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Applications of quantum representations of mapping class groups

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1. Introduction and motivation

Quantum representations of mapping class groups are finite dimensional representations of mapping class groups that are obtained from Witten Reshetikhin-Turaev TQFT's. In these notes we will see how these representations enjoy some exotic properties which allow interesting applications to geometric group theory. To a surface Σ and an integer p, is associated a finite dimensional complex vector space $V_p(\Sigma)$ equipped with a projective action ρ_p of $\mathrm{Mod}(\Sigma)$ (the mapping class group of Σ).

First, for any Dehn twist $T \in \mathrm{Mod}(\Sigma)$, $\rho_p(T)$ has finite order for all p. Although for a fixed p the representation ρ_p has kernel, Andersen proved that if φ is non central in $\mathrm{Mod}(\Sigma)$, $\rho_p(\varphi)$ is not trivial for p big enough. This strong property is often referred to as asymptotic faithfulness of the quantum representations.

Moreover for all p, the projective action ρ_p preserves a sesquilinear, non degenerate, positive definite form on $V_p(\Sigma)$ and ρ_p have coefficients in the number field $\mathbb{Q}[e^{i\pi/p}]$. Note that any finite dimensional representation with finite image will automatically preserve a unitary form and will have its coefficients in a number field. Funar and separately Masbaum proved that, except for few surfaces, the image of ρ_p is infinite for all but finitely p.

The richest properties appear when p is prime. Indeed it was proved by Roberts that in this case ρ_p is irreducible. Larsen and Wang proved that if p is a big enough prime, then the image of ρ_p is dense in $\mathrm{PU(V_p(\Sigma))}$. Finally Gilmer and Masbaum prove that when p is prime, up to a suitable choice of basis, ρ_p has coefficients in the ring $\mathbb{Z}[e^{i\pi/p}]$. This prime setting is usually the one that brings the most surprising results concerning mapping class groups.

Let us give a little historical background on this theory. In 1989, Witten gave a physical interpretation of the Jones polynomial in terms of Feynman path integral and Chern-Simons gauge theory (see [W89]). One year before Witten introduced in [W88] the notion of Topological Quantum Field Theories, often called TQFT's, in our case we are working with a $(2+1)$-TQFT which is a specific kind of functor from a category of cobordisms between surfaces to the category of vector spaces. Such theories are described using the so-called Aityah-Segal axioms. Although we will not describe these very rich objects, we note that a $(2+1)$-TQFT gives naturally representations of mapping class groups and invariants of closed 3-manifold. The main problem with Witten's approach was the fact that the Feynman path integral he used is not well defined. A proper construction of this TQFT was given in 1991 by Reshetikhin and Turaev using the category of semi-simple representations of the universal enveloping algebra for the quantum lie algebra for $\mathrm{SL}(2)_q$ (see [RT91]). Then Blanchet, Habegger, Masbaum and Vogel re-built these objects from the Kauffman bracket (see [BHMV95]) using the universal construction applied to the invariants of closed 3-manifolds introduced by Lickorish (see [L91]).

In these notes, we will use the Kauffman bracket point of view from [BHMV95]. Section 2 will explain how to build quantum representations but with some proofs left. The goal is to show how explicit computations can be made. Section 3 is devoted to the important properties of quantum representations. Masbaum's proof that the images are infinite and Freedman-Walker-Wang's proof of asymptotic faithfulness will be given with details. In section 4, we will see how Masbaum and Reid used the integral structure of these representations to prove that all finite groups are involved in the mapping class group. We will also talk about the work in [KS16] and [KS16b] where applications to surface groups and finite covers of surfaces are given. Finally in Section 5 we will discuss some open problems such as property (T) for the mapping class group, Ivanov's question concerning the abelianization of finite index subgroups of the mapping class group, the kernel of quantum representations and the AMU conjecture.

2. Construction of quantum representations

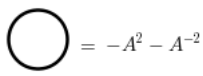
2.1. Skein module of 3-manifolds

Let Σ be a compact oriented surface without boundary. A set of banded points on Σ is an oriented sub-manifold in Σ diffeomorphic to a finite disjoint union of intervals $[0,1]$.

Definition 1. Let M be a compact oriented 3-manifold. Let $\mathcal{P}$ be a set of banded points of ∂M. A banded tangle in $(M,\mathcal{P})$ is an orientation preserving embedding $T : l \times [0,1] \to M$ where l is a one dimensional manifold (maybe with boundary and not necessarily connected) such that $\partial M \cap T(l \times [0,1]) = T(\partial l \times [0,1]) = \mathcal{P}$. Moreover we require this intersection to be transverse and that the orientation of $l \times [0,1]$ is the same as the of $\mathcal{P}$.

Definition 2. For $A \in \mathbb{C}$ we define the field $K_A = \{P(A)/Q(A) \,|\, P, Q \in \mathbb{Q}[X],\ Q(A) \neq 0\}$.

Definition 3. Let M be a compact oriented 3 manifold, $\mathcal{P}$ be a set of banded points on ∂M and A be a non zero complex number. We define the Skein Module of $S_A(M,\mathcal{P})$ to the K_A vector space with basis given by isotopy classes of banded tangles in $(M,\mathcal{P})$ modulo the following local relations :



When $\mathcal{P}$ is empty we simple write $S_A(M)$ for $S_A(M,\mathcal{P})$.

Lemma 2.1. *$S_A(S^3) = K_A$ and is generated by the empty link.*

Example 1. $S_A(D^2 \times S^1) = K_A[z]$, where z^nn is the link in $D^2 \times S^1$ which consists in n parallel copies of the core of $D^2 \times S^1$.

Example 2. We consider the ball B^3 equipped a set $\mathcal{P}_n$ of $n \geq 0$ banded points on its boundary. From the definition we see that if n is odd then $S_A(B^3,\mathcal{P}_n) = 0$, hence we suppose that n is even. Let us consider a plane intersecting B^3 in a disc D. We can suppose $\mathcal{P}_n \subset \partial D$. Given a banded tangle T in $(B^3,\mathcal{P}_n)$ we see that after resolving all crossing and after eliminating all trivial circles, T can be written a linear combination of diagrams in $(D, \partial P_n)$ without crossings. Up to isotopy, the number of such diagrams is given the Catalan numbers :

$$\dim(S_A(B^3,\mathcal{P}_{2n})) = \frac{1}{n+1}\binom{2n}{n}$$

Example 3. Let Σ be a connected, compact oriented surface with negative Euler characteristic. Let $\mathcal{C}$ be pants decomposition of Σ. Let H be the handlebody built by gluing 3-balls to each pair of pants defined by $\mathcal{C}$, notice that $\partial H = \Sigma$. For each $\gamma \in \mathcal{C}$ we chose a disc $D_\gamma \subset H$ with $\partial D_\gamma = \gamma$. Let $\Gamma \subset H$ be a banded trivalent graph dual to $\mathcal{C}$, in particular Γ intersects D_γ transversally for each $\gamma \in \mathcal{C}$. A banded link L in H is said to be in normal form if

1. $L \subset \Gamma$,

2. the number of intersection points of intersection of L with D_γ is minimal among the isotopy class of L for each $\gamma \in \mathcal{C}$,

3. no connected component of L is contained in an embedded 3-ball.

The set of links in normal form is a basis of $S_A(H)$. Starting from any banded link L in H, we see that after resolving all crossings, eliminating all trivial components and using isotopy, L can be written in $S_A(H)$ as a linear combination of banded links in normal forms. For L a link in normal form and $\gamma \in \mathcal{C}$ we define $i_\gamma(L)$ the number of connected components of $L \cap D_\gamma$. Notice that if $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{C}$ and L is a banded link in normal form :

$$i_{\gamma_1}(L) \leq i_{\gamma_2}(L) + i_{\gamma_3}(L), \quad i_{\gamma_2}(L) \leq i_{\gamma_1}(L) + i_{\gamma_3}(L), \quad i_{\gamma_3}(L) \leq i_{\gamma_1}(L) + i_{\gamma_2}(L)$$

and $i_{\gamma_1}(L)+i_{\gamma_2}(L)+i_{\gamma_3}(L)$ even. Remark that those conditions are sufficient, in the sense that if all such inequalities are true then there is a link in normal form with those intersection numbers.

2.2. Jones-Wenzl idempotents

Let $n \geq 0$ and $\mathcal{P}_n$ a set of n banded points on the disc D^2. Fix a non zero complex number A. The vector space $S_A(D^2 \times [0,1], -\mathcal{P}_n \times \{0\} \cup \mathcal{P}_n \times \{1\})$ is endowed with a product given by staking banded tangles from bottom to top, the given algebra is called the Temperley-Lieb algebra and is denoted by $T_n(A)$.

As we saw before a basis of $T_n(A)$ is given by planar tangles without crossings. The unit element of $T_n(A)$ is $\mathcal{P}_n \times [0,1]$ which is an element of the canonical basis. We define $\epsilon_n : T_n(A) \to K_A$ as the K_A-linear map sending the unit to 1 and other planar diagrams without crossings to 0.

For $k \in \mathbb{N}$, we define :

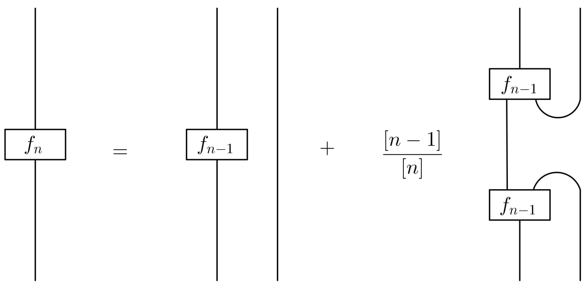
$$[k] = \frac{A^{2k} - A^{-2k}}{A^2 - A^{-2}}$$

Theorem 2.2. *If $[1], \dots, [n-1]$ are different from zero then there exists a non zero element $f_n \in T_n(A)$ such that*

$$xf_n = f_n x = \epsilon_n(x) f_n$$

for all $x \in T_n(A)$. Moreover f_n is unique up to the multiplication by a non zero element of K_A.

There is one preferred such element $f_n \in T_n(A)$ called the Jones-Wenzl idempotent (or projector) which can be computed using the following recursive formula :



Notice that if A is not a root of unity then all Jones-Wenzl idempotents exist.

An important step to construct quantum representations is to choose A to be a $2p$-th primitive root of unity. Two very similar theories appear, the SU(2) theory when p is even and the SO(3) theory when p is odd.

In what follows, we choose $p \geq 3$ an integer and set A to be a $2p$-th primitive of unity. Let

$$r = \begin{cases} p/2, & \text{if } p \text{ is even} \\ p, & \text{if } p \text{ is odd} \end{cases}$$

We see that the Jones-Wenzl idempotents $f_0, \dots, f_{r-1}$ exist as $[n]$ vanishes if and only if A is a $4n$-th root of unity and $A^4 \neq 1$.

Definition 4. Let M be a 3-manifold compact oriented with non empty boundary. Let $\mathcal{P} \subset \partial M$ be a set of banded points. Suppose $\mathcal{P} = p_1 \sqcup \cdots \sqcup p_n$ with each p_k connected. A coloring of $\mathcal{P}$ is just a map $c : \{p_1, \ldots, p_n\} \to \{0, \ldots, r-2\}$.

With Jones-Wenzl idempotents we can now define the colored skein module of a 3-manifold with boundary.

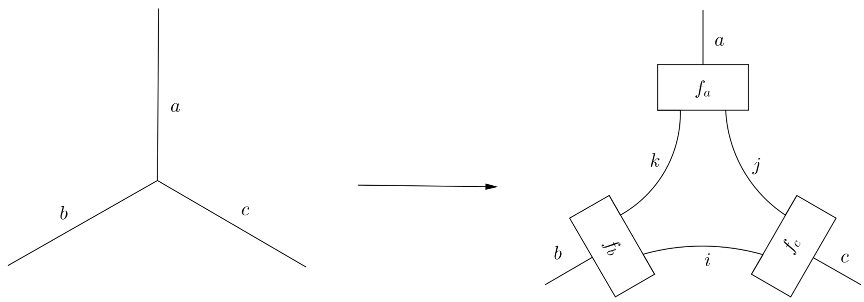
Definition 5. Let M be a 3-manifold with colored banded points $(\mathcal{P}, c)$ on its boundary. Again we write $\mathcal{P} = p_1 \sqcup \cdots \sqcup p_n$ with each p_k connected. By replacing each p_k by $c(p_k)$ copies (in a small enough neighborhood), we get a new set of banded points on ∂M that we denote by $\mathcal{P}^c$. Now in we can modify the element $\mathcal{P}^c \times [0,1] \in S_A(\partial M \times [0,1], -\mathcal{P}^c \times \{0\} \cup \mathcal{P}^c \times \{1\})$, by inserting in the middle of each parallel copies of $p_k \times [0,1]$ the idempotent $f_{c(p_k)}$, an element $\Phi \in S_A(\partial M \times [0,1], -\mathcal{P}^c \times \{0\} \cup \mathcal{P}^c \times \{1\})$. This element can be viewed as an endomorphism of $S_A(M, \mathcal{P}^c)$ and we define $S_A(M, \mathcal{P}, c)$ to be the image of this endomorphism.

Definition 6. Let a, b, c three integers. We say that the triple a, b, c is admissible if $a+b+c$ is even and $a \leq b+c$, $b \leq a+c$, $c \leq a+b$. Moreover, we say that the triple a, b, c is p-admissible if it is admissible and $a+b+c \leq 2r-4$ and a, b, c are between 0 and $r-2$.

Important computations for quantum representations will be done via trivalent graphs. Let M be a 3-manifold whose boundary is equipped with a finite set of colored banded points $(\mathcal{P}, c)$. Let Γ be a banded uni-trivalent graph in the interior of M whose univalent vertices are attached to $\mathcal{P}$. A p-admissible coloring of Γ is a map $\sigma : E(\Gamma) \to \{0, \ldots, r-2\}$ such that

1. if e is an univalent edge connected to a banded point $p \subset \mathcal{P}$ then $\sigma(e) = c(p)$,
2. when p is odd, $c(e)$ is even for all $e \in E(\Gamma)$,
3. if three edges e_1, e_2, e_3 meet at a trivalent vertex then $\sigma(e_1), \sigma(e_2), \sigma(e_3)$ are p-admissible.

Suppose that Γ is equipped with a p-admissible coloring σ. We produce the following element in $S_A(M, \mathcal{P}, c)$ denoted by Γ_σ by replacing each edge e of Γ by the Jones-Wenzl idempotent $f_{\sigma(e)}$ and replacing each trivalent vertex by the following :



where $i = \dfrac{b+c-a}{2}$, $j = \dfrac{a+c-b}{2}$ and $k = \dfrac{b+c-a}{2}$. Notice that the admissibility condition for the triple a, b, c ensures that i, j, k are in $\mathbb{N}$, moreover it is the unique planar way of connecting three Jones-Wenzl idempotents without any turn back.

2.3. The TQFT vector space

Let Σ be a connected, compact oriented surface equipped with a finite set of colored banded points $(\mathcal{P}, c)$. Let H be a handlebody with boundary Σ. In this section will see how to understand the vector space on which the mapping class group of Σ acts via the quantum representation. There are many ways to understand this space, for us it will be convenient to see it as a certain quotient of $S_A(H, \mathcal{P}, c)$. First let us state the following lemma whose proof can be found in [HP95].

Lemma 2.3. *For any $g \geq 0$, $S_A(\#_g S^2 \times S^1) = K_A.\emptyset$. Where $\emptyset$ is the empty link.*

Recall that $-H \underset{\Sigma}{\sqcup} H = \#_g S^2 \times S^1$ where g is the genus of Σ. Lemma 2.3 tells us that for $x \in S_A(-H \underset{\Sigma}{\sqcup} H)$ there is exists a unique $\langle x \rangle_A \in K_A$ such that $x = \langle x \rangle_A \emptyset$. We can then define for L, L' two banded tangles in $(H, \mathcal{P}, c)$, the quantity $\langle L, L' \rangle_A = \langle L \cup L' \rangle_A$ which in turn can be extended to a sesquilinear form (linear on the left and antilinear on the right) on $S_A(H, \mathcal{P}, c)$. Let

$$I_A(H, \mathcal{P}, c) = \{x \in S_A(H, \mathcal{P}, c) \mid \langle x, y \rangle_A = 0 \quad \forall y \in S_A(H, \mathcal{P}, c)\}$$

be the left kernel $\langle , \rangle_A$. Let us define

$$V_A(H, \mathcal{P}, c) = S_A(H, \mathcal{P}, c)/I_A(H, \mathcal{P}, c)$$

Theorem 2.4. *The following properties hold :*

1. *the K_A vector space $V_A(H, \mathcal{P}, c)$ is finite dimensional,*

2. *the dimension of $V_A(H, \mathcal{P}, c)$ only depends on $(\Sigma, \mathcal{P}, c, p)$, in particular it does not depend on H.*

Suppose there is exists a connected banded uni-trivalent graph $\Gamma \subset H$ (whose univalent vertices are attached to $\mathcal{P}$) such that H retracts to Γ. Such a graph will be called a spine of $(H, \mathcal{P})$. Spines are very useful to describe basis of $V_A(H, \mathcal{P}, c)$.

Theorem 2.5. *Let Γ be a spine of $(H, \mathcal{P})$. A basis of $V_A(H, \mathcal{P}, c)$ is given by*

$$\{\Gamma_\sigma \mid \sigma \ p-\text{admissible}\}$$

Remark 2.1. By construction $\langle , \rangle_A$ is non-degenerate on $V_A(H, \mathcal{P}, c)$, in fact the basis $\{\Gamma_\sigma\}_{\sigma\ p-\text{admissible}}$ is orthogonal with respect to $\langle , \rangle_A$. More details will be given in 3.1.

The dimension of $V_A(H, \mathcal{P}, c)$ is given by the number of p-admissible colorings of a spine of $(H, \mathcal{P})$. The Verlinde formula gives a beautiful way to express this quantity. Let us give this formula when $\mathcal{P}$ is empty :

$$\dim(V_A(\Sigma)) = \left(\frac{p}{4}\right)^{g-1} \sum_{k=1}^{\lfloor \frac{p-1}{2} \rfloor} \left(\sin \frac{2k\pi}{p}\right)^{2-2g}$$

here g is the genus of Σ.

2.4. Curve operators and fusion rules

Let Σ be a compact connected oriented surface. Chose a handlebody H with boundary Σ and chose a finite set of colored banded points $(\mathcal{P}, c)$ on Σ. For $\gamma \subset \Sigma - \mathcal{P}$ a simple closed curve, we define

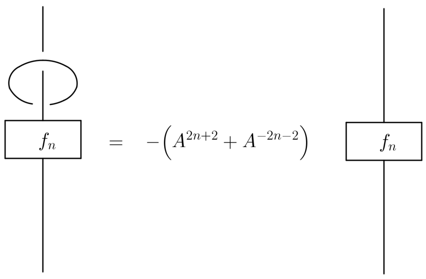
$$Z_p(\gamma) = \gamma \times [1/4, 3/4] \sqcup (\mathcal{P} \times [0,1])^c \in S_A^r(\Sigma \times [0,1], -\mathcal{P} \times \{0\} \sqcup \mathcal{P} \times \{1\}, c \times \{0,1\})$$

where $(\mathcal{P} \times [0,1])^c$ is the element obtained by cabling each banded point $x \subset \mathcal{P}$ by the Jones-Wenzl idempotent $f_{c(x)}$. Gluing $\Sigma \times [0,1]$ with H along Σ, we can see $Z_p(\gamma)$ acts on $S_A(H, \mathcal{P}, c)$, moreover we have that for all $u, v \in S_A(H, \mathcal{P}, c)$

$$\langle Z_p(\gamma) u, v \rangle_A = \langle u, Z_p(\gamma) v \rangle_A$$

Hence $Z_p(\gamma)$ preserves $I_A(H, \mathcal{P}, c)$ and admits an induced action on $V_A(H, \mathcal{P}, c)$. The endomorphism $Z_p(\gamma) \in \text{End}(V_A(H, \mathcal{P}, c))$ is called the curve operator associated to γ. Understanding curve operators is key to define an action of the mapping class group on $V_A(H, \mathcal{P}, c)$. Let $\Gamma \subset H$ be a spine of $(H, \mathcal{P})$, by Theorem 2.5, a basis $S_A^r(H, \mathcal{P}, c)$ is given by the Γ_σ for σ a p-admissible coloring of Γ.

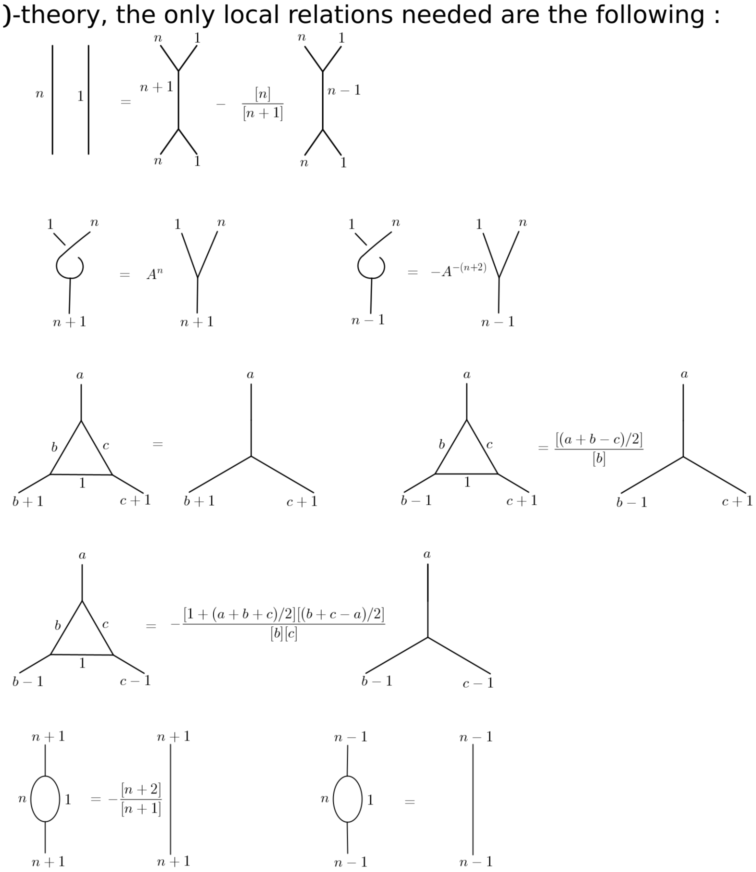
Suppose that γ encircles an edge e of Γ. In this case the $Z_p(\gamma)$ is easy to compute in the basis $(\Gamma_\sigma)_\sigma$. From the properties of the Jones-Wenzl projectors, we can prove that for all n between 0 and $r-2$



This graphical computation is left to the reader as an exercise and implies that

$$Z_p(\gamma)\Gamma_\sigma = -\left(A^{2\sigma(e)+2} + A^{-2\sigma(e)-2}\right)\Gamma_\sigma \tag{2.1}$$

We see that $Z_p(\gamma)$ is diagonal in the basis $(\Gamma_\sigma)_\sigma$, when γ does not encircle an edge of Γ, the computation is more complicated and involves the so-called fusion rules from [MV94]. In the SU(2)-theory, the only local relations needed are the following :



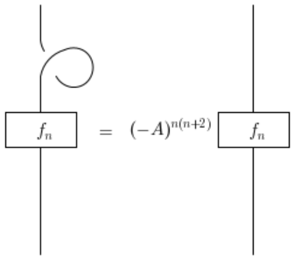
The curve γ has to be colored by one, the element $Z_p(\gamma)\Gamma_\sigma$ in the handlebody H is then a graph with two connected components. Using the above local relations, $Z_p(\gamma)\Gamma_\sigma$ can be expressed as a linear combination of admissible colorings of Γ. In the SO(3)-theory, things are more complicated since the admissibility condition only involves even colors. In this case we need to color γ by $p-1$, we will not describe here the equivalent local rules needed.

2.5. Action of the mapping class group

Let Σ be a compact connected oriented surface. Chose a handlebody H with boundary Σ, a finite set of colored banded points $(\mathcal{P}, c)$ on Σ and Γ a spine of $(H, \mathcal{P})$. Our goal is to find an action of the mapping class group of Σ on $V_A(H, \mathcal{P}, c)$. Before that, let's recall some basic definitions. Let $\mathrm{Mod}(\Sigma, \mathcal{P})$ be the group $\{f \in \mathrm{Homeo}^+(\Sigma) \,|\, f_{|\mathcal{P}} = \mathrm{Id}\}$ up to isotopy (considering isotopies that fix the banded points). For each connected component $p \in \mathcal{P}$ we chose a point $x_p \in p$. Let $\mathcal{P}^0$ be the finite set of all x_p for p connected component of $\mathcal{P}$. We will consider also the group $\mathrm{Mod}(\Sigma, \mathcal{P}^0)$ which is the group $\{f \in \mathrm{Homeo}^+(\Sigma) \,|\, f(x) = x,\ \forall x \in \mathcal{P}^0\}$ up to isotopy. There is a natural surjection

$$\mathrm{Mod}(\Sigma, \mathcal{P}) \to \mathrm{Mod}(\Sigma, \mathcal{P}^0)$$

whose kernel is the abelian group generated by Dehn twists along curves encircling a connected component of $\mathcal{P}$. In particular this kernel is isomorphic to $\mathbb{Z}^{|\mathcal{P}^0|}$. For γ a simple close curve on $\Sigma - \mathcal{P}$ we denote by t_γ the Dehn twist along γ. Notice that the handlebody group $\mathrm{Mod}(H, \mathcal{P})$ acts naturally on $V(H, \mathcal{P}, c)$. The action of $\mathrm{Mod}(\Sigma, \mathcal{P})$ on $V(H, \mathcal{P}, c)$ is meant to extend this action. The first step is to understand the action of t_γ where γ is a curve on $\Sigma - \mathcal{P}$ encircling an edge of Γ. The following graphical equality can be obtained the fusion rules shown in Section 2.4.



(2.2)

This implies that

$$\rho_p(t_\gamma)\Gamma_\sigma = (-A)^{\sigma(e)(\sigma(e)+2)}\Gamma_\sigma \tag{2.3}$$

for any p-admissible coloring of Γ. Comparing this equation with Equation (2.1), we see that $Z_p(\gamma)$ and $\rho_p(t_\gamma)$ are both diagonal in the basis $(\Gamma_\sigma)_\sigma$. Moreover we have that $-(A^{2n+2} + A^{-2n-2}) \neq -(A^{2m+2} + A^{-2m-2})$ for $0 \leq n \neq m \leq r-2$. Therefore

$$Q_p(Z_p(\gamma)) = \rho_p(t_\gamma)$$

where Q_p is the unique polynomial in $K_A[X]$ of degree $r-1$ such that

$$Q_p(-A^{2n+2} - A^{-2n-2}) = (-A)^{n(n+2)}$$

for all $0 \leq n \leq r-2$.

Now let γ be any simple closed curve, the curve operator $Z_p(\gamma)$ is well defined so let us define $\rho_p(\gamma)$ as

$$\rho_p(t_\gamma) = Q_p(Z_p(\gamma))$$

Recall that $\mathrm{Mod}(\Sigma, \mathcal{P})$ is generated by Dehn twists along simple closed curves. At first glance, it is hard to believe that such a definition of the action a Dehn twist should lead to a group action of $\mathrm{Mod}(\Sigma, \mathcal{P})$. Surprisingly enough, it works!

Theorem 2.6. *The followings hold :*

1. *For any simple closed curve γ, $\rho_p(t_\gamma) \in \mathrm{GL}(V_A(H, \mathcal{P}, c))$,*

2. *If $\gamma_1, \ldots, \gamma_n$ are simple closed curves such that $t_{\gamma_1}^{\epsilon_1} \cdots t_{\gamma_n}^{\epsilon_n} = 1$ in $\mathrm{Mod}(\Sigma, \mathcal{P})$ for some $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ then there exists $\lambda \in K_A$ with absolute value one such that $\rho_p(t_{\gamma_1})^{\epsilon_1} \cdots \rho_p(t_{\gamma_n})^{\epsilon_n} = \lambda \mathrm{Id}$.*

Theorem 2.6 implies that ρ_p extends to a projective representation

$$\rho_p : \mathrm{Mod}(\Sigma, \mathcal{P}) \to \mathrm{PGL}(V_A(H, \mathcal{P}, c))$$

Theorem 2.7. *The isomorphism class of ρ_p does not depend on H and only depends on $(\Sigma, \mathcal{P}, c)$.*

We can now forget the handlebody H and simply write $V_p(\Sigma, \mathcal{P}, c)$ for the space $V_A(H, \mathcal{P}, c)$.

Lemma 2.8. *ρ_p induces a projective representation of $\mathrm{Mod}(\Sigma, \mathcal{P}^0)$.*

Proof. Let γ be a curve that encircles a banded point x colored by $c(x)$. The Dehn twist along γ acts as

$$\rho_p(t_\gamma) = (-A)^{c(x)(c(x)+2)}\mathrm{Id}$$

which is trivial in $\mathrm{PGL}(V_p(\Sigma, \mathcal{P}, c))$. □

Remark 2.2. Let A and A' be two $2p$-th primitive roots of unity, let $\Psi \in \mathrm{Gal}(K_A, \mathbb{Q})$ such that $\Psi(A) = A'$. Let $\varphi \in \mathrm{Mod}(\Sigma, \mathcal{P})$, let M (resp. M') be the matrix of the action of φ on $V_A(H, \mathcal{P}, c)$ (resp. $V_{A'}(H, \mathcal{P}, c)$) on the basis $(\Gamma_\sigma)_\sigma$. The formulas for the actions of Dehn twists say that $M' = \Psi(M)$.

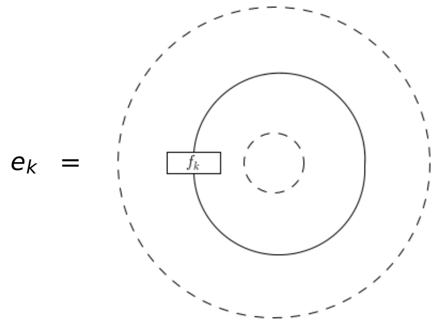
Finally there is a nice equivariant action of $\mathrm{Mod}(\Sigma)$ on curve operators.

Proposition 2.9. *If γ is a closed curve and $\mathrm{Mod}(\Sigma, \mathcal{P})$ then*

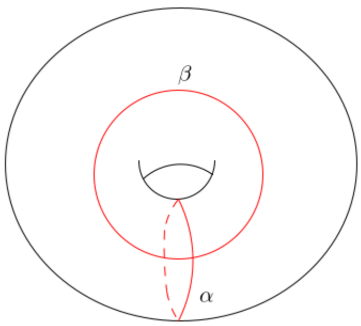
$$\rho_p(\varphi) Z_p(\gamma) \rho_p(\varphi)^{-1} = Z_p(\varphi(\gamma))$$

2.6. Example of the torus in SU(2)-theory

Consider the genus one surface $\mathbb{T}^2$ without banded points and p even, recall that in this case $r = p/2$. Let H be the solid torus $D^2 \times S^1$. By Theorem 2.5 a basis of $V_p(\mathbb{T}^2)$ is given by



where $k \in \{0, \ldots, r-2\}$. The mapping class group of the torus is generated by the Dehn twists t_α and t_β where α and β are the following curves on the torus :



It is known $\mathrm{Mod}(\mathbb{T}^2)$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$ via the map that sends t_α to $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and t_β to $\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$. Let $S = t_\alpha t_\beta t_\alpha = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, we can check that t_α and S generate $\mathrm{Mod}(\mathbb{T}^2)$. After computation we have

$$\rho_p(t_\alpha)e_k = (-A)^{k(k+2)}e_k$$
$$\rho_p(S)e_k = \lambda\sum_{l=0}^{r-2}(-1)^{k+l}[(k+1)(l+1)]e_l$$

For some explicit $\lambda \in K_A$. It is possible to show that this representation factors through $\mathrm{SL}(2,\mathbb{Z}/4r)$ and hence has finite image.

3. Important properties

3.1. Hermitian form

Let Σ be a genus $g \geq 0$ surface equipped with a set of colored banded points $(\mathcal{P}, c)$. Let H be a handlebody with boundary Σ. As mentioned in Section 2.3, the space $V_A(H,\mathcal{P},c)$ is endowed with a non degenerate sesquilinear form $\langle\, ,\,\rangle_A$.

Theorem 3.1. *The form $\langle\, ,\,\rangle_A$ is invariant by the action of the mapping class group : for any $\varphi \in \mathrm{Mod}(\Sigma,\mathcal{P})$ and $x, y \in V_A(H,\mathcal{P},c)$, $\langle \rho_p(\varphi)x, \rho_p(\varphi)y\rangle_A = \langle x,y\rangle_A$.*

A strategy to prove this theorem could be to prove the invariance by a finite number of Dehn twists generating the mapping class group. This strategy seems quite tedious and the only elegant proof I know relies on the TQFT hidden behind this construction. If Γ is a spine of $(H,\mathcal{P})$ and explicit computation of the form $\langle\, ,\,\rangle_A$ is possible on the basis $(\Gamma_\sigma)_\sigma$.

Theorem 3.2. *If σ and σ' are two p-admissible colorings of Γ, $\langle \Gamma_\sigma, \Gamma_{\sigma'}\rangle_A = 0$ when $\sigma \neq \sigma'$ and*

$$\langle \Gamma_\sigma, \Gamma_\sigma\rangle_A = \frac{\prod_{v\in V(\Gamma)} \langle \sigma(e_v), \sigma(e'_v), \sigma(e''_v)\rangle_p}{\prod_{e\in E(\Gamma)} (-1)^{\sigma(e)}[\sigma(e)+1]} \tag{3.1}$$

Here for a vertex $v \in V(\Gamma)$ we denote by e_v, e'_v, e''_v the three edges meeting at v. For a, b, c an p-admissible triple, the quantity $\langle a,b,c\rangle_p$ is defined by

$$\langle a,b,c\rangle_p = (-1)^{\frac{a+b+c}{2}}\frac{[\frac{a+b+c}{2}+1]![\frac{b+c-a}{2}]![\frac{a+c-b}{2}]![\frac{a+b-c}{2}]!}{[a]![b]![c]!}$$

where $[n]! = [n][n-1]\cdots[1]$.

Notice that $[n+1] \neq 0$ for all n between 0 and $r-2$, therefore Equation (3.1) makes sense. Moreover if a, b, c is p-admissible triple then $\langle a,b,c\rangle_p \neq 0$, in particular $\langle \Gamma_\sigma, \Gamma_\sigma\rangle_A \neq 0$. From analyzing the sign of $\langle \Gamma_\sigma, \Gamma_\sigma\rangle_A$, we deduce the following :

Corollary 3.3. *The form $\langle\, ,\,\rangle_A$ is positive definite when $A = e^{i\pi/p}$ for p even and $A = ie^{i\pi/2p}$ for p odd.*

Corollary 3.4. *For all $\varphi \in \mathrm{Mod}(\Sigma,\mathcal{P})$, the endomorphism $\rho_p(\varphi)$ is diagonalizable.*

Proof. Let $\varphi \in \mathrm{Mod}(\Sigma,\mathcal{P})$ and let M be the matrix of the action of φ on $V_{A_0}(H,\mathcal{P},c)$ on the basis $(\Gamma_\sigma)_\sigma$, where A_0 is a value for which $\langle\, ,\,\rangle$ is positive definite. By Theorem 3.1, M is a unitary matrix and hence is diagonalizable. Now let A be a $2p$-th primitive root of unity and let M' be the matrix of the action of φ on $V_A(H,\mathcal{P},c)$ on the basis $(\Gamma_\sigma)_\sigma$. There exists $\Psi \in \mathrm{Gal}(K_A,\mathbb{Q})$ such that $M' = \Psi(M)$. It is a small exercise to conclude that M' is diagonalizable. □

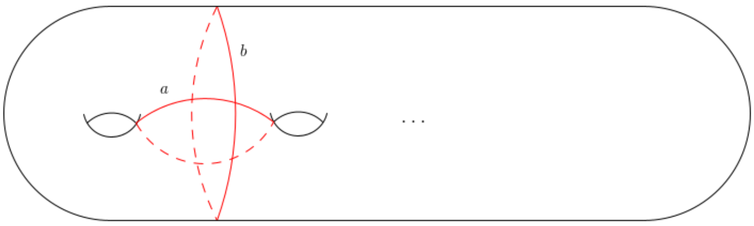
3.2. Infinite image

As we saw before, if the surface Σ is a torus then the image of ρ_p is finite for all p.

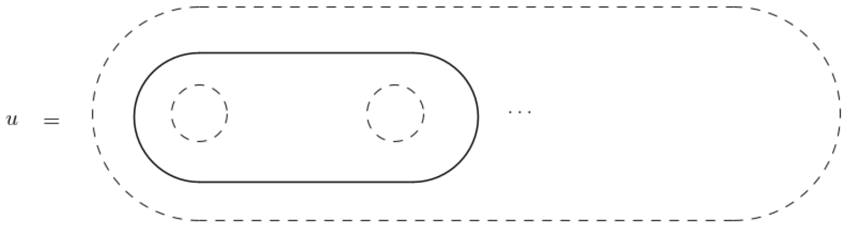
Theorem 3.5 ([F99],[M98])**.** *Let Σ be closed surface of genus at least 2. For p even, the group $\rho_p(\mathrm{Mod}(\Sigma))$ is infinite provided that $p \notin \{6, 8, 12, 20\}$.*

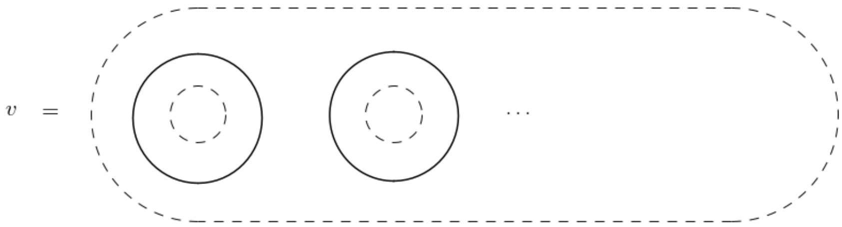
This Theorem was proved separately by Masbaum (in [M98]) and Funar (in [F99]). A very similar statement for p odd can be deduced. Let us present the proof given by Masbaum which is quite straightforward.

Proof of Theorem 3.5 from [M98]*.* Let a and b the two following curves on Σ :



Let u, v the two following banded tangles in H (the standard handlebody bounding Σ) :





Using the fusion rules it is possible to show that these two elements are linearly independent in $V_p(\Sigma)$. Let V be the K_A vector space generate by u and v. An easy computation shows that $\rho_p(t_a)$ and $\rho_p(t_b)$ stabilize V, moreover in the basis (u, v) we have :

$$\rho_p(t_a)_{|V} = \begin{pmatrix} 1 & A^6 - A^2 \\ 0 & A^8 \end{pmatrix} \quad \text{and} \quad \rho_p(t_b)_{|V} = \begin{pmatrix} A^8 & 0 \\ A^6 - A^2 & 1 \end{pmatrix}$$

An easy computation shows that the matrix $\rho_p(t_a t_b^{-1})_{|V}$ has infinite order for $r \notin \{6, 8, 12, 20\}$. □

Remark 3.1. As far as I know the representation ρ_p is the only unitary representation of the mapping class group having infinite image.

Corollary 3.6 ([F99])**.** *Let Σ be a surface of genus at least 2. The normal subgroup generated by k-th powers of Dehn twists has infinite index in $\mathrm{Mod}(\Sigma)$ if $k \notin \{1, 2, 3, 4, 6, 8, 12\}$.*

Partial proof. Let r be an integer different from $3, 4, 6, 10$. Let $N(t^{4r})$ be the normal subgroup generated by $4r$-th powers of Dehn twists. Let t be a Dehn twist, we know that $\rho_{2r}(t)$ is diagonalizable with eigenvalues $(-A)^{j^2+2j}$ where A is $4r$-th primitive root of unity and j is an integer. Therefore $\rho_{2r}(t^{4r})$ is the identity, hence $N(t^{4r}) \subseteq \ker(\rho_{2r})$. Now from the previous theorem we know that $\rho_{2r}(\mathrm{Mod}(\Sigma))$ is infinite so $N(t^{4r})$ has infinite index in $\mathrm{Mod}(\Sigma)$. To prove the statement for $N(t^k)$ when k is odd or not divisible by 4, one needs to use a version of Theorem 3.5 for the SO(3)-theory, in this case Dehn twist act as odd order matrices. □

This answer a question raised by Birman in [B74]. The motivation comes from the congruence property in $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$. For q an integer, it is possible to prove that the normal subgroup generated by $(\mathrm{Id}_n + E_{i,j})^q$, for $i \neq j$ is the kernel of the map $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/q)$ and hence is finite index. Therefore it is natural to ask a similar question replacing $\mathrm{SL}_n(\mathbb{Z})$ by $\mathrm{Mod}(\Sigma)$ and the elementary matrices $\mathrm{Id}_n + E_{i,j}$ by Dehn twists.

3.3. Asymptotic faithfulness

If t is Dehn twist, we have seen that $\rho_p(t)$ has finite order for all p. This says that none of the representations ρ_p is faithful. The following theorem tells that for a given element φ in the mapping class group, $\rho_p(\varphi)$ is not trivial provided that p is big enough.

Theorem 3.7 ([A06])**.** *Let Σ be a genus $g \geq 1$ surface and let $\varphi \in \mathrm{Mod}(\Sigma)$ be a non-central element. There exists p_φ such that for all $p \geq p_\varphi$ even we have $\rho_p(\varphi)$ non trivial.*

This theorem was orginally proved in the geometric setting by Andersen (see [A06]). It was then proved in the Skein theoretical setting by Freedman, Walker and Wang (see [FWW02]). Finally an other proof in the Skein setting was given by Marché and Narimannejad (see [MN08]) by analyzing the limit of $\mathrm{Ad}(\rho_p)$ in the Fell topology.

Note that the center of the $\mathrm{Mod}(\Sigma)$ is trivial if $g \geq 3$ and isomorphic to $\mathbb{Z}/2$ for $g = 1, 2$.

Theorem 3.7 is stated in SU(2)-theory, a fairly straightforward change to Freedman-Walker-Wang's proof leads to the same statement in SO(3)-theory.

Corollary 3.8 ([FWW02])**.** $\mathrm{Mod}(\Sigma)$ *is residually finite.*

Proof. Recall that a group G is said to be residually finite if for any non trivial element $g \in G$, there exists Q a finite group and $\Psi : G \to Q$ a group morphism such that $\Psi(g) \neq 1$. Recall also a Theorem by Malcev saying that any finitely generated linear group is residually finite. Now let $\varphi \in \mathrm{Mod}(\Sigma)$ non central for p big enough $\rho_p(\varphi)$ is non trivial in $\rho_p(\mathrm{Mod}(\Sigma))$ which is finitely generated linear group, we can conclude using Malcev's theorem. Now if $g = 1$ or 2, the center of $\mathrm{Mod}(\Sigma)$ is detected by the action on the integral homology modulo 3. □

This corollary was originally proved by Grossman. Although Grossman proof is elementary, I find it interesting that quantum representations say something important about the structure of the mapping class group.

Now let us present the idea of the proof of Theorem 3.7 given by Freedman Walker and Wang in [FWW02].

Proof of Theorem 3.7 from [FWW02]. Let (γ, α) be a pair non-isotopic simple closed curves on Σ. We admit the following technical lemma : we find a pants decomposition $\mathcal{C} = \{\alpha_1, \ldots, \alpha_{3g-3}\}$ of Σ such that :

1. $\alpha_1 = \alpha$,

2. for any $\alpha_j, \alpha_k, \alpha_l$ bounding a pair of pants, $i(\alpha_j, \gamma) + i(\alpha_k, \gamma) \geq i(\alpha_l, \gamma) \geq |i(\alpha_j, \gamma) - i(\alpha_k, \gamma)|$.

Now let φ a non-central element of $\mathrm{Mod}(\Sigma)$, since Dehn twists generate $\mathrm{Mod}(\Sigma)$, there exists a curve α such that $\varphi(\alpha) \neq \alpha$. Applying the previous lemma to the pair $(\varphi(\alpha), \alpha)$ we find a pants decomposition $\mathcal{C} = \{\alpha_1, \dots, \alpha_{3g-3}\}$ satisfying the properties above mentioned. The goal is to prove that for p big enough $Z_p(\alpha) \neq Z_p(\varphi(\alpha))$, by Proposition 2.9 it would imply that $\rho_p(\varphi)$ does not commute with $Z_r(\alpha)$ for p big enough and hence is not a scalar multiple of the identity.

Let H be the handlebody defined by $\mathcal{C}$ and Γ be a spine of H dual to $\mathcal{C}$ (each curve of $\mathcal{C}$ encircles an edge of Γ). If $\varphi(\alpha) \in \mathcal{C} - \{\alpha\}$ then it is easy to see that $Z_p(\alpha) \neq Z_p(\varphi(\alpha))$ for any p.

Suppose now that $\varphi(\alpha) \notin \mathcal{C} - \{\alpha\}$. Let σ_0 be the zero coloring of Γ, we have

$$Z_p(\alpha)\Gamma_{\sigma_0} = -(A^2 + A^{-2})\Gamma_{\sigma_0}$$

we wish to prove that for p big enough, $Z_p(\varphi(\alpha))\Gamma_{\sigma_0}$ is not proportional to Γ_{σ_0}.

Let $\sigma : E(\Gamma) \to \mathbb{N}$ defined by $\sigma(e) = i(\alpha_k, \varphi(\alpha))$ where α_k is the unique curve in $\mathcal{C}$ encircling the edge e. Notice that by the previous lemma, it is an admissible coloring of Γ. Let

$$r_\varphi = 2 + \frac{1}{2}\mathrm{Max}\{i(\alpha_j, \gamma) + i(\alpha_k, \gamma) + i(\alpha_l, \gamma) \,|\, \alpha_j, \alpha_k, \alpha_l \in \mathcal{C} \text{ bounding a pair of pants}\}$$

Let us fix $p \geq 2r_\varphi$ and notice that σ is a r-admissible coloring of Γ. Using the fusion rules, we can express $Z_p(\varphi(\alpha))\Gamma_{\sigma_0}$ in the basis of r-admissible colorings of Γ as

$$Z_p(\varphi(\alpha))\Gamma_{\sigma_0} = \lambda \Gamma_\sigma + \sum_{\sigma' < \sigma} \lambda_{\sigma'} \Gamma_{\sigma'}$$

where $\lambda \neq 0$. Here for σ' a p-admissible coloring of Γ, $\sigma' < \sigma$ means that for all $e \in E(\Gamma)$, $\sigma'(e) \leq \sigma(e)$ and there exists $e_0 \in E(\Gamma)$ such that $\sigma'(e_0) < \sigma(e_0)$. Notice that the condition $\varphi(\alpha) \notin \mathcal{C} - \{\alpha\}$ ensures that σ is not the zero coloring. We conclude that $Z_p(\varphi(\alpha))\Gamma_{\sigma_0}$ is not proportional to Γ_{σ_0}.

□

3.4. Irreducibility, Zarisky density and integral TQFT

A difficult problem is to understand the decompositions into irreducible representations of quantum representations. Such a decomposition exists because quantum representations are unitary representations. Roberts answers this question when p is a prime number.

Theorem 3.9 ([R01])**.** *For any surface equipped with a finite set of colored banded points, ρ_p is irreducible when p is a prime or twice a prime.*

The original proof of Roberts only works for surfaces without banded points. The extension to surfaces with colored banded points is done by adding an easy extra argument.

In section 3.1, we mentioned that when p is odd, the representation ρ_p preserves a unitary form when $A = ie^{i\pi/p}$. An important result by Larsen and Wang gives the closure of the image of quantum representations when p is prime.

Theorem 3.10 ([LW05])**.** *Let Σ be a surface of genus at least 2 without banded points. When p is a prime integer and for $A = ie^{i\pi/p}$, the group $\rho_p(\mathrm{Mod}(\Sigma))$ is dense in $\mathrm{PU}(V_p(\Sigma))$.*

Remark 3.2. In SU(2)-theory, when p is twice a prime and $A = e^{i\pi/p}$, the closure of the group $\mathrm{Im}(\rho_p)$ can be described explicitly and it is not the full projective unitary group of the underlying space.

Finally an important source of applications is the integral structure preserved by these representations. So far we know the matrices of the representations ρ_p lie in the number field $K_A = \mathbb{Q}(A)$. Gilmer and Masbaum proved that when p is prime, these matrices can be conjugated to matrices in the integral ring $\mathbb{Z}[A^2]$.

Theorem 3.11 ([GM07])**.** *When p is a prime integer, there exists a $\mathbb{Z}[A^2]$-lattice $S_p(\Sigma, \mathcal{P}, c) \subset V_p(\Sigma, \mathcal{P}, c)$ stable by $\rho_p(\mathrm{Mod}(\Sigma, \mathcal{P}))$.*

4. Applications

4.1. All finite groups are involved in the mapping class group

The first striking application of integrability properties of quantum representations come from the work Masbaum and Reid in [MR12]. Let Σ_g be a surface of genus $g \geq 3$. Their main theorem is

Theorem 4.1 ([MR12])**.** *If G is a finite group, then there exists a finite index subgroup $\Gamma \subseteq \mathrm{Mod}(\Sigma_g)$ and a surjection from Γ to G.*

We can rephrase this theorem by saying that any finite group can be viewed as a quotient of a finite index subgroup of $\mathrm{Mod}(\Sigma_g)$. As the mapping class group is finitely generated, it is easy to see that some finite groups cannot be viewed as quotients of $\mathrm{Mod}(\Sigma_g)$ it self. This theorem answers a question raised by U. Hamenstadt in her talk at the 2009 Georgia Topology Conference. Even today finite index subgroups of the mapping class group are not well understood, I believe Theorem 4.1 was a good step in this direction.

Theorem 4.1 is well known for genus one and two. Note that a similar result was proved later by Grunewald, Larsen, Lubotzky and Malestein in [GLLM15] using Prym representations. Now let us explain the main ideas of the proof.

Let p be a prime number and A be a $2p$-th primitive root of unity. By Theorem 3.11 we can see the quantum representation as a homomorphism

$$\rho_p : \mathrm{Mod}(\Sigma_g) \to \mathrm{PGL}(d_p, \mathbb{Z}[A^2])$$

where d_p is dimension of $V_p(\Sigma_g)$. By using a central extension of $\mathrm{Mod}(\Sigma_g)$ on which ρ_p lifts to a linear representation (non projective) and using that the abelianization of $\mathrm{Mod}(\Sigma_g)$ is trivial (recall that $g \geq 3$), we can prove that $\rho_p(\mathrm{Mod}(\Sigma_g)) \subset \mathrm{PSL}(d_p, \mathbb{Z}[A^2])$. We refer to [MR12, Section 3] for more details on this fact.

Working with a representation whose coefficients live in the cyclotomic ring $\mathbb{Z}[A^2]$ allows us to build many finite quotients of the image of ρ_p by taking finite quotients of $\mathbb{Z}[A^2]$. For instance the quotient of $\mathbb{Z}[A^2]$ by the ideal generated by $h = 1 - A^2$ is isomorphic to the finite field $\mathbb{F}_p$. The kernel of

$$\mathrm{Mod}(\Sigma_g) \to \mathrm{PSL}(d_p, \mathbb{F}_p)$$

gives an interesting finite index subgroup of the mapping class group. More generally, a well know fact in number theory says that for infinitely many prime q, there exists an ideal $\mathcal{I}$ of $\mathbb{Z}[A^2]$ such that $\mathbb{Z}[A^2]/\mathcal{I}$ is isomorphic to $\mathbb{F}_q$. Such a prime q is said to be a rational prime that splits completely to $\mathbb{Q}[A]$, and in our context it gives a map

$$\rho_{p,q} : \mathrm{Mod}(\Sigma_g) \to \mathrm{PSL}(d_p, \mathbb{F}_q)$$

One the main technical result in [MR12] is the following

Theorem 4.2. *For infinitely many rational prime q that splits completely to $\mathbb{Q}[A]$, the image of the map $\rho_{p,q}$ is $\mathrm{PSL}(d_p, \mathbb{F}_q)$.*

The proof of this theorem uses Theorem 4.2 about the density of the image of ρ_p and the so-called *strong approximation*. We will not give much details about the proof but we will explain how from this we can deduce Theorem 4.1.

Lemma 4.3. *Let G be a finite group, for N big enough, G can be embedded in $\mathrm{PSL}(N, \mathbb{F}_q)$ for all odd prime q.*

This Lemma can be deduced from the fact that any finite group is a subgroup of a finite symmetric group which can be embedded in $\mathrm{PSL}(N, \mathbb{F}_q)$ via permutation matrices for N big enough and any q odd prime.

Now let G be a finite group, as $d_p \to +\infty$ when p goes to infinity, by Lemma 4.3 there exists a prime p such that G can be viewed as a subgroup of $\mathrm{PSL}(d_p, \mathbb{F}_q)$ for any q odd prime. Now let q be such that $\rho_{p,q}$ exists and is surjective. We see $\Gamma = \rho_{p,q}^{-1}(G)$ is a finite index subgroup of $\mathrm{Mod}(\Sigma_g)$ and by construction G is a quotient of Γ. This concludes the proof.

4.2. Representations of surface groups

Let Σ be a surface equipped with a non-empty finite set of banded points $\mathcal{P}$. We choose a banded point $x \subset \mathcal{P}$ and we set $\mathcal{P}_x = \mathcal{P} - x$. Suppose that $\Sigma - \mathcal{P}_x$ has negative Euler characteristic and let x_0 be a point on x. The idea behind quantum representations of surface groups starts with the Birman exact sequence which gives the following :

$$1 \to \pi_1(\Sigma - \mathcal{P}_x, x_0) \to \mathrm{Mod}(\Sigma - \mathcal{P}_x, x_0) \to \mathrm{Mod}(\Sigma - \mathcal{P}_x) \to 1$$

Here the arrow $\mathrm{Mod}(\Sigma - \mathcal{P}_x, x_0) \to \mathrm{Mod}(\Sigma - \mathcal{P}_x)$ is given by the forgetful map and $\pi_1(\Sigma - \mathcal{P}_x, x_0) \to \mathrm{Mod}(\Sigma - \mathcal{P}_x, x_0)$ is given by the point pushing map, we refer to [FM11, Section 4.2] for more details. The key point is that $\pi_1(\Sigma - \mathcal{P}_x, x_0)$ can be viewed as a subgroup of $\mathrm{Mod}(\Sigma - \mathcal{P}_x, x_0)$, hence any representation of $\mathrm{Mod}(\Sigma - \mathcal{P}_x, x_0)$ restricts to a representation of $\pi_1(\Sigma - \mathcal{P}_x, x_0)$. We wish to apply this to the representations ρ_p's, the first problem is that for quantum representations we deal with banded points and not points. A way to bypass this is to consider another version of the point pushing map, often referred to as boundary pushing map, indeed it is possible to see $\pi_1(U(\Sigma - \mathcal{P}_x), x)$ (where $U(\Sigma - \mathcal{P}_x)$ is the unit tangent bundle of $\Sigma - \mathcal{P}_x$ for any Riemannian metric) as a subgroup of $\mathrm{Mod}(\Sigma, \mathcal{P})$. Therefore one get an projective action of $\pi_1(U(\Sigma - \mathcal{P}_x), x)$ on $V_p(\Sigma, \mathcal{P}, c)$ (where c is a coloring of $\mathcal{P}$) via ρ_p. Moreover we have that $\pi_1(U(\Sigma - \mathcal{P}_x), x)$ is a central extension of $\pi_1(\Sigma - \mathcal{P}_x, x_0)$ where the center of $\pi_1(U(\Sigma - \mathcal{P}_x), x)$ can be viewed as the subgroup generated by the Dehn twist along a curve δ_x encircling x. Finally as the Dehn twist t_{δ_x} acts as a scalar multiple of the identity via ρ_p, we see that the projective action of $\pi_1(U(\Sigma - \mathcal{P}_x), x)$ on $V_p(\Sigma, \mathcal{P}, c)$ induces a representation

$$\rho_p : \pi_1(\Sigma - \mathcal{P}_x, x_0) \to \mathrm{PGL}(V_p(H, \mathcal{P}, c))$$

The existence of such quantum representations of surface groups lead to some applications in geometric group theory.

Theorem 4.4 ([KS16])**.** *The following hold :*

1. *for any simple loop* $\gamma \in \pi_1(\Sigma - \mathcal{P}_x, x_0)$ *and for any* p *integer,* $\rho_p(\gamma)$ *has finite order,*

2. *the group* $\rho_p(\pi_1(\Sigma - \mathcal{P}_x, x_0))$ *is infinite for* p *big enough.*

In this theorem a simple loop is loop that is freely homotopic to a simple closed curve. Theorem 4.4 answers a question raised by Kisin and McMullen, the existence of surface groups representations sending all simple loops to finite order elements and having infinite images were not known. The first point of this theorem is an easy consequence of the fact that a simple loop in $\pi_1(\Sigma - \mathcal{P}, x_0)$ is send via the point pushing map to the product of two commuting Dehn twist (one twist times the inverse of another one to be more precise). The second point is proved by exhibiting an explicit loop sent by ρ_p to an infinite order matrix for p big enough. This is in similar fashion as the proof given by Masbaum of Theorem 3.5.

One exotic consequence of Theorem 4.4 is the following

Corollary 4.5. *There exists a finite cover* $\Sigma' \to \Sigma - \mathcal{P}_x$ *such that no simple closed curve on* $\Sigma - \mathcal{P}_x$ *lifts to a closed curve on* Σ'.

Proof. Let p such that $\rho_p(\pi_1(\Sigma - \mathcal{P}_x))$ is infinite. As $\rho_p(\pi_1(\Sigma - \mathcal{P}_x))$ is a finitely generated infinite linear group, by Selberg's lemma, it admits a finite index torsion free subgroup $H \subset \rho_p(\pi_1(\Sigma - \mathcal{P}_x))$. Let $G = \rho_p^{-1}(H)$ (it is a finite index subgroup of $\pi_1(\Sigma - \mathcal{P}_x)$) and let $\Sigma' \to \Sigma - \mathcal{P}_x$ be the associated finite cover. As H is torsion free and because of the first point in 4.4, no simple loop in $\pi_1(\Sigma - \mathcal{P}_x)$ belongs to G. Therefore no simple closed curve on $\Sigma - \mathcal{P}_x$ lift to a closed curve on Σ'. □

The next application concerns the homology of fine covers of surfaces.

Question 1. Can we find a finite cover $\Sigma' \to \Sigma - \mathcal{P}_x$ such that the homology of Σ' is not generated by lifts of simple closed curves of Σ?

This seemingly elementary question was raised in different contexts. For instance it was asked by Looijenga when studying Prym representations of mapping class groups (see [L15]). This question is also very important Ivanov's problem concerning the abelianization of finite index subgroups of mapping class groups, we refer to the paper from Putman and Wieland (see [PW13]). Let us rephrase this question more precisely. Let $\Sigma' \to \Sigma - \mathcal{P}_x$ be a finite cover of $\Sigma - \mathcal{P}_x$. For $\gamma \in \pi_1(\Sigma - \mathcal{P}_x)$ a simple loop, we define $n(\gamma)$ to be the smallest integer such that $\gamma^{n(\gamma)} \in \pi_1(\Sigma')$. Let $\pi_1^s(\Sigma')$ be the subgroup of $\pi_1(\Sigma')$ generated by

$$\{\gamma^{n(\gamma)} \,|\, \gamma \in \pi_1(\Sigma - \mathcal{P}_x) \text{ simple loop}\}$$

We define $H_1^s(\Sigma', \mathbb{Z}) \subseteq H_1(\Sigma', \mathbb{Z})$ to be the $\mathbb{Z}$-submodule generated by $\pi_1^s(\Sigma')$ via the abelianization. Another way to define $H_1^s(\Sigma', \mathbb{Z})$ is to say that it is the $\mathbb{Z}$-submodule of $H_1(\Sigma', \mathbb{Z})$ generated by connected components of preimages of simple closed curves in $\Sigma - \mathcal{P}_x$ via the covering map. Finally we define

$$H_1^s(\Sigma', \mathbb{Q}) = H_1^s(\Sigma', \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \subseteq H_1(\Sigma', \mathbb{Q})$$

In this setting Question 1 can be rephrased :

Question 2. Can we find a finite cover $\Sigma' \to \Sigma - \mathcal{P}_x$ such $H_1^s(\Sigma', \mathbb{Z}) \neq H_1(\Sigma', \mathbb{Z})$?

And the stronger version :

Question 3. Can we find a finite cover $\Sigma' \to \Sigma - \mathcal{P}_x$ such $H_1^s(\Sigma', \mathbb{Q}) \neq H_1(\Sigma', \mathbb{Q})$?

A first positive answer to Question 2 was given in [KS16] using quantum representations of mapping class groups. Later and using different technics, Malestein and Putman (see [MP19]) gave a positive answer to Question3 for surfaces with at least one puncture (in our setting $\mathcal{P}_x \neq \emptyset$). As far as I know Question 3 remains open when $\mathcal{P}_x = \emptyset$. Let us now explain how to answer Question 2 using quantum representations.

Let p be a prime number such that $\rho_p(\pi_1(\Sigma - \mathcal{P}_x))$ is infinite. By Theorem 3.11, a certain $\mathbb{Z}[A^2]$-lattice of $V_p(\Sigma, \mathcal{P}, c)$ is stable by the action of the mapping class group, by restriction this is still true for the action of $\pi_1(\Sigma - \mathcal{P}_x)$. Hence we can see ρ_p has a homomorphism :

$$\rho_p : \pi_1(\Sigma - \mathcal{P}_x) \to \mathrm{PGL}(d, \mathbb{Z}[A^2])$$

where d is the dimension of $V_p(\Sigma, \mathcal{P}, c)$. Let $h = 1 - A^2$ which is a prime in $\mathbb{Z}[A^2]$. Let $N \geq 1$ be integer, we have that $\mathbb{Z}[A^2]/h^N\mathbb{Z}[A^2]$ is a finite ring. For instance, for $N = 1$, we can check that $\mathbb{Z}[A^2]/h\mathbb{Z}[A^2]$ is isomorphic to the finite field of cardinality p. We denote by

$$\rho_{p,N} : \pi_1(\Sigma - \mathcal{P}_x) \to \mathrm{PGL}(d, \mathbb{Z}[A^2]/h^N\mathbb{Z}[A^2])$$

the projection of the coefficients of ρ_p into $\mathbb{Z}[A^2]/h^N\mathbb{Z}[A^2]$. The kernel of $\rho_{p,N}$ is now a finite index subgroup of $\pi_1(\Sigma - \mathcal{P}_x)$ and hence can be seen as the fundamental group of a covering $\Sigma'_{p,N}$ of $\Sigma - \mathcal{P}_x$. More precisely we just built a tower of coverings :

$$\Sigma'_{p,\infty} \to \cdots \to \Sigma'_{p,N+1} \to \Sigma'_{p,N} \to \cdots \to \Sigma - \mathcal{P}_x$$

where $\Sigma'_{p,\infty} \to \Sigma - \mathcal{P}_x$ is the infinite covering associated to $\ker(\rho_p) \subset \pi_1(\Sigma - \mathcal{P}_x)$. For N big enough, these covering actually give a positive answer to Question 2

Theorem 4.6 ([KS16])**.** *For N big enough, the index of $H_1^s(\Sigma'_{p,N}, \mathbb{Z})$ inside $H_1(\Sigma'_{p,N}, \mathbb{Z})$ is at least p^e where*

$$e = \left\lfloor \frac{N}{p-1} \right\rfloor + 1$$

4.3. Algorithm to detect non simple loops

In this subsection we will see how the quantum representation of surface group help to detect non simple loops in surface groups. We will summarize the paper [KS16b].

Let Σ_g of genus $g \geq 2$ and let x_0 be a marked point of Σ_g. The goal is to solve the following problem :

Goal. Given S a set of generators of $\pi_1(\Sigma_g)$. We want to build an algorithm that takes a word w in $S \cup S^{-1}$ as an input and answers if the loop associated to w is simple or not.

As mentionned in Theorem 4.4, the image $\rho_p(\pi_1(\Sigma_g, x_0))$ is infinite for p big enough. A more precise statement is the following

Proposition 4.7. *If $e \in \pi_1(\Sigma_g, x_0)$ is a figure eight loop, then $\rho_p(e)$ has infinite order for $p \geq 5$.*

Here a figure eight loop e is a loop freely homotopic to a wedge of two circles and such the two subloops of e generate a copy of F_2 inside $\pi_1(\Sigma_g, x_0)$. The proof Proposition 4.7 is done by an explicit computation. We denote by $V_5(\Sigma_g, 2)$ the vector space associated to (Σ_g, x_0) for $p = 5$ where x_0 has be colored by 2. We denote by $d(g)$ the dimension of $V_5(\Sigma_g, 2)$. Let

$$\eta_g : \pi_1(\Sigma_g, x_0) \to \mathrm{PGL}(d(g), \mathbb{C})$$

be the associated representation. We can now restate previous ideas in the direction of our goal.

Proposition 4.8. *The sequence of representations η_g, depending on the genus g satisfies that*

1. *If $\gamma \in \pi_1(\Sigma_g, x_0)$ is a power of a simple loop then $\eta_g(\gamma)$ has order* 5*.*
2. *If $\gamma \in \pi_1(\Sigma_g, x_0)$ is a figure eight loop then $\eta_g(\gamma)$ has infinite order.*

At the moment we have an algorithm that just allows us to check if a loop is maybe a power of a simple loop or maybe a figure eight loop. To make this algorithm stronger we use a Theorem by Scott that states that $\pi_1(\Sigma_g, x_0)$ is LERF (see [S78]). LERF stands for *locally extended residually finite*, it means that subgroups are close for the profinite topology. We will not here explain more about the profinite topology but the important consequence is the following property :

Proposition 4.9. *If $\gamma \in \pi_1(\Sigma_g, x_0)$ is not the power of a simple loop then there exists a covering $\Sigma' \to \Sigma_g$ such that γ lifts to Σ' to a figure eight loop.*

Moreover by the work of Patel (see [P14]) the index of $\pi_1(\Sigma') \subseteq \pi_1(\Sigma_g)$ can be explicitely bounded :

Theorem 4.10 (Patel)**.** *Let S be a generating set of $\pi_1(\Sigma_g)$ and l_S be the word metric on $\pi_1(\Sigma_g)$ induced by S. There exist an explicit function $f : \mathbb{N} \to \mathbb{N}$ such that if $\gamma \in \pi_1(\Sigma_g, x_0)$ is not the power of a simple loop then there exists a covering $\Sigma' \to \Sigma_g$ with $[\pi_1(\Sigma_g) : \pi_1(\Sigma')] \leq f(l_S(\gamma))$ such that γ lifts to Σ' to a figure eight loop.*

We are finally ready to describe our algorithm :

1. Choose S a generating set of $\pi_1(\Sigma_g, x_0)$.
2. Choose $w \in \pi_1(\Sigma_g, x_0)$ written as a word in $S \cup S^{-1}$.
3. Consider $\mathcal{C}_w$ the set of all finite index subgroups of $\pi_1(\Sigma_g)$ with index at most $f(l_S(w))$.
4. Note that $\mathcal{C}_w$ is a finite set and each $H \in \mathcal{C}_w$ can be written as $H = \pi_1(\Sigma_k)$ where k explicitely depends on the index of H.

5. For each k such that $\pi_1(\Sigma_k) \in \mathcal{C}_w$, let $\hat{\eta}_k : \pi_1(\Sigma_g) \to \mathrm{PGL}(\hat{d}(g,k), \mathbb{C})$ be the representation $\eta_k : \pi_1(\Sigma_k) \to \mathrm{PGL}(d_k, \mathbb{C})$ induced on $\pi_1(\Sigma_g)$. In particular $\hat{d}(g,k) = [\pi_1(\Sigma_g) : \pi_1(\Sigma')]d_k$.

6. If for all k with $\pi_1(\Sigma_k) \in \mathcal{C}_w$, $\hat{\eta}_k(w)$ has order 5 then we know that w never lifts to a figure eight loop hence we know that it is the power of a simple loop.

7. If there exists k with $\pi_1(\Sigma_k) \in \mathcal{C}_w$, such that $\hat{\eta}_k(w)$ has infinite order we know that w is not the power of a simple loop.

Note that because we have to enumerate all finite index subgroups of a surface group up to a certain index, this algorithm has a super-exponential complexity. This algorithm is very difficult to implement but can viewed has representation theoretical characterization of simple loops.

5. Open problems

5.1. Property (T)

In this subsection we will discuss some open problem regarding the mapping class group and Kazhdan's property (T). We first start with some general definitions.

Definition 7. Let Γ a finitely generated discrete group, $\mathcal{H}$ be a Hilbert space and $\rho : \Gamma \to U(\mathcal{H})$ be a unitary representation.

1. We say that ρ has a fixed vector if there exists $u \in \mathcal{H}$ such that $\rho(g)u = u$ for all $g \in \Gamma$.

2. Let $\mathcal{S}_{\mathcal{H}}$ be the unit sphere in $\mathcal{H}$ and S be a finite generating set of Γ. We say that ρ has almost fixed vectors if $\forall \epsilon > 0$, $\exists u \in \mathcal{S}_{\mathcal{H}}$ such that $||\rho(g)u - u|| \leq \epsilon$ for all $g \in S$.

Another way to formulate that ρ has almost fixed vectors is to say that the infimum of $||\rho(g)u - u||$ when $u \in \mathcal{S}_{\mathcal{H}}$ and $g \in S$ is zero. This property is actually independent of the generating set S. The final definition :

Definition 8. Let Γ a finitely generated discrete group, we say that Γ has Kazhdan's property (T) if any unitary action on a Hilbert space having almost fixed vectors has a fixed vector.

Knowing that a given discret group has Kazhdan's property (T) has deep implication in geometric group theory and dynamics. This question is widely open for mapping class groups except for genus one and two where it is known not to have property (T). In the context of quantum representation we can look at a weaker, but still very difficult, problem.

Let Σ_g be a genus $g \geq 1$ surface, let $p \geq 5$ be a prime number and let ρ_p be the quantum representation of $\mathrm{Mod}(\Sigma_g)$ for $A = ie^{i\pi/p}$. Recall that for this choice of A, the representation ρ_p is unitary. As Kazhdan's property (T) is related to linear representations and not projective representations, we consider the adjoint representation :

$$\mathrm{Ad}(\rho_p) : \mathrm{Mod}(\Sigma_g) \to \mathrm{GL}(\mathrm{End}_0(V_p(\Sigma_g)))$$

Recall that for $g \in \mathrm{Mod}(\Sigma_g)$ and $X \in \mathrm{End}(V_p(\Sigma_g))$:

$$\mathrm{Ad}(\rho_p)(g)X = \rho_p(g)X\rho_p(g)^{-1}$$

The unitary structure on $V_p(\Sigma_g)$ induces a unitary structure on $\mathrm{End}(V_p(\Sigma_g))$ which is preserved by $\mathrm{Ad}(\rho_p)$. Note that by Theorem 3.9, the representation ρ_p is irreducible (as p is prime). Therefore by the Schur's lemma $\mathrm{Ad}(\rho_p)$ does not have any invariant vector. Of course, since this representation is finite dimensional, it does not have almost invariant vectors. We can consider the infinite dimensional unitary representation :

$$\tilde{\rho} = \bigoplus_{p \geq 5 \text{ prime}} \mathrm{Ad}(\rho_p)$$

To be more precise we need to look at the action on the completion of the space (to have an Hilbert space). Recall that $\tilde{\rho}$ does not have any invariant vector, we can naturally ask the following :

Question 4. Does $\tilde{\rho}$ have almost invariant vectors?

Clearly if $\tilde{\rho}$ has almost invariant vectors, $\mathrm{Mod}(\Sigma_g)$ does not have Property (T). For $g = 1$ it is known that $\mathrm{Mod}(\Sigma_1)$ is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$ which is virtually free and hence does not have Property (T). Freedman and Krushkal moreover proved that $\tilde{\rho}$ does not have invariant vectors (see [FK06]). For $g \geq 2$, Question 4 is at the moment unanswered.

5.2. First Betti number of some finite index subgroups

Let Σ_g be a genus $g \geq 3$ surface. It is well know that the abelianization of $\mathrm{Mod}(\Sigma_g)$ is trivial. The following question was raised by Ivanov :

Question 5. Can we find a finite index subgroup of $\mathrm{Mod}(\Sigma_g)$ with infinite abelianization?

We say that a finitely generated group virtually surjects onto $\mathbb{Z}$ if one of its finite index subgroup surjects onto $\mathbb{Z}$. Ivanov's question is equivalent to ask if $\mathrm{Mod}(\Sigma_g)$ virtually surjects onto $\mathbb{Z}$. Remark that $\mathbb{Z}$ does not have Property (T) and having Property (T) passes to finite index subgroups. Hence a finitely generated group that virtually surjects onto $\mathbb{Z}$ does not have Property (T).

In genus one $\mathrm{Mod}(\Sigma_1) = \mathrm{SL}(2, \mathbb{Z})$ and the subgroup generated by the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ has finite index and is isomorphic to the free group F_2. Clearly the free group on two generators surjects onto $\mathbb{Z}$.

Before discussing about the genus two case, recall that $\mathrm{Mod}(\Sigma_g)$ acts on $H_1(\Sigma_g, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ by preserving the intersection form (which is a symplectic form). Therefore we have a homomorphism :

$$\mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$$

This map is often referred to as the homological representation of the mapping class group. It is actually surjective and its kernel is called the Torelli group.

For the genus two case, Taherkhani in [T00] considered the map

$$p_2 : \mathrm{Mod}(\Sigma_2) \to \mathrm{Sp}(4, \mathbb{Z}/2)$$

obtained by the composition the homological representation with the natural projection $\mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{Sp}(4, \mathbb{Z}/2)$. As the target group is finite, the kernel Γ_2 of p_2 is a finite index subgroup of $\mathrm{Mod}(\Sigma_2)$. Taherkhani used computer to prove that Γ_2 has infinite abelianization.

For $g \geq 3$ few results are known about Question 5. McCarthy in [Mc01] proved that any finite index subgroup of $\mathrm{Mod}(\Sigma_g)$ (we still work with $g \geq 3$) containing the Torelli group has finite abelianization. Similar Ershov and He in [EH18] proved that, any finite index subgroup of $\mathrm{Mod}(\Sigma_g)$ containing certain terms in the Johnson filtration has finite abelianization. A natural question in the quantum setting is the following

Question 6. Let $g \geq 3$ and let H be a finite index subgroup of $\mathrm{Mod}(\Sigma_g)$. Suppose there exist p such that H contains $\ker(\rho_p)$. Can H have infinite abelianization?

A weaker and maybe more accessible question comes from the natural finite index subgroups of $\mathrm{Mod}(\Sigma_g)$ coming from the integral structure of quantum representations. Let $p \geq 5$ be a prime and A be a $2p$-th primitive root of unity. Recall that $\rho_p : \mathrm{Mod}(\Sigma_g) \to \mathrm{PGL}(d_p, \mathbb{Z}[A^2])$ for some integer d_p. For $\mathcal{I} \subset \mathbb{Z}[A^2]$ an ideal, we define $\Gamma_{p,\mathcal{I}}$ to be the kernel of the map $\mathrm{Mod}(\Sigma_g) \to \mathrm{PGL}(d_p, \mathbb{Z}[A^2]/\mathcal{I})$.

Question 7. Can $\Gamma_{p,\mathcal{I}}$ have a infinite abelianization?

5.3. Dehn twist power p and kernel

Let p be an integer and A be a $2p$-th primitive root of unity. Let Σ_g be a surface of genus $g \geq 0$ equipped with a finite set of colored banded points $(\mathcal{P}, c)$. For γ an essential simple close curve of Σ_g which does not encircle any banded point in $\mathcal{P}$, we denote by t_γ the Dehn twist along γ, we also denote by $o_p(\gamma)$ the order of $\rho_p(t_\gamma)$. By Formula 2.2, we have that if p is odd then $o_p(\gamma) = p$ and if p is even then $o_p(\gamma) = p$ when γ is non-separating and $o_p(\gamma) = p/2$ when γ is separating. Let D_p be the subgroup of $\mathrm{Mod}(\Sigma_g, \mathcal{P})$ generated by $t_\gamma^{o_p(\gamma)}$ where γ is an essential simple close curve which does not encircle a banded point in $\mathcal{P}$. As the conjugate of a Dehn twist is still a Dehn twist, we see that D_p is a normal subgroup of $\mathrm{Mod}(\Sigma_g, \mathcal{P})$. Moreover, D_p is included in $\ker(\rho_p)$.

Question 8. Do we have $D_p = \ker(\rho_p)$?

There are certain case where this question has a clear negative answer, for instance in genus zero where $|\mathcal{P}| \in \{0, 1, 2, 3\}$. We must also excluded the case $g = 1$ and $\mathcal{P} = \emptyset$, in this case D_p is of infinite index in $\mathrm{Mod}(\Sigma_1) = \mathrm{SL}(2, \mathbb{Z})$ whereas $\ker(\rho_p)$ has finite index.

Although this question seems pretty naive, the kernels of the quantum representations are well understood in only few cases. In an unpublished work, Masbaum answered this question in two cases :

Theorem 5.1 (Masbaum)**.** *In* SO(3)*-theory for the torus equipped with one banded point colored by $c = (p-5)/2$ and in* SU(2)*-theory for the sphere equipped with four banded points colored by one, $D_p = \ker(\rho_p)$.*

In both of these cases, the proof relies on the fact that the quantum representation is two-dimensional and that image of ρ_p can be identified with a hyperbolic triangle group. We must also cite the work of Deroin and Marché :

Theorem 5.2 ([DM22])**.** *Suppose $p = 5$ and that all banded points in $\mathcal{P}$ are colored by two. If $(g, |\mathcal{P}|) \in \{(0, 5), (1, 2), (1, 3), (2, 1)\}$ then D_p has finite index in $\ker(\rho_p)$.*

This theorem is obtained by studying geometric structures on certain moduli spaces of curves. The paper by Deroin and Marché involves some fine arguments from algebraic geometry.

Knowing if the kernel of a quantum representation is exactly the normal subgroup generated by certain powers of Dehn twist can be useful for Ivanov's question. To simplify the setting let us suppose that $g \geq 3$ and $\mathcal{P} = \emptyset$. Let Γ be a finite index subgroup of $\mathrm{Mod}(\Sigma_g)$. As Γ has finite index in $\mathrm{Mod}(\Sigma_g)$, for any Dehn twist $t_\gamma \in \mathrm{Mod}(\Sigma_g)$ there exists $n_\gamma \geq 1$ such that $t_\gamma^{n_\gamma} \in \Gamma$. We need the following theorem from Putman :

Theorem 5.3 ([P10])**.** *For any morphism $\varphi : \Gamma \to \mathbb{Z}$, $\varphi(t_\gamma^{n_\gamma}) = 0$.*

We note that for a certain p, $D_p \subset \Gamma$, a consequence of Putman's theorem is that

$$H^1(\Gamma, \mathbb{Z}) = H^1(\Gamma/D_p, \mathbb{Z})$$

In other words morphisms from Γ to $\mathbb{Z}$ are given by morphisms from Γ/D_p to $\mathbb{Z}$. Hence a positive answer to Question 8 would imply that $\mathrm{Mod}(\Sigma_g)$ does not virtually surject onto $\mathbb{Z}$ if and only if the image of the quantum representation ρ_p does not virtually surject onto $\mathbb{Z}$ for all p.

5.4. AMU conjecture and relation to volume conjecture

Let Σ be a compact connected oriented surface and $\mathcal{P} \subset \Sigma$ be finite set of banded points. Suppose that $\chi(\Sigma - \mathcal{P}) < 0$ and that $\Sigma - \mathcal{P}$ is not a thrice punctured sphere. The conjecture we are going to discuss in this subsection concerns the Nielsen-Thurston classification in the mapping class group. We refer to [FM11] for more details.

Definition 9. $\varphi \in \mathrm{Mod}(\Sigma, \mathcal{P})$ is reducible if it fixes the isotopy class of a multicurve which is not null-homotopic and does not encircle a single point in $\mathcal{P}$.

Definition 10. An element in $\varphi \in \mathrm{Mod}(\Sigma, \mathcal{P})$ is pseudo-Anosov if it has infinite order and is irreducible.

Definition 10 is not the real definition of pseudo-Ansov elements in the mapping class group, the real definition involves measured foliations on $\Sigma - \mathcal{P}$. The definition we use here is a consequence of Thurston's theorem on classification of diffeomorphisms of surfaces. Pseudo-Ansov elements have the richest dynamic when acting on various geometric spaces associated to the surface, for instance the Teichmuller space or the curve complex. We will not give much details about these geometric aspects but let us cite a topological example to illustrate these interesting dynamical properties. For this example let us suppose that Σ has genus at least two and that $\mathcal{P} = \emptyset$, to a pseudo-Anosov element $\varphi \in \mathrm{Mod}(\Sigma)$ is associated a real number $\lambda_\varphi > 1$ called the stretching factor of φ with the following property : if α, β are isotopy classes of non null-homotopic curves :

$$\lim_{n \to +\infty} \sqrt[n]{i(\varphi^n(\alpha), \beta)} = \lambda_\varphi$$

Definition 11. An element $\varphi \in \mathrm{Mod}(\Sigma, \mathcal{P})$ has a pseudo-Anosov piece if there is a diffeomorphism f of Σ in the class φ and a strict subsurface $\Sigma_0 \subset \Sigma$ such that $f(\Sigma_0) = \Sigma_0$ and the isotopy class of $f_{|\Sigma_0}$ is pseudo-Anosov in $\mathrm{Mod}(\Sigma_0)$.

The following conjecture was stated by Andersen, Masbaum and Ueno in [AMU06].

Conjecture 5.1 ([AMU06])**.** $\varphi \in \mathrm{Mod}(\Sigma, \mathcal{P})$ has a pseudo-Anosov piece if and only if there exists $p_0(\varphi)$ such that $\rho_p(\varphi)$ has infinite order for $p \geq p_0(\varphi)$. Here ρ_p is the action on

$$\bigoplus_c V_p(\Sigma, \mathcal{P}, c)$$

This conjecture is often referred to as the AMU conjecture. We note that one implication in this conjecture is easy, as if φ has no pseudo-Anosov piece then up to some power it is a product of Dehn twists along disjoint curves which always acts with finite order via ρ_p for all p.

Remark 5.1. Note that the AMU conjecture implies a dichotomy for orders of elements in $\rho_p(\mathrm{Mod}(\Sigma, \mathcal{P}))$. Indeed, an element $\varphi \in \mathrm{Mod}(\Sigma, \mathcal{P})$ would satisfy that $\rho_p(\varphi)$ has finite order for all p or $\rho_p(\varphi)$ has infinite order for p big enough.

The conjecture is known to be true for the sphere with four marked points (see [AMU06]) and for the torus with one market point (see [S12]). In general for genus at least two surfaces, it is very difficult to find even a single φ such that $\rho_p(\varphi)$ has infinite order for p big enough.

An interesting approach for the AMU conjecture can be the use of the Witten Reshetikhin-Turaev invariants of 3-manifolds. Given a an oriented compact 3-manifold M, $p \geq 3$ an integer and A a $2p$-th primitive of unity, Reshetikhin and Turaev defined in [RT91] an invariant $Z_p(M) \in \mathbb{C}$ using quantum groups. This invariant was later re-interpreted later by Blanchet, Habegger, Masbaum and Vogel using skein theory (see [BHMV92]). For simplicity, let us explain the relation between quantum representations and these invariants for a surface without banded point. For f a diffeomorphism of Σ, let

$$M_f = M \times [0,1]/(x,0) \sim (f(x),1)$$

be the mapping torus of f. Two isotopic diffeomorphisms of Σ will have homeomorphic mapping tori, hence we can talk about the mapping torus associated with an element $\varphi \in \mathrm{Mod}(\Sigma)$ which will denoted by M_φ. As the quantum representation ρ_p fits in a much deeper structure called TQFT, we have that

$$|\mathrm{Tr}(\rho_p(\varphi))| = |Z_p(M_\varphi)|$$

Here $\varphi \in \mathrm{Mod}(\Sigma)$ and the choice of $2p$-th primitive root of unity to define ρ_p and Z_p is the same. Notice that the automorphism $\rho_p(\varphi)$ is well defined up to the multiplication by a complex number with absolute value one, therefore the absolute value of its trace is well defined. Not let us cite the famous volume conjecture.

Conjecture 5.2 (Chen-Yang/Kashaev)**.** Let M is compact hyperbolic manifold without boundary. For a odd integer, let $Z_p(M)$ be the Witten Reshetikhin-Turaev invariant associated to M where the $2p$-th primitive root chosen is $-e^{i\pi/p}$. Then

$$\lim_{p\to+\infty} \frac{\pi \ln |Z_p(M)|}{p} = \mathrm{Vol}(M)$$

where $\mathrm{Vol}(M)$ is the hyperbolic volume of M.

Here we say that a compact 3-manifold is hyperbolic if it admits a Riemannian metric with -1 sectional curvature. By Mostow rigidity theorem, if such a metric exists, it is unique up to deformation, hence the volume is well defined. We see that the volume conjecture predicts an exponential growth for SO(3) quantum invariants of hyperbolic 3-manifolds. To see the relation with the AMU conjecture, we need the following famous theorem by Thurston.

Theorem 5.4 (Thurston)**.** $\varphi \in \mathrm{Mod}(\Sigma)$ *is a pseudo-Anosov element if and only if* M_φ *is hyperbolic.*

Now the trick is notice that the dimension of the space $V_p(\Sigma)$ is polynomial in p. Hence if the volume conjecture holds for M_φ, then for appropriate choice of roots of unity :

$$\frac{|\mathrm{Tr}(\rho_p(\varphi))|}{\dim(V_p(\Sigma))} = \frac{|Z_p(M_\varphi)|}{\dim(V_p(\Sigma))} \underset{p\to+\infty}{\longrightarrow} +\infty$$

If a complex matrix has a trace whose absolute value is strictly bigger than the dimension of the space, then it has a eigenvalue whose absolute value is strictly bigger than one, hence it has infinite order. Therefore if the Witten Reshetikhin-Turaev invariant of M_φ has exponential growth (with the appropriate choice of roots of unity), then $\rho_p(\varphi)$ has infinite order for p big enough.

This strategy was used by Detcherry and Kalfagianni in [DK19] to build infinitely many non-conjugate pseudo-Anosov mapping classes satisfying the AMU conjecture.

5.5. AMU conjecture for surface groups

Let $p \geq 3$ an integer and Σ be a surface without boundary with genus at least two. Suppose that Σ is equipped with a single banded point x (for simplicity). Recall that using the Birman exact sequence, we can consider the representation

$$\rho_p : \pi_1(\Sigma) \to \mathrm{PGL}(V_p(\Sigma, x))$$

here $V_p(\Sigma, x)$ is a short hand for $\bigoplus_c V_p(\Sigma, \mathcal{P}, c)$. A theorem by Kra (see [K81]) tells us how to read the Nielsen-Thurston classification in the point pushing subgroup of the mapping class group.

Theorem 5.5. *An element* $\gamma \in \pi_1(\Sigma)$ *is not the power of a simple loop if and only if its image in* $\mathrm{Mod}(\Sigma, x)$ *by the Birman exact sequence has a pseudo-Anosov piece.*

From this one can immediately deduce the following consequence of the AMU conjecture :

Conjecture 5.3 (AMU conjecture for surface groups)**.** $\gamma \in \pi_1(\Sigma)$ is not the power of a simple loop if and only if $\rho_p(\gamma)$ has infinite order for p big enough.

This conjecture says that the quantum representations would somehow detect non-simple loops in the fundamental group of a surface. In [MS21] an algorithmic method was found to check if a given loop $\gamma \in \pi_1(\Sigma)$ satisfies or not the AMU conjecture. More precisely, let U be the set of elements in $\mathbb{Z}[X^{\pm 1}]$ of the form $\pm X^m$ for $m \in \mathbb{Z}$. We defined for a given loop

$\gamma \in \pi_1(\Sigma)$, a Laurent polynomial $P_\gamma \in \mathbb{Z}[X^{\pm 1}]$ well defined up to the multiplication by an element in U. The main result concerning the AMU conjecture was the following :

Theorem 5.6 (Marché-S.)**.** *For $\gamma \in \pi_1(\Sigma)$ if $P_\gamma \notin U$ then $\rho_p(\gamma)$ has infinite order for p big enough.*

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