We will assume all surfaces to be connected and orientable. Unless otherwise stated, surfaces have no boundary. Notable exceptions are *subsurfaces*, which are always assumed to be closed subsets of the ambient surface. By *curve* on a surface we will mean the homotopy class of a simple closed curve (i.e. the image of an embedding of S^1). Note that by work of Baer (see [FM12, Chapter 1]), homotopy and isotopy classes of simple closed curves coincide. Usually, curves are assumed to be *essential*, that is, neither contractible nor bounding a once-punctured disk. Given a surface S, we denote by $\mathcal{C}(S)$ the collection of essential curves on S. A collection of curves (or subsurfaces) is *locally finite* if for every compact set $K \subset S$, all but finitely many curves (or subsurfaces) of the collection can be homotoped to be disjoint from K.

The *mapping class group* $\mathrm{MCG}(S)$ of a surface S is the group of homotopy classes of orientation-preserving homeomorphisms:

$$\mathrm{MCG}(S) = \mathrm{Homeo}^+(S)/_{\text{homotopy}}.$$

By work of Baer and Munkres (see [FM12, Chapter 1]), it coincides with the group of isotopy classes of orientation-preserving homeomorphisms and also with the group of orientation-preserving diffeomorphisms modulo smooth homotopy or isotopy.

Arguably, the simplest surfaces are the closed ones, which are classified by their genus. They are examples of *finite-type* surfaces, that is, surfaces whose fundamental group is finitely generated. Concretely, finite-type surfaces are compact surface with finitely many (possibly zero) points removed (punctures). Finite-type surfaces are classified by their signature: the pair (g, n), where g is the genus and n the number of punctures.

Surfaces whose fundamental group is not finitely generated are said *of infinite type*. One can show that the fundamental group of any infinite-type surface is the free group on countably infinitely generators.

Let's first look at some examples of surfaces of infinite type (see also Figure 2.1).

Example 2.1. The *Loch Ness monster surface* M is a surface of infinite genus, obtained by gluing infinitely many two-holed tori to a one-holed torus

Example 2.2. Let C be a Cantor set in the sphere S^2; $S^2 \smallsetminus C$ is an infinite-type surface, homeomorphic[1] to the boundary of a tubular neighborhood of an infinite d-valent tree (for any chosen $d \geq 2$) embedded in $\mathbb{R}^3$.

Example 2.3. The plane with points with integer coordinates removed, $\mathbb{R}^2 \smallsetminus \mathbb{Z}^2$, is a planar surface of infinite type.

Example 2.4. The surface L obtained by gluing infinitely many two-holed one-punctured tori T_n, for $n \in \mathbb{Z}$, where each T_n is glued to T_{n-1} and T_{n+1}, is another example of surface of infinite genus.

There are many many more infinite-type surfaces — actually, uncountably many, up to homeomorphisms. This statement will follow from the classification of surfaces of (finite- and) infinite-type (Theorem 2.6). To state the result, we need to talk about ends of surfaces.

An *admissible descending chain* of a surface S is a nested sequence $U_1 \supset U_2 \supset \dots$, where the U_i are unbounded connected open subsets with compact boundary, with the property that for every compact subset K of S, the U_i are eventually disjoint from K. Two admissible descending chains $U_1 \supset U_2 \supset \dots$ and $V_1 \supset V_2 \supset \dots$ are equivalent if they are eventually contained in each other, that is: for every $n \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that $V_N \subset U_n$ and $U_N \subset V_n$. An *end* is an equivalence class of admissible descending chains. Informally, an end should be thought as a way of "going to infinity", i.e. leaving every compact subset. In the examples above:

[1] This will follow from the classification of surfaces (Theorem 2.6).

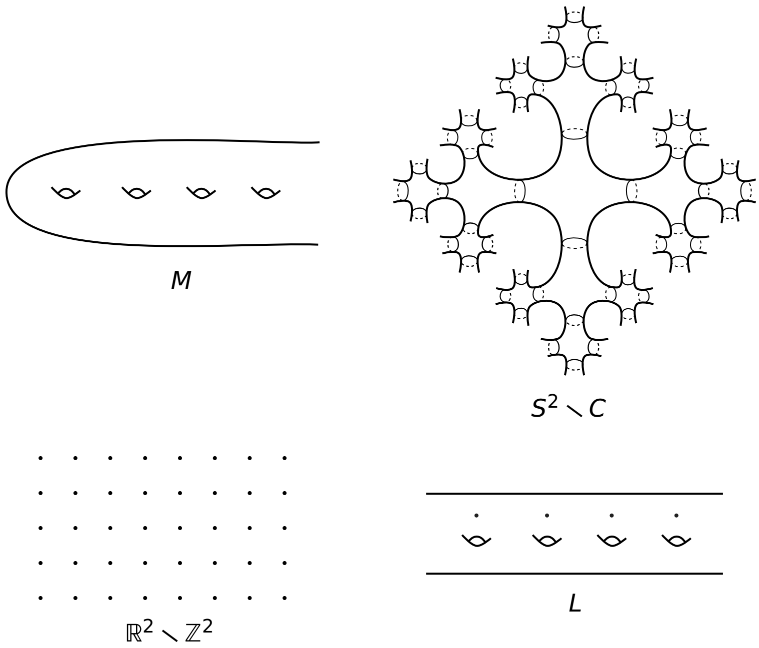

Figure 2.1: Four examples of infinite-type surfaces

- the Loch Ness monster has a single end;
- $S^2 \smallsetminus C$ has uncountably many ends, corresponding to the Cantor set points;
- $\mathbb{R}^2 \smallsetminus \mathbb{Z}^2$ has countably many ends: one per point with integer coordinates, together with an extra point "at infinity" (identify $\mathbb{R}^2$ with $\mathcal{C}$; the point at infinity is the extra point added to obtain the Riemann sphere $\hat{\mathbb{C}}$);
- L has one end per puncture, together with two other ends, which correspond to walking away from every compact by following the T_n for $n \to \infty$ or $n \to -\infty$.

One can notice a difference between the end of the Loch Ness monster and the ends of the sphere with a Cantor set removed: there is genus "arbitrarily far out" into the end of the Loch Ness monster, while this is not true for $S^2 \smallsetminus C$. This idea can be formalized as follows. We say that an end $[U_1 \supset U_2 \supset \dots]$ is *planar* if eventually all U_i are planar surfaces (i.e. they can be embedded in the plane), otherwise it is *nonplanar*. In the examples above, the end of the Loch Ness monster is nonplanar, while the sphere with a Cantor set removed has only planar ends. Both can appear on the same surface, as Example 2.4 shows: there, the punctures are planar ends and the other two ends are nonplanar.

It is also natural to think of the set of ends of $S^2 \smallsetminus C$ as having the topology induced by the one on the sphere, and similarly for $\mathbb{R}^2 \smallsetminus \mathbb{Z}^2$. Even when there is no obvious way to think of a surface and its ends as a subset of a closed surface, we can still define a topology on the set of ends, which coincides with the natural one in the case of $S^2 \smallsetminus C$ or $\mathbb{R}^2 \smallsetminus \mathbb{Z}^2$. A basis for this topology is given by

$$\{U^* \mid U \subset S \text{ open}\},$$

where

$$U^* := \{e = [U_1 \supset U_2 \supset \dots] \mid \exists n \in \mathbb{N} : U_n \subset U\}.$$

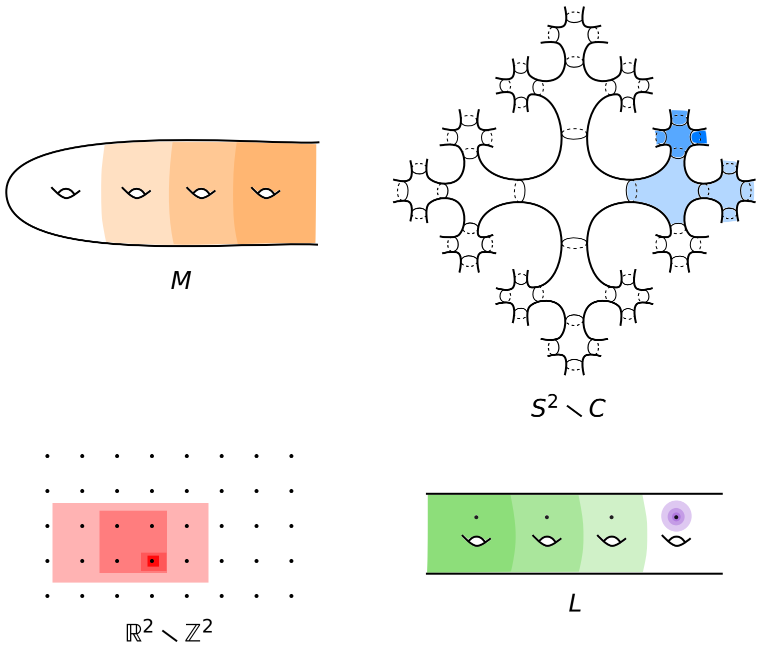

Figure 2.2: Some admissible descending chains

We denote by $\mathrm{Ends}(S)$ the *space of ends* of the surface, i.e. the set of ends with the topology described above. The *space of nonplanar ends* is the subspace $\mathrm{Ends}_g(S)$ given by all nonplanar ends. A nice exercise is to show that $\mathrm{Ends}_g(S)$ is a closed subset of $\mathrm{Ends}(S)$.

In the examples above, we have:

- $\mathrm{Ends}(M) = \mathrm{Ends}_g(M)$ is a singleton;
- $\mathrm{Ends}(S^2 \smallsetminus C)$ is homeomorphic to a Cantor set and there are no nonplanar ends;
- $\mathrm{Ends}(\mathbb{R}^2 \smallsetminus \mathbb{Z}^2)$ is homeomorphic to $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$, and $\mathrm{Ends}_g(\mathbb{R}^2 \smallsetminus \mathbb{Z}^2) = \varnothing$;
- $\mathrm{Ends}(L)$ is homeomorphic to $\left\{\pm\left(1 - \frac{1}{n}\right) \mid n \in \mathbb{N}\right\} \cup \{-1, 1\}$, and $\mathrm{Ends}_g(L)$ corresponds to $\{-1, 1\}$.

One can show that (see [AS60, Chapter 1]):

Proposition 2.5. *The endspace of a surface is totally disconnected, second countable and compact. In particular, it is homeomorphic to a closed subset of the Cantor set.*

The genus of a surface can be defined in multiple ways; for our purposes, we can say that the genus of a surface is the maximum cardinality of a locally finite collection of pairwise disjoint curves which do not disconnect the surface. In particular, the genus is either a non-negative integer or it is (countably) infinite.

We now can state the classification of surfaces, due to Kerékjártó [Ker23] and Richards [Ric63]:

Theorem 2.6 (Kerékjártó, Richards)**.** *Two (connected, orientable) surfaces (without boundary) S_1 and S_2 are homeomorphic if and only if they have the same genus and there is a homeomorphism of pair of topological spaces:*

$$(\mathrm{Ends}(S_1), \mathrm{Ends}_g(S_1)) \simeq (\mathrm{Ends}(S_2), \mathrm{Ends}_g(S_2)).$$

Moreover:

1. *for any two surfaces S_1 and S_2 with the same genus and any homeomorphism*
$$\theta : (\mathrm{Ends}(S_1), \mathrm{Ends}_{\mathrm{g}}(S_1)) \simeq (\mathrm{Ends}(S_2), \mathrm{Ends}_{\mathrm{g}}(S_2)),$$
there is a homeomorphism $f : S_1 \to S_2$ which induces θ;

2. *given any $g \in \{0, 1, 2, \dots\} \cup \{\infty\}$, any closed subset E of the Cantor set and any closed subset F of E such that $g = \infty$ if and only if $F \neq \emptyset$, there is a surface S of genus g such that*
$$(\mathrm{Ends}(S), \mathrm{Ends}_{\mathrm{g}}(S)) \simeq (E, F).$$

A few remarks are in order:

Remark 2.7*.* In part (2), the condition $g = \infty$ if and only if $F \neq \emptyset$ is necessary, since it corresponds to saying that the genus is infinite if and only if there are nonplanar ends. Indeed, if a surface S has infinite genus and $K_1 \subset K_2 \subset \dots$ is a compact exhaustion of S, then $S \smallsetminus K_1$ has finitely many components, so at least one of them, denoted U_1, has infinite genus. Similarly, $U_1 \smallsetminus K_2$ has finitely many components, and one of them, denoted U_2, has infinite genus. By repeating the argument we construct a chain defining a nonplanar end. Conversely, if $[U_1 \supset U_2 \supset \dots]$ is a nonplanar end and the genus of U_1 were finite, then for sufficiently large i, U_i would have genus zero and hence would be planar, a contradiction.

Remark 2.8*.* A consequence of part (1) is that the natural map
$$\mathrm{MCG}(S) \to \mathrm{Homeo}((\mathrm{Ends}(S), \mathrm{Ends}_{\mathrm{g}}(S)))$$
is surjective. Its kernel is the *pure mapping class group* PMCG(S). If the surface has nonempty boundary, we define the pure mapping class group to be the subgroup of the kernel given by maps that fix the boundary components pointwise.

Remark 2.9*.* The theorem recovers the classification of surfaces of finite type: indeed, the endspace of a surface of genus g and n punctures is a set of n points with the discrete topology, and its space of nonplanar ends is empty. So the existence of a homeomorphism between the endspaces is equivalent to saying that the surfaces have the same number of punctures.

Let us look at some examples of mapping classes:

Example 2.10*.* Given any compact subsurface $K \subset S$ and any $f \in \mathrm{PMCG}(K)$, we get a mapping class on S given by extending f to the identity outside K. Mapping classes obtained like this are called *compactly supported* and they form the *compactly supported mapping class group* $\mathrm{MCG}_c(S)$. Concrete examples of compactly supported mapping classes are Dehn twists: informally, given an essential curve α, the *Dehn twist* τ_α about α is the mapping class obtained by cutting S along α, twisting one side by 360° and gluing back. More precisely, if we identify a closed regular neighborhood of α with the annulus $S^1 \times [0, 1]$, where $S^1 = \mathbb{R}/\mathbb{Z}$, the Dehn twist is the homotopy class of a homeomorphism which acts as
$$S^1 \times [0, 1] \ni (t, s) \mapsto (t + s, s) \in S^1 \times [0, 1]$$
on the neighborhood and is the identity outside (see also [FM12, Chapter 3]).

Example 2.11*.* Suppose $\{\alpha_i \mid i \in I\}$ is a locally finite collection of pairwise disjoint distinct curves. Given any collection of integers $\{n_i \in \mathbb{Z} \mid i \in I\}$, we can construct a mapping class, called *multitwist*, given by the (possibly countably infinite) product
$$\prod_{i \in I} \tau_{\alpha_i}^{n_i}.$$
Different integer sequences yield different mapping classes (we can find curves with different images). A consequence is that big mapping class groups are uncountable (while mapping class groups of finite-type surfaces are countable).

Example 2.12. If the surface has infinite genus, its mapping class group contains maps called *handle shifts*. A handle shift is a map supported on a strip with handles indexed by $\mathbb{Z}$; on the strip, the map acts as a translation on the handles (see Figure 2.3 for two examples). The precise definition of handle shifts can be found in the paper by Patel and Vlamis [PV18], who first introduced them. Similarly, Lanier and Loving [LL20] introduced *puncture shifts*, where handles are replaced by punctures.

Example 2.13. Symmetries of surfaces also give mapping classes: an example is the surface in Figure 2.3, with three ends, all nonplanar, which has an order-three rotation.

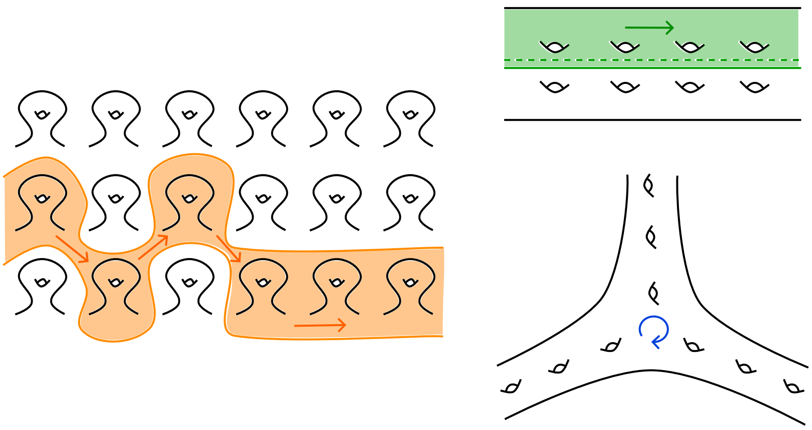

Figure 2.3: Two handle shifts and an order-three rotation

3. Topological aspects

Mapping class groups are endowed with a natural topology, induced by the compact-open topology on the the group of orientation-preserving homeomorphisms of the surface. One can check that the topology is compatible with the group operations and hence mapping class groups can be thought of as topological groups.

Alternatively, we can consider the topology whose basis is given by sets of the form $f \cdot U_A$, where f is a mapping class, A a finite collection of curves,

$$U_A := \{g \in \mathrm{MCG}(S) \mid g(\alpha) = \alpha \ \ \forall \alpha \in A\}$$

and

$$f \cdot U_A := \{f \circ g \mid g \in U_A\}.$$

Essentially, the idea is that two mapping classes are considered to be “close to each other” if they agree on many curves. Note that the U_A form a neighborhood basis of the identity.

It turns out that the two topologies are the same. To prove it, one uses the fact that a mapping class that fixes all curves (up to homotopy) is the identity. This is a consequence of the Alexander method (see [FM12, Chapter 2]), which has been shown to hold for surfaces of infinite type by Hernández Hernández, Morales and Valdez in [HHMV19].

Note that if the surface is of finite type, there is a *finite* collection of curves so that, if a mapping class fixes all curves in the collection, it is the identity (see [FM12, Chapter 2]). Thus the mapping class group of a finite-type surface has the discrete topology. On the other hand, the topology is never discrete on a surface of infinite type: indeed, given any finite collection of curves A, U_A is never reduced to the identity — for instance, it contains any Dehn twist about a curve disjoint from all curves in A. Since the topology is not trivial, it is interesting to see what are its properties. We start by proving two basic ones, following [Vla][2].

Proposition 3.1. *Let S be a surface of infinite type. Then:*

1. *any compact subset of* $\mathrm{MCG}(S)$ *has empty interior. In particular,* $\mathrm{MCG}(S)$ *is not locally compact;*

2. $\mathrm{MCG}(S)$ *is not compactly generated.*

Proof.

1. Let $K \subset \mathrm{MCG}(S)$ be compact and by contradiction suppose that it has non-empty interior. Up to translating, we can assume that the identity is in the interior of K. By definition of the topology then there is a finite set of curves A such that $U_A \subset K$. If α is a curve disjoint from all curves in A, then $\{\tau_\alpha^n \mid n \in \mathbb{N}\}$ is a sequence contained in U_A, and hence in K, which does not admit any converging subsequence, contradicting compactness.

2. Let $K \subset \mathrm{MCG}(S)$ be compact. Endow $\mathcal{C}(S)$ with the discrete topology. Then $\mathrm{MCG}(S)$ acts continuously on $\mathcal{C}(S)$, so for every $\alpha \in \mathcal{C}(S)$ and for every $n \in \mathbb{N}$, $K^n \cdot \alpha$ is a compact subset $\mathcal{C}(S)$. As $\mathcal{C}(S)$ is discrete, $K^n \cdot \alpha$ is finite.

   Pick a locally finite collection of curves $\{\alpha_n, \beta_n \mid n \in \mathbb{N}\}$ such that:

   - $i(\alpha_n, \beta_n) \neq 0$ for every n, and
   - $i(\alpha_n, \alpha_m) = i(\alpha_n, \beta_m) = 0$ for every $n \neq m$.

   For every n, the collection $\{\tau_{\beta_n}^k(\alpha_n) \mid k \in \mathbb{Z}\}$ is infinite, so there is $k_n \geq 0$ such that

   $$\tau_{\beta_n}^{k_n}(\alpha_n) \notin K^n \cdot \alpha.$$

   Now, let $f = \prod_{n \in \mathbb{N}} \tau_{\beta_n}^{k_n}$. If $f \in K^n$, then

   $$\tau_{\beta_n}^{k_n}(\alpha_n) = f(\alpha) \in K^n \cdot \alpha,$$

   which is impossible. So $f \notin K^n$ for every n, that is, K does not generate $\mathrm{MCG}(S)$.

□

We also mention the following result, whose proof can be found in [AV20]:

Proposition 3.2. *Let S be an infinite-type surface. Then* $\mathrm{MCG}(S)$ *is a Polish group, homeomorphic to $\mathbb{R} \setminus \boldsymbol{Q}$ and to the Baire space $\mathbb{N}^{\mathbb{N}}$.*

Recall that a topological group is *Polish* if it is separable and completely metrizable, or equivalently second countable and completely metrizable.

[2] The second property was proven by a group of participants of the AIM workshop *Surfaces of infinite type*, which took place at the American Institute of Mathematics in San Jose, California, in 2019.

3.1. Topology and geometric group theory

The idea of geometric group theory is to study groups either by considering them as geometric objects, or by looking at their actions on metric spaces. Such an approach has been very useful in the case of mapping class groups of surfaces of finite type, which are finitely generated and admit interesting geometric actions on well-behaved spaces — see for instance Bestvina's survey [Bes23] for a selection of results.

The reason why finite generation is useful is the following. If a group G is finitely generated and we fix a finite generating set F, we can think of G as a metric space: the distance between two group elements g and h is the smallest integer n such that $g = hf_1 \dots f_n$, where $f_i \in F$. Of course, this metric depends on the generating set, but its *coarse geometry* does not. More precisely: recall that two metric spaces (X, d_X) and (Y, d_Y) are *quasi-isometric* if there are a map $\varphi : X \to Y$ and constants $C \geq 1$ and $K, D \geq 0$, so that:

- for every $x_1, x_2 \in X$,

$$\frac{1}{C} d_X(x_1, x_2) - K \leq d_Y(\varphi(x_1), \varphi(x_2)) \leq C d_X(x_1, x_2) + K,$$

- for every $y \in Y$ there is $x \in X$ so that $d_Y(y, \varphi(x)) \leq D$.

Given a finitely generated group, one can show that any two metrics induced by finite generating sets are quasi-isometric, so there is a well-defined quasi-isometry class of metrics associated to the group. The idea of geometric group theory is then to use properties of the group as a geometric space to deduce algebraic properties. A similar approach works if the group is locally compact and compactly generated (see [CdlH16]).

It is then natural to wonder whether such an approach can be adapted to big mapping class groups. We will quickly mention some of the results that have been obtained so far and we refer the interested reader to the references below for more details. Quite recently, Rosendal [Ros22] started developing a geometric group theory approach for Polish groups, and Mann and Rafi [MR23] began to study big mapping class groups from this perspective. They proved in particular that a large class of big mapping class groups has a nontrivial quasi-isometry type, so they in particular admit isometric continuous actions on some metric spaces. Explicit such actions have been constructed by many authors ([Bav16, AFP17, DFV18, AV18, Ras20, Sch20, BNV23, HQR22]), allowing in some cases to deduce algebraic properties of the group (e.g. the existence of unbounded quasimorphisms [Bav16, HQR22]). On the other hand, they also proved that some mapping class groups of surfaces of infinite type (such as the sphere with a Cantor set removed or the Loch Ness monster) are quasi-isometric to a point from Rosendal's viewpoint, which means that they do not admit continuous isometric actions with unbounded orbits on any metric space. In some cases, such as the sphere with a Cantor set removed, the continuity assumption is not necessary, as shown by Vlamis [Vla24]. In particular, for some surfaces it is much less clear how to study the mapping class groups through geometric group theory and actions on spaces, as for most applications it is important to have unbounded orbits.

4. Algebraic properties

In this section we will concentrate on the study of some algebraic properties of big mapping class groups. First, we describe properties which do not depend on the underlying surface, as long as it is of infinite type. We then move onto properties which depend on the underlying surface. Finally, we talk about the problem of generating big mapping class groups.

4.1. Properties of all big mapping class groups

We have already seen that no big mapping class group is finitely generated, contrary to mapping class groups of finite-type surfaces. We will now show, following work of Patel and Vlamis [PV18], that no big mapping class group is residually finite.

A group G is *residually finite* if the intersection of all its proper finite-index subgroups is reduced to the identity element:

$$\bigcap_{\substack{\{1\}\neq H<G,\\ [G:H]<\infty}} H = \{1\}.$$

It is well know that G is residually finite if an only if either (and hence both) of the following conditions holds:

1. $\displaystyle\bigcap_{\substack{\{1\}\neq H\triangleleft G,\\ [G:H]<\infty}} H = \{1\}$;

2. for every $g \in G \setminus \{1\}$, there is a homomorphism $\varphi : G \to F$, where F is a finite group, so that $\varphi(g) \neq 1$.

In particular, subgroups of residually finite groups are residually finite as well — said differently, if a group contains a non-residually finite subgroup, it is not residually finite.

In [Gro75], Grossmann proved that mapping class groups of finite-type surfaces are residually finite. In [PV18], Patel and Vlamis proved:

Theorem 4.1 (Patel–Vlamis)**.** *Let S be a surface of infinite type. Then* $\mathrm{MCG}(S)$ *is not residually finite.*

The idea of the proof is to show the existence of a subgroup which is not residually finite; for surfaces of finite genus, the subgroup is the infinite braid group, while for infinite-genus surfaces, it is the compactly supported mapping class group.

Before proving the theorem, let us define the infinite braid group and show it is not residually finite.

For any $n \geq 2$, the *braid group on n strands B_n* is the mapping class group of D_n, the closed disk with n points removed from its interior, or equivalently the group given by the following generators and relations:

$$B_n = <\sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \;\forall i,j : |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \;\forall i > .$$

There are natural inclusions $B_n < B_{n+1}$ given by sending the generators of B_n to the first $n-1$ generators of B_{n+1}. We can then talk about the direct limit of the B_n, which is the *infinite braid group B_∞*:

$$B_\infty := \varinjlim_n B_n.$$

Again, there are explicit generators and relations defining B_∞:

$$B_\infty = <\sigma_i, i \in \mathbb{N} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \;\forall i,j : |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \;\forall i > .$$

We will show:

Lemma 4.2. *If $\varphi : B_\infty \to G$ is a surjective homomorphism on a finitely generated group G, then G is cyclic.*

Proof. Suppose f and G are as in the statement. Since G is finitely generated, there is some $N \geq 2$ so that $\varphi|_{B_N}$ is surjective. For every $n > N$ and ever $m < N$, σ_n and σ_m commute, which implies that $\varphi(\sigma_n)$ is in the center of G for every $n > N$.

We claim that if $\varphi(\sigma_n)$ is in the center of G, then $\varphi(\sigma_n) = \varphi(\sigma_{n-1})$. If the claim holds, by induction every generator of B_∞ has the same image, which implies that G is cyclic.

Let us then prove the claim. Suppose that $\varphi(\sigma_n)$ is in the center of G. Then

$$\varphi(\sigma_n)\varphi(\sigma_{n-1}) = \varphi(\sigma_n\sigma_{n-1}) \overset{(*)}{=} \varphi(\sigma_{n-1}^{-1}\sigma_n\sigma_{n-1}\sigma_n) =$$
$$= \varphi(\sigma_{n-1}^{-1})\varphi(\sigma_n)\varphi(\sigma_{n-1})\varphi(\sigma_n) \overset{(**)}{=} \varphi(\sigma_n)\varphi(\sigma_n)$$

and thus $\varphi(\sigma_{n-1}) = \varphi(\sigma_n)$, where $(*)$ is given by the relations between the generators and for $(**)$ we use the fact that $\varphi(\sigma_n)$ is in the center of G. □

A consequence of the lemma is the non-residual finiteness of B_∞:

Corollary 4.3. *B_∞ is not residually finite.*

Proof. Suppose H is a proper finite-index subgroup of B_∞. Then we have a projection $B_\infty \to B_\infty/H$, where B_∞/H is finite. By Lemma 4.2, B_∞/H is cyclic, and in particular abelian. Thus $B_\infty \to B_\infty/H$ factors through the abelianization of B_∞, i.e. $[B_\infty, B_\infty] \subset H$.

As a consequence,

$$\bigcap_{\substack{\{1\}\neq H \lhd G,\\ [G:H]<\infty}} H \supset [B_\infty, B_\infty] \neq \{1\},$$

i.e. B_∞ is not residually finite. □

With these results at hand, let us prove that big mapping class groups are not residually finite.

Proof of Theorem 4.1. Suppose first that the genus of S is finite. Then S has infinitely many planar ends. If S has infinitely many isolated planar end $\{p_1, p_2, p_3 \dots\}$, we can choose a nested sequence of subsurfaces Σ_n, where Σ_n is a disk punctured exactly at $\{p_1, \dots, p_n\}$. Then

$$B_\infty = \varinjlim_n B_n \simeq \varinjlim_n \mathrm{MCG}(\Sigma_n) < \mathrm{MCG}(S).$$

If S has only finitely many isolated planar ends, by the classification of closed subsets of the Cantor set (see [Kec95, Theorems 6.4 and 7.4]) we know that

$$\mathrm{Ends}(S) = C \sqcup F,$$

where C is a Cantor set and F a finite (possibly empty) collection of points. In particular we can find a disk $\Sigma \subset S$ containing C in its interior and disjoint from F, and

$$B_\infty < \mathrm{MCG}(\Sigma) < \mathrm{MCG}(S),$$

where the first containment is shown by Funar and Kapoudjian in [FK04]. In any case, we can find B_∞ in the mapping class group of an infinite-type surface of finite genus, which is therefore not residually finite.

We can now assume that S has infinite genus. We fix an exhaustion F_n of S by finite-type surfaces; by the assumption on the genus, $g(F_n) \to \infty$. Let m_n be the smallest index of a proper subgroup of $\mathrm{PMCG}(F_n)$. By a result of Paris [Par10], $m_n \to \infty$. Suppose that $H \lhd \mathrm{MCG}_c(S)$ has finite index $k > 1$; since

$$\mathrm{MCG}_c(S) = \varinjlim_n \mathrm{PMCG}(F_n),$$

there is some $N > 0$ so that for every $n \geq N$, $\mathrm{PMCG}(F_n) \not< H$. This implies that $H \cap \mathrm{PMCG}(F_n)$ is a proper subgroup of $\mathrm{PMCG}(F_n)$, and

$$k \geq [\mathrm{PMCG}(F_n) : H \cap \mathrm{PMCG}(F_n)] \geq m_n \to \infty,$$

a contradiction. This means that $\mathrm{MCG}_c(S)$ is not residually finite, and hence neither is $\mathrm{MCG}(S)$. □

Other examples of properties which hold for all big mapping class groups are:

- for any infinite-type surface S,
$$\mathrm{Comm}(\mathrm{MCG}^{\pm}(S)) \simeq \mathrm{Aut}(\mathrm{MCG}^{\pm}(S)) \simeq \mathrm{MCG}^{\pm}(S),$$
where Comm is the abstract commensurator and $\mathrm{MCG}^{\pm}(S)$ is the *extended* mapping class group, i.e. the group of all homeomorphisms, up to homotopy [BDR20]. The same holds for finite-type surfaces [Iva97, Kor99, Luo00a];

- any normal subgroup of any big mapping class group has trivial center [LL20]. The same holds for finite-type surfaces [BM19];

- no big mapping class group is acylindrically hyperbolic [BG18], contrary to the finite-type case [BF02, Bow06];

- every big mapping class group fails the Tits alternative [All21], contrary to the finite-type case [Iva84, McC85].

4.2. Properties which depend on the surface

So far we discussed properties which hold for all big mapping class groups. On the other hand, Bavard, Dowdall and Rafi proved in [BDR20] that non-homeomorphic subsurfaces have non-isometric mapping class groups. So there must be properties which depend on the underlying surface. Even more, we might hope to characterize topological features of a surface via algebraic properties of the associated mapping class group. An example of such a result is the following, due to Patel and Vlamis [PV18]:

Theorem 4.4 (Patel–Vlamis)**.** *Let S be a surface. Then* $\mathrm{PMCG}(S)$ *is residually finite if and only if S has finite genus.*

Note that $\mathrm{PMCG}(S)$ contains $\mathrm{MCG}_c(S)$, and if the genus of S is infinite $\mathrm{MCG}_c(S)$ is shown to be not residually finite in the proof of Theorem 4.1; therefore we already know that if $\mathrm{PMCG}(S)$ is residually finite, then the genus of the surface is finite. Theorem 4.4 shows that the converse implication holds too.

Of course, knowing the genus of a surface is not enough to determine it. So we might ask if we can characterize surfaces just by looking at some list of algebraic invariants of their mapping class group. Recall that, in the finite-type setting, the virtual cohomological dimension and the algebraic rank of $\mathrm{MCG}(S)$ allow us to recover the topology of the surface (see [Har86, BLM83]).

While for infinite-type surfaces we don't yet know a list of algebraic properties of mapping class groups which allow us to determine the underliyng surfaces, Aramayona, Patel and Vlamis showed [APV20] how to detect a certain countable class of surfaces:

Theorem 4.5 (Aramayona–Patel–Vlamis)**.** *Let Ω_n be the surface with n ends, all nonplanar. Then $S \simeq \Omega_n$ if and only if the following conditions hold:*

1. $\mathrm{PMCG}(S)$ *is not residually finite,*

2. $\mathrm{PMCG}(S)$ *is not circularly orderable,*

3. $\mathrm{H}^1(\mathrm{PMCG}(S);\mathbb{Z})$ *has rank* $n-1$*, and*

4. $\mathrm{PMCG}(S) < \mathrm{Aut}(\mathrm{PMCG}(S))$*.*

One of the ingredients for this theorem is their computation of the rank of $\mathrm{H}^1(\mathrm{PMCG}(S);\mathbb{Z})$ in terms of the number of nonplanar ends of the surface:

Theorem 4.6 (Aramayona–Patel–Vlamis)**.** *Let S be a surface. Then the rank of* $\mathrm{H}^1(\mathrm{PMCG}(S);\mathbb{Z})$ *is:*

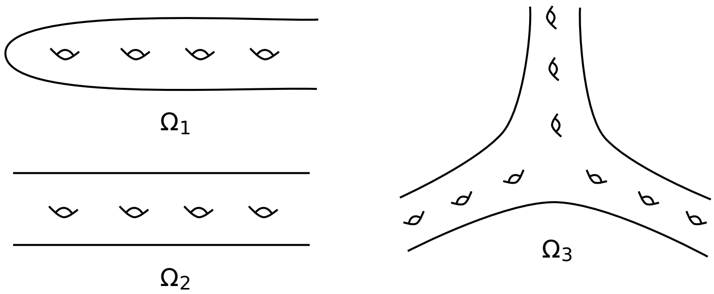

Figure 4.1: Three of the surfaces from Theorem 4.5

- *zero, if* $|\mathrm{Ends_g}(S)| \leq 1$*;*
- n*, if* $\infty > |\mathrm{Ends_g}(S)| = n+1 \geq 2$*;*
- *infinite otherwise.*

Theorem 4.5 also makes reference to circular orderability. Roughly speaking, a group is *circularly orderable* if there is an invariant circular order on the group elements — see e.g. [Cal04] for a precise definition. We just note here that subgroups of circularly orderable groups are circularly orderable (it is enough to restrict the order to the subgroup), and finite groups are circularly orderable if and only if they are cyclic. In particular, it is enough to find finite, non-cyclic subgroups to show that a group is not circularly orderable. While there is so far no complete classification of which big mapping class groups are circularly orderable, we do have some partial results:

Theorem 4.7 (Bavard–Walker, Calegari–Chen, Aougab–Patel–Vlamis)**.** *Let S be an infinite-type surface.*

1. *If S has a unique isolated planar end,* $\mathrm{MCG}(S)$ *is circularly orderable.*
2. *If S has infinite genus and either no planar ends, or a self-similar endspace, then* $\mathrm{MCG}(S)$ *is not circularly orderable.*

A surface is said to have *self-similar endspace* if for every decomposition

$$\mathrm{Ends}(S) = U_1 \sqcup \cdots \sqcup U_k,$$

where the U_i are clopen subsets, there are an index $i \in \{1, \ldots, k\}$ and an open subset $V \subset U_i$ so that

$$(V, V \cap \mathrm{Ends_g}(S)) \simeq (\mathrm{Ends}(S), \mathrm{Ends_g}(S)).$$

The notion is supposed to capture some features of the auto-similarity of Cantor sets, and indeed $S^2 \smallsetminus C$ has self-similar endspace. The same holds for the Loch Ness monster, while it is not true for the surfaces in Examples 2.3 and 2.4.

To prove part (1) of Theorem 4.7, the idea is to construct an action of the mapping class group on a circle — this is done by Bavard and Walker in [BW23] and Calegari and Chen in [CC20]. We can then use this action to deduce that the mapping class group is a subgroup of $\mathrm{Homeo}^+(S^1)$, which is circularly orderable. Part (2) instead is proven by showing the existence of non-cyclic finite subgroups, which is a consequence of the following result by Aougab, Patel and Vlamis [APV21]:

Theorem 4.8 (Aougab–Patel–Vlamis)**.** *Let S be an infinite-genus surface.*

1. *If S has no nonplanar ends, given any finite group G there is a hyperbolic metric X on S so that* $\mathrm{Isom}^+(X) \simeq G$.

2. *If S has self-similar endspace, given any countable group G there is a hyperbolic metric X on S so that* $\mathrm{Isom}^+(X) \simeq G$.

If X is a hyperbolic metric on S, we can see $\mathrm{Isom}^+(X)$ as a subgroup of $\mathrm{MCG}(S)$, so Theorem 4.8 implies that for all the surfaces as in part (2) of Theorem 4.7, the mapping class group contains any finite group as a subgroup.

The idea of the proof of Theorem 4.8 comes from previous work by Allcock [All06], who proved that for every countable group G, there is a hyperbolic surface X whose group of orientation preserving isometries is isomorphic to G. To prove this fact, Allcock builds a surface modeled on a Cayley graph of G; in particular, he has very little control on the topology of the surface he constructs. The difficulty in Aougab, Patel and Vlamis' work is that in their case, the surface is fixed, so they need to ingeniously tweak Allcock's construction to obtain a surface of the fixed topological type.

As a consequence of Theorem 4.8, one can show that infinite-genus surfaces with self-similar endspaces do not have many algebraic properties:

Corollary 4.9. *Let S be a surface of infinite genus and self-similar endspace. Let (P) be an algebraic property of groups which passes to subgroups and such that there is a countable group G which does not satisfy (P). Then $\mathrm{MCG}(S)$ does not have (P). For instance, $\mathrm{MCG}(S)$ is not linear, nor (cyclically or linerarly) orderable, nor coherent.*

We end this section with a mention of surfaces with a non-displaceable subsurface of finite type. A subsurface Σ of a surface S is called *non-displaceable* if, for every $f \in \mathrm{MCG}(S)$, $f(\Sigma) \cap \Sigma \neq \varnothing$ — by which we mean that Σ and its image cannot be homotoped to be disjoint. Surfaces admitting a finite-type non-displaceable subsurface were first considered in [MR23], and Horbez, Qing and Rafi [HQR22] proved that their mapping class groups satisfy algebraic properties such as having infinite-dimensional second bounded cohomology group or containing uncountably many normal subgroups. They actually proved that containing such a subsurface can be detected at the mapping class group level:

Theorem 4.10 (Horbez–Qing–Rafi)**.** *Let S be an infinite-type surface. Then S contains a non-displaceable surface of infinite type if and only if $\mathrm{MCG}(S)$ contains a non-abelian normal free subgroup.*

4.3. Generating mapping class groups

Given a group, a natural question is whether one can find a "good" generating set (where the meaning of "good" depends on the context). For a topological group, we can also talk about *topological generating sets*: a topological group G is *topologically generated* by $F \subset G$ if the closure of the group (algebraically) generated by F is the whole group.

If S is a finite-type surface, we saw in Section 3 that $\mathrm{MCG}(S)$ has the discrete topology, so topological or algebraic generation are the same. Moreover, we mentioned that in this case the mapping class group is finitely generated, and it is a by now classical result that it can be generated by very simple elements [Lic64, Hum79, Bir69].

Theorem 4.11 (Birman, Dehn, Lickorish, Humphries)**.** *Let S be a finite-type surface. Then $\mathrm{PMCG}(S)$ is generated by finitely many Dehn twists, and $\mathrm{MCG}(S)$ is generated by finitely many Dehn twists and half twists.*

Recall (see Remark 2.8) that we have the following short exact sequence:

$$1 \to \mathrm{PMCG}(S) \to \mathrm{MCG}(S) \to \mathrm{Homeo}((\mathrm{Ends}(S), \mathrm{Ends}_{\mathrm{g}}(S))) \to 1.$$

Thus any generating set of $\mathrm{MCG}(S)$ induces a generating set of the group of homeomorphisms of $(\mathrm{Ends}(S), \mathrm{Ends}_{\mathrm{g}}(S))$. We have also seen that the endspace of a surface can be very complicated, so it seems very difficult to find a “reasonable” generating set for the mapping class group of every surface. One question which seems more approachable is: what is a (topological/algebraic) generating set of the pure mapping class group of a surface of infinite type?

The pure mapping class group contains the compactly supported elements, so in particular all Dehn twists. By Theorem 4.11, it is not hard to see that the collection of all Dehn twists generates $\mathrm{MCG}_c(S)$. By taking closures, and since limits of compactly supported mapping classes still act trivially on the ends, we have:

$$\overline{\langle\{\tau_\alpha \mid \alpha \in \mathcal{C}(S)\}\rangle} = \overline{\mathrm{MCG}_c(S)} < \mathrm{PMCG}(S).$$

An interesting question is whether the last inclusion is strict; it was show by Patel and Vlamis [PV18] that the inclusion is an equality only for certain surfaces:

Theorem 4.12 (Patel–Vlamis)**.** *Let S be a surface. The collection of Dehn twists topologically generates the pure mapping class group of S if and only if S has at most one nonplanar end. If S has more than one nonplanar ends,* $\mathrm{PMCG}(S)$ *is topologically generated by Dehn twists and handle shifts.*

We remark that the statement has been strenghtened by Aramayona, Patel and Vlamis [APV20], who proved that if $|\operatorname{Ends_g}(S)| \geq 2$, we have

$$\mathrm{PMCG}(S) = \overline{\mathrm{MCG}_c(S)} \rtimes \prod_{i=1}^{r} \langle h_i \rangle,$$

where the h_i are pairwise commuting handle shifts, and

$$r = \begin{cases} n-1, & \text{if } |\operatorname{Ends_g}(S)| = n \\ |\mathbb{N}|, & \text{otherwise.} \end{cases}$$

Sketch of proof. We prove here that $\mathrm{PMCG}(S) \neq \overline{\mathrm{MCG}_c(S)}$ if S has more than one nonplanar end and we then give some indications on how to prove the remaining statements.

Fix a separating curve α such that both complementary components contain at least one nonplanar end. Then there is a handle shift h such that $h(\alpha) \neq \alpha$ and $i(\alpha, h(\alpha)) = 0$. Note that $h \in \mathrm{PMCG}(S)$ — any handle shift is a pure mapping class.
Claim: for every $f \in \mathrm{MCG}_c(S)$, if $i(\alpha, f(\alpha)) = 0$, then $\alpha = f(\alpha)$.

The claim is enough to conclude: indeed, it implies that

$$\mathrm{MCG}_c(S) \subset \mathrm{PMCG}(S) \smallsetminus h \cdot U_{\{\alpha\}},$$

and $\mathrm{PMCG}(S) \smallsetminus h \cdot U_{\{\alpha\}}$ is a closed proper subset of $\mathrm{PMCG}(S)$. In particular,

$$\overline{\mathrm{MCG}_c(S)} \subset \mathrm{PMCG}(S) \smallsetminus h \cdot U_{\{\alpha\}} \neq \mathrm{PMCG}(S).$$

Let us then prove the claim. Given any $f \in \mathrm{MCG}_c(S)$, there is a compact subsurface Σ containing α so that:

1. the support of f is contained in Σ, so f induces a pure mapping class of Σ;
2. every boundary component of Σ is separating;
3. the closure of each component of $S \smallsetminus \Sigma$ contains an end.

Note that, by (1), α and $f(\alpha)$ are contained in Σ. Let U be a component of $\Sigma \smallsetminus \alpha$; then $f(U)$ is a component of $\Sigma \smallsetminus f(\alpha)$, homeomorphic to U. Moreover, let γ be a boundary component of Σ; by (2) and (3), and since f does not permute ends, $f(\gamma) = \gamma$. This implies that $\gamma \subset U$ if and only if $\gamma = f(\gamma) \subset f(U)$, or, said differently:

$$U \cap \partial\Sigma = f(U) \cap \partial\Sigma.$$

If $\alpha \neq f(\alpha)$, then $U \neq F(U)$. Suppose by contradiction that $i(\alpha, f(\alpha)) = 0$. Then either U $f(U)$ or $f(U)$ U, which is impossible since U and $f(U)$ are homeomorphic.

The idea of the rest of the proof is the following:

- if α is a curve which does not separate nonplanar ends, one can show that for every $f \in \mathrm{PMCG}(S)$ there is $f_\alpha \in \mathrm{MCG}_c(S)$ such that $f(\alpha) = f_\alpha(\alpha)$;

- if α is a curve which separates nonplanar ends, one can show that for every $f \in \mathrm{PMCG}(S)$ there are $f_\alpha \in \mathrm{MCG}_c(S)$ and a handle shift h_α such that $f(\alpha) = f_\alpha(h_\alpha(\alpha))$.

Note that if S has at most one nonplanar end, there is no curve separating nonplanar ends.

We then fix a pants decomposition (with good properties) $\{P_n\}_{n\in\mathbb{N}}$ and define $\Sigma_n = \bigcup_{i=1}^n P_i$. Given a pure mapping class f, we use the result above to show that there is $f_n \in \mathrm{MCG}_c(S)$, if $|\mathrm{Ends}_g(S)| \leq 1$, or $f_n \in \langle \mathrm{MCG}_c(S), \{\text{handle shifts}\}\rangle$ otherwise, so that

$$f|_{\Sigma_n} = f_n|_{\Sigma_n}.$$

This means that f_n converges to f. □

Patel and Vlamis' result gives a very nice answer to the topological generation question. Unfortunately, there is no good generating set which is known for big (pure) mapping class groups. By Theorem 4.12, if we want a good generating set of $\mathrm{PMCG}(S)$ it is enough to provide a generating set of $\overline{\mathrm{MCG}_c(S)}$ and then add handle shifts. Since $\overline{\mathrm{MCG}_c(S)}$ is uncountable, we cannot hope it to be generated by Dehn twists (which form a countable family). The next natural set to consider seems to be the set of multitwists (see Section 2), which is uncountable. Together with George Domat and Sebastian Hensel [DFH23], we show:

Theorem 4.13 (Domat–F.–Hensel)**.** *Let S be a surface of infinite type. Then the collection of multitwists does not generate $\overline{\mathrm{MCG}_c(S)}$.*

An obvious question which stems from Theorem 4.13 is: what is the subgroup generated by the collection of multitwists? Is there an alternative, more explicit description of its elements?

Moreover, one might wonder what could be a good generating set. The proof of Theorem 4.13 should be easily generalized to show that $\overline{\mathrm{MCG}_c(S)}$ is not generated by elements which are products of maps supported on a locally finite collection of pairwise disjoint compact subsurfaces of bounded complexity. If we do not ask the complexity to be bounded, it is possible that we can recover a generating set, but one might start to wonder whether such a collection of maps is too complicated to be useful.

There are other natural generating sets one could consider. In particular, for surfaces of finite type, the mapping class group is generated by torsion elements, or even involutions (except in a few cases — see [BF04, Kas03, Luo00b]). Generation by torsion elements has been investigated for surfaces of infinite type as well; even if we don't have a complete classification, it is known for some surfaces the mapping class group is generated by its torsion elements (e.g. $S_g \setminus C$, where C is a Cantor set [CC22] or the surface with only nonplanar ends, which form a Cantor set [MT21]), while for other surfaces it isn't (e.g. $\mathbb{R}^2 \setminus \mathbb{Z}^2$ [MT21] or the Loch Ness monster [DD22]).

Note also that, by a result of Vlamis [Vla24], if F is a generating set of $\mathrm{MCG}(S^2 \setminus C)$, there is some $N > 0$ so that every mapping class of $S^2 \setminus C$ is the product of at most N elements in F. In a way, we can think of this fact as saying that a generating set is almost as complicated as the whole group. So it is also plausible that there is no way to describe "good" generating sets for all infinite-type surfaces.

Sketch of proof of Theorem 4.13. The idea of the proof is to quite explicitly construct a mapping class in $\overline{\mathrm{MCG}_c(S)}$ which is not a finite product of multitwists.

The first claim is that, on any infinite-type surface different from the Loch Ness monster, there is a collection Σ_n of pairwise disjoint finite-type subsurfaces which is locally finite and such that for every $f \in \overline{\mathrm{MCG}_c(S)}$, $f(\Sigma_n) \cap \Sigma_n \neq \varnothing$. We then use the projection complex machinery of Bestvina, Bromberg and Fujiwara to show that there are mapping classes f_n supported on Σ_n and quasimorphisms $\varphi_n : \overline{\mathrm{MCG}_c(S)} \to \mathbb{R}$ of defect at most 12 such that as $k \to \infty$,

$$|\varphi_n(f_n^k)| \to \infty,$$

and whose absolute value is at most 12 when evaluated on a multitwist of a map whose support is disjoint from Σ_n. Recall that a *quasimorphism* of a group G is a map $\varphi : G \to \mathbb{R}$ such that

$$\Delta := \sup_{g,h\in G} |\varphi(gh) - \varphi(g) - \varphi(h)| < \infty;$$

Δ is the *defect* of the quasimorphism.

Then for any product $\tau_1 \dots \tau_k$ of k multitwists and for every n,

$$\varphi_n(\tau_1 \dots \tau_k) \leq 24k,$$

while if $f = \prod_{n\in\mathbb{N}} f_n^{k_n}$, for powers k_n growing sufficiently fast, then $|\varphi_n(f)| \to \infty$. So f is not the product of finitely many multitwists.

If the surface is the Loch Ness monster, we cannot find the collection of surfaces with the required properties, but we can deduce the result from the one for the Loch Ness monster with a point removed, by using the Birman exact sequence. □

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