Surgery equivalence relations for 3-manifolds

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Abstract

By classical results of Rochlin, Thom, Wallace and Lickorish, it is well-known that any two 3-manifolds (with diffeomorphic boundaries) are related one to the other by surgery operations. Yet, by restricting the type of the surgeries, one can define several families of non-trivial equivalence relations on the sets of (diffeomorphism classes of) 3-manifolds. In this expository paper, which is based on lectures given at the school “Winter Braids XI” (Dijon, December 2021), we explain how certain filtrations of mapping class groups of surfaces enter into the definitions and the mutual comparison of these surgery equivalence relations. We also survey the ways in which concrete invariants of 3-manifolds (such as finite-type invariants) can be used to characterize such relations.

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Introduction

It is a classical result of Rochlin and Thom, dating back to the early 50’s, that any closed oriented 3-manifold $M$ is the boundary of a compact oriented 4-manifold $W$. By elementary differential topology arguments (considering a handle decomposition of $W$), it follows that $M$ is obtained from the 3-sphere $S^3$ by finitely many knot surgeries. Here a “knot surgery” in a
3-manifold $V$ merely consists in removing a regular neighborhood $N(K)$ of a knot $K$ in $V$ and gluing it back while exchanging the meridian with a parallel curve of $K$ on $\partial N(K)$.

Here is another (equivalent) way of viewing any closed oriented 3-manifold $M$ as the result of “modifying” $S^3$ in some way. Consider a Heegaard splitting of $M$, i.e. the decomposition of $M = H \cup H'$ into two handlebodies $H, H'$ of the same genus $g$ such that $H \cap H' = \partial H = \partial H'$: the existence of such a decomposition arises again from elementary differential topology (considering, this time, a handle decomposition of $M$ itself). Since there also exists a Heegaard splitting of $S^3$ of genus $g$, and since any two handlebodies of genus $g$ are diffeomorphic, one can find a compact oriented surface $T$ in $S^3$ and a self-diffeomorphism $t$ of $T$ such that $M$ is obtained from the 3-sphere $S^3$ by cutting along $T$ and gluing back with $t$. We call this operation a twist along $T$ by $t$.

Since “knot surgeries” and “twists” (as defined above) are thus too general to define interesting relations between 3-manifolds, it is natural to impose some conditions on these operations. For instance, if one desires a twist to preserve the homology type of 3-manifolds, we should require the gluing diffeomorphism to act trivially in homology; similarly, one can ensure that a knot surgery preserves the homology type by requiring the knot to be null-homologous and by choosing the parallel in a convenient way. Stronger conditions on knot surgeries or twists can guarantee preservation of stricter features of the 3-manifolds: for instance, their “nilpotent homotopy types”, or, their invariance under certain families of topological invariants. It turns out that, in the past 40 years, several families of highly non-trivial equivalence relations have been defined for 3-manifolds by restricting the type of the “knot surgeries” or “twists.”

In this expository paper, we aim at surveying the study of such surgery equivalence relations which, for some of them, have been introduced several times in the literature with different descriptions. More specifically, via the above notion of “twists”, we shall review how certain filtrations of mapping class groups of surfaces enter into the definitions and the mutual comparison of these equivalence relations. Furthermore, we will survey the ways in which concrete invariants of 3-manifolds (such as finite-type invariants) can be used to characterize such relations.

This expository paper is based on lectures given at the school “Winter Braids XI”, which was held at the IMB (Dijon) in December 2021. So, in §1, we start with preliminary contents for readers who might not be so familiar with certain constructions of differential topology (e.g. handle decompositions) or basic results of low-dimensional topology (including the generation of the mapping class groups in relation with the above-mentioned theorem of Rochlin [90] and Thom [98]). Next, in §2, we review the definitions of three families of surgery equivalence relations: the $k$-equivalence relations defined by Cochran, Gerges & Orr [11], the $Y_k$-equivalence relations defined under different names by Goussarov [27] and Habiro [30], and the $J_k$-equivalence relations which arise naturally from the study of the latter. It follows from their definitions that all these relations are “hierarchized” as follows:

\[
\begin{align*}
Y_1\text{-eq.} & \iff Y_2\text{-eq.} \iff Y_3\text{-eq.} \iff \cdots \iff Y_k\text{-eq.} \iff Y_{k+1}\text{-eq.} \iff \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
J_1\text{-eq.} & \iff J_2\text{-eq.} \iff J_3\text{-eq.} \iff \cdots \iff J_k\text{-eq.} \iff J_{k+1}\text{-eq.} \iff \cdots \\
\downarrow & \downarrow \\
2\text{-eq.} & \iff 3\text{-eq.} \iff \cdots \iff k\text{-eq.} \iff (k+1)\text{-eq.} \iff \cdots
\end{align*}
\]

For instance, $Y_1$-equivalence (resp. 2-equivalence) is generated by the twists (resp. the knot surgeries) of the above-mentioned kinds that preserve the homology type of 3-manifolds. We give particular emphasis on the $Y_k$-equivalence relations: indeed, their definition and their study are closely tied to those of the lower central series of the subgroup of the mapping class group acting trivially in homology, namely the Torelli group of a surface. The main advantage of the $Y_k$-equivalence, with respect to the $J_k$-equivalence and the $k$-equivalence, consists in the existence of a kind of “surgery calculus” — known as clasper calculus — which is very efficient to describe the associated quotient sets of 3-manifolds.
The final section, §3, is devoted to the problem of characterizing all these equivalence relations. We start by reviewing a result of Matveev [68] which classifies $Y_1$-equivalence for closed 3-manifolds, and we extract from the literature several results for the characterization of the other equivalence relations in low degree $k$. We also consider the problem of characterizing them in arbitrary degree $k$: in the case of the $Y_k$-equivalence relations, such a problem is connected to the theory of finite-type invariants which we briefly outline. In fact, the exactizing them in arbitrary degree $k$.

Our exposition will be mainly directed towards closed oriented 3-manifolds and homology cylinders over a compact oriented surface $\Sigma$. The latter constitute a particular, but very important, class of compact oriented 3-manifolds with boundary parametrized by $\partial(\Sigma \times [-1, +1])$: in fact, homology cylinders even constitute a monoid into which the Torelli group of $\Sigma$ naturally embeds via the mapping cylinder construction, and which is essentially the monoid of $\mathbb{Z}$-homology 3-spheres in the case $\Sigma := D^2$. Since the works of Goussarov [27], Habiro [30] and Garoufalidis & Levine [26], most of the study on surgery equivalence relations for 3-manifolds have been focused on monoids of homology cylinders in relation with the theory of finite-type invariants and the algebraic structure of mapping class groups.

The case of 3-manifolds with arbitrary boundary is not so much developed in the literature, although we should mention the notable exception of knots and (string-)links exteriors. In the study of knots and (string-)links, the $Y_k$-equivalence relations are replaced by the more specific “$C_k$-equivalence relations” (which can be formulated in purely knot-diagrammatic terms), and the role played by the lower central series of the Torelli group for 3-manifolds is played by the lower central series of the pure braid group (which is much better understood): then, the study in this case turns out to be rather particular, but it also shares many similarities and connections with the general case. This study started in relation with the theory of Vassiliev invariants through the works of Stanford [95] and Habiro [30], before being developed and generalized in several directions (see [69] and references therein). Yet, for a better delimitation of the problems, the present survey will not consider the specific case of knots and (string-)links.

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1. Basics about 3-manifolds and mapping class groups

We start this expository paper by reviewing basic facts and constructions for 3-manifolds and mapping class groups of surfaces.

**Conventions.** All manifolds are assumed to be smooth and, unless otherwise stated, they are connected and oriented. For any integer $n \geq 0$, $D^n \subset \mathbb{R}^n$ is the $n$-dimensional euclidean disk and $S^n := \partial D^{n+1}$ is the $n$-dimensional sphere.

1.1. Surgeries and handle decompositions

We first recall the general definitions of surgeries and handle decompositions in any dimension $m \geq 1$, before illustrating these constructions by specializing to the dimension $m = 3$.

Let $M$ be a (possibly disconnected) $m$-manifold, let $k \in \{1, 2, \ldots, m\}$ and let $i : S^{k-1} \times D^{m+1-k} \hookrightarrow \text{int}(M)$ be an embedding. The $m$-manifold $M' := (M \setminus \text{int}(i(S^{k-1} \times D^{m+1-k}))) \cup_{i'} (D^k \times S^{m-k})$ where $i' := i|_{S^{k-1} \times S^{m-k}}$. 

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is said to be obtained from $M$ by the surgery of index $k$ along $i$. Observe that, reversely, $M$ is obtained from $M'$ by a surgery of index $(m + 1 - k)$.

**Example 1.1.** In dimension $m := 3$, we get the following operations $M \to M'$:

1. **Index $k = 1$**: we consider the disjoint union $S^0 \times D^3$ of two balls in $M$ and replace it by $D^1 \times S^2$; thus the two balls are deleted and their boundaries are identified one to the other in an orientation-preserving way.

2. **Index $k = 2$**: we consider a solid torus $S^1 \times D^2$ in $M$ and replace it by another one $D^2 \times S^1$; “meridians” and “parallels” of solid tori are exchanged during this process.

3. **Index $k = 3$**: we consider a thickened sphere $S^2 \times D^1$ in $M$ and we fill each of the two spheres $S^2 \times S^0$ with a ball.

Thus, a surgery of index 1 can be of two types in dimension 3: if the two balls $S^0 \times D^3$ belong to the same connected component of $M$, then $M' \cong M \# (S^1 \times S^2)$ which can also be obtained by surgery of index 2 along a solid torus $S^1 \times D^2 \subset M$ such that $S^1 \times \{0\}$ bounds a disk; otherwise, $M'$ is obtained from $M$ by taking the connected sum of two of its connected components.

Similarly, a surgery of index 3 can be of two types: if the thickened sphere $S^2 \times D^1$ is separating, then $M$ is reversely obtained from $M'$ by taking the connected sum of two of its connected components; otherwise, we have $M \cong M' \# (S^1 \times S^2)$.

We conclude that, in dimension 3, it is enough to consider surgeries of index 2. For later use, we reformulate them in knot-theoretical terms. Let $K \subset \text{int}(M)$ be a knot; a parallel of $K$ is a simple closed curve in the boundary $\partial N(K)$ of the regular neighborhood $N(K)$ of $K$, that is isotopic to $K$ inside $N(K)$; the meridian of $K$ is the simple closed curve $\mu(K)$ in $\partial N(K)$ that bounds a disk in $N(K)$ but not in $\partial N(K)$; up to isotopy in $\partial N(K)$, the meridian is unique but there are infinitely many possibilities for a parallel. See Figure 1.1.

![Figure 1.1: A knot $K$ (black) in its regular neighborhood $N(K)$, together with the meridian (red) and a parallel (blue)](image)

We now assume that $K$ is framed in the sense that a parallel $\rho(K)$ has been specified; then the 3-manifold obtained from $M$ by surgery along $K$ is

$$M_K := (M \setminus \text{int} N(K)) \cup_{\phi} (D^2 \times S^3)$$

where $\phi : S^1 \times S^1 \to \partial N(K)$ is a diffeomorphism mapping $\{1\} \times S^1$ to $\mu(K)$ and $S^1 \times \{1\}$ to $\rho(K)$. The manifold $M_K$ is well-defined only up to orientation-preserving diffeomorphisms, and the surgery $M \to M_K$ is the same as a surgery $M \to M'$ of index 2, where the embedding $i : S^1 \times D^2 \leftarrow \text{int}(M)$ has image $N(K)$ and maps $S^1 \times \{0\}$ (resp. $S^1 \times \{1\}$) to $K$ (resp. to $\rho(K)$).

Very often, a framed knot $K$ in a 3-manifold $M$ is given by drawing on the blackboard a knot diagram, which represents the image of a generic projection of the knot on a planar surface $B \subset M$ onto which (part of) $M$ deformation retracts: we keep track of the “over/under” crossing information at each double point and the parallel of $K$ is given by lifting the curve
parallel to the projection of $K$ in $B$. This is called the "blackboard framing convention". For instance, here are three diagrams of the trivial knot $U \subset S^3$ showing three different framings:

then the resulting manifold $S^3_U$ is $S^1 \times S^2$, $S^3$ and $\mathbb{RP}^3$, respectively. (To be specific, the knots are given in $\mathbb{R}^3 \subset S^3$ and the planar surface $B$ onto which we project is an affine plane of $\mathbb{R}^3$.)

A surgery of index $k$ is only the tip of the iceberg of a higher-dimensional operation. Let $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$. A $k$-handle in dimension $n$ is a copy of $D^k \times D^{n-k}$; its boundary can be decomposed into two parts:

$$\partial(D^k \times D^{n-k}) = (S^{k-1} \times D^{n-k}) \cup (D^k \times S^{n-k-1})$$

Let $W$ be an $n$-manifold with boundary. Attaching a $k$-handle to $W$ means to specify an embedding $i : S^{k-1} \times D^{n-k} \hookrightarrow \partial W$ to construct the new $n$-manifold

$$W' = W \cup_i (D^k \times D^{n-k}).$$

Then $\partial W'$ is obtained from $\partial W$ by a surgery of index $k$.

**Remark 1.2.** Technically speaking, the new manifold $W'$ has "corners" but there exists a standard procedure to round those "corners". Alternatively, one can give a smooth model of the attachment of a $k$-handle that arises from Morse theory (see below). For instance, here are schematic images (with or without corners) of a 1-handle attached in dimension 2:

Two closed $m$-manifolds $M$ and $M'$ are cobordant if there exists a compact $(m+1)$-manifold $W$ such that $\partial W \cong (-M) \sqcup M'$. Then, $W$ is called a cobordism from $M$ to $M'$. Of course, any compact $n$-manifold $W$ with boundary can be viewed as a cobordism from $\emptyset$ to $\partial W$ and, in particular, any closed $n$-manifold $W$ can be viewed as a cobordism from $\emptyset$ to $\emptyset$.

**Definition 1.3.** The $m$-th cobordism group is the quotient set

$$\Omega_m := \frac{\{\text{closed } m\text{-manifolds}\}}{\text{cobordism}}$$

equipped with the disjoint union $\sqcup$ operation.

Thom [100] studied those abelian groups for all integers $m \geq 1$: he described them as kinds of stable homotopy groups, he showed that they constitute the coefficient modules of a generalized homology theory, he computed $\Omega_m$ up to degree $m = 7$ and gave, among other things, an explicit computation of the ring $\Omega_* \otimes \mathbb{Q}$ . . . For this pioneering work, Thom received the Fields Medal in 1958.

**Example 1.4.** As soon as one knows the classification of closed $k$-manifolds for $k \in \{0, 1, 2\}$, it is pretty clear that

$$\Omega_0 \cong \mathbb{Z}, \quad \Omega_1 \cong \Omega_2 = \{0\}.$$  

However, it is much less obvious that $\Omega_3 = \{0\}$ as well: we shall prove it in §1.3.
Let $W$ be an $n$-dimensional cobordism from $M$ to $M'$. A handle decomposition of $W$ is an increasing sequence

$$W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_n = W$$

where $W_{-1} \cong M \times [-1 - \epsilon, -1 + \epsilon]$ and $W_i$ is obtained from $W_{i-1}$ by attaching finitely many $i$-handles. Note that $-W$ is a cobordism from $M'$ to $M$ and has a dual handle decomposition, consisting of one handle of index $n-i$ for every handle of index $i$ in $W$.

**Fact.** Morse theory tells us that any cobordism $W$ has a handle decomposition. Specifically, any Morse function $f : W \to [-1 - \epsilon, n + \epsilon]$ such that

- for each $i \in \{0, 1, \ldots, n\}$, all critical points of $f$ of index $i$ are in $f^{-1}(i)$,
- $(1 - \epsilon)$ and $(n + \epsilon)$ are regular values of $f$,
- $f^{-1}([1 - \epsilon, 1 + \epsilon]) = M$ and $f^{-1}([n - \epsilon, n + \epsilon]) = M'$,

defines a handle decomposition of $W$ by setting $W_i := f^{-1}([1 - \epsilon, i + \epsilon])$. Furthermore, there is one handle of index $i$ for every critical point of $f$ of index $i$.

We recommend Milnor’s textbooks [70, 71] for an introduction to Morse theory. As a complement to this, Cerf theory can also tell us how any two handle decompositions of the same cobordism are related one to the other by some operations (namely, creation/annihilation of two handles of consecutive indices, and handle slidings). But we shall not need that in these lectures.

It follows from the above fact that, in particular, any closed $n$-manifold $W$ has a handle decomposition $W_0 \subset W_1 \subset \cdots \subset W_n = W$ where $W_0$ consists of 0-handles, $W_1$ is obtained from $W_0$ by attaching 1-handles, and so on, to finish by gluing $n$-handles to get $W$. As is easily checked, we can assume that

- $W_0$ consists of a single 0-handle $D^0 \times D^n$,
- dually, $W_n$ is obtained from $W_{n-1}$ by attaching a single $n$-handle $D^n \times D^0$.

**Example 1.5.** Let $M$ be a closed 3-manifold. According to what has been recalled above, $M$ has a handle decomposition

$$M_0 \subset M_1 \subset M_2 \subset M_3 = M$$

with a single 0-handle and a single 3-handle. Thus, there is an integer $g \geq 0$ such that $M_1$ is diffeomorphic to

$$H_g := \Sigma_g := \partial H_g$$

which is a called the standard handlebody of genus $g$, and whose boundary

$$\Sigma_g := \partial H_g$$

is the standard closed (oriented) surface of genus $g$. Dually, there is an integer $g'$ such that $M'_1 := M \setminus \text{int}(M_1)$ is diffeomorphic to $H_{g'}$. Since $M_1$ and $M'_1$ share the same boundary, we must have $\Sigma_g = \Sigma_{g'}$: hence $g = g'$. We conclude that any closed 3-manifold $M$ can be decomposed as

$$M \cong H_g \cup_f (\Sigma_g)$$

where $f : \Sigma_g \to \Sigma_g$ is an orientation-preserving diffeomorphism. Such a decomposition is called a Heegaard splitting of $M$ of genus $g$.

In the rest of these notes, we restrict our attention to 3-manifolds.
1.2. Mapping class groups of surfaces

The Heegaard splittings, which have been described in Example 1.5, reveal that all closed 3-manifolds can be efficiently presented in terms of diffeomorphisms of surfaces. The following lemma adds that, being only interested in 3-manifolds up to diffeomorphisms, we only have to consider diffeomorphisms of surfaces up to isotopy.

**Lemma 1.6.** Let \( g \in \mathbb{N} \). The (oriented) diffeomorphism type of \( M_f := H_g \cup_f (-H_g) \) only depends on the isotopy class of \( f \).

**Proof.** For any orientation-preserving diffeomorphisms \( E : H_g \to H_g \) and \( f : \Sigma_g \to \Sigma_g \), we clearly have
\[
M_f \circ E |_{\Sigma_g} \sim M_f |_{\Sigma_g} = M_{E \circ f} |_{\Sigma_g}.
\]
Assume that \( f' : \Sigma_g \to \Sigma_g \) is another orientation-preserving diffeomorphism which is isotopic to \( f \). Then \( e = f^{-1} \circ f' \) is isotopic to the identity, and we can use a collar neighborhood of \( \Sigma_g \) in \( H_g \) to construct a diffeomorphism \( E : H_g \to H_g \) such that \( E |_{\Sigma_g} = e \). We conclude that \( M_f = M_{E \circ f} \).

Thus we are led to consider the *mapping class group* of the surface \( \Sigma_g \), which is defined by
\[
M(\Sigma_g) := \{ \text{orientation-preserving diffeomorphisms } \Sigma_g \to \Sigma_g \}/\text{isotopy}.
\]

We refer to the textbooks [6, 19] for an exposition of mapping class groups. For the moment, we just need to review the simplest examples and give explicit generating systems for those groups.

**Example 1.7.** The group \( M(\Sigma_0) \) is trivial. Besides, through its action on the abelian group \( H_1(\Sigma_1; \mathbb{Z}) \cong \mathbb{Z}^2 \), the group \( M(\Sigma_1) \) is isomorphic to \( \text{SL}(2; \mathbb{Z}) \). See the above-mentioned textbooks, or [64, §2] for a direct treatment of these examples.

Let \( \alpha \) be a simple closed curve in \( \Sigma_g \). We identify a regular neighborhood \( N(\alpha) \) of \( \alpha \) with the annulus \( S^1 \times [0, 1] \), in such a way that orientations are preserved. The *Dehn twist* along \( \alpha \) is the diffeomorphism \( T_\alpha : \Sigma_g \to \Sigma_g \) defined by
\[
T_\alpha(x) = \begin{cases} x & \text{if } x \not\in N(\alpha) \\ (e^{2i\pi \theta} r, \theta) & \text{if } x = (e^{2i\pi \theta}, r) \in N(\alpha) = S^1 \times [0, 1]. \end{cases}
\]

Because of the choice of \( N(\alpha) \) and its “parametrization” by \( S^1 \times [0, 1] \), the diffeomorphism \( T_\alpha \) is only defined up to isotopy. But the isotopy class \([T_\alpha] \in M(\Sigma_g) \) only depends on the isotopy class of the curve \( \alpha \). Here is the effect of \( T_\alpha \) on a curve \( \rho \) which crosses transversely \( \alpha \) in a single point:

![Diagram](image)

**Theorem 1.8 (Dehn 1938).** In any genus \( g \geq 1 \), the group \( M(\Sigma_g) \) is generated by finitely many Dehn twists.
Dehn’s generating system [13] can be written explicitly. It consists of 2 twists in genus $g = 1$, and 5 twists in genus $g = 2$: see Figure 1.2. In genus $g > 2$, $\mathcal{M}(\Sigma_g)$ is generated by the Dehn twists along the $2g(g-1)$ curves shown in Figure 1.3: the curves $\alpha_i$ (blue, for all $i \in \{1, \ldots, g\}$), $\beta_i$ (red, for all $i \in \{1, \ldots, g\}$), $\delta_i$ (purple, for all $i \in \{1, \ldots, g\}$), $\gamma_{ij}$ (green, for any pair $\{i,j\}$ of two elements in $\{1, \ldots, 2g\}$ that are of distance at least three in the cyclic order).

Figure 1.2: Dehn’s generators in genus 1 and 2

Figure 1.3: Dehn’s generators in genus $g > 2$

In the sequel, we shall only need the following information about Dehn’s generating system of $\mathcal{M}(\Sigma_g)$:

(1.2) In genus $g > 1$, the group $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists along simple closed curves, each avoiding a sub-handlebody of genus 1 of $H_g$.

Here $\Sigma_g$ is regarded as the boundary of the standard handlebody $H_g$, and a sub-handlebody of genus $k$ of $H_g$ is the image of $H_k$ under some diffeomorphism $H_k \#_3 H_{g-k} \cong H_g$.

Remark 1.9. In the sixties, Lickorish rediscovered and simplified Dehn’s generating system of the mapping class group [57]. He proved that $\mathcal{M}(\Sigma_g)$ is actually generated by the Dehn twists along the simple closed curves

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}$$

shown below:

Afterwards, Humphries [49] showed that $2g + 1$ Dehn twists are enough to generate $\mathcal{M}(\Sigma_g)$: specifically, those are the twists along $\beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}, \alpha_1, \alpha_2$. ■
1.3. Triviality of $\Omega_3$

Let $\mathcal{V}(\emptyset)$ be the set of diffeomorphism classes of closed 3-manifolds. (Recall that, unless otherwise stated, 3-manifolds are always oriented.)

**Theorem 1.10** (Rochlin 1951, Thom 1951, Wallace 1960, Lickorish 1964). The following four statements are equivalent, and hold true:

1. we have $\Omega_3 = \{0\}$, i.e. any two $M, M' \in \mathcal{V}(\emptyset)$ are cobordant;

1'. any $M \in \mathcal{V}(\emptyset)$ is the boundary of a compact 4-manifold $W$;

2. for any $M, M' \in \mathcal{V}(\emptyset)$, there is a sequence of surgeries along framed knots $M = M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_r = M'$;

2' for any $M \in \mathcal{V}(\emptyset)$, there is a framed link $L \subset S^3$ such that $S^3_L \cong M$.

**Proof(s).** The equivalence between (1) and (1’) is clear. Assuming (1), let $M \in \mathcal{V}(\emptyset)$: there is a compact 4-manifold $W$ such that $\partial W \cong (-S^3) \cup M$; let $W := W \cup D^4$ where $D^4$ is glued along the $S^3$ boundary component of $W$; then $\partial W \cong M$. Assuming (1’), let $M, M' \in \mathcal{V}(\emptyset)$: then $(-M) \cup M' \in \mathcal{V}(\emptyset)$ and there is a compact 4-manifold $W$ with boundary $(-M) \cup M'$.

The equivalence between (2) and (2’) is also easy. Assuming (2), let $M \in \mathcal{V}(\emptyset)$; there is a sequence of surgeries along framed knots $S^3 = M_0 \hookrightarrow \cdots \hookrightarrow M_r = M$; for each $i$, we can assume that the framed knot $K_i \subset M_i$ along which we do the surgery to get $M_{i+1}$ is disjoint from the glued solid tori that correspond to the previous surgeries, hence we can view $K_i$ as a knot in the initial manifold $S^3$; then the framed link $L := K_0 \cup \cdots \cup K_{r-1}$ is such that $S^3_L \cong M$. Assuming (2’), let $M, M' \in \mathcal{V}(\emptyset)$; there is a framed link $L \subset S^3$ such that $S^3_L \cong M$; by doing the surgeries along the components of $L$ stepwisely, we obtain a first sequence of surgeries $S^3 = M_0 \hookrightarrow \cdots \hookrightarrow M_r = M$; similarly, we find a second sequence of surgeries $S^3 = M_0' \hookrightarrow \cdots \hookrightarrow M_r' = M'$; thus, by reversing the first sequence, we get a sequence of surgeries producing $M'$ from $M$.

The equivalence between (1) and (2) is a result of Wallace [106]. Indeed Wallace proved that, in any dimension $m \geq 1$, two closed $m$-manifolds $M$ and $M'$ are cobordant if and only if there is a sequence

$$M = M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_r = M'$$

where $M_i \hookrightarrow M_{i+1}$ stands for a surgery of index $k_i$ and the sequence $(k_i)_i$ is not decreasing. (In [106], surgeries are called spherical modifications.) This equivalence follows from the existence of handle decompositions for cobordisms and the relation between surgery and attachment of handles. Observing that, in dimension $m = 3$, only surgeries of index 2 do matter (see Example 1.1), Wallace assumes (1) to deduce (2’) thus answering a question of Bing [3].

Indeed, statement (1) had been proved independently by Rochlin [90] and Thom [98, 99, 100]. Actually, Thom gave three proofs of very different natures: let us expose the proof that came chronologically first and is sketched in [98]. It uses Heegaard splittings of 3-manifolds, the key idea being that the subset

$$B_g := \{[f] \in \mathcal{M}(\Sigma_g) : M_f = H_g \cup_f (-H_g) \text{ bounds a compact 4-manifold}\}$$

is a subgroup of the mapping class group, for every $g \in \mathbb{N}$:

1. $1 \in B_g$ because $M_{d|g}$ is diffeomorphic to $\#^0(S^1 \times S^2)$ which, for instance, is the boundary of $\#^0(D^2 \times S^2)$;

2. if $f \in B_g$, then $f^{-1} \in B_g$ because $M_{f^{-1}} \cong -M_f$;
if \( f, f' \in B_g \), then \( f'f \in B_g \) because, given a compact 4-manifold \( W \) bounded by \( M_f \) and a compact 4-manifold \( W' \) bounded by \( M_{f'} \), the 3-manifold \( M_{f'}f \) is the boundary of the 4-manifold obtained by gluing \( W \) and \( W' \) along the “left side” handlebody \( H_g \) of \( M_f \) and the “right side” handlebody \(-H_g\) of \( M_{f'} \).

Since any 3-manifold has a Heegaard splitting, the triviality of \( \Omega_3 \) will follow from the fact that, for any \( g \geq 1 \), we have \( B_g = \mathcal{M}(\Sigma g) \) or, equivalently, that each of Dehn’s generators of \( \mathcal{M}(\Sigma g) \) belongs to \( B_g \). This is proved by induction on \( g \). In genus \( g = 1 \), there are two generators \( \tau \) (see Figure 1.2): the corresponding 3-manifold \( M_\tau \) is either \( S^3 = \partial B^4 \) or \( S^1 \times S^2 = \partial (D^2 \times S^2) \); hence \( B_1 = \mathcal{M}(\Sigma 1) \). Assume that \( B_{g-1} = \mathcal{M}(\Sigma g-1) \). According to (1.2), each Dehn generator of \( \mathcal{M}(\Sigma g) \) is a Dehn twist \( \tau \) along a simple closed curve avoiding a sub-handlebody of genus 1 of \( H_g \); therefore \( M_\tau \) is diffeomorphic to \( (S^1 \times S^2) \# M_h \) for some \( h \in \mathcal{M}(\Sigma g-1) \); hence \( M_\tau \) is related to \( M_h \), by a surgery of index \( 1 \), so that \( M_\tau \) and \( M_h \) are cobordant; by the induction hypothesis, \( M_h \) bounds, and so does \( M_\tau \); hence \( \tau \in B_g \).

Being not aware of Dehn’s work [13], Lickorish re-proves in [57] that \( \mathcal{M}(\Sigma g) \) is generated by finitely many Dehn twists (see Remark 1.9), and he shows statement (2) in a direct way.

The key idea in his argument is the following:

**Lickorish’s trick.** Let \( U \) and \( V \) be compact 3-manifolds whose boundaries are identified. Let \( \gamma \subset \partial V \) be a simple closed curve, and let \( K \subset \text{int}(V) \) be the knot obtained by slightly “pushing” \( \gamma \). Then we have

\[
U \cup_{\tau} (-V) \cong U \cup_{\text{id}} (-V_K)
\]

where \( \tau := T_\gamma \) is the Dehn twist along \( \gamma \), and \( V_K \) is obtained from \( V \) by surgery along \( K \) framed with the parallel differing from \( \gamma \) by a meridian of \( K \).

This trick is easily verified using the definitions of a surgery and a Dehn twist. Let \( g \in \mathbb{N} \) and \( f \in \mathcal{M}(\Sigma g) \). Decomposing \( f \) as a product of Dehn twists (or their inverses), Lickorish’s trick implies that \( M_f = H_g \cup_f (-H_g) \) can be transformed into \( M_{\text{id}} = \#^g(S^1 \times S^2) \) by finitely many surgeries along framed knots. The same is true about \( S^3 \), since we have \( S^3 = M_i \) for some \( i \in \mathcal{M}(\Sigma g) \) and whatever \( g \) is. Hence, \( M_f \) can be transformed into \( S^3 \) by finitely many surgeries.

**Remark 1.11.** Rourke gave in [91] yet another proof of statement (2) of Theorem 1.10, which is also based on the presentations of 3-manifolds by their Heegaard splittings. But, in contrast with Thom’s and Lickorish’s arguments, his proof does not need knowledge about the generation of the mapping class group. It is both tricky and elementary.

We can be more general and consider 3-manifolds with boundary. Let \( R \) be a closed surface, which may be disconnected. A compact 3-manifold \( M \) has boundary parametrized by \( R \), if \( M \) comes with a map \( m : R \rightarrow M \) which is an orientation-preserving diffeomorphism onto \( \partial M \).

Our convention will always be to denote the boundary parametrization with the lower-case letter.

Two manifolds with parametrized boundary \( M \) and \( M' \) are considered **diffeomorphic** if there is an orientation-preserving diffeomorphism \( f : M \rightarrow M' \) such that \( f \circ m = m' \). We denote by \( \mathcal{V}(R) \) the set of diffeomorphism classes of compact 3-manifolds with boundary parametrized by the surface \( R \).

**Corollary 1.12.** For any \( M, M' \in \mathcal{V}(R) \), there is a sequence of surgeries along framed knots \( M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_r = M' \).

**Proof.** Denote by \( (R_i)_i \) the family of connected components of \( R \) and, for each \( i \), fix an identification of \( R_i \) with the standard surface \( \Sigma g_i \) where \( g_i \) is the genus of \( R_i \). Fix in \( S^3 \) a copy \( H \) of the disjoint union \(-\sqcup_i H_{g_i}\) of standard handlebodies. Then \( S^3 \setminus \text{int}(H) \) with the obvious boundary parametrization defines a “preferred” element of \( \mathcal{V}(R) \).
We shall prove that $M$ can be transformed into $S^3 \setminus \text{int}(H)$ by surgery along a framed link $L$. To this purpose, we consider the closed 3-manifold

$$\overline{M} := M \cup_m (-\bigcup_i H_{g_i}).$$

By Theorem 1.10, there is a framed link $L \subset \overline{M}$ and an orientation-preserving diffeomorphism $\phi : \overline{M} \to S^3$; furthermore, we can assume that $L$ is contained in $M$ after an isotopy. The image $H' \subset S^3$ of $\bigcup_i H_{g_i} \subset \overline{M}$ by $\phi$ is a disjoint union of handlebodies. Of course, we have an a priori $H \neq H'$. Then, we think of $H$ and $H'$ as regular neighborhoods in $S^3$ of some knotted framed graphs $G$ and $G'$, respectively, of the same topological type. After finitely many “crossing changes” and “framing changes”, $G'$ can be transformed to $G$ since they have the same topological type. Each of these “crossing changes” and “framing changes” can be realized by surgery along a framed trivial knot and, after an isotopy, we can assume that each such knot does not meet the part of $S^3 = \phi(\overline{M})$ where the surgery along $L$ took place. Therefore, after addition of some components to the framed link $L$, we can assume that $H = H'$ as subsets of $S^3$. Hence $\phi$ restricts to an orientation-preserving diffeomorphism $M_L \to S^3 \setminus \text{int}(H)$. This diffeomorphism may not be compatible with the boundary parametrizations of $M$ and $S^3 \setminus \text{int}(H)$. However, since $\mathcal{T}(R_i)$ is generated by Dehn twists and since every Dehn twist can be realized by a surgery along a knot (using Lickorish’s trick), we can assume this compatibility at the price of adding to $L$ yet other components. We conclude that $M_L$ and $S^3 \setminus \text{int}(H)$ represent the same class in $\mathcal{I}(R)$.

\[ \square \]

2. Surgery equivalence relations: definitions and first properties

We have seen in §1 that the surgery operations arising directly from differential topology are too general in dimension three: any two compact 3-manifolds (with the same parametrized boundary, if any) can be related one to the other by such operations. Thus, to relate 3-manifolds in an interesting way, we need to consider more restrictive modifications and one reasonable restriction is to require that they preserve the homology type of 3-manifolds. So, we are led to consider the subgroup of the mapping class group that acts trivially in homology.

2.1. Torelli groups of surfaces

Let $S$ be a compact surface with, at most, one boundary component. As a generalization of (1.1), the mapping class group of $S$ is defined by

$$\mathcal{M}(S) = \begin{cases} \{ \text{orientation-preserving diffeomorphisms } S \to S \} & \text{if } \partial S = \emptyset, \\ \{ \text{diffeomorphisms } S \to S \text{ that are the id on } \partial S \} & \text{if } \partial S \neq \emptyset. \end{cases}$$

**Definition 2.1.** The Torelli group of $S$ is the subgroup $\mathcal{I}(S)$ of $\mathcal{M}(S)$ that acts trivially on $H := H_1(S; \mathbb{Z})$.

The study of the Torelli group, from algebraic and topological viewpoints, was initiated by Birman in her early works, in particular [4, 5]. Then it was developed considerably by Johnson in the eighties: see his survey [41]. Here we shall simply review a generating system of $\mathcal{I}(S)$.

**Remark 2.2.** According to Example 1.7, the Torelli group is not interesting in genus 0 and 1: hence we shall assume that the genus of $S$ is at least 2.

First of all, let us determine the action of a Dehn twist in homology. For this, we need the (homology) intersection form of the surface $S$

$$\omega : H_1(S; \mathbb{Z}) \times H_1(S; \mathbb{Z}) \to \mathbb{Z}$$

I-11
which is defined as follows: if $a = [\alpha] \in H_1(S; \mathbb{Z})$ and $b = [\beta] \in H_1(S; \mathbb{Z})$ are represented by smooth oriented closed curves $\alpha$ and $\beta$, in transverse position, then

$$\omega([\alpha], [\beta]) := \sum_{x \in \alpha \cap \beta} \begin{cases} +1, & \text{if } (\bar{\alpha}_x, \bar{\beta}_x) \text{ is direct} \\ -1, & \text{otherwise} \end{cases}. $$

Note that the pairing $\omega$ is bilinear, skew-symmetric and non-singular: thus, $\omega$ is a symplectic form on $H = H_1(S; \mathbb{Z})$.

**Lemma 2.3.** Let $\alpha \subset S$ be a simple closed curve. The action of the Dehn twist $T_\alpha$ in homology is given by the following formula:

$$\forall x \in H, \quad (T_\alpha)_*(x) = x + \omega([\alpha], x) \cdot [\alpha]. \quad (2.1)$$

In other words, $(T_\alpha)_*$ is the transvection defined by the vector $[\alpha]$ and the linear form $\omega([\alpha], -)$. Formula (2.1) is easily deduced from the definition of a Dehn twist. Here are two immediate consequences of the transvection formula (2.1):

(i) for a simple closed curve $\alpha \subset S$, we have $T_\alpha \in I(S)$ if and only if we have $[\alpha] = 0 \in H$ (i.e. $\alpha$ is separating in $S$);

(ii) for any simple closed curves $\alpha, \beta$ in $S$ such that $\alpha \cap \beta = \emptyset$ and $[\alpha] = [\beta] \in H$ (i.e. $\alpha$ and $\beta$ cobound a subsurface of $S$) we have $T_\alpha^{-1}T_\beta \in I(S)$.

Following Johnson, we call an element $T_\alpha$ of type (i) a BSCC map (for “Bounding Simple Closed Curve”), and its genus is the genus of the subsurface of $S$ bounded by $\alpha$. (If $\partial S = \emptyset$, then there are two such subsurfaces and we take the minimal genus of those two.). Besides, we call an element $T_\alpha^{-1}T_\beta$ of type (ii) a BP map (for “Bounding Pair”), and its genus is the genus of the subsurface of $S$ with boundary $\alpha \cup \beta$. (If $\partial S = \emptyset$ and $[\alpha] \neq 0$, then there are two such subsurfaces and we take the minimal genus of those two.).

The following is a combination of several works, namely [4, 88, 38].

**Theorem 2.4** (Birman 1971, Powell 78, Johnson 1979). The Torelli group $I(S)$ has the following generating sets, whose nature depends on the genus $g$ and the number $n$ of boundary component of $S$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$g = 2$</th>
<th>$g \geq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>BSCC maps of genus 1</td>
<td>BP maps of genus 1</td>
</tr>
<tr>
<td>1</td>
<td>BSCC maps of genus 1 &amp; BP maps of genus 1</td>
<td>BP maps of genus 1</td>
</tr>
</tbody>
</table>

One of the major accomplishments from Johnson’s works in the 80’s is the fact that the group $I(S)$ is finitely generated in genus at least 3 [40], but we will not need this fact in these lectures. Note that $I(S)$ is not finitely generated in genus 2 [58].

### 2.2. Torelli twists in 3-manifolds

We fix a closed surface $R$, which may be disconnected.

**Definition 2.5.** Let $M \in \mathcal{V}(R)$, let $S \subset \text{int}(M)$ be a compact surface with one boundary component and let $s \in I(S)$. The 3-manifold obtained from $M$ by a Torelli twist along $S$ with $s$ is

$$M_s := (M \setminus \text{int}(N(S))) \cup_{\bar{s}} N(S). \quad (2.2)$$

where $N(S)$ is a regular neighborhood of $S$ in $M$ identified to $S \times [-1, 1]$, and $\bar{s}$ is the self-diffeomorphism of $\bar{s}(S \times [-1, 1])$ given by $s$ on $S \times \{1\}$ and the identity elsewhere. With the obvious boundary parametrization $m_s : R \rightarrow M_s$ induced by $m$, we get $M_s \in \mathcal{V}(R)$. ■
Equivalently, $M_s$ is obtained by cutting open $M$ along $S$ and gluing back with $s$:

![Diagram](image)

**Definition 2.6.** Let $M, M' \in \mathcal{V}(R)$. We say that $M$ and $M'$ are Torelli–equivalent if there is a compact surface $S \subset \text{int}(M)$ and an $s \in \mathcal{I}(S)$ such that $M_s \cong M'$. 

**Lemma 2.7.** The Torelli-equivalence is a non-trivial equivalence relation on $\mathcal{V}(R)$.

**Proof.** The Torelli-equivalence is clearly reflexive and symmetric as a relation in $\mathcal{V}(R)$. We verify the transitivity by considering a first Torelli twist $M \rightarrow M_s = M'$ along $S \subset M$ and a second one $M' \rightarrow M'_s$ along $S' \subset M'$. Since $S'$ deformation retracts onto a 1-dimensional subcomplex and since the part $N(S) \subset M_s$ of the decomposition (2.2) is a handlebody which also retracts to a 1-dimensional subcomplex, we can isotope $S'$ in $M'$ so that it lies in the part $M \setminus \text{int}(S) \subset M_s$ of the decomposition (2.2). Hence we can view $S'$ as a subsurface of $M$, disjoint from $S$. We attach to $S \cup S'$ a 1-handle, inside $M$, to get a larger subsurface $T := S \#_3 S'$ of $M$. We have $t := s \#_3 s' \in \mathcal{I}(T)$ and $M'' \cong M_t$. Hence $M''$ is Torelli-equivalent to $M$.

To prove that the Torelli-equivalence is a non-trivial relation, we observe that a Torelli twist $M \rightarrow M_s$ induces a unique isomorphism in homology such that the following diagram is commutative:

\[
\begin{array}{ccccc}
H_1(M; \mathbb{Z}) & \xrightarrow{\phi_s} & H_1(M_s; \mathbb{Z}) & \xrightarrow{\text{incl}_*} & H_1(M \setminus \text{int}(S); \mathbb{Z}) \\
\text{incl}_* & & \text{incl}_* & & \\
& & & & \\
\end{array}
\]

(The unicity follows from the surjectivity of the homomorphism $\text{incl}_*$ induced by the inclusion $M \setminus \text{int}(S) \hookrightarrow M$, and the existence is justified using the Mayer-Vietoris theorem.) Hence two manifolds in $\mathcal{V}(R)$ with different homotopy types cannot be Torelli-equivalent. 

We now give another description of the Torelli-equivalence. Let $M \in \mathcal{V}(R)$. A **Y-graph** in $M$ is a surface $G \subset \text{int}(M)$ consisting of one “node”, three “edges” and three “leaves” as shown on the left side of Figure 2.1. The regular neighborhood of $G$ is a handlebody of genus 3, inside which $G$ can be replaced by the 6-component framed link shown on the right side of Figure 2.1 (using the blackboard framing convention): to get this link, the node of $G$ is replaced by one copy of the borromean rings, and each leaf of $G$ becomes a knot “clasping” one of those three rings. We define $M_G$ to be the 3-manifold obtained from $M$ by surgery along this framed link, and we call the move

\[M \rightarrow M_G\]

a **Y-surgery**. This operation is equivalent to the “borromean surgery” move that Matveev considered in [68]. Under this form, this operation was introduced by Goussarov [27] and Habiro [30] as part of a much larger package which is now known as “clasper calculus”: see §2.5 below.

**Proposition 2.8.** Two manifolds $M, M' \in \mathcal{V}(R)$ are Torelli–equivalent if, and only if, there is a sequence of Y-surgeries $M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_r = M'$.

**Sketch of the proof.** In the definition of a Torelli twist $M \rightarrow M_s$ along $S \subset M$, we can assume that the genus of $S$ is arbitrary high: indeed, we can always take the boundary-connected sum of $S$ with another subsurface $U$ of $M$ (with $\partial U \cong S^1$) and extend $s$ by the identity to a diffeomorphism of $S \#_3 U$. Besides, we know from Theorem 2.4 that $\mathcal{I}(S)$ is generated by BP
Remark 2.9. Lemma 5.1] or [63, Fig. 6.2].

Then the rest of the argument consists in showing that surgery along the parallel given by the curve \( h \) of genus \( \alpha \) given by the two curves \( \alpha \cup \beta \subset S \) that define the BP map \( s \), with the appropriate framings. Then the rest of the argument consists in showing that surgery along \( \alpha \cup \beta \) is equivalent to a Torelli twist maps of genus \( g \) if the genus of \( S \) is at least 3. Hence it is enough to show that a \( Y \)-surgery is equivalent to a Torelli twist \( M \to M_v \) defined by a BP map \( s \) of genus 1.

Using Lickorish’s trick, we see that \( M_v \cong M_{AUB} \) where \( A \cup B \) is the 2-component link in \( M \) given by the two curves \( \alpha \cup \beta \subset S \) that define the BP map \( s \), with the appropriate framings. Then the rest of the argument consists in showing that surgery along \( A \cup B \) is equivalent to the surgery along a 6-component framed link defining a \( Y \)-surgery: this is explained in [22, Lemma 5.1] or [63, Fig. 6.2].

\[ \square \]

Remark 2.9. A blink of genus \( h \) in a compact 3-manifold \( M \) is a compact surface \( B \subset \text{int}(M) \) of genus \( h \) with two boundary components \( \partial B = B^+ \cup \{-B^-\} \): the knot \( B^\pm \) is framed with the parallel given by the curve \( \partial \text{int}(B^\pm) \cap B \) and corrected by the meridian \( \pm \mu(B^\pm) \). Surgeries along blinks have been considered in [35, 68] and [23], where the term “blink” was coined.

As in the proof of Proposition 2.8, we deduce from Lickorish’s trick that surgery along a blink is equivalent to a Torelli twist with a BP map of the same genus. Thus two manifolds \( M, M' \in \mathcal{V}(R) \) are Torelli–equivalent if, and only if, one can find a disjoint union \( B = \bigsqcup B_i \) of blinks in \( M \) such that \( M_B \cong M' \).

Finally, we give another description of the Torelli–equivalence in terms of Heegaard splittings. However, we only formulate this description for the two instances of a surface \( R \) that we shall consider later:

(i) \( R = \emptyset \): then \( \mathcal{V}(R) \) consists of closed 3-manifolds;

(ii) \( R = \partial(S \times [-1, 1]) \) where \( S \) is a compact surface with \( \partial S \cong S^1 \): then \( \mathcal{V}(R) \) consists of cobordisms (with “vertical” boundary) from \( S \) to \( S \).

The notion of “Heegaard splitting” in the case (i) has been seen in Example 1.5, and it can be reformulated as follows. A \( \text{Heegaard splitting} \) of genus \( g \) of a closed 3-manifold \( M \) is a decomposition \( M = M_- \cup M_+ \) where \( M_-, M_+ \) are two copies of the handlebody \( H_g \) in \( M \) such that \( M_- \cap M_+ = \partial M \) (which is called the Heegaard surface).

Likely, the notion of “Heegaard splitting” in the case (ii) is defined as follows. Let \( M \) be a cobordism from \( S \) to \( S \). We set \( \partial_M := m(S \times \{ \pm 1 \}) \), and we denote a collar neighborhood of \( \partial_- M \) (resp. \( \partial_+ M \)) simply by \( \partial_- M \times [-1, 0] \) (resp. \( \partial_+ M \times [0, 1] \)). A \( \text{Heegaard splitting} \) of \( M \) of genus \( g \) is a decomposition \( M = M_- \cup M_+ \) where \( M_- \) is obtained from \( \partial_- M \times [-1, 0] \) by adding \( g \) 1-handles along \( \partial_+ M \times \{ 0 \} \), \( M_+ \) is obtained from \( \partial_+ M \times [0, 1] \) by adding \( g \) 1-handles along \( \partial_- M \times \{ 0 \} \), and we have \( M_- \cap M_+ = \partial M_- \cap \partial M_+ \) (which is called the Heegaard surface).

The existence of Heegaard splittings in this situation (cobordisms with “vertical” boundary) is again an application of Morse theory.

Proposition 2.10. Assume that \( R \) is of one of the above types (i) and (ii). Two manifolds \( M, M' \in \mathcal{V}(R) \) are Torelli–equivalent if, and only if, there is a Heegaard splitting \( M = M_- \cup M_+ \) with Heegaard surface \( S \) and an \( s \in \mathcal{I}(S) \) such that \( M' \cong M_- \cup s M_+ \).

Proof. We only prove the proposition in the case (i), the case (ii) being similar and a little bit more technical (see [67, Lemma 2.1] for instance). It is enough to show that, given a...
closed 3-manifold \(M\) and a surface \(E \subset M\) with one boundary component, we can always find a Heegaard splitting \(\tilde{M} = M_{\sim} \cup M_{+}\) whose Heegaard surface contains a subsurface that is isotopic to \(E\) in \(M\).

Let \(N(E)\) be a regular neighborhood of \(E\) in \(M\) and set \(\tilde{M} := M \setminus \text{int}(N(E))\). Viewing \(\tilde{M}\) as a cobordism from \(\emptyset\) to \(\partial N(E)\), we can find a handle decomposition

\[
\tilde{M}_0 \subset \tilde{M}_1 \subset \tilde{M}_2 = \tilde{M}
\]

where \(\tilde{M}_0\) consists of a single 0-handle, \(\tilde{M}_1\) is obtained from \(\tilde{M}_0\) by attaching 1-handles and \(\tilde{M}_2\) is obtained from \(\tilde{M}_1\) by attaching 2-handles. The latter can be viewed, dually, as 1-handles attached to \(N(E)\) inside \(M\). Hence there is a Heegaard splitting \(M = M_{\sim} \cup M_{+}\) where

\[
M_{\sim} := \tilde{M}_1 \quad \text{and} \quad M_{+} := (\tilde{M}_2 \setminus \text{int}(\tilde{M}_1)) \cup N(E).
\]

Observe that \(E\) can be isotoped in \(N(E)\) onto \(\partial N(E)\); furthermore, since \(E\) deformation retracts onto a 1-dimensional subcomplex, we can next isotope it in \(\partial N(E)\) to make it disjoint from the attaching locus of the 1-handles attached to \(N(E)\). Thus we have isotoped \(E\) to a subsurface of the Heegaard surface. \(\square\)

### 2.3. Filtrations on the Torelli groups

We will define surgery equivalence relations for 3-manifolds which are much stronger than the Torelli-equivalence and arise from certain filtrations of the Torelli group.

To define these filtrations, we first recall that the lower central series of a group \(G\) is the decreasing sequence of subgroups

\[
G = \Gamma_1 G \supset \Gamma_2 G \supset \Gamma_3 G \supset \cdots
\]

that are defined inductively by \(\Gamma_{i+1} G := [\Gamma_i G, G]\) for all \(i \geq 1\). Let \(S\) be a compact surface with one boundary component, and fix a base-point \(\ast \in \partial S\). The canonical action of \(I(S)\) on the fundamental group \(\pi := \pi_1(S, \ast)\) induces, for every integer \(k \geq 1\), a group homomorphism

\[
\rho_k : I(S) \longrightarrow \text{Aut}(\pi/\Gamma_{k+1} \pi)
\]

since \(\Gamma_{k+1} \pi\) is a characteristic subgroup of \(\pi\). Defining \(J_k I(S) := \ker(\rho_k)\) for every \(k \geq 1\), we get a filtration of the Torelli group

\[
I(S) = J_1 I(S) \supset J_2 I(S) \supset J_3 I(S) \supset \cdots
\]

which is nowadays refered to as the Johnson filtration of \(I(S)\). The study of the Johnson filtration on its whole started in Morita’s seminal work [75], and it is still an active field of research. (See [92] for a survey.)

**Example 2.11.** Johnson made a deep study of the second term of the filtration

\[
\mathcal{K}(S) := J_2 I(S)
\]

in [42, 43], so much that this group is called the Johnson subgroup (or the Johnson kernel). In particular, Johnson proved that \(\mathcal{K}(S)\) is generated by BSCC maps. \(\blacksquare\)

One of the main reasons to be interested in this filtration is that it has a trivial intersection

\[
\bigcap_{k \geq 1} J_k I(S) = \{1\}
\]

as can be easily checked from the following two classical facts:

(i) (Baer 1928) the canonical action of \(I(S)\) on \(\pi\) is faithful [2];

(ii) (Magnus 1937) the lower central series of \(\pi\) has a trivial intersection, because \(\pi\) is free [59].
Thus, one of the main objectives of the study of the Johnson filtration is to fully understand its associated graded, namely

$$\text{Gr}^l \mathcal{I}(S) = \bigoplus_{k \geq 1} J_{k+l} \mathcal{I}(S).$$

Another interesting feature of the Johnson filtration is that it is strongly central in the sense that

$$\forall k, l \in \mathbb{N}^*, \quad [J_k \mathcal{I}(S), J_l \mathcal{I}(S)] \subset J_{k+l} \mathcal{I}(S)$$

(see [75, Prop. 4.1]). Consequently, the commutator operation in the group $\mathcal{I}(S)$ induces a Lie ring structure on $\text{Gr}^l \mathcal{I}(S)$, which opens the door to Lie-theoretical methods in the study of $\mathcal{I}(S)$. (Again, see [92] for a survey.)

The Johnson filtration has also been much studied in relation with the lower central series $\mathcal{I}(S) = \Gamma_3 \mathcal{I}(S) \supset \Gamma_2 \mathcal{I}(S) \supset \Gamma_1 \mathcal{I}(S) \supset \cdots$ of the Torelli group. Indeed, (2.6) implies that the latter is contained in the former:

$$\forall k \in \mathbb{N}^*, \quad \Gamma_k \mathcal{I}(S) \subset J_k \mathcal{I}(S).$$

The associated graded of the lower central series of the Torelli group

$$(2.8) \quad \text{Gr}^l \mathcal{I}(S) = \bigoplus_{k \geq 1} \Gamma_{k+l} \mathcal{I}(S)$$

has been determined with rational coefficients by Hain [34], as part of the stronger result of identifying the Malcev Lie algebra of $\mathcal{I}(S)$. For a comparison between $\text{Gr}^l \mathcal{I}(S) \otimes \mathbb{Q}$ and $\text{Gr}^l \mathcal{I}(S) \otimes \mathbb{Q}$ in low degrees, see [76, 77].

Remark 2.12. Hain also obtained in [34] that the inclusion reciprocal to (2.7) is not true: specifically, there is no $d \in \mathbb{N}^*$ such that $J_d \mathcal{I}(S) \subset \Gamma_3 \mathcal{I}(S)$.

The above paragraphs only give a brief and limited overview of what is known about the Johnson filtration and the lower central series of the Torelli group. We conclude this subsection with an informal “comparison table” between those two filtrations:

<table>
<thead>
<tr>
<th>trivial intersection?</th>
<th>lower central series $(\Gamma_k \mathcal{I}(S))_k$</th>
<th>Johnson filtration $(J_k \mathcal{I}(S))_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>testing elements?</td>
<td>given $h \in \mathcal{I}(S)$ and $k \in \mathbb{N}^*$, it is hard to decide whether $h \in \Gamma_k \mathcal{I}(S)$ unless $k$ is small (say $k \leq 3$)</td>
<td>given $h \in \mathcal{I}(S)$ and $k \in \mathbb{N}^*$, it is easy to decide whether $h \in J_k \mathcal{I}(S)$ using “Johnson homomorph.”</td>
</tr>
<tr>
<td>explicit generators?</td>
<td>it is easy to deduce an explicit generating syst. in any degree $k$ from a generating syst. of $\mathcal{I}(S)$</td>
<td>it seems difficult to construct an explicit generating syst. in a given degree $k$</td>
</tr>
<tr>
<td>finitely generated?</td>
<td>yes, in any degree $k$: if $g \geq 3$ for $k = 1$ [40] if $g \geq 4$ for $k = 2$ [16, 10] if $g \geq 2k - 1$ for $k \geq 3$ [10]</td>
<td>yes, in any degree $k$: if $g \geq 3$ for $k = 1$ [40] if $g \geq 4$ for $k = 2$ [16, 10] if $g \geq 2k - 1$ for $k \geq 3$ [10]</td>
</tr>
</tbody>
</table>

2.4. Stronger surgeries in 3-manifolds

We are now in position to introduce two families of surgery equivalence relations that refine the Torelli-equivalence. We fix a closed surface $R$, which may be disconnected.

Definition 2.13. Let $k \in \mathbb{N}^*$. Two 3-manifolds $M, M' \in \mathcal{V}(R)$ are $J_k$-equivalent (resp. $Y_k$-equivalent) if $M'$ can be obtained from $M$ by a Torelli twist $M \rightarrow M_s$ along a surface $S \subset \text{int}(M)$ with an $s \in J_k \mathcal{I}(S)$ (resp. an $s \in \Gamma_k \mathcal{I}(S)$).

Of course, the $J_1$-equivalence and $Y_1$-equivalence are just the same as the Torelli-equivalence.

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Lemma 2.14. For every $k \in \mathbb{N}^*$, the $J_k$-equivalence (resp. the $Y_k$-equivalence) is an equivalence relation in $\mathcal{V}(R)$.

Proof. We come back to the proof of Lemma 2.7, using the same notations.

If we have $s \in J_k \mathcal{I}(S)$ and $s' \in J_k \mathcal{I}(S')$, then $s \equiv_2 s'$ belongs to $J_k \mathcal{I}(S \equiv_2 S')$ as can be checked from the fact that $\pi_1(S \equiv_2 S')$ is the free product of $\pi_1(S)$ and $\pi_1(S')$. This proves the transitivity of the $J_k$-equivalence.

If we have $s \in \Gamma_k \mathcal{I}(S)$ and $s' \in \Gamma_k \mathcal{I}(S')$, then $s \equiv_2 s'$ belongs to $\Gamma_k \mathcal{I}(S \equiv_2 S')$ as follows from the fact that $s \equiv_2 s' = (s \equiv_2 s') \circ (\text{id}_3 s')$. This proves the transitivity of the $Y_k$-equivalence.

Remark 2.15. Proposition 2.10 can also be refined to reformulate the $J_k$-equivalence (resp. the $Y_k$-equivalence) in terms of Heegaard splittings.

We deduce from (2.7) the following “ladder” of equivalence relations:

$\begin{align*}
Y_1 & \Leftrightarrow Y_2 \Leftrightarrow Y_3 \Leftrightarrow \cdots \Leftrightarrow Y_k \Leftrightarrow Y_{k+1} \Leftrightarrow \cdots \\
J_1 & \Leftrightarrow J_2 \Leftrightarrow J_3 \Leftrightarrow \cdots \Leftrightarrow J_k \Leftrightarrow J_{k+1} \Leftrightarrow \cdots
\end{align*}$

Note that the converse of the implication “$Y_k \Rightarrow J_k$” is not true. Specifically, there is no $d \in \mathbb{N}^*$ such that “$J_d \Rightarrow Y_3$”: this can be easily deduced from Hain’s result mentioned in Remark 2.12.

After $Y_1 = J_1$, the next equivalence relation to consider is the $J_2$-equivalence. Let us give an alternative description in terms of surgeries along knots. Given $M \in \mathcal{V}(R)$ and a null-homologous knot $K \subset \text{int}(M)$, there is a unique parallel $\rho_0(K) \subset \partial N(K)$ that is null-homologous in $M \setminus K$: for any $n \in \mathbb{Z}$, the knot $K$ is said to be $n$-framed if it is equipped with the unique parallel $\rho_n(K)$ that represents the homology class $n[\mu(K)] + [\rho_0(K)] \in H_1(\partial N(K); \mathbb{Z})$. (Here, we fix an orientation of $K$, we orient $\rho_0(K)$ compatibly with $K$ and orient $\mu(K)$ with the right-hand rule using the orientation of $M$.) Following Cochran, Gerges and Orr [11], we say that an $M \in \mathcal{V}(R)$ is 2-surgery equivalent to an $M' \in \mathcal{V}(R)$ if there is a finite sequence

$M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_r = M'$

of surgeries along null-homologous $(\pm 1)$-framed knots.

Proposition 2.16. The $J_2$-equivalence is the same as the 2-surgery equivalence. In particular, the 2-surgery equivalence is an equivalence relation in $\mathcal{V}(R)$.

Proof. Assume that $M, M'$ are $J_2$-equivalent: then there is a surface $S \subset \text{int}(M)$ and an $s \in J_2 \mathcal{I}(S)$ such that $M' \cong M_s$. According to what has been mentioned in Example 2.11, $s$ decomposes as a product of BSCC maps (or their inverses). Thus, by considering parallel copies of $S$, we find a finite sequence

$M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_r = M'$

where each move $M_i \rightarrow M_{i+1}$ is a Torelli twist defined by a BSCC map (or its inverse). By Lickorish’s trick, such a move can be interpreted as a surgery along a null-homologous $(\pm 1)$-framed knot. So $M$ is 2-surgery equivalent to $M'$.

Assume now that $M$ is 2-surgery equivalent to $M'$. We wish to prove that $M$ and $M'$ are $J_2$-equivalent. By transitivity of $J_2$, we can assume that $M'$ is obtained from $M$ by a single surgery along a null-homologous $(\pm 1)$-framed knot $K \subset M$. There is a Seifert surface for $K$ in $M$, i.e. a compact surface $\Sigma$ such that $\partial \Sigma = K$. The regular neighborhood $N(\Sigma)$ is a handlebody, in which $K$ can be viewed as a push-out of a bounding simple closed curve $\gamma \subset \partial N(\Sigma)$. Then, by Lickorish’s trick, $M' = M_K$ is diffeomorphic to $(M \setminus \text{int}(N(\Sigma))) \cup_{\Sigma} N(\Sigma)$ where $\tau := T_\gamma$. Hence $M'$ is the result of the Torelli twist $M \rightarrow M_{\gamma}$ along the surface $S$ obtained from $\partial N(\Sigma)$ by cutting a small open disk, with $s := T_{\gamma} \in J_2 \mathcal{I}(S)$.

Remark 2.17. A boundary link in a compact 3-manifold $M$ is a framed link $L = \bigcup L_i$ for which there exists a compact surface $S = \bigcup S_i \subset \text{int}(M)$ with as many connected components as $L$, such that $\partial S_i = L_i$ and the parallel of $L_i$ differs from the curve $\partial N(L_i) \cap S_i$ by $\pm \mu(L_i)$. Surgeries
along boundary links have been considered in [68, 23, 11], for instance. The argument used in the proof of Proposition 2.16 shows that surgery along a boundary link is equivalent to the simultaneous realization of Torelli twists by BSCC maps on pairwise-disjoint surfaces. Thus two manifolds $M, M' \in \mathcal{V}(R)$ are $J_2$-equivalent if, and only if, one can find a boundary link $L$ in $M$ such that $M_L \cong M'$.

In general, Cochran, Gerges and Orr make in [11] the following definition for any integer $k \geq 2$.

**Definition 2.18.** A manifold $M \in \mathcal{V}(R)$ is $k$-surgery equivalent to an $M' \in \mathcal{V}(R)$ if there is a finite sequence $M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_r = M'$ where each move $M_i \rightarrow M_{i+1}$ is the surgery along a $(\pm 1)$-framed knot $K_i$ that is trivial in $\Gamma_k \pi_1(M_i)$.

It turns out that the $k$-surgery equivalence is indeed an equivalence relation [11, Cor. 2.2 & Prop. 2.3]. But $k$-surgery equivalence is very different from $J_k$-equivalence in higher degree $k$: while the former is rather well understood, the latter still remains unexplored (see §3.5). In fact, since one does not know explicit generating systems for the Johnson filtration, it seems that one does not know generators for the $J_k$-equivalence relation for $k > 2$.

### 2.5. Clasper calculus

In contrast with the $J_k$-equivalence, explicit generators are known for the $Y_k$-equivalence: these are defined in terms of “surgeries” along certain framed graphs, and generalize in degree $k > 1$ the $Y$-surgeries that have been recalled in §2.2. These surgery techniques were developed independently by Goussarov [27, 28] and Habiro [30].

We give a very brief overview of those techniques, using Habiro’s terminology and conventions. Let $M \in \mathcal{V}(R)$. A graph clasper in $M$ is a (possibly disconnected) compact surface $G \subset \text{int}(M)$, which is decomposed into leaves, nodes and edges. Leaves are copies of the annulus $S^1 \times D^1$ and nodes are copies of the disc $D^2$. Edges are 1-handles (i.e. copies of $D^1 \times D^1$) connecting those leaves and nodes; the ends of an edge constitute the attaching locus of the 1-handle (i.e. $S^0 \times D^1$). There are two rules to respect in the attachment: each leaf receives exactly one end of an edge, and each node receives exactly three ends of edges. The degree of $G$ is the number of its nodes. The shape of $G$ is the abstract graph, whose vertices have valency 1 or 3, onto which $G$ deformation retracts after deletion of all of its leaves.

**Example 2.19.** Graph claspers of degree 0 (and shape I) are called basic claspers and consist of only one edge and two leaves:

A connected graph clasper of degree 1 (and shape $Y$) is a $Y$-graph, as shown in Figure 2.1. Here is an example of a connected graph clasper of degree 3:

Surgery along a graph clasper $G \subset \text{int}(M)$ is defined as follows. We first replace each node with three leaves in a “Borromean rings” fashion:
This results in a disjoint union of basic claspers, which we replace by 2-component framed links as follows:

(For instance, if we start from a $Y$-graph $G$, then we recover the 6-component framed link shown in Figure 2.1.) Then, the surgery $M \to M_G$ along $G$ is defined as the surgery along the resulting framed link in $M$, and we have the following generalization of Proposition 2.8:

**Proposition 2.20** (Habiro 2000). For any integer $k \geq 1$, the $Y_k$-equivalence relation is generated by surgeries along connected graph claspers of degree $k$.

See [30], and the appendix of [63] for a proof. Note that the $Y_k$-equivalence appears in the works of Goussarov and Habiro under different names: it is named “$(k-1)$-equivalence” in [27] and “$A_k$-equivalence” in [30].

There exists a clasper calculus, which has been developed in [28, 30, 22]. This calculus can be regarded as a braided version of the commutator calculus in groups or, to be more accurate, an instance of the braided Hopf-algebraic calculus. In the setting of [30], there is a notion of “clasper”, which is more general than the above notion of “graph clasper”, and there are 12 “moves” which can be applied to claspers without changing the diffeomorphism types of the resulting manifolds.

Thanks to Proposition 2.20, this clasper calculus can be used to show that certain operations $G \to G'$ on graph claspers will not change the $Y_\ell$-equivalence class of the resulting manifold, for $\ell$ large enough depending on the degrees of the components of $G$ and the nature of the operation. Thus, these operations are very useful tools to study sets of $Y_k$-equivalence classes up to $Y_\ell$-equivalence for some $\ell > k$.

Here are some instances of such operations on graph claspers, taking place in a manifold $M \in V(R)$ which we fix from now on:

(O_0) **Cutting an edge.** Any graph clasper $G$ can be transformed to a graph clasper $G'$ (of the same degree, but not the same shape) by insertion of a Hopf link of two leaves at the middle of an edge:

\[ G \quad \to \quad G' \]

(In fact, this operation is Habiro’s “Move 2” [30].)

(O_1) **Developing a node.** Any graph clasper $G$ of degree $k+1$, showing one node incident to two leaves, can be transformed to a graph clasper $G'$ of degree $k$ by the following transformation:
(In fact, this operation is essentially Habiro’s “Move 9” [30].)

(O2) **Sliding an edge.** If \( G \) is a connected graph clasper of degree \( k \) in \( M \) and if \( G' \) is obtained from \( G \) by sliding one of its edges along a disjoint framed knot \( K \), then we have \( M_G \sim_{Y_{k+1}} M_{G'} \):

\[
\begin{align*}
& \xymatrix{
G \\
& K \\
\ar@{.}[] & & & & \\
& G' \\
\end{align*}
\sim_{Y_{k+1}}
\]

(O3) **Cutting a leaf.** If \( G \) is a connected graph clasper of degree \( k \) in \( M \) with a leaf \( L \) decomposed as \( L = L_1 \# L_2 \), then we have \( M_G \sim_{Y_{k+1}} M_{G_1 \cup G_2} \) where \( G_l \) is \( G \) with the leaf \( L \) replaced by the “half-leaf” \( L_l \) and \( G_1 \cup G_2 \) is a disjoint union of \( G_1 \) and \( G_2 \):

\[
\begin{align*}
& \xymatrix{
G \\
& L \\
\ar@{.}[] & & & & \\
& K \\
\ar@{.}[] & & & & \\
& G' \\
\end{align*}
\sim_{Y_{k+1}}
\]

(O4) **Crossing a leaf with a leaf.** If \( G_1 \cup G_2 \) is the disjoint union of two connected graph claspers in \( M \) of degrees \( k_1 \) and \( k_2 \), respectively, and if \( G_1' \cup G_2' \) is obtained from \( G_1 \cup G_2 \) by crossing a leaf of \( G_1 \) with a leaf of \( G_2 \), then we have \( M_{G_1 \cup G_2} \sim_{Y_{k_1+k_2}} M_{G_1' \cup G_2'} \).

(O5) **Half-twisting an edge.** If \( G \) is a connected graph clasper of degree \( k \) in \( M \) and if \( G^- \) is obtained from \( G \) by adding a half-twist to an edge, then there is a disjoint union \( G \cup G^- \) of \( G \) and \( G^- \) in \( M \) such that \( M_{G \cup G^-} \sim_{Y_{k+1}} M \).

**Remark 2.21.** References for the above operations on graph claspers include [30] (in the case of links instead of 3-manifolds), [28], [22], [21], [85, §E] and [62].

In the rest of this subsection, we outline the general strategy to study the \( Y_{k} \)-equivalence relations using the above techniques of clasper calculus. So, let us assume that we have been able to classify the \( Y_{k} \)-equivalence relation on \( \mathcal{V}(R) \) for some \( k \geq 1 \), and that we now wish to classify the \( Y_{k+1} \)-equivalence on a specific \( Y_{k} \)-equivalence class \( V_0 \subset \mathcal{V}(R) \).

For this, we fix a 3-manifold \( V \in \mathcal{V}_0 \) and we consider the free abelian group \( \mathbb{Z} \cdot C_k \) generated by the set

\[
C_k := \{ \text{connected graph claspers in } V \text{ of degree } k \}/\text{isotopy}.
\]
Then we consider the map
\[ \psi_k : Z \cdot C^k \longrightarrow V_0 \rightarrow Y_{k+1}, \quad \sum_i \epsilon_i G_i \longrightarrow [V_{(\cup_i G_i^i)}], \]
where, for a family \((G_i)_i\) of connected graph claspers of degree \(k\) in \(V\) weighted by a family of signs \((\epsilon_i)_i\), we choose an arbitrary disjoint union \(\cup_i G_i^i\) of the graph claspers \(G_i^i\) using the convention that \(G_i^- := (G_i\text{ with a half-twist on a edge})\) and \(G_i^+ := G_i\). That \(\psi_k\) is well-defined follows from the operations \((O_2), (O_4), (O_5)\).

Let us show that \(\psi_k\) is surjective. Any \(M \in V_0\) is \(Y_k\)-equivalent to \(V\) and, so, by Proposition 2.20, there is a sequence \(V = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_r = M\) where each move \(M_i \rightarrow M_{i+1}\) is either a surgery along a connected graph clasper of degree \(k\), or the inverse of such a surgery; furthermore, thanks to \((O_0)\), we can assume that each graph clasper involved in the sequence is tree-shaped. Now, any surgery \(W \rightarrow W_T\) along a tree-shaped graph clasper \(T\) in a 3-manifold \(W\) has the following properties:

- it is reversible, in the sense that there is a graph clasper \(I\) (of the same shape as \(T\)) in \(W_T\), such that \((W_T)_I \cong W\);
- there is a \(t \in M(\partial N(T))\) such that \(W_T \cong (W \setminus \text{int}(N(T)) \cup_t N(T))\), hence any graph clasper in \(W_T\) can be isotoped into the subset \(W \setminus \text{int}(N(T))\) of \(W_T\).

It follows that there exists a disjoint union \(G = \cup_i G_i\) of (tree-shaped) connected graph claspers of degree \(k\) in \(V\) such that \(G_0 \cong M\). We deduce that \(\psi_k(\sum_i G_i) = M\).

Thus, we would like to understand the equivalence relation \(\sim\) on \(Z \cdot C^k\) such that the map \(\psi_k\) factorizes to a bijection on the quotient set:

\[
\begin{array}{ccc}
Z \cdot C^k & \xrightarrow{\psi_k} & V_0 \\
\sim & & \sim \\
& Y_{k+1} & \\
\end{array}
\]

For instance, it follows from \((O_2)\) that we must have \(G' \sim G^+\) for any graph claspers \(G\) and \(G'\) in \(M\) which have the same shape and the same leaves. Besides, there are other instances of the relation \(\sim\) that deal with leaves and result from \((O_0), (O_1)\) and \((O_3)\). Finally, using other operations on graph claspers (not in the above list), we obtain other instances of the relation \(\sim\) that do not affect the leaves but change the shape: one such example is the so-called “IHX relation”. Once we have a candidate for the relation \(\sim\), the difficulty is then to show the injectivity of the resulting map \(\bar{\psi}_k : Z \cdot C^k/\sim \rightarrow V_0/Y_{k+1}\). This is proved by finding sufficiently enough topological invariants on \(V(R)\) — or, at least, on its subset \(V_0\) — that are unchanged by \(Y_{k+1}\)-surgery and constitute a left-inverse \(Z_k\) to \(\bar{\psi}_k\) when they are conveniently assembled all together:

\[
\begin{array}{ccc}
Z \cdot C^k & \xrightarrow{\bar{\psi}_k} & V_0 \\
\sim & & \sim \\
& Y_{k+1}/id & \\
\end{array}
\]

At the end of this process, we conclude that \(\bar{\psi}_k\) is injective and, so, bijective, thus obtaining a combinatorial description of the quotient set \(V_0/Y_{k+1}\), and concluding that the invariant \(Z_k\) classifies the \(Y_{k+1}\)-equivalence relation on \(V_0\).

**Remark 2.22.** In all the few situations that the author knows, the relation \(\sim\) on \(Z \cdot C^k\) happens to be always defined by a subgroup of \(Z \cdot C^k\): hence \(V_0/Y_{k+1}\) has a structure of abelian group, although \(V_0\) may not have (a priori) a natural operation. If \(V_0\) does have a natural operation compatible with the \(Y_{k+1}\)-equivalence and if we know that \(V_0/Y_{k+1}\) inherits a structure of abelian group, it is often much easier to carry on the above process with rational coefficients in order to get a combinatorial description of the vector space \((V_0/Y_{k+1}) \otimes Q\).
The above "general strategy", to study inductively the $Y_1$-equivalence relations by clasper calculus, will be mentioned in the next sections in a few examples.

### 2.6. Other kinds of surgeries

To conclude this section, we mention yet other surgery equivalence relations. Some of them are just alternative descriptions of the relations that have been introduced in the previous subsections, but other ones are quite different. We fix a closed surface $R$, which may be disconnected.

1. **LP surgeries.** A homology handlebody of genus $g$ is a compact 3-manifold $C'$ with the same homology type as $H_g$; the Lagrangian of $C'$ is the kernel $L_{C'}$ of the homomorphism $H_1(aC'; Z) \rightarrow H_1(C'; Z)$ induced by the inclusion $aC' \hookrightarrow C'$: this is a Lagrangian subgroup of $H_1(aC'; Z)$ with respect to the intersection form. Following Auclair and Le-scop [1], we call LP-pair a couple $C = (C', C'')$ of two homology handlebodies whose boundaries are identified $aC' = aC''$ in such a way that $L_{C'} = L_{C''}$. (The acronym "LP" is for "Lagrangian-Preserving".) Given an $M \in \mathcal{V}(R)$ and an LP pair $C = (C', C'')$ such that $C' \subset M$, one can replace in $M$ the submanifold $C'$ by $C''$ to obtain a new 3-manifold

$$M_C := (M \setminus \text{int}(C')) \cup_3 C''.$$ The move $M \to M_C$ in $\mathcal{V}(R)$ is called an LP-surgery.

A Torelli twist $M \to M_\ell$ can be interpreted as an LP-surgery since a regular neighborhood of a surface $S \subset M$ is a handlebody. Conversely, an LP-surgery can be realized by finitely many $Y$-surgeries because, for any LP pair $C$, the homology handlebodies $C'$ and $C''$ are Torelli-equivalent. (See Remark 3.9 below.) Therefore, LP-surgery equivalence is the same as Torelli-equivalence.

There is also a rational version of the LP-surgery using $H_1(\ast; Q)$ instead of $H_1(\ast; Z)$, which has been considered by Moussard [78]. However, rational LP-surgery equivalence is coarser than Torelli-equivalence as a relation.

2. **Torelli surgeries.** Let $M \in \mathcal{V}(R)$, let $C \subset M$ be a handlebody and let $c \in \mathcal{I}(aC)$. Following Kuperberg and Thurston [50], we say that

$$M_C := (M \setminus \text{int}(C)) \cup_3 C$$

is obtained from $M$ by a Torelli surgery along $C$. Clearly, a Torelli surgery can be realized by a Torelli twist (by choosing a small open disk $D \subset aC$ and isotoping $c$ so that it fixes $D$ pointwisely); conversely, a Torelli twist can be realized by a Torelli surgery (because a regular neighborhood of a surface with non-empty boundary is a handlebody). Thus, the $Y_k$-equivalence and $j_k$-equivalence relations can be reformulated in terms of Torelli surgeries.

3. **Lagrangian Torelli surgeries.** Let $C$ be a handlebody. The Lagrangian Torelli group of $S := aC \setminus$ (small open disk) is defined by

$$\mathcal{I}^+(S) := \left\{ f \in \mathcal{M}(S) : f_*(L_C) \subset L_C \text{ and } f_* \text{ is the id on } \frac{H_1(S; Z)}{L_C} \right\}$$

where $L_C$ is the Lagrangian of $C$. A Lagrangian Torelli surgery is defined in a way similar to a Torelli surgery using the Lagrangian Torelli group instead of the Torelli group. Clearly, a Lagrangian Torelli surgery is a special case of an LP surgery; therefore, the equivalence relation defined by Lagrangian Torelli surgeries is again the Torelli-equivalence.

Nevertheless, following Faes [17, §A], we can define a new family of equivalence relations on $\mathcal{V}(R)$ by considering the following filtration on the Lagrangian Torelli group
of a handlebody $C$. Let $L_C$ denote the kernel of the homomorphism $\rho : \pi_1(S) \to \pi_1(C)$ induced by the inclusion $S \hookrightarrow C$ and consider, for any integer $k \geq 1$, the subset

$$L_k I^k(S) := \{ f \in I^k(S) : pf_*(L_C) \subset \Gamma_{k+1} \pi_1(C) \}$$

of the mapping class group of $S$. According to Levine [55, 56], the filtration

$$I^k(S) = L_1 I^k(S) \supset L_2 I^k(S) \supset L_3 I^k(S) \supset \cdots$$

is a decreasing sequence of subgroups of the Lagrangian Torelli group, which contains the Johnson filtration of the Torelli group $I(S)$. But, in contrast with the latter, the intersection of the former is not trivial: its intersection is the subgroup of $I^2(S)$ consisting of all diffeomorphisms that extend to the full handlebody $C$; hence this intersection is irrelevant for Lagrangian Torelli surgeries.

Thus, it is interesting to consider the following relation for any $k \in \mathbb{N}^*$: we say that $M, M' \in \mathcal{V}(R)$ are $L_k$-equivalent if $M'$ can be obtained from $M$ by a Lagrangian Torelli surgery $M \to M'$ along a handlebody $C \subset \text{int}(M)$ with $c \in L_k I^2(S)$. Clearly, we have $J_k \Rightarrow L_k$ for any $k \geq 1$. We have already mentioned the equality of relations $L_1 = J_1$, and it follows essentially from Levine’s results that $L_2 = J_2$. However, the $L_3$-equivalence is strictly weaker than the $J_3$-equivalence as a relation [17, §A].

### 3. Surgery equivalence relations: their characterization

In this section, we review several results from the middle 1970’s to nowadays, which provide characterizations of the $J_k$-equivalence, the $Y_k$-equivalence, and the $k$-surgery equivalence relations in terms of topological invariants (for some or all values of $k \in \mathbb{N}^*$).

#### 3.1. Two case studies to consider

Let $R$ be a compact surface. The problem of characterizing surgery equivalence relations in $\mathcal{V}(R)$ is very much dependent on the choice of $R$. So we shall restrict ourselves to the two cases that we have already mentioned on page 14:

(i) $R = \emptyset$;  
(ii) $R = \partial(\Sigma \times [-1, 1])$ where $\Sigma$ is a compact surface with $\partial \Sigma \cong S^1$.

Actually, in case (i), our interest in the set of closed 3-manifolds $\mathcal{V}(\emptyset)$ will quickly specialize to the class

$$\mathcal{S} := \{ M \in \mathcal{V}(\emptyset) : H_*(M; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z}) \}$$

of homology 3-spheres. Of course, this is a strong restriction but, as we shall see, $\mathcal{S}$ is still a very rich set-up for studying surgery equivalence relations.

**Remark 3.1.** The set $\mathcal{S}$ with the connected sum operation $\#$ is a monoid, whose neutral element is $S^3$. $\blacksquare$

Similarly, in case (ii), our interest in the set $\mathcal{V}(\partial(\Sigma \times [-1, 1]))$ of cobordisms will be restricted to the subset $IC(\Sigma)$ of homology cylinders over $\Sigma$. Those are cobordisms $(C, c)$ from $\Sigma$ to $\Sigma$ such that the boundary parametrizations

$$c_+ := c|_{\Sigma \times \{+1\}} : \Sigma \to C \quad \text{and} \quad c_- := c|_{\Sigma \times \{-1\}} : \Sigma \to C$$

induce isomorphisms in homology and satisfy $c_+, c_- : H_1(\Sigma; \mathbb{Z}) \to H_1(C; \mathbb{Z})$:

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{c_+} & C \\
\downarrow & & \downarrow \\
\Sigma & \xrightarrow{c_-} & C
\end{array}
\]
Two cobordisms \((C, c)\) and \((D, d)\) from \(\Sigma\) to \(\Sigma\) can be multiplied by gluing \(D\) “on the top of” \(C\), using the boundary parametrizations \(d_−\) and \(c_+\) to identify \(d_−(\Sigma)\) with \(c_+(\Sigma)\):

\[
\begin{array}{ccc}
\Sigma & \circ & \Sigma \\
\hline
C & \longrightarrow & D \\
\hline
\Sigma & \circ & \Sigma \\
\hline
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma & \circ & \Sigma \\
\hline
C & \longrightarrow & D \\
\hline
\Sigma & \circ & \Sigma \\
\hline
\end{array}
\]

It is easily checked that \(C \circ D \in IC(\Sigma)\) if \(C, D \in IC(\Sigma)\). Hence the set \(IC(\Sigma)\) with this operation \(\circ\) is a monoid, whose neutral element is the trivial cylinder \(U := \Sigma \times [-1, 1]\) (with the obvious boundary parametrization).

**Proposition 3.2.** The “mapping cylinder” construction defines a monoid homomorphism \(cyl : I(\Sigma) \rightarrow IC(\Sigma)\), which is injective and surjective onto the group of units of \(IC(\Sigma)\).

**About the proof.** A diffeomorphism \(f : \Sigma \rightarrow \Sigma\) defines a cobordism \(cyl(f)\) from \(\Sigma\) to \(\Sigma\) whose underlying 3-manifold is the trivial cylinder \(U\) and whose boundary parametrization \(\partial(\Sigma \times [-1, 1]) \rightarrow \partial U\) is given by \(f\) on the top surface \(\Sigma \times \{+1\}\) and by the identity elsewhere. Clearly, the diffeomorphism class of \(cyl(f)\) only depends on the isotopy class of \(f\) and, obviously, \(cyl(f)\) is a homology cylinder if \(f\) induces the identity in homology. Thus we obtain a map \(cyl : I(\Sigma) \rightarrow IC(\Sigma)\). Clearly it is multiplicative, and it is injective for the following reason: two diffeomorphisms \(\Sigma \rightarrow \Sigma\) are isotopic rel \(\partial\Sigma\) if and only if they are homotopic rel \(\partial\Sigma\), by the classical result of Baer [2] that we have already alluded to at page 15. The image of \(cyl\) is determined in [32, Prop. 2.4], for instance.

The following is easily checked.

**Proposition 3.3.** The map \(\iota : S \rightarrow IC(\Sigma)\) defined by \(\iota(M) := M \ast U\) is an injection of monoids, and it is an isomorphism for \(\Sigma = D^2\).

Thus, the monoid of homology cylinders \(IC(\Sigma)\) can be viewed as a simultaneous generalization of the Torelli group \(I(\Sigma)\) and the monoid \(S\).

### 3.2. Characterization of the Torelli–equivalence

The most fundamental result is the characterization of the Torelli–equivalence, which has been obtained for closed 3-manifolds by Matveev [68]. To state his result, we recall that the linking number

\[
\text{Lk}(K, L) \in \mathbb{Q}
\]

of two disjoint oriented knots \(K, L\) in a closed 3-manifold \(M\) is defined when \(K\) and \(L\) are rationally null-homologous: let \(n \in \mathbb{N}^*\) be such that \(n[K] = 0 \in H_3(M; \mathbb{Z})\) and let \(\Sigma \subset M\) be a surface transverse to \(L\) such that \(\partial \Sigma\) consists of \(n\) parallel copies of the knot \(K\); then

\[
\text{Lk}(K, L) := \frac{1}{n} \Sigma \cdot L
\]

where \(\Sigma \cdot L \in \mathbb{Z}\) denotes the algebraic intersection number. It can be verified that the class of \(\text{Lk}(K, L)\) modulo 1 only depends on the integral homology classes of \(K\) and \(L\). Hence we get a map

\[
\lambda_M : \text{Tors} H_1(M; \mathbb{Z}) \times \text{Tors} H_1(M; \mathbb{Z}) \longrightarrow Q/\mathbb{Z}, ([K], [L]) \mapsto (\text{Lk}(K, L) \mod 1)
\]

which is called the (torsion) linking pairing of \(M\) and is one of the eldest invariants of closed 3-manifolds [94, §77]. The map \(\lambda_M\) is bilinear, symmetric and non-singular (see [64, Lemma 6.7], for instance).

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We conclude that can be completed inside the re-glued handlebody to get a surface ∂ and (3.3)

\[
\psi \Rightarrow \text{Surgery equivalence relations for } 3\text{-manifolds}
\]

**Theorem 3.4** (Matveev 1987). Two manifolds \( M, M' \in \mathcal{V}(\emptyset) \) are Torelli–equivalent if, and only if, there is an isomorphism \( \psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z}) \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Tors } H_1(M; \mathbb{Z}) \times \text{Tors } H_1(M; \mathbb{Z}) & \lambda_M & \mathbb{Q}/\mathbb{Z} \\
\psi \times \psi \downarrow & & \\
\text{Tors } H_1(M'; \mathbb{Z}) \times \text{Tors } H_1(M'; \mathbb{Z}) & \lambda_{M'} & \\
\end{array}
\]

**Sketch of proof.** Assume that \( M \) and \( M' \) are Torelli–equivalent. Hence there is a Torelli twist \( M \to M_\delta \) along a surface \( S \subset M \) such that \( M_\delta \cong M' \). This surgery induces an isomorphism \( \psi := \psi_\delta \) in homology, as described by (2.3). Using the notations of (3.2) and setting \( x := [K] \) and \( y := [\lambda] \), we have

\[
\lambda_M(x, y) = \frac{1}{n} \Sigma \bullet L \mod 1.
\]

Since the handlebody \( N(S) = S \times [-1, 1] \) deformation retracts onto a 1-dimensional subcomplex, we can isotope \( K \) and \( L \) in \( M \) to make them disjoint from \( N(S) \); hence, as subsets of \( M \setminus \text{int} N(S) \), \( K \) and \( L \) can also be regarded as knots in \( M_\delta = M' \); so we have \( \psi(x) = [K] \) and \( \psi(y) = [\lambda] \) in \( H_1(M'; \mathbb{Z}) \). Furthermore, we can isotope \( \Sigma \) so that it cuts the handlebody \( N(S) \) transversely along meridional disks of \( N(S) \): in particular, the boundary \( \partial \Sigma' \subset \partial N(S) \) of \( \Sigma' := \Sigma \cap (M \setminus \text{int} N(S)) \) is null-homologous in \( N(S) \). Recall that \( M' \) is obtained from \( M \setminus \text{int} N(S) \) by re-gluing \( N(S) \) using a diffeomorphism \( \hat{\psi} : \partial N(S) \to \partial N(S) \) that acts trivially in homology: hence \( \partial \Sigma' \) is still null-homologous in the re-glued handlebody of \( M' \), so that \( \Sigma' \subset M \setminus \text{int} N(S) \) can be completed inside the re-glued handlebody to get a surface \( \Sigma' \subset M' \) satisfying \( \partial \Sigma' = nK \).

We conclude that

\[
\lambda_{M'}(\psi(x), \psi(y)) = \left( \frac{1}{n} \Sigma' \bullet L \mod 1 \right) = \left( \frac{1}{n} \Sigma \bullet L \mod 1 \right) = \lambda_M(x, y).
\]

Assume now that there is an isomorphism \( \psi \) in homology satisfying (3.3). According to Theorem 1.10, \( M \) has a surgery presentation in \( S^3 \): i.e., there is an \( n \)-component framed link \( L \subset S^3 \) such that \( M = S^3_L \). We now recall the way of computing \( \lambda_M \) from the linking matrix of \( L \), which is the \( n \times n \) matrix

\[
\text{Lk}(L) := \left( \text{Lk}(L_i, L_j) \right)_{ij}.
\]

(Here we have chosen an orientation for each component \( L_i \) of \( L \), and the linking number \( \text{Lk}(L_i, L_j) \) is an integer because \( H_1(S^3; \mathbb{Z}) \) is trivial; by convention, \( \text{Lk}(L_i, L_i) := \text{Lk}(L_i, \rho(L_i)) \) is the linking number of \( L_i \) and its parallel \( \rho(L_i) \).)

Let \( H := \mathbb{Z}^n \), let \( f : H \times H \to \mathbb{Z} \) be the symmetric bilinear map whose matrix in the canonical basis \( (e_i) \) is \( \text{Lk}(L) \), and let \( f : H \to \text{Hom}(H, \mathbb{Z}) \) be the adjoint of \( f \). We consider the symmetric bilinear form

\[
\lambda_f : G_f \times G_f \to \mathbb{Q}/\mathbb{Z}
\]

defined on the finite abelian group \( G_f := \text{Tors} \left( \text{Coker} f \right) \) by

\[
\forall \{u\}, \{v\} \in G_f \subset \text{Hom}(H, \mathbb{Z}), \quad \lambda_f(\{u\}, \{v\}) := (f_Q(\check{u}, \check{v}) \mod 1)
\]

where \( f_Q \) is the extension of \( f \) to rational coefficients and where \( \check{u}, \check{v} \) are antecedents of \( u_Q, v_Q : H \otimes \mathbb{Q} \to \mathbb{Q} \) by the adjoint \( f_Q : H \otimes \mathbb{Q} \to \text{Hom}(H \otimes \mathbb{Q}, \mathbb{Q}) \). It is easily verified that \( \lambda_f \) is non-singular.

This algebraic construction from the matrix \( \text{Lk}(L) \) has the following topological interpretation in terms of the 4-manifold \( W_L \) obtained from \( D^4 \) by attaching 2-handles along \( L \):

- \( H \cong H_2(W_L; \mathbb{Z}) \) and \( -f \) then corresponds to the intersection form of \( W_L \);
- hence \( \text{Coker} f \cong H_1(M; \mathbb{Z}) \) and \( -\lambda_f \) then corresponds to \( \lambda_M \).

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We proceed similarly with $M'$ to get a symmetric bilinear form $f'$ on a finitely-generated free abelian group $H'$. By assumption, we have $(G_f, \lambda_f) \simeq (G_{f'}, \lambda_{f'})$ and it follows from early works in knot theory [46, 48] and algebra [105, 15] that the pairs $(H, f)$ and $(H', f')$ are stably equivalent, meaning that there exist integers $n_\pm, n'_\pm \geq 0$ such that

$$(H, f) \oplus (Z, +1)^{\oplus n_+} \oplus (Z, -1)^{\oplus n_-} \simeq (H', f') \oplus (Z, +1)^{\oplus n'_+} \oplus (Z, -1)^{\oplus n'_-}. $$

The direct sum with $(Z, \pm 1)$ can be realized, at the level of surgery presentations, by the disjoint union with the $(\pm 1)$-framed unknot, and this does not change the 3-manifold after surgery. Besides, an automorphism of $H$ can be decomposed into finitely many “elementary” automorphisms which, in terms of the basis $(e_i)_i$ of $H$, are given by the operations $e_i \leftrightarrow e_j$, $e_i \rightarrow -e_i$ or $(e_i, e_j) \rightarrow (e_i + e_j, e_i)$; these “elementary” automorphisms can be realized, at the level of surgery presentations, by the renumbering $i \leftrightarrow j$ of the components of $L$, the change of orientation $L_i \rightarrow -L_i$ or the operation $(L_i, L_j) \rightarrow (L_i \# L_j, L_j)$, respectively. All these elementary operations on links (which constitute the so-called “Kirby calculus” [45]) do not affect the 3-manifold after surgery: it is obvious for the first two operations and, for the third operation, it is justified by sliding the attaching locus of a handle of $W_L$ along another handle.

Therefore, we can assume without restriction of generality that $M$ and $M'$ are presented by surgery in $S^3$ along framed links with the same linking matrix:

$$\text{Lk}(L) = \text{Lk}(L').$$

Then a result of Murakami & Nakanishi [80] asserts that $L$ and $L'$ are related one to the other, by isotopies and finitely many local moves of the following type:

$$\text{(3.4)}$$

Such a local move (called a $\Delta$-move in [80]) can be realized by surgery along a $Y$-graph: see [30, Fig. 34 (b)], for instance. We conclude that, up to diffeomorphisms, $M$ and $M'$ are related one to the other by finitely many $Y$-surgeries. Hence they are Torelli–equivalent. \hfill \Box

**Remark 3.5.** The proof of Theorem 3.4 given in [68] is not detailed, and the knot-theoretical ingredient in terms of linking matrices [80] is actually posterior to [68]. By refining this proof, [60] and [14] extend Theorem 3.4 to the setting of 3-manifolds with spin and complex spin structures, respectively: these extensions involve quadratic forms which refine the linking pairing and depend on the (complex) spin structures. See also [79] for a detailed proof of Matveev’s theorem and additional contents.

As an immediate consequence of Theorem 3.4, we obtain the following result about $S$ which dates back to [5] and is proved there with Heegaard splittings. The formulation in terms of blinks (see Remark 2.9) appears in [35].

**Corollary 3.6** (Birman 1974). *Any homology 3-sphere is Torelli–equivalent to $S^3$.*

By refining the proof of Theorem 3.4, we can also prove the following refinement of Corollary 3.6 which generalizes [80].

**Corollary 3.7.** Let $M, M' \in S$ and let $L \subset M, L' \subset M'$ be framed oriented $n$-component links. The pairs $(M, L)$ and $(M', L')$ are Torelli–equivalent if, and only if, we have $\text{Lk}(L) = \text{Lk}(L')$.

Let $\Sigma$ be a compact surface with $\partial \Sigma \cong S^1$. We now turn to homology cylinders over $\Sigma$ (whose definition has been given in §3.1). The following, which appears in [30], states that $\text{IC}(\Sigma)$ constitutes a Torelli–equivalence class.
Proposition 3.8 (Habiro 2000). Any homology cylinder over $\Sigma$ is Torelli–equivalent to the trivial cylinder $U = \Sigma \times [-1, 1]$.

**Sketch of the proof.** Fix a system of meridians and parallels in the surface $\Sigma$, i.e. a system of simple oriented closed curves $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ having the following intersection pattern:

Let $(C, c) \in IC(\Sigma)$: recall that $C$ is viewed as a cobordism from the “top” surface $\partial_+ C := c_+(\Sigma)$ to the “bottom” surface $\partial_- C := c_- (\Sigma)$. By gluing one 2-handle along each curve $c_-(\alpha_i)$ on $\partial_- C$ and one 2-handle along each curve $c_+(\beta_j)$ on $\partial_+ C$, the homology cylinder $C$ is turned into a homology 3-ball $C'$. Next, by adding a 3-handle to $C'$, we get a homology 3-sphere $\overline{C'}$. Each 2-handle $D^2 \times D^1$ has a co-core, which is the image of $\{0\} \times D^1$ after attachment of the 2-handle: hence the above procedure has also produced a framed oriented $(2g)$-component tangle $(\gamma_1, \ldots, \gamma_g, \overline{\gamma}_1, \ldots, \overline{\gamma}_g)$ in $C'$, which is called a bottom-top tangle. Now, we can connect the two extremities of each component $\gamma_j^+$ (resp. $\gamma_j^-$) by a small arc on the “top” (resp. “bottom”) boundary of $C'$ to get an oriented framed knot $G_j^+$ (resp. $G_j^-$) in $\overline{C'}$. It can be deduced from the equality $c_{\ast} = c_{- \ast} = H_1(\Sigma; Z) \rightarrow H_1(C; Z)$ that the linking matrix of the framed oriented link $G := (G_1^+, \ldots, G_g^+, G_1^-, \ldots, G_g^-)$ is

$$\text{Lk}(G) = \begin{pmatrix} 0_g & I_g \\ I_g & 0_g \end{pmatrix}$$

so that, in particular, it does not depend on $C \in IC(\Sigma)$.

If we apply the above constructions to the trivial cylinder $U$ instead of $C$, we obtain $U' \cong D^3$ and, inside $\overline{U'} \cong S^3$, we obtain a link $T$ with $\text{Lk}(T) = \text{Lk}(G)$. It follows then from Corollary 3.7 that the pair $(\overline{C'}, G)$ is Torelli–equivalent to $(\overline{U'}, T)$ and, therefore, $C$ is Torelli–equivalent to $U$.

We refer to [9, Cor. 7.7] for a more general result and more detailed arguments. \(\square\)

**Remark 3.9.** Recall that $H_k$ is the standard handlebody of genus $k$, with boundary $\Sigma_k$. A manifold $C \in \Upsilon(\Sigma_k)$ is a homology handlebody of genus $k$ if it has the same homology type as $H_k$. Using the same method of proof as for Proposition 3.8, we can show the following characterization due to Habegger [29]: two homology handlebodies $C', C''$ of genus $k$ are Torelli–equivalent if, and only if, they have the same Lagrangians:

$$\ker (\alpha' : H_1(\Sigma_k; Z) \rightarrow H_1(C'; Z)) = \ker (\alpha'' : H_1(\Sigma_k; Z) \rightarrow H_1(C''; Z))$$

See also Auclair & Lescop [1, Lemma 4.11]. \(\blacksquare\)

### 3.3. Characterization of $J_k$ and $Y_k$ at low $k$ for closed manifolds

The $j_1$-equivalence on $\Upsilon(\mathbb{O})$ being perfectly understood thanks to Theorem 3.4, we now turn to the $j_2$-equivalence. Recall from Proposition 2.16 that the $j_2$-equivalence coincides with the 2-surgery equivalence. The latter has been characterized in [11].

In addition to the linking pairing $\lambda_M$ of a closed 3-manifold $M$, the characterization of the 2-surgery equivalence involves the cohomology ring of $M$. It follows from Poincaré duality that all the (co)homology groups of $M$ are determined by $H_1(M; Z)$. Furthermore, the cohomology ring $H^*(M; Z_r)$ is determined for any $r \in \mathbb{N}$ by the triple-cup product form

$$u^r_M : H^1(M; Z_r) \times H^1(M; Z_r) \times H^1(M; Z_r) \rightarrow Z_r,$$
which is the trilinear and skew-symmetric form defined by
\[ \forall x, y, z \in H^1(M; \mathbb{Z}_r), \quad u_M^{(r)}(x, y, z) := \langle x \cup y \cup z, [M] \rangle \in \mathbb{Z}_r. \]

It turns out that all these forms can be encoded by a single invariant: the abelian (oriented) homotopy type of \( M \), which is defined as the homology class
\[ (3.5) \quad \mu_1(M) := f_*([M]) \in H_3(H_1(M)). \]

Here, homology groups are taken with \( \mathbb{Z}_r \)-coefficients, \( f : M \to K(H_1(M), 1) \) is a continuous map in an Eilenberg–MacLane space that induces the canonical homomorphism \( \pi_1(M) \to H_1(M) \) at the level of \( \pi_1 \), and the homology of the space \( K(H_1(M), 1) \) is identified to the homology of the (abelian) group \( H_1(M) \).

**Theorem 3.10** (Cochran–Gerges–Orr 2001). Let \( M, M' \in \forall(\emptyset) \). The following three statements are equivalent:

1. \( M \) and \( M' \) are \( J_2 \)-equivalent;
2. there is an isomorphism \( \psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z}) \) such that \( \lambda_M \) corresponds to \( \lambda_{M'} \) through \( \psi \), and \( u_M^{(r)} \) corresponds to \( u_{M'}^{(r)} \) through \( \text{Hom}(\psi, \mathbb{Z}_r) \) for all \( r \in \mathbb{N} \);
3. there is an isomorphism \( \psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z}) \) such that the induced map \( \psi_* : H_3(H_1(M; \mathbb{Z})) \to H_3(H_1(M'; \mathbb{Z})) \) maps \( \mu_1(M) \) to \( \mu_1(M') \).

**About the proof.** In fact, the results of [11] give a fourth, equivalent condition:

4. there is a cobordism \( W \) from \( M \) to \( M' \) such that the maps incl\(_{M}^* : H_1(M; \mathbb{Z}) \to H_1(W; \mathbb{Z}) \) and incl\(_{M'}^* : H_1(M'; \mathbb{Z}) \to H_1(W; \mathbb{Z}) \) induced by the inclusions are isomorphisms.

Some of the implications are not too difficult to prove, like

- (1) \( \Rightarrow \) (4) and (4) \( \Rightarrow \) (1) working with the formulation of the \( J_2 \)-equivalence in terms of 2-surgeries;
- (4) \( \Rightarrow \) (3) using the canonical map \( \Omega_3(K(H_1(M), 1)) \to H_3(H_1(M)) \) defined on the third cobordism group relative to \( K(H_1(M), 1) \);
- (3) \( \Rightarrow \) (2) using that the forms \( \lambda_M \) and \( u_M^{(r)} \) are defined by (co)homology operations, which also exist in the category of groups.

Some other implications like (2) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (4) are much more involved. We recommend the reading of [11] where techniques of low-dimensional topology, differential topology and algebraic topology intertwine in a rich manner.

As an immediate consequence of Theorem 3.10, we obtain the following result about \( S \). It appeared priorly in [68], in its formulation with boundary links (see Remark 2.17).

**Corollary 3.11** (Matveev 1987). Any homology 3-sphere is \( J_2 \)-equivalent to \( S^3 \).

Although the first publication of Corollary 3.11 seems to be [68], it appears that the result was known from Johnson as early as 1977 [37]. It has been reproved (in its formulation with 2-surgeries) by Casson in order to give a surgery description of his invariant [8].

Morita [73] gave yet another proof of Corollary 3.11 using Heegaard splittings. By extending Morita’s techniques and after long computations, Pitsch obtained the following in [87]:

**Theorem 3.12** (Pitsch 2008). Any homology 3-sphere is \( J_3 \)-equivalent to \( S^3 \).

In a very recent paper [18], Faes proved the next step for \( S \). But, in contrast with Pitsch’s proof of Theorem 3.12, his arguments require the classification of the \( Y_k \)-equivalence on \( S \) for \( k \in \{ 2, 3, 4 \} \), which was obtained by Habiro [30].
Theorem 3.13 (Faes 2022). Any homology 3-sphere is $J_4$-equivalent to $S^3$.

Hence we now return to the family of $Y_k$-equivalence relations and, for this purpose, we review a few 3-manifold invariants whose nature is very different from the linking pairing or the cohomology ring. Recall that the set of spin structures on an $n$-manifold $V$ (with $n \geq 2$) is defined in terms of its bundle $FV$ of oriented frames $GL_n(\mathbb{R}; n) \rightarrow E(FV) \rightarrow V$ by

$$
\text{Spin}(V) := \{ \sigma \in H^1(E(FV); \mathbb{Z}_2) : |\text{fiber} \neq 0 \in H^1(GL_n(\mathbb{R}; n); \mathbb{Z}_2) \}.
$$

When it is non-empty (i.e. when the second Stiefel–Whitney class $w_2(V) \in H^2(V; \mathbb{Z}_2)$ vanishes), the set Spin($V$) is an affine space over $H^1(M; \mathbb{Z}_2)$, the action being given by $x \cdot \sigma = \sigma + p^*(x)$ for any $x \in H^1(M; \mathbb{Z}_2)$ and $\sigma \in \text{Spin}(M)$.

Any closed 3-manifold $M$ has a trivial tangent bundle and, so, it admits spin structures. Given $\sigma \in \text{Spin}(M)$, the Rochlin invariant of $(M, \sigma)$ is defined by

$$
R_M(\sigma) := \text{sgn}(W) \mod 16
$$

where $W$ is a compact 4-manifold bounded by $M$ to which $\sigma$ extends, and sgn($W$) denotes the signature of its intersection form on $H_2(W; \mathbb{Z})$. That $R_M(\sigma)$ is well-defined follows from the vanishing of $Q^\text{Spin}_3$ (a refinement of Theorem 1.10), the fact (due to Rochlin) that the signature of a spinable closed 4-manifold is divisible by 16, and the fact (due to Novikov) that the signature is additive under full-boundary gluing. (See [45] for these classical results on 4-dimensional topology.) Hence there is a map $R_M : \text{Spin}(M) \rightarrow Z_{16}$ attached to any closed 3-manifold $M$.

Besides, according to [51, 72], we can associate to any $\sigma \in \text{Spin}(M)$ a quadratic form over the linking pairing $\lambda_M$, which means a map $q_{M,\sigma} : \text{Tors}H_1(M; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$ satisfying

$$
\forall x, y \in \text{Tors}H_1(M; \mathbb{Z}), \quad q_{M,\sigma}(x + y) = q_{M,\sigma}(x) + q_{M,\sigma}(y) + \lambda_M(x, y).
$$

Hence there is also a map $q_M : \text{Spin}(M) \rightarrow \text{Quad}(\lambda_M)$ whose target is the set of quadratic forms over $\lambda_M$. (This is the refinement of the linking pairing that has been evoked in Remark 3.5.)

We can now state the characterization of $Y_2$ on $\mathcal{V}(\emptyset)$ given in [61].

Theorem 3.14 (Massuyeau 2003). Two manifolds $M, M' \in \mathcal{V}(\emptyset)$ are $Y_2$-equivalent if, and only if, there is an isomorphism $\psi : H_1(M; \mathbb{Z}) \rightarrow H_1(M'; \mathbb{Z})$ and a bijection $\Psi : \text{Spin}(M') \rightarrow \text{Spin}(M)$ satisfying the following:

1. $\lambda_M$ corresponds to $\lambda_{M'}$ through $\psi$, and $u^{(r)}_{M'}$ corresponds to $u^{(r)}_M$ through $\text{Hom}(\psi, Z_2)$ for any $r \in \mathbb{N}$;

2. $R_{M'}$ corresponds to $R_M$ through $\Psi$;

3. $\psi$ and $\Psi$ are compatible in the sense that $\Psi$ is affine over $\text{Hom}(\psi, Z_2)$ and we have the commutative diagram:

$$
\begin{array}{ccc}
\text{Spin}(M) & \xrightarrow{q_M} & \text{Quad}(\lambda_M) \\
\Psi \downarrow & \cong & \cong \Psi^* \\
\text{Spin}(M') & \xrightarrow{q_{M'}} & \text{Quad}(\lambda_{M'}). \\
\end{array}
$$

About the proof. Assume a Torelli twist $M \rightarrow M_s$ along a surface $S \subset M$ such that $M_s \cong M'$. This surgery induces an isomorphism $\psi_s$ in homology as we have seen at (2.3). Furthermore, as shown in [60], the surgery $M \rightarrow M_s$ induces a canonical bijection $\Psi_s : \text{Spin}(M_s) \rightarrow \text{Spin}(M)$, which is affine over

$$
\text{Hom}(\psi_s, Z_2) : \text{Hom}(H_1(M_s); \mathbb{Z}_2) \cong H^1(M_s; \mathbb{Z}_2) \rightarrow H^1(M; \mathbb{Z}_2) \cong \text{Hom}(H_1(M), \mathbb{Z}_2)
$$

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Specifically, it is the unique map that fits into the following commutative diagram:

\[
\begin{array}{ccc}
\text{Spin}(M) & \xrightarrow{\text{incl}^*} & \text{Spin}(M \setminus \text{int}(N)) \\
\downarrow \Psi_{3,1} & & \downarrow \Psi_{3,1} \\
\text{Spin}(M_s) & \xrightarrow{\text{incl}^*} & \text{Spin}(M \setminus \text{int}(S))
\end{array}
\]

We have seen in the proof of Theorem 3.4 that the linking pairing is preserved by the Torelli twist \( M \rightarrow M_s \), but this is not true anymore neither for the cohomology ring or for the Rochlin function. Nonetheless, we can explicitly compute how those two invariants change after a single \( Y \)-surgery, and thus observe that there is no variation if the \( Y \)-graph has a 0-framed null-homologous leaf: hence, using the operation \((\otimes_1)\) at page 19, we see that there is no variation by surgery along a connected graph clasper of degree 2. Using Proposition 2.20, we deduce that the isomorphism class of the triplet (linking pairing, cohomology rings, Rochlin function) is invariant under \( Y_2 \)-equivalence.

To prove the converse, we apply the “general strategy” by clasper calculus, which has been sketched on page 21 (with \( k := 1 \)). Thus, although Theorem 3.10 and Theorem 3.14 show similarities in their statements, their proofs are very different and logically independent. □

As an immediate consequence of Theorem 3.14, we obtain the following result for homology 3-spheres which appeared priorly in [30]. Note that an \( M \in S \) has a unique spin structure \( \sigma_0 \), and it turns out that \( R(M, \sigma_0) \) can only be 0 or 8 modulo 16: in this case, the Rochlin invariant of \( M \) refers to \( R(M, \sigma_0)/8 \in \mathbb{Z}_2 \).

**Corollary 3.15** (Habiro 2000). Two homology 3-spheres are \( Y_2 \)-equivalent if, and only if, they have the same Rochlin invariant.

The paper [30] also contains the characterization of \( Y_3 \) and \( Y_4 \). To state this, let us recall that the **Casson invariant**

\[ \lambda(M) \in \mathbb{Z} \]

of an \( M \in S \) is an integral lift of the Rochlin invariant \( R(M, \sigma_0)/8 \in \mathbb{Z}_2 \). In some sense, \( \lambda(M) \) is defined to count the number of conjugacy classes of irreducible representations of \( \pi_1(M) \) in the Lie group \( \text{SU}(2) \) using a Heegaard splitting of \( M \) [8]. Casson also provided a surgery formula for \( \lambda \) in terms of the Alexander polynomial of knots, which makes this invariant very computable: see, for instance, the textbook [93]. By means of this surgery formula, Morita could prove that \( \lambda \) behaves like a “quadratic” function on the Torelli group [73, 74], and Lescop generalized Morita’s result in a broader situation [54] (namely, Walker’s extension of the Casson invariant to rational homology 3-spheres). This quadraticity of \( \lambda \) is an expression of its property to be a finite-type invariant of degree 2 (see §3.5 below), and this is precisely the property of \( \lambda \) that is needed for the following result.

**Theorem 3.16** (Habiro 2000). Two homology 3-spheres are \( Y_3 \)-equivalent (resp., \( Y_4 \)-equivalent) if, and only if, they have the same Casson invariant.

The characterization of \( Y_3 \) (and, a fortiori, \( Y_4 \)) in the general case of closed 3-manifolds does not seem to appear in the literature. Neither is the characterization of \( J_3 \) (and, a fortiori, \( J_4 \)).

**Remark 3.17.** At this point of our discussion, it is important to focus on the nature of the results that we have presented so far for closed 3-manifolds. Each of them is concerned with a certain surgery equivalence relation \( \sim \) and states that

\[ \forall M, M' \in \mathbb{V}() \quad M \sim M' \iff I(M) \sim I(M') \]

where \( I : \mathbb{V}() \rightarrow A \) is a certain “package” of algebraic invariants with values in an appropriately-defined set where there is a notion of isomorphism \( \sim \). But such a characterization of \( \sim \) is not yet a classification result, since it continues with two other problems:
\(\diamond\) **Realization:** Does one know what is the image of \(I\) in \(A\)?

\(\diamond\) **Isomorphism:** Is the isomorphism problem solved in \(A\)?

So, let us reconsider the above characterizations of surgery equivalence relations under this new angle:

<table>
<thead>
<tr>
<th>Characterization</th>
<th>Torelli–equivalence</th>
<th>(J_2)-equivalence</th>
<th>(Y_2)-equivalence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Realization</td>
<td>Theorem 3.4</td>
<td>Theorem 3.10</td>
<td>Theorem 3.14</td>
</tr>
<tr>
<td>Isomorphism</td>
<td>solved [104]</td>
<td>solved [96, 102]</td>
<td>solved [102]</td>
</tr>
</tbody>
</table>

Wall showed that any non-singular bilinear pairing on a finite abelian group can be realized as the linking pairing of a closed 3-manifold [104]. He also gave a partial description (by generators and relations) of the abelian monoid of isomorphism classes of such pairings (where the operation is the direct sum \(\oplus\)). His work has been completed later on by Kawauchi & Kojima [44].

Sullivan proved in [96] that any trilinear alternate form on a finitely-generated free abelian group can be realized as the triple-cup product form of a closed 3-manifold: it is interesting to note that, in the middle of the 70’s and in order to prove this result, Sullivan was already using a surgery operation equivalent to the \(Y\)-surgery.

There exist several kinds of relations between the linking pairing, the triple-cup product forms and the Rochlin function. For instance, the triple-cup product forms \(\eta^r_M\) and \(\eta^s_M\) with coefficients in \(\mathbb{Z}_r\) and \(\mathbb{Z}_s\), respectively, are related in an obvious way if \(r\) divides \(s\). But there are also other, more delicate, relations: for instance, the third “discrete” differential of the Rochlin function \(R_M\) is given by \(\eta^2_M\). In fact, Turaev described in [102] all such possible relations, and he thus solved the realization problem for the triplet (linking pairing, cohomology rings, Rochlin function). However, since the isomorphism problem for trilinear skew-symmetric forms does not seem to be solved (even for coefficients in \(\mathbb{Q}\)), there is currently no procedure to decide (in general) whether two closed 3-manifolds are \(J_2\)-equivalent. Consequently, the same applies to the \(Y_2\)-equivalence relation.

\[\Box\]

### 3.4. Characterization of \(J_k\) and \(Y_k\) at low \(k\) for homology cylinders

We now consider the case of homology cylinders over a compact surface \(\Sigma\) (with one boundary component).

We start with some generalities about the structure added by the sequence of \(J_k\)-equivalence relations on the monoid \(\mathcal{IC}(\Sigma)\). For every \(k \in \mathbb{N}^*\), denote by \(Y_k\mathcal{IC}(\Sigma)\) the subset of homology cylinders that are \(Y_k\)-equivalent to the trivial cylinder \(U\). Hence, we get a decreasing sequence

\[\mathcal{IC}(\Sigma) = Y_1\mathcal{IC}(\Sigma) \supset Y_2\mathcal{IC}(\Sigma) \supset Y_3\mathcal{IC}(\Sigma) \supset \cdots\]

of submonoids, which is called the \(Y\)-filtration. Goussarov [28] and Habiro [30] proved that, for any integer \(k \geq 0\), the quotient monoid

\[\frac{\mathcal{IC}(\Sigma)}{Y_{k+1}}\]

is a group and, that, for any integers \(i, j \geq 1\), the inclusion

\[\left[\frac{Y_i\mathcal{IC}(\Sigma)}{Y_{k+1}}, \frac{Y_j\mathcal{IC}(\Sigma)}{Y_{k+1}}\right] \subset \frac{Y_{i+j}\mathcal{IC}(\Sigma)}{Y_{k+1}}\]

holds true in that group. In particular, \(Y_k\mathcal{IC}(\Sigma)/Y_{k+1}\) is an abelian group for all \(k \geq 1\), and the direct sum of abelian groups

\[\text{Gr} Y^{\mathcal{IC}(\Sigma)} := \bigoplus_{k \geq 1} \frac{Y_k\mathcal{IC}(\Sigma)}{Y_{k+1}}\]

has the structure of a graded Lie ring. The following is easily checked.
On the one hand, we fix an embedding \( j : \Sigma \to S^3 \) such that \( j(\Sigma) \) is a Heegaard surface of \( S^3 \) (deprived of a small open disk), and we identify \( N(j(\Sigma)) \) with \( \Sigma \times [-1, 1] \) via \( j \). Then the Casson invariant induces a map
\[
\lambda_j : IC(\Sigma) \to \mathbb{Z}, \quad M \mapsto \lambda((S^3 \setminus \text{int}(\Sigma \times [-1, 1]))) \cup M,
\]
which constitutes an invariant of homology cylinders. It depends on the choice of \( j \), of course, but this dependency can be managed as Morita did in the case of the Torelli group \([74]\). On the other hand, we can consider the homology \( H_1(M, \partial M; \mathbb{Z}[H]) \) of \( M \) relative to its “bottom”

**Proposition 3.18.** The “mapping cylinder” construction \( cyl : I(\Sigma) \to IC(\Sigma) \) induces a morphism of graded Lie rings \( Gr(cyl) : Gr^r IC(\Sigma) \to Gr^r IC(\Sigma) \).

Thus the “Lie algebra of homology cylinders” \( Gr^r IC(\Sigma) \) is highly related to the “Torelli Lie algebra” \( Gr^r I(\Sigma) \), which has been reviewed at (2.8). We refer to the works \([30, 26, 29, 9, 31, 67, 81, 82]\); see also the end of §3.5 in this connection.

In this subsection, we only deal with the low-degree parts of \( Gr^r IC(\Sigma) \). We start with the characterization of the \( Y_2 \)-equivalence, which needs two invariants of homology cylinders. The first invariant is the action of \( IC(\Sigma) \) on the second nilpotent quotient \( \pi/\Gamma_3 \pi \) of \( \pi = \pi_1(\Sigma, *) \). Indeed, as observed in \([26]\), the group homomorphism (2.5) can be extended (for any \( k \in \mathbb{N}^* \))

\[
\begin{array}{ccc}
I(\Sigma) & \xrightarrow{\rho_k} \text{Aut}(\pi/\Gamma_{k+1} \pi) \\
\downarrow \text{cyl} & \rotatebox{90}{\ldots} & \rho_k \\
IC(\Sigma) & \end{array}
\]

The second invariant of homology cylinders that we need is the Birman–Craggs homomorphism, which originates from constructions of Birman & Craggs \([7]\) on the Torelli group and was studied by Johnson \([39]\). In our setting, the most efficient way to define it is as follows:

\[
\beta : IC(\Sigma) \to \text{Map}(\text{Spin}(\Sigma), Z_2), \quad M \mapsto \frac{1}{8} R_M
\]

Here, we associate to any \( M \in IC(\Sigma) \) the closed 3-manifold
\[
\tilde{M} := M \cup_m (-\Sigma \times [-1, 1]),
\]
we identify \( \text{Spin}(\Sigma) \) to \( \text{Spin}(\tilde{M}) \) via the map \( m_k : \Sigma \to M \to \tilde{M} \), and we use the fact that the Rochlin function \( R_M \) takes values in \( \{0,8\} \) (because \( H_1(\tilde{M}; \mathbb{Z}) \) is torsion-free). The following is a generalization of Corollary 3.15 in genus \( g > 0 \).

**Theorem 3.19** (Habiro 2000, Massuyeau–Meilhan 2002). Two homology cylinders \( M, M' \) are \( Y_2 \)-equivalent if, and only if, \( \beta(M) = \beta(M') \) and \( \rho_2(M) = \rho_2(M') \).

**About the proof.** This characterization is announced in \([30]\) and proved in \([66]\). It preceded Theorem 3.14 and uses the same techniques for its proof. Note that the situation of homology cylinders is simpler than the situation of closed manifolds for two reasons: the first homology groups of homology cylinders are torsion-free (hence there is no linking pairing to deal with), and they come with a natural parametrization by an abelian group independent of the manifold (namely \( H_1(\Sigma; \mathbb{Z}) \)).

**Remark 3.20.** Actually, the results in \([66]\) give an explicit computation of the abelian group \( IC(\Sigma)/Y_2 \) and, thanks to Johnson’s computation of the abelianized Torelli group \([43]\), this implies that the degree 1 part

\[
Gr_1(cyl) : I(\Sigma)/[I(\Sigma), I(\Sigma)] \to IC(\Sigma)/Y_2
\]

of the “mapping cylinder” construction is an isomorphism.

To state now the characterization of the \( Y_3 \)-equivalence, we need still more invariants. On the one hand, we fix an embedding \( j : \Sigma \to S^3 \) such that \( j(\Sigma) \) is a Heegaard surface of \( S^3 \) (deprived of a small open disk), and we identify \( N(j(\Sigma)) \) with \( \Sigma \times [-1, 1] \) via \( j \). Then the Casson invariant induces a map

\[
\lambda_j : IC(\Sigma) \to \mathbb{Z}, \quad M \mapsto \lambda((S^3 \setminus \text{int}(\Sigma \times [-1, 1]))) \cup M,
\]

which constitutes an invariant of homology cylinders. It depends on the choice of \( j \), of course, but this dependency can be managed as Morita did in the case of the Torelli group \([74]\). On the other hand, we can consider the homology \( H_1(M, \partial M; \mathbb{Z}[H]) \) of \( M \) relative to its “bottom”
boundary” \( \partial \Sigma = m_*(\Sigma) \), with coefficients twisted by \( m_{\Sigma, *}: H_1(M; \mathbb{Z}) \to H := H_1(\Sigma; \mathbb{Z}) \); the order of this \( \mathbb{Z}[H] \)-module
\[
\Delta(M, \partial \Sigma) := \text{ord} \, H_1(M, \partial \Sigma; \mathbb{Z}[H]) \in \mathbb{Z}[H]
\]
is a relative version of the Alexander polynomial. With this definition, \( \Delta(M, \partial \Sigma) \) is only defined up to multiplication by a unit of the ring \( \mathbb{Z}[H] \), i.e. an element of \( \pm H \); however, by using Turaev’s refinement of the Reidemeister torsion, this indeterminacy can be fixed. Next, it is possible to “expand” \( \Delta(M, \partial \Sigma) \) as an element of (the degree completion of) the symmetric algebra \( S(H) \), and to keep only the degree 2 part of that expansion:
\[
\alpha(M) \in S^2(H).
\]
The following is a generalization of Theorem 3.16 in genus \( g > 0 \).

**Theorem 3.21** (Massuyeau–Meilhan 2013). Two homology cylinders \( M, M' \) are \( Y_3 \)-equivalent if, and only if, we have \( \lambda_j(M) = \lambda_j(M') \), \( \rho_3(M) = \rho_3(M') \) and \( \alpha(M) = \alpha(M') \).

**About the proof.** This theorem is proved in [67]. An important step in the proof consists in identifying the abelian group \( Y_3 IC(\Sigma)/Y_3 \) and, for this, the “general strategy” by clasper calculus (see page 21) is applied (with \( k := 2 \)). But, the difficulty is to assemble all three invariants that are expected to characterize the \( Y_3 \)-equivalence (namely \( \lambda_j, \rho_3, \alpha \)) into a single homomorphism \( Z_2 \) defined on \( Y_2 IC(\Sigma)/Y_3 \). This role of “unifying invariant” is played by the degree 2 part of the LMO homomorphism \( Z \) [9, 31], whose behaviour under \( Y_2 \)-surgery is well-understood. (See also the end of §3.5 in this connection.)

It is also explained in [67] how to deduce from Theorem 3.19 and Theorem 3.21 characterizations of the \( J_2 \)-equivalence and \( J_3 \)-equivalence, respectively. Specifically, \( J_2 \) is classified by \( \rho_2 \) and \( J_3 \) is classified by the couple \( (\rho_3, \alpha) \). In genus \( g = 0 \), Theorem 3.12 is thus recovered with a completely different proof than [87]. Besides, the same strategy of proof (i.e., use \( Y_k \) to understand \( J_k \)) is used in [18] for proving Theorem 3.13.

**Remark 3.22.** Nozaki, Sato and Suzuki [81] have determined the abelian group \( Y_3 IC(\Sigma)/Y_4 \). Their description too involves a “clasper surgery” map \( \psi_k \) of the type described on page 21 (with \( k := 3 \)), and their arguments involve some (reductions of) higher-degree parts of the LMO homomorphism \( Z \). It still remains to deduce from their result a characterization of the \( Y_4 \)-equivalence relation on the full monoid \( IC(\Sigma) \).

**Remark 3.23.** In contrast with the case of closed 3-manifolds, the above characterizations of \( Y_k \)-equivalence and \( J_k \)-equivalence relations for homology cylinders do not lead to “isomorphism problems” of the type mentioned in Remark 3.17.

### 3.5. Characterization in higher degrees

To conclude, we now survey what is known about the characterization in arbitrary high degrees of the three main families of relations that have been considered in these notes: namely the \( k \)-surgery equivalence, the \( J_k \)-equivalence and the \( Y_k \)-equivalence.

First of all, we consider the family of \( k \)-surgery equivalence relations on \( \nu(\emptyset) \). We start with an easy observation.

**Proposition 3.24.** Any homology 3-sphere \( M \) is \( k \)-surgery equivalent to \( S^3 \), for every \( k \geq 1 \).

**Proof.** By Corollary 3.11, there is a sequence
\[
S^3 = M_0 \to M_1 \to \cdots \to M_r = M
\]
where each move \( M_i \to M_{i+1} \) is a \( (\pm 1) \)-framed surgery along a knot \( K_i \) in a homology 3-sphere \( M_i \). Since \( \pi_1(M_i) \) has trivial abelianization, we have \( \pi_1(M_i) = \Gamma_k \pi_1(M_i) \) for all \( k \geq 1 \); hence the move \( M_i \to M_{i+1} \) can be viewed as a \( k \)-surgery for every \( k \geq 1 \).
Nevertheless, as was shown in [11], the family of $k$-surgery relations is very interesting for $3$-manifolds that are homologically non-trivial. Following Turaev [103], we define the $k$-th nilpotent (oriented) homotopy type of a closed $3$-manifold $M$ as
\[
\mu_k(M) := f_*([M]) \in H_3 \left( \frac{\pi_1(M)}{\Gamma_{k+1} \pi_1(M)} ; \mathbb{Z} \right)
\]
where $f : M \rightarrow K(\pi_1(M)/\Gamma_{k+1} \pi_1(M), 1)$ is a continuous map in an Eilenberg–MacLane space induc-\ ing the canonical homomorphism $\pi_1(M) \rightarrow \pi_1(M)/\Gamma_{k+1} \pi_1(M)$ at the level of $\pi_1$. (Of course, for $k := 1$, we recover what we called in (3.5) the “abelian homotopy type” of $M$.)

One can view $\mu_k(M)$ as an approximation of the (oriented) homotopy type of $M$ since, according to [101, 97], the latter is encoded by $\pi_1(M)$ and the image of the fundamental class $[M]$ in $H_3(\pi_1(M) ; \mathbb{Z})$. Then we have the following generalization of the equivalence (1)$\iff$(3) in Theorem 3.10.

**Theorem 3.25** (Cochran–Gerges–Orr 2001). Let $k \in \mathbb{N}^*$. Two closed $3$-manifolds $M$ and $M'$ are $(k + 1)$-surgery equivalent if, and only if, there is an isomorphism
\[
\psi : \pi_1(M)/\Gamma_{k+1} \pi_1(M) \longrightarrow \pi_1(M')/\Gamma_{k+1} \pi_1(M')
\]
mapping $\mu_k(M)$ to $\mu_k(M')$.

Although the realization problem for nilpotent homotopy types of $3$-manifolds has been (formally) solved in [103], it seems to be really difficult to classify the $k$-surgery equivalence relations, especially because the third homology groups of finitely-generated nilpotent groups do not seem to be well understood. Yet, Cochran, Gerges & Orr have been able to apply Theorem 3.25 in one particular case: using a good knowledge [36] of the third homology group of finitely-generated free-nilpotent groups, they prove that a closed $3$-manifold $M$ is $k$-surgery equivalent to $\#^m(S^3 \times S^2)$ if, and only if, we have $H_1(M) \simeq Z^m$ and all Massey products of $M$ of length $\leq 2k − 1$ vanish. (For $k := 2$, this is an instance of the equivalence “(1)$\iff$(2)” in Theorem 3.10.)

Here is another consequence of Theorem 3.25, which does not seem to have been observed before.

**Corollary 3.26.** Let $M, M' \in \mathcal{V}(\mathcal{Q})$ and let $k \geq 2$ be an integer. If $M$ and $M'$ are $J_{2k−2}$-equivalent, then they are $k$-equivalent.

**Proof.** Let $j \in \mathbb{N}^*$ and assume a Torelli twist $M \rightarrow M_s$ along a surface $S \subset M$ with an $s \in J/I(S)$. The Seifert–Van Kampen theorem shows the existence of a unique isomorphism
\[
\psi_s : \pi_1(M)/\Gamma_{j+1} \pi_1(M) \xrightarrow{\sim} \pi_1(M_s)/\Gamma_{j+1} \pi_1(M_s)
\]
that fits into the commutative diagram:
\[
\begin{array}{ccc}
\pi_1(M) & \xrightarrow{\psi_s} & \pi_1(M_s) \\
\| & \| & \| \\
\Gamma_{j+1} \pi_1(M) & \xrightarrow{\sim} & \Gamma_{j+1} \pi_1(M_s) \\
\| & \| & \| \\
\Gamma_{j+1} \pi_1(M) & \xrightarrow{\sim} & \Gamma_{j+1} \pi_1(M_s) \\
\end{array}
\]
In order to compare $\mu_j(M)$ and $\mu_j(M_s)$ via $\psi_s$, we consider the mapping torus of $s$ which, with the notation (3.6), can be defined as
\[
\text{tor}(s) := \text{cyl}(s)
\]
where $\text{cyl}(s) \in J\mathcal{C}(S)$ denotes the mapping cylinder of $s$. This is a closed $3$-manifold whose $j$-th nilpotent fundamental group can be identified to that of $S$ by the isomorphism
\[
\varphi_s : \pi_1(S)/\Gamma_{j+1} \pi_1(S) \xrightarrow{\sim} \pi_1(\text{tor}(s))/\Gamma_{j+1} \pi_1(\text{tor}(s))
\]
that is induced by the inclusion $S = S \times 1 \rightarrow \text{tor}(s)$. Besides, the inclusion $S \hookrightarrow M$ induces a homomorphism

$$\iota : \pi_1(S)/\Gamma_{j+1}\pi_1(S) \xrightarrow{\sim} \pi_1(M)/\Gamma_{j+1}\pi_1(M).$$

Then, a simple homological computation in a singular 3-manifold that contains the three of $M$, $M_s$ and $\text{tor}(s)$ shows that

$$\psi^{-1}_x(\mu_j(M)) = \mu_j(M) + \iota_* \psi^{-1}_x(\mu_i(\text{tor}(s))).$$

This variation formula for the $j$-th nilpotent homotopy type is established in the introduction of [65], generalizing [26, Theorem 2] and [33, Theorem 5.2].

The same formula shows that, given a compact surface $\Sigma$ with $\partial \Sigma \cong S^1$, the following map is a group homomorphism:

$$M_j : j_2\Sigma(\Sigma) \rightarrow H_3\left(\pi_1(\Sigma)_{j+1}\pi_1(\Sigma); \mathbb{Z}\right), f \mapsto \mu_f(\text{tor}(f)).$$

This is essentially the $j$-th Morita homomorphism, introduced in [75] as a refinement of the “$j$-th Johnson homomorphism”. As shown by Heap in [33], the kernel of $M_j$ is $\mathbb{J}_2/\mathbb{I}(\Sigma)$. Therefore, if $M'$ is the result of a Torelli twist $M \rightarrow M_s$ with an $s \in \mathbb{J}_{2(k-1)}(\Sigma)$, we have $\mu_{k-1}(\text{tor}(s)) = 0$. So, we conclude thanks to (3.7) that $M$ and $M'$ are k-surgery equivalent.

**Remark 3.27.** It would be interesting to have a direct proof of Corollary 3.26, which would apply to $\nu(R)$ for any compact surface $R$. Indeed, surgery along a connected graph clasper of degree $2k - 2$ can always be realized as a sequence of three $k$-surgeries (see [66, Fig. 3] for $k = 2$); therefore, by Proposition 2.20, the $Y_{2k-2}$-equivalence is stronger than the $k$-surgery equivalence [30]. Given that “$Y_{2k-2} \Rightarrow J_{2k-2}$”, it is likely that Corollary 3.26 is true in $\nu(R)$ for any $R$.

The following question now arises for the family of $J_k$-equivalence relations: can we expect a result analogous to Theorem 3.25? This seems to be currently out of reach, as revealed already by the case of homology 3-spheres. Indeed, the methods for proving the triviality of the $J_3$-equivalence (resp., $J_4$-equivalence) in [87] (resp., in [18]) seem to be hard to adapt to arbitrary high degrees.

**Remark 3.28.** So, in view of Proposition 3.24, we can hardly imagine a kind of converse to Corollary 3.26.

In contrast with the $J_k$-equivalence, we know (at least, theoretically) how to characterize the $Y_k$-equivalence relation in any degree $k \geq 1$ by means of a certain family of topological invariants of 3-manifolds. In the sequel, we fix a compact surface $R$ and a $Y_1$-equivalence class $\nu_0$ in $\nu(R)$.

**Definition 3.29.** Let $A$ be an abelian group. A map $F : \nu_0 \rightarrow A$ is a **finite-type invariant** of degree at most $d$ if, for any $M \in \nu_0$, for any pairwise-disjoint compact surfaces $S_0, S_1, \ldots, S_d \subset \text{int}(M)$ with $\partial S_i \cong S^1$, and for all $s_0 \in \mathcal{I}(S_0), s_1 \in \mathcal{I}(S_1), \ldots, s_d \in \mathcal{I}(S_d)$, we have

$$\sum_{j \in \{0, 1, \ldots, d\}} (-1)^{|j|} \cdot F(M_j) = 0 \in A$$

where $M_j$ is obtained from $M$ by twist along $\bigcup_{j \in |j|} S_j$ with $\bigcup_{j \in S_j}$.

**Remark 3.30.** The notion of “finite-type invariants” for homology 3-spheres has been introduced by Ohtsuki in [84], as an analogue of the notion of “Vassiliev invariants” for knots and links in $S^3$. This notion has been extended and studied by Cochran & Melvin [12], who considered arbitrary 3-manifolds. In this Ohtsuki–Cochran–Melvin theory, the basic operation is the 2-surgery instead of the Torelli twist.

The rich interplay between the theory of finite-type invariants and the study of mapping class groups was firstly considered by Garoufalidis & Levine [20, 23, 24, 25]. Next, came the “clasper calculus” of Goussarov and Habiro [28, 30], which offered very efficient tools to study and enumerate finite-type invariants. Their works also revealed that the Torelli twist
Assume that

\[ I := \ker(\varepsilon : Z[G] \to Z) \]

where the augmentation \( \varepsilon \) is the ring homomorphism mapping any \( g \in G \) to \( 1 \in Z \). The \( I \)-adic filtration of \( Z[G] \) is the sequence \( Z[G] = I^0 \supset I = I^1 \supset I^2 \supset \cdots \) defined by the powers of \( I \). The following classical fact relates this to the lower central series (2.4) of \( G \).

**Lemma 3.31.** Let \( k \in \mathbb{N}^* \). For any \( g \in \Gamma_k G \), we have \( (g - 1) \in I^k \).

**Proof.** The statement is obviously true for \( k = 1 \). Next, for any \( k \in \mathbb{N}^* \), an element of \( \Gamma_{k+1} G \) is (by definition) a product of commutators of the form \([x, y]\) or \([y, x]\) where \( x \in G \) and \( y \in \Gamma_k G \). Besides, we have the following identities in \( Z[G] \), for any \( g, h \in G \):

\[
gh - 1 = ((g - 1) - (h^{-1} - 1)) \cdot h
\]

\[
[g, h] - 1 = ((g - 1)(h - 1) - (h - 1)(g - 1))g^{-1}h^{-1}.
\]

Hence the statement is justified by an induction on \( k \geq 1 \). \( \square \)

We can now prove the following.

**Proposition 3.32.** Let \( M, M' \in V_0 \) and let \( d \in \mathbb{N} \). If \( M \) and \( M' \) are \( Y_{d+1} \)-equivalent, then \( F(M) = F(M') \) for any finite-type invariant \( F : V_0 \to A \) of degree at most \( d \).

**Proof.** Assume that \( M \to M' \cong M' \) by a Torelli twist along \( S \subset \text{int}(M) \) with \( s \in \Gamma_{d+1} I(S) \). Consider the map \( f : \tilde{I}(S) \to A \) defined by \( f(u) := F(M_u) \) and extend it by additivity to \( f : \tilde{Z}[\tilde{I}(S)] \to A \).

The fact that \( F \) is of finite type of degree at most \( d \) implies that \( f \) vanishes on all elements of the form \((s_0 - 1)(s_1 - 1)\cdots(s_d - 1)\) with \( s_0, s_1, \ldots, s_d \in \tilde{I}(S) \). Since those elements generate \( I^{d+1} \) additively, we have \( f(I^{d+1}) = 0 \). We conclude using the fact that \((s - 1) \in I^{d+1} \) by Lemma 3.31. \( \square \)

If Proposition 3.32 had a converse, then we would get (at least, theoretically) a characterization of the \( Y_k \)-equivalence relation. Indeed, the converse is true for the class \( V_0 := S \).

**Theorem 3.33** (Habiro 2000). Any two homology 3-spheres are \( Y_{d+1} \)-equivalent if, and only if, they are not distinguished by finite-type invariants of degrees at most \( d \).

Thus, Corollary 3.15 and Theorem 3.16 are proved by identifying all (the few) finite-type invariants of homology 3-spheres of degrees 1, 2 and 3.

**About the proof of Theorem 3.33.** The theorem is announced in [30] and it is proved there in the analogous case of knots in \( S^3 \). See [62] for a proof, which involves clasper calculus. \( \square \)

Let \( \Sigma \) be a compact surface with one boundary component, and consider now the class \( V_0 := \tilde{Z}C(\Sigma) \) of homology cylinders over \( \Sigma \). Except in the case \( \Sigma = D^2 \), it is not known whether the converse to Proposition 3.32 holds true for \( \tilde{Z}C(\Sigma) \).

**Goussarov–Habiro Conjecture (GHC).** Let \( d \in \mathbb{N}^* \). Any two homology cylinders over \( \Sigma \) are \( Y_{d+1} \)-equivalent if, and only if, they are not distinguished by finite-type invariants of degree at most \( d \).
Currently, the GHC is only known to be true up to degree \( d = 4 \), the most recent result in this direction being obtained in [82]. By comparing Lemma 3.31 to Proposition 3.32, we see that the GHC is an analogue of the following problem in group theory, which can be stated for any group \( G \).

**Dimension Subgroup Problem (DSP).** Let \( k \in \mathbb{N}^* \). Determine the gap between \( \Gamma_k G \) and \((1 + t^k) \cap G\) in \( \mathbb{Z}[G] \).

It had been conjectured during a long time that the inclusion \( \Gamma_k G \subseteq (1 + t^k) \cap G \) should be an equality, until Rips found the first counter-example for \( k = 4 \) and a finite 2-group \( G \) [89].

In fact, the DSP can be generalized replacing the lower central series of \( G \) by any series \( G = N_1 G \supset N_2 G \supset N_3 G \supset \cdots \) of subgroups which is strongly central (i.e. \( [N_i G, N_i G] \subseteq N_{i+1} G \) for all \( i, j \in \mathbb{N}^* \)), and by replacing the \( I \)-adic filtration by an appropriate filtration of \( \mathbb{Z}[G] \). Furthermore, some versions of the DSP can be formulated in the group algebra \( F[G] \) for any commutative field \( F \), rather than in the group ring \( \mathbb{Z}[G] \), and these versions of the problem have an explicit solution whose nature depends on the characteristic of \( F \). (See, for instance, the monograph [86].)

It is observed in [63] that some results of Goussarov [28] and Habiro [30] about the \( Y \)-filtration on \( \mathcal{IC}(\Sigma) \) can be interpreted as follows: the GHC in degree \( d \) is an instance of the DSP for the group \( G := \mathcal{IC}(\Sigma)/Y_{d+1} \). Thus, analogues of the GHC for finite-type invariants with values in commutative fields are obtained in [63], and the following weak version of the GHC is then derived:

**Theorem 3.34** (Massuyeau 2007). Let \( d \in \mathbb{N}^* \). There exists an integer \( D \), depending on \( d \) and the topological type of \( \Sigma \), with the following property: if two homology cylinders are not distinguished by finite-type invariants of degree at most \( D \), then they are \( Y_{d+1} \)-equivalent.

We mention the following corollary: two homology cylinders are not distinguished by finite-type invariants if, and only if, they are \( Y_k \)-equivalent for any integer \( k \geq 1 \). Actually, it is conjectured that finite-type invariants classify homology cylinders (and, in particular, homology 3-spheres).

We conclude with two questions which naturally arise from our discussion on Theorem 3.33 and its expected generalization, namely the GHC.

- **Does one know well enough all finite-type invariants of a given degree \( d \)?** For homology 3-spheres, one can construct infinite series of finite-type invariants following Ohtsuki’s original idea [83], by appropriate expansions of quantum invariants. Furthermore, there is a very powerful invariant of homology 3-spheres: the LMO invariant [53], which is known to be universal among \( \mathcal{Q} \)-valued finite-type invariants [52] and to dominate large families of quantum invariants [47]. For homology cylinders too, there is a universal \( \mathcal{Q} \)-valued finite-type invariant: the LMO homomorphism defined on the monoid \( \mathcal{IC}(\Sigma) \), which allows for an explicit diagrammatic description of the Lie algebra \( \mathfrak{Gr}^f \mathcal{IC}(\Sigma) \) with rational coefficients [9, 31]. (See [32] for a survey.) But computing those universal invariants is a challenge in high degrees (despite their combinatorial construction) and, moreover, it is not known whether they dominate all finite-type invariants (including those with values in torsion abelian groups). Nevertheless, recent works of Nozaki, Sato & Suzuki provide encouraging perspectives [81, 82].

- **Can we hope an analogue of Theorem 3.33 for arbitrary closed 3-manifolds?** The answer is trivially “yes” in degree 0, but it is certainly “no” in higher degrees: for instance, \( 4^3(S^1 \times S^2) \) and \( (S^1 \times S^1 \times S^1) \# (S^1 \times S^2) \) are not \( Y_2 \)-equivalent (because their cohomology rings are not isomorphic), although they are not distinguished by finite-type invariants of degree at most one [63, Ex. 3.4]. Yet, this negative answer is not necessarily disappointing. It rather suggests that the notion of finite-type invariant (as given in Definition 3.29) is not appropriate for homologically non-trivial 3-manifolds:
the notion probably needs to be refined, by adding a kind of homological structures to 3-manifolds, like a (complex) spin structure or a parametrization of its first homology group.

References


Course n° I— Surgery equivalence relations for 3-manifolds


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