WBLN

Winter Braids Lecture Notes

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Braids in Contact 3–manifolds

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Contents

1. Introduction

This is a lecture note for the lectures given at the 2017 Winterbraids conference in Caen. The aim of the lectures was to introduce 3–dimensional contact geometry to the audience. This subject is strongly related to braids, and one of its main theorems (the Bennequin's theorem, which I show in the first lecture) uses braids as its main tool. Contact geometry, although originated in the works of Sophus Lie and Huygens, has been living its renaissance since the early 2000s, when Giroux introduced topological tools to study contact structures. The aim of the last two lectures was to see these tools in action, through examples and proofs. In these notes, though, I only explain the first two lectures, as the third was based on a paper, that has already appeared since then [EV17].

There are several excellent notes and books on contact structures [Etn03, Etn06, Gei06, OS04] that give a more complete introduction to some of the subjects in these notes. However the topics of braid foliations and open book foliations mostly appear in papers, so I spent more time explaining some of the details, and I believe that these notes are a good way to start learning about these subjects.

Acknowledgments

I would like to thank the organisers of the conference for the invitation, and the opportunity to talk about my research area to a different audience. It was eye-opening to see which subjects were interesting to the participants, and I was very happy to get all the feedbacks on my talk. I would also like to thank John Etnyre for helping me prepare the lectures, and Paolo Ghiggini for looking at my notes, and giving invaluable feedback.

2. Contact Structures

To put everything in perspective, we start this section with the general definition of contact structures, but we will soon specialise to the 3–dimensional case, where there are explicit models one can work with. So, the reader unexperienced in differential geometry should not get discouraged by this section; everything will get very concrete staring from the next subsection. For a complete introduction to contact structures see [Gei06, OS04] or [Etn03].

A *contact structure* on an odd dimensional manifold M^{2n+1} is a totally non integrable hyperplane field ξ. This means, that there is no greater than n dimensional hypersurface (not even locally) whose tangents lie in ξ. Locally any such hyperplane field can be given as the kernel of a 1-form α (i.e. $\xi = \ker \alpha$), and the non integrability condition then translates to

$$\alpha \wedge d\alpha^n > 0,$$

where $d\alpha^n$ is the wedge-product of $d\alpha$ with itself n times. In the above case α is called a *contact form* for ξ. If α can be given globally, then the contact structure is *cooriented*, and then we require the $(2n+1)$-form $\alpha \wedge d\alpha^n$ to be a positive multiple of the volume form of M.

Example 2.1. The standard example for contact structures is given by $\mathbb{R}^{2n+1}$ with coordinates

$$(z, x_1, y_1, \ldots, x_n, y_n)$$

and contact form

$$\alpha_{\mathrm{st}} = dz - \sum_{i=1}^{n} y_i dx_i.$$

See Figure 2.1 for the $2n + 1 = 3$ case.

By Darboux's Theorem [Gei06, Theorem 2.5.1] the standard contact structure gives a local model for all contact structures, in the sense that we can always choose local coordinates so that the contact structure is the standard $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ in those coordinates.

The maximal dimension of an embedded manifold that can be tangent to the contact structure is n, and we call an embedded n–dimensional submanifold L *Legendrian* if it is tangent to ξ at all points (i.e. $T_pL \subset \xi_p$). For example in the standard contact structure all the n–planes with fixed z and y_i-coordinates are Legendrian.

In 3–dimensions closed connected Legendrian submanifolds are called *Legendrian knots*, and they are $\mathcal{C}^0$-dense:

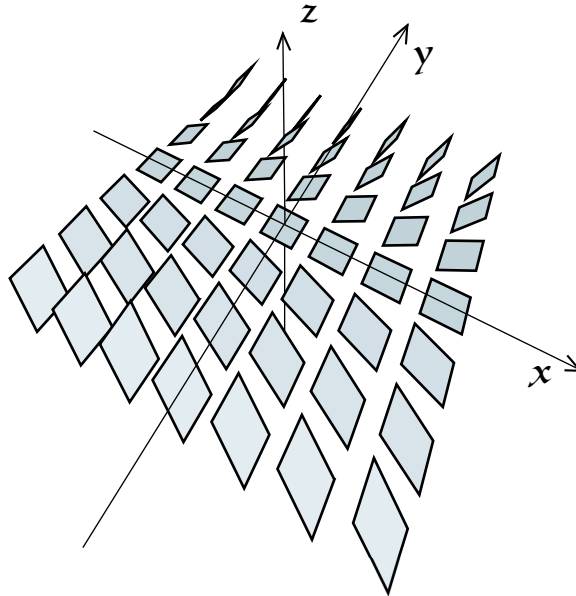**Proposition 2.2.** *Any knot K can be $\mathcal{C}^0$-approximated by a Legendrian knot L.*

Figure 2.1: The standard contact structure $\xi_{st} = \ker(dz - ydx)$ on $\mathbb{R}^3$. (Figure by Stephan Schönenberger).

Sketch of Proof. It is enough to prove this statement locally, thus we can work with an arc A embedded in the standard contact structure $(\mathbb{R}^3, \xi_{st})$. An arc L is Legendrian in the standard contact structure if $T_pL \in \xi_{st}$ or in other words if $\alpha_{st}(T_pL) = 0$. For the coordinates (x, y, z) of L this means $dz - ydx = 0$ or $y = \frac{dz}{dx}$ on L. Thus the y–coordinate of a Legendrian arc L can be recovered from the projection of L to the (x, z)-plane as the slope of the projection.

The y-coordinate of A does not have this property, but as in Figure 2.2 we can approximate its (x, z)-projection with an arc L' whose slope is "close" to the y-coordinate of A. Then the Legendrian lift of L' gives a good approximation for A. $\qquad\square$
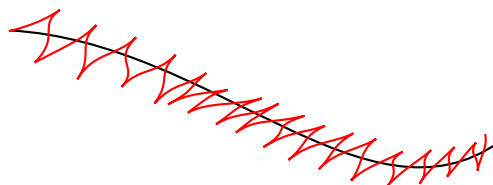

Figure 2.2: Approximation by a Legendrian arc in the standard contact structure as seen in the projection to the (x, z)-plane.

Example 2.3. Using the above Proposition we can now give a real life appearance of contact structures. Consider a "skate" or the front wheel of a car moving on the plane $\mathbb{R}^2$. Its configuration space can be described by its position, $(x, y) \in \mathbb{R}^2$ and the angle $\vartheta \in S^1$ of the front wheel. If the car does not slide, then all motions of its front wheel satisfies the equation $\tan \vartheta = \frac{dy}{dx}$. In other words, if we take the contact structure $\xi = \ker(\cos \vartheta \, dy + \sin \vartheta \, dx)$ on $\mathbb{R}^2 \times S^1$, then (non-sliding) motions of the front wheel are in

one-to-one correspondance with Legendrian paths in $(\mathbb{R}^2 \times S^1, \xi)$. Now Proposition 2.2 translates to the fact that "you can always parallel park your car" to a space that is bigger than your car: simply take any path in the plane, that brings the front wheel to the right position, and then approximate it with a Legendrian arc. The approximating arc describes how to park the car.

Other applications of contact geometry can be found in partial differential equations, Riemannian geometry, optics and Thermodynamics.

3. Bennequin bound

On $\mathbb{R}^3$ in addition to the standard contact structure $\xi_{\mathrm{st}} = \ker(dz - ydx)$ there are several contact structures one can define:

Example 3.1. The symmetric contact structure ξ_{sym} in $\mathbb{R}^3$, given by the contact form

$$\alpha_{\mathrm{sym}} = dz - ydx + xdy,$$

or in polar coordinates (z, r, ϑ):

$$\alpha_{\mathrm{sym}} = dz - r^2 d\vartheta.$$

This contact structure agrees with ξ_{st} on the y-axis, but it is rotational symmetric.

Example 3.2. The *overtwisted contact structure* ξ_{OT} on $\mathbb{R}^3$ is given as the kernel of the 1-form:

$$\cos r\, dz + r \sin r\, d\vartheta.$$

It is called overtwisted because it has an embedded disc

$$D = \{(z, r, \vartheta) : z = 0, r \leq \pi, \vartheta \in S^1\},$$

which is tangent to ξ_{OT} at the boundary.

The dichotomy of overtwisted and *tight* (non overtwisted) contact structures has been discovered by Eliashberg, noticing, that overtwisted contact structures can be classified using only homological data. Tight contact structures are generally hard to classify.

In the following we will investigate if the other contact structures ξ_{st} and ξ_{sym} have such an embedded disc, or more generally which of the above three contact structures are "different". We define two contact structures (M_0, ξ_0) and (M_1, ξ_1) to be *contactomorphic* if there is a diffeomorphism $\psi : M_0 \to M_1$ that brings ξ_0 to ξ_1 i.e. $\psi_* \xi_0 = \xi_1$. Note that the contact structures ξ_{st} and ξ_{sym} are contactomorphic through the diffeomorphism

$$\psi : (x, y, x) \mapsto (x, \frac{y}{2}, z + \frac{xy}{2}).$$

Thus the remaining question is whether ξ_{sym} and ξ_{OT} are contactomorphic to each other. Bennequin [Ben83] gave a very clever negative answer to the above question using braids. In the remaining of this section we will build up our language to state and prove his theorem.

3.1. Transverse knots

In dimension 3, we distinguish another type of knots that respects the contact structure, called *transverse knots*. The tangent of these knots are positively transverse to the contact structures (i.e. $T_p K \pitchfork \xi_p$, and $T_p K$ coorients ξ_p, or simply $\alpha_p(T_p K) > 0$). These knots are closely related to Legendrian knots, in the sense, that transverse knots are classified by the set of Legendrian knots that are $\mathcal{C}^0$-close to them [EFM01]. More importantly to our discussion, braids around the z-axis are naturally transverse in ξ_{sym} (just isotop the braid far from the axis, and notice that as $r \to \infty$ then the notions "having positive ϑ-derivative" and "being positively transverse to ξ_{sym}" agree). See Subsection 4.3 for a more complete discussion on the relation of braids and transverse knots.

3.1.1. Self linking number

The *self linking number* is an invariant of homologically trivial transverse knots. We first pick a Seifert surface Σ for K. As $H^2(\Sigma) = 0$ the planefield $\xi|_\Sigma$ is trivialisable, so we can choose a vector field v over Σ that gives a section of the bundle $\xi|_\Sigma \to \Sigma$ (i.e. $v_p \in \xi_p$ for all $p \in \Sigma$). Let $K' = K + \varepsilon v$ be a small push off of K in the direction given by v. Then the *self linking number* of K in ξ with respect to Σ is defined as:

$$sl(K; \Sigma) = lk(K, K')$$

Clearly the above definition does not depend on the section v of $\xi|_\Sigma$. Moreover if $H_2(M) = 0$, then it is also independent on Σ, and in this case we write $sl(K)$ for $sl(K; \Sigma)$. If $H^2(M) = 0$, then we can choose the section v globally, not only over Σ.

There is another useful way to compute the self linking number as an obstruction of extending a framing of K given by $T\Sigma \cap \xi$ over Σ. More precisely let w be a framing of K given as vectors in the lines $T_p\Sigma \cap \xi_p$ pointing outward of Σ. Then the self linking number can be computed as

(3.1) $$sl(K; \Sigma) = -\langle e(\xi|_\Sigma; w), [\Sigma]\rangle.$$

Now the statement of Bennequin's Theorem is the following:

Theorem 3.3 (Bennequin). *If K is a transverse knot in $(\mathbb{R}^3, \xi_{sym})$ and Σ a Seifert surface, then*

$$sl(K) \leq -\mathcal{X}(\Sigma).$$

3.1.2. Self linking number in $(\mathbb{R}^3; \xi_{\mathrm{sym}})$.

If K is given as the closure of a braid B around the z-axis then the self linking number can be computed in terms of the combinatorial data encoded in B. To see this first notice that $v = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ gives a section of ξ_{sym}. Then the linking of K and $K' = K + \varepsilon v$ can be computed as half the signed number of intersections of the projections of K and K' for example to the xy-plane. As one can see in Figure 3.1 the projections intersect each other negatively on the far left and far right for every strand of B, and twice around every crossing of B with sign agreeing of that of the crossing. Thus
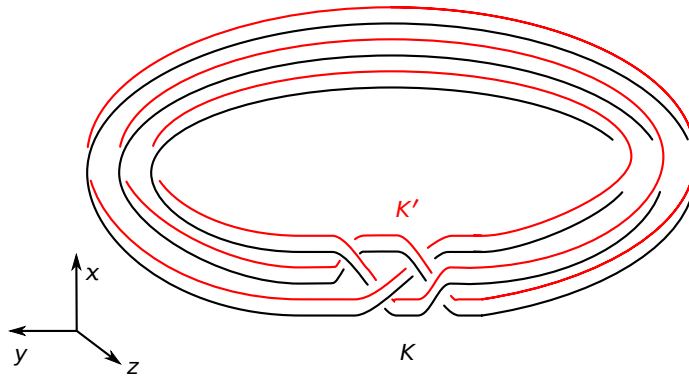

Figure 3.1: Computing the self linking number of a braid in ξ_{sym} in the projection to the xy-plane.

$$sl(K) = lk(K, K') = a(B) - n(B),$$

where $n(B)$ is the number of strands of B, $a(B)$ is the *algebraic length* of B (i.e. for a braid $B = \prod \sigma_{n_i}^{\varepsilon_i}$ we have $a(B) = \Sigma \varepsilon_i$).

Note that for positive braids (i.e. if $\varepsilon_i = +1$ for all i) on one hand one can construct a Seifert surface Σ for B just by gluing $|B|$ (positively) twisted bands to $n(B)$ discs (See Figure 3.2), thus $\mathcal{X}(\Sigma) = n(B) - |B|$. On the other hand $a(B) = |B|$. So we conclude that for positive braids we have the equality: $sl(B) = -\mathcal{X}(\Sigma)$.
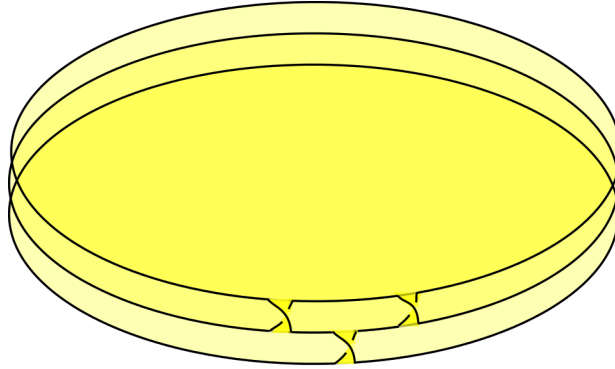

Figure 3.2: A Seifert surface with $\chi(\Sigma) = n(B) - |B|$ for a positive braid.

3.2. Characteristic foliations.

Equation (3.1) gives another way to compute the self linking number of a transverse knot using the characteristic foliation. Let Σ be any surface for K, then w can be extended to Σ as follows. The *characteristic foliation* $\mathcal{F}_\xi(\Sigma) = \mathcal{F}_\xi$ of a surface Σ is an oriented singular foliation given as the integral curves of the oriented line field $T_p\Sigma \cap \xi_p$. Note that since Σ is oriented and ξ_p is co-oriented the intersection $T_p\Sigma \cap \xi_p$ has a natural orientation (whenever $T_p\Sigma \neq \xi_p$) given by a vector $w_p \in T_p\Sigma \cap \xi_p$ with the property that w_p and a vector $v_p \in T_p\Sigma$ with $\alpha_p(v_p) > 0$ gives the positive orientation of $T_p\Sigma$.

Characteristic foliations in contact structures can be easily recognised. Remember that the divergence of a vector field w is defined by the equation $\mathcal{L}_X\omega = div_\omega w \cdot \omega$. Then

Lemma 3.4. [Gir00] *Let ω be an area form for Σ, then the vector field w over Σ defines a characteristic foliation on $\Sigma \hookrightarrow (M, \xi)$ for some (M, ξ) if and only if $div_\omega w \neq 0$ at the zeros of w.* $\qquad\square$

The above lemma for example, implies that characteristic foliations cannot have centers (depicted on the left side of Figure 3.3). Isolated singularities of a generic characteristic foliation are elliptic (see the first two figure of Figure 3.3) or hyperbolic (see the third picture of Figure 3.3). At an isolated singular point the sign of $div_\omega w$ is positive (resp. negative) if the orientation of ξ_p and $T_p\Sigma$ agree (resp. disagree). This means, that positive elliptic points are sources, and negative elliptic points are sinks. See Figure 3.3. The signs of hyperbolic points cannot be directly read from a picture. Let $e_\pm$ denote the number of positive (resp. negative) elliptic points, and similarly $h_\pm$ denote the number of hyperbolic points with the given signs. Now we have $(e_+ + e_-)$ many elliptic points and $(h_+ + h_-)$ many hyperbolic points, thus:

Theorem 3.5 (Poincaré–Hopf)**.** *With the notations above*

$$\mathcal{X}(\Sigma) = (e_+ + e_-) - (h_+ + h_-). \quad \square$$

Remember that the above Theorem follows from the fact that the Euler characteristic can be computed as the signed count of zeros of a generic section of TM that agrees with TK along K:

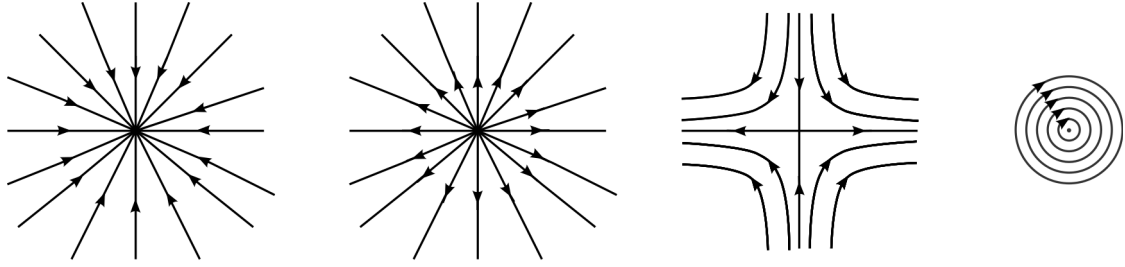$$\mathcal{X}(\Sigma) = \langle e(T\Sigma, TK), [\Sigma] \rangle,$$

Figure 3.3: Possible singularities of a generic singular foliation. The first two are elliptic points (sink and source), the third is a hyperbolic or saddle point and the fourth is a center

and since the sections TK and w of $T\Sigma$ along K give the same framing, we have that the above equals to

$$\mathcal{X}(\Sigma) = \langle e(T\Sigma, w|_K), [\Sigma]\rangle.$$

The singular vector field w then gives such a generic section of $T\Sigma$. Note that w also gives a section of $\xi|_\Sigma$, and clearly has the same zeros as in the previous computation. The signs of these zeros, however, are different: at positive singularities the orientation of $T_p\Sigma$ and ξ_p agree, thus the signs agree, and similarly, at negative singularities the signs disagree. Thus

(3.2) $$sl(K) = -\langle e(\xi|_\Sigma, w|_K), [\Sigma]\rangle = (e_- - h_-) - (e_+ - h_+).$$

By Theorem 3.5 and Equation (3.2) Bennequin's inequality can be written as:

$$(e_- - h_-) - (e_+ - h_+) = sl(K) \leq -\mathcal{X}(\Sigma) = -(e_+ - h_+) - (e_- - h_-)$$

which is equivalent to

(3.3) $$e_- \leq h_-.$$

Thus ideally we would like to prove that for any transverse knot we can always find a Seifert surface whose characteristic foliation has no negative elliptic points (i.e. $e_- = 0$). We won't be able to prove this statement, but we will be able to reduce e_- as long as $sl(K) > -\mathcal{X}(\Sigma)$. To achieve this goal we need to introduce a new foliation $\mathcal{F}_b$ on some embedding of Σ, which on the one hand is topologically conjugate to $\mathcal{F}_\xi$ (i.e. Σ has a homeomorphism that brings the two foliations into each other); thus can be used to compute the self linking number, and on the other hand is more rigid, so that we have a good understanding on how changes of the embedding $\Sigma \hookrightarrow \mathbb{R}^3$ affect the foliation $\mathcal{F}_b$.

3.3. Braid foliations

Consider the S^1-family of half planes given (in cylindrical coordinates) by

$$H_\vartheta = \{(z, r, \vartheta) : (z, r) \in \mathbb{R} \times \mathbb{R}_{\geq 0}\} \subset \mathbb{R}^3$$

for a fixed $\vartheta \in S^1$. These half planes intersect each other in the z-axis, and otherwise cover every point of the space exactly once. See Figure 3.4. The intersection of a surface Σ with the half planes $\{H_\vartheta\}_{\vartheta \in S^1}$ gives a singular foliation $\mathcal{F}_b(\Sigma) = \mathcal{F}_b$ on Σ. Similarly to characteristic foliations the foliation $\mathcal{F}_b$ inherits an orientation from the coorientation of H_ϑ given by $\frac{\partial}{\partial \vartheta}$. Also notice that the coorientation $\frac{\partial}{\partial \vartheta}$ descends as a coorientation of the leaves of $\mathcal{F}_b$. Generically $\mathcal{F}_b$ can be put in a nice position:

-Transversely pointing outward at $\partial\Sigma$: Since K is in braid position, the orientation convention for the leaves tells us, that $\mathcal{F}_b$ is transvere to K and points out the surface.
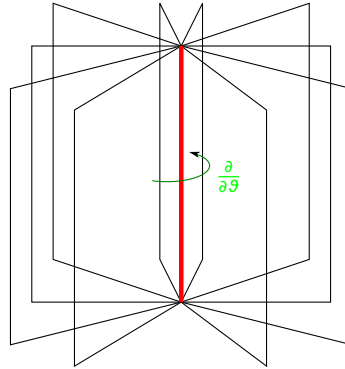
Figure 3.4: The planes H_ϑ and the z-axis.

-Elliptic points: By a $\mathcal{C}^\infty$-small isotopy we can assume that Σ is transverse to the z-axis, thus $\mathcal{F}_b$ has only finitely many intersections with the z-axis, each of which is an elliptic point. Positive intersections give sources and negative ones are sinks. The coorientation is counterclockwise along positive elliptic points, and clockwise along negative ones. See Figures 3.5 and 3.6.

-Morse-like singularities: Away from the z-axis we have a well defined map $\vartheta \colon \mathbb{R}^3 \setminus \{z\text{-axis}\} \to S^1$, thus by a $\mathcal{C}^\infty$-small isotopy we can arrange that $\vartheta|_\Sigma \colon \Sigma \to S^1$ is a circle-valued Morse function with singularities of distinct critical values. The local minima and maxima of $\vartheta|_\Sigma$ correspond to centers of the foliation $\mathcal{F}_b$, while the index 1 saddle points give hyperbolic singularities. The sign of such critical points are determined by whether the orientation of H_ϑ and $T\Sigma$ agree or disagree. Thus minima are positive, maxima are negative, while the sign of saddle points are determined by the coorientation of the leaves around them. See Figure 3.5 and 3.6.

-Leaves are arcs or circles: By the previous point at a regular value of $\vartheta|_\Sigma$ the level set $H_\vartheta \cap \Sigma$ is compact, thus it consists of circles and properly embedded arcs.

-No saddle -saddle connection: Since the critical values are all distinct, every half plane, H_ϑ, contains at most one critical point, thus the arcs cannot connect saddles. They can run between a saddle and an elliptic point, or a point in $\partial\Sigma$, and between two elliptic points of different signs, or between positive elliptic points and a point of $\partial\Sigma$.

The above properties can all be achieved by a $\mathcal{C}^\infty$-small isotopy, and in the following we will always assume them when we talk about braid foliations. We show, that by possibly changing Σ we can also achieve that there are no circle leaves.

Proposition 3.6. *Any Seifert surface Σ of K with minimal genus can be isotoped so that its braid foliation has no circle leaves.*

Sketch of Proof. Suppose that $\mathcal{F}_b$ has a circle leaf. Since $\partial\Sigma = K$ is a braid (in particular Σ is not a torus or sphere, and the foliation has at least one elliptic point), then following one of the transverse directions $\pm\frac{\partial}{\partial\vartheta}$ from a point on the circle we get to a hyperbolic point p with a separatrix that forms a loop. See Figure 3.7. Let $D \subset H_\vartheta$ be the disc bounded by this loop. If the interior of D is disjoint from Σ, then as it is shown on Figure 3.7 we can surger Σ along D to get a new Seifert surface Σ' for K union some closed surface. After isotopy the braid foliation on Σ' has less loop separatrices than the braid foliation on Σ, thus the above process will eventually end.
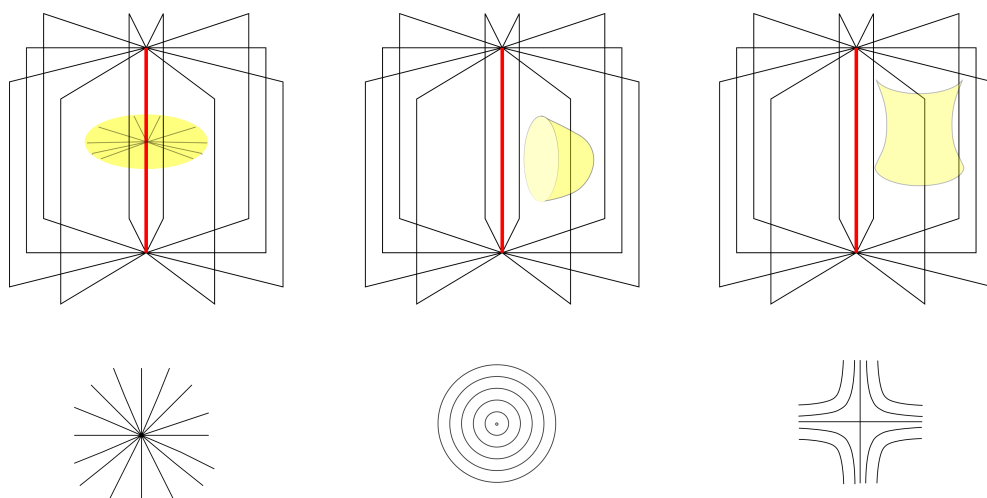
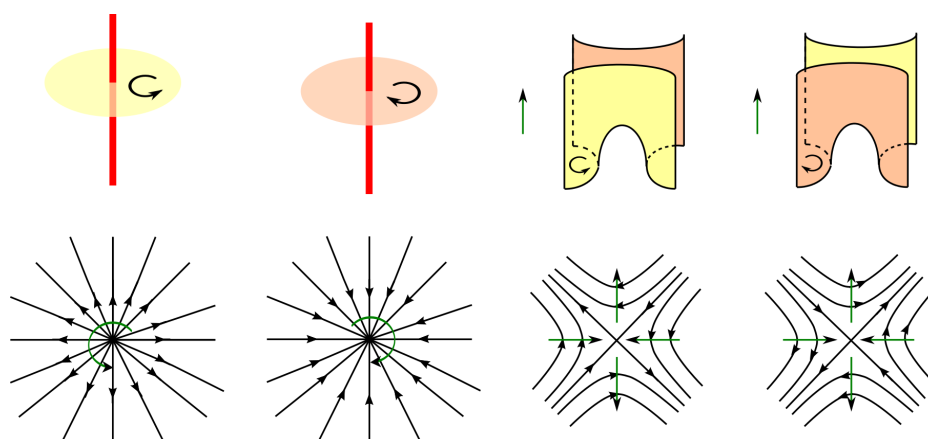Figure 3.5: Singularities of $\mathcal{F}_b$ as embedded in $\mathbb{R}^3$.

Figure 3.6: Sign of the singularities of a braid foliations. In order: a positive and a negative elliptic point, and a positive and a negative saddle point. The green arrows indicate the $\frac{\partial}{\partial \vartheta}$ direction, while the curly arrows indicate the orientation of Σ.

Since the only singularity on H_ϑ is p, the interior of D intersects Σ only in circles. We choose an innermost circle, and following $\pm \frac{\partial}{\partial \vartheta}$ we end up with a hyperbolic point p' and a loop separatrix bounding a disc D' on $H_{\vartheta'}$. Since D' is isotopic to a disc (a subset of D) disjoint from Σ and an annulus part of Σ (foliated by circles of $\mathcal{F}_b$) we can isotop Σ to be disjoint from D', and proceed as above.

Note that, if Σ has minimal genus, each of the above steps separates off spheres, that bound balls in $\mathbb{R}^3$, and thus Σ' is isotopic to Σ. So any minimal genus Seifert surface can be isotoped rel. K so that the braid foliation does not contain a circle leaf. □

As a consequence of Proposition 3.6 we can remove center singularities:

Corollary 3.7. *K has a Seifert surface Σ with braid foliation that has no center singularities.* □
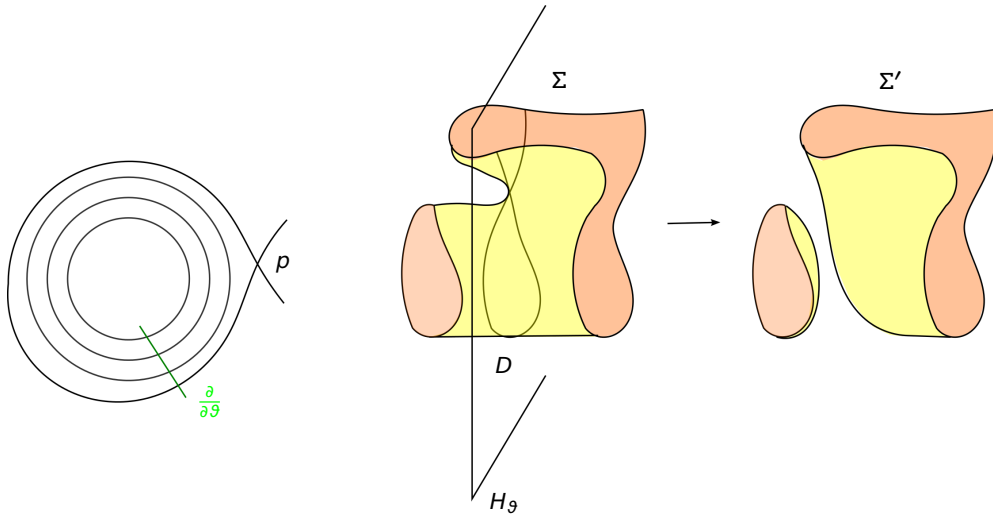
Figure 3.7: Left: separatrix of the hyperbolic point p forming a loop. Right: surgery along D.

Since characteristic foliations do not have center singularites, this was an essential step to prove conjugacy of $\mathcal{F}_\xi$ and $\mathcal{F}_b$.

Proposition 3.8. *Suppose that Σ has a braid foliation with no center singularities. Then there is an ambient isotopy of Σ, so that $\mathcal{F}_\xi$ and $\mathcal{F}_b$ are topologically conjugate.*

Sketch of Proof. First we push up Σ a bit near the intersections with the z-axis, so that Σ is horizontal at the elliptic points. This makes sure that, $\mathcal{F}_b$ and $\mathcal{F}_\xi$ have the same elliptic points with the same signs.

Then we isotop Σ by deforming the r-direction with a smoothing of the function that has slope 1 at $[0, \varepsilon] \cup [2\varepsilon, \infty)$ and has slope C at $[\varepsilon, 2\varepsilon]$. Where ϵ is sufficiently small, so that $\mathcal{F}_b$ has no singularity in the $\{\varepsilon < r < 2\varepsilon\}$ area, and C is so large that H_ϑ and ξ are close enough. This will ensure, that the hyperbolic singularities of $\mathcal{F}_b$ and $\mathcal{F}_\xi$ are close to each other.

Since $\mathcal{F}_b$ has no circle leaves, the surface Σ can be obtained as the union of the neighborhoods of elliptic points, neighborhoods of separatrices and regularly foliated discs. The statement then follows from the fact that for each of these discs D, the foliations $\mathcal{F}_b$ and $\mathcal{F}_\xi$ are topologically conjugate rel. ∂D. This is obviously true, for the neighborhoods of the elliptic points and the regularly foliated discs. As for the neighborhood of the separatrices assume, without loss of generality, that D is the neighborhood of (the two) stable separatrices at a hyperbolic point p of $\mathcal{F}_b$. See Figure 3.8. Then $\mathcal{F}_b$ points outward along ∂D, thus
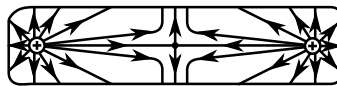

Figure 3.8: Neighborhood of a stable separatrix. Note that instead of the elliptic points, the separatrix could run into $\partial \Sigma$ on one or both sides.

∂D is positively braided around the z-axis. Thus it is a transverse (un)knot. This implies that $\mathcal{F}_\xi$ also points outward along ∂D, thus since $\mathcal{F}_b|_D$ and $\mathcal{F}_\xi|_D$ have the same singularites they are indeed conjugate rel. ∂D. $\qquad\square$

Proposition 3.9. *Let Σ be a Seifert surface for the transverse knot K. Then either $sl(K) \leq -\mathcal{X}(\Sigma)$ or K is transverse isotopic to a transverse knot that has a Seifert surface with braid foliation with no negative elliptic points.*

Sketch of Proof. Let s be a negative elliptic point. The *star D_s* of s is defined as the closure of the union of the leaves flowing into s. Since s is a negative elliptic point, the leaves ending at s can start at positive elliptic points or hyperbolic points. Let us first consider the set of hyperbolic points $\{h_1, \ldots, h_n\}$ that are connected to s. Each of them has ingoing separatrices, that also belong to D_s, and connect h_i with two positive elliptic points that are also connected with s. Thus we can (re)number the hyperbolic and positive elliptic points connected to s to form a circle $h_1, e_1, h_2, \ldots, h_n, e_n$ with ingoing separatrices connecting e_i and e_{i+1} to h_i (here the indices are cyclical). See Figure 3.9. Note that since each hyperbolic point
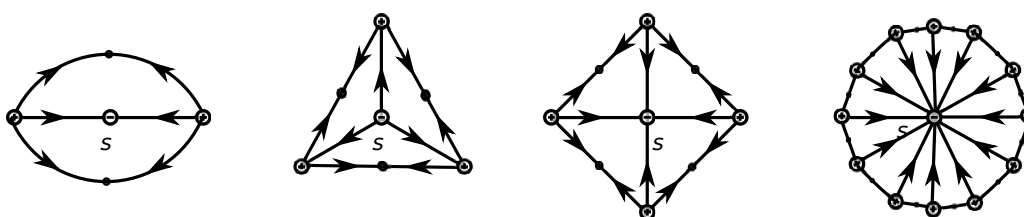

Figure 3.9: The star of a negative elliptic point, with $n = 2, 3, 4$ and 12.

is connected to s, they must have different ϑ-coordinate, thus this circle is embedded. In fact the circle (together with the stable separatrices) forms the boundary of D_s. Note that n must be greater than 1, as for $n = 1$, h_1 would be connected with two ingoing separatrices to e_1. This is impossible, as all incoming leaves of e_1 have different ϑ-coordinate, while the separatrices of h_1 all happen in the same time.

When $n = 2$, then a small neighbourhood N of D_s is depicted in Figure 3.10, and the foliation determines (up to isotopy) a unique embedding of N into $\mathbb{R}^3$, shown also on Figure 3.10. (Note also, that the fact that D_s is coming from a braid foliation forces one of the hyperbolic points to be positive and the other one to be negative.) Let's consider the curve U of Figure 3.10 separating off a negative and a positive elliptic point
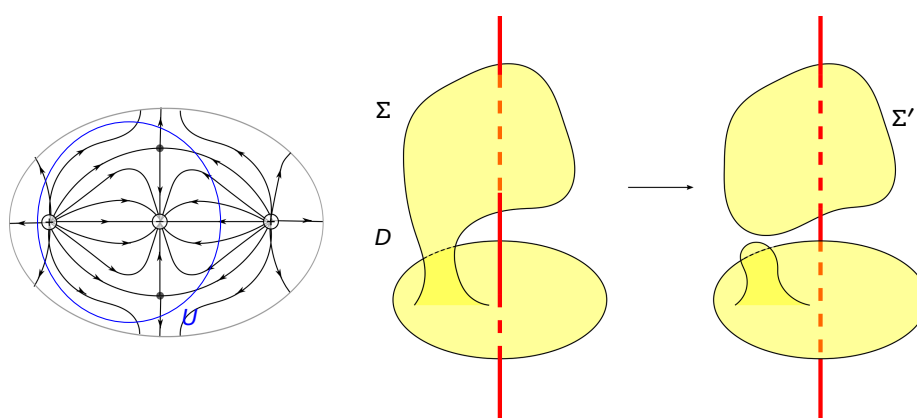

Figure 3.10: Embedding N into $\mathbb{R}^3$. The right picture shows the embedding after the surgery along D.

and the two hyperbolic points from ∂N. Then U bounds a disc D in $\mathbb{R}^3$ that does not intersect the z-axis. If

D is disjoint from Σ, then we can perform surgery on Σ along D to obtain a new Seifert surface Σ' that has one less negative elliptic point.

If D intersects Σ, then it intersects it in arcs (with endpoints on $K \cap D$) and circles. We first remove the intersection of K with D (which then automatically removes the arcs) by *exchange moves* as on Figure 3.11. (See [BF98] for a precise description of the process.) This move induces an isotopy of K and consequently of Σ. As one can see on the right hand side of Figure 3.11, this move does not change $a(B)$ and $n(B)$ for
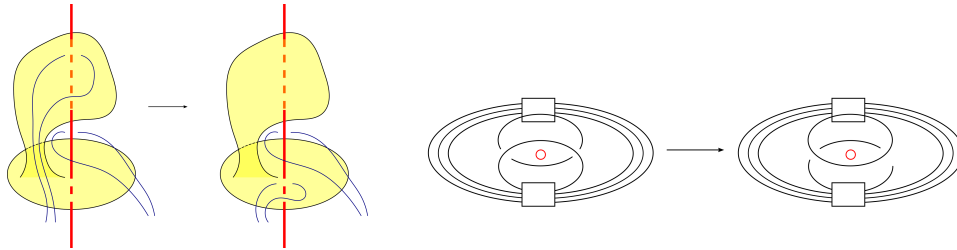

Figure 3.11: Left: an exchange move. The blue strands indicate (possibly more than 1) strand(s) of K. Right: the effect of an exchange move on the braid presentation.

a braid. In particular an exchange move produces a transverse knot K' with the same self linking number. This observation would be enough to prove Bennequin's Theorem, but we will use the fact that an exchange move does not change the transverse isotopy type of K and although during the isotopy the foliation might change, $\mathcal{F}_b$ at the beginning and at the end are conjugate.

To remove circle intersections of Σ and D, take an innermost one c, bounding a disc $D' \subset D$ disjoint from Σ, and do surgery on Σ along D'. We then obtain a new Seifert surface Σ' (containing D_s) and a closed surface, which we will remove. (Again, if we assumed that Σ had minimal genus, then Σ' would be isotopic to Σ.) By repeating this process we can remove all circle intersections.

Now suppose that $n > 2$ for all negative elliptic points s. If $sl(K) > -\mathcal{X}(\Sigma)$, then by Equation (3.3) we also have $e_- > h_-$, thus there is at least one negative elliptic point s, so that D_s contains more positive than negative hyperbolics points on its boundary. Then there must be two consecutive positive hyperbolic points, say h_1 and h_2 in D_s. Take a small neighborhood D of a transverse arc connecting h_1 and h_2 and intersecting leaves coming from e_1 only. See Figure 3.12. Then the foliation of D defines (up to isotopy) a unique embedding of D into $\mathbb{R}^2$. By an isotopy of this embedding, shown on the bottom of Figure 3.12, we can modify the foliation of D as on the top right of Figure 3.12. This modification is essentially exchanging the order of critical points for the circle-valued Morse function ϑ. See [BF98] for a precise statement. This change in the foliation reduces the number of hyperbolic points in D_s, thus by induction we can find a negative elliptic point s with $n = 2$, and proceed as above. So eventually we can isotop Σ to reduce the number of negative elliptic points, and then again by induction we can achieve that the foliation has no negative elliptic points, as required. $\qquad \square$

Proof of Bennequin's Theorem. Take a transverse braid representation of K with a Seifert surface Σ, then either $sl(K) \leq -\mathcal{X}(\Sigma)$, or by Proposition 3.9 we can isotop Σ to have a braid foliation with no negative elliptic points. Then by Proposition 3.8 $\mathcal{F}_b$ and $\mathcal{F}_\xi$ are conjugate, thus they have the same number of critical points from each type. In particular $\mathcal{F}_\xi$ has no negative elliptic points. Remember that by the discussion after Equation (3.2) this is equivalent to Bennequin's inequality. $\qquad \square$

Theorem 3.10. ξ_{sym} *is not contactomorphic to* ξ_{OT}.

Proof. For this, one needs to notice, that the unknot $U = \{z = 0, r = \pi + \varepsilon\}$ in ξ_{OT} violates Bennequin's inequality. The characteristic foliation $\mathcal{F}_{\xi_{OT}}$ on the disc $D = \{r \leq \pi + \varepsilon\}$ is depicted on the right hand side
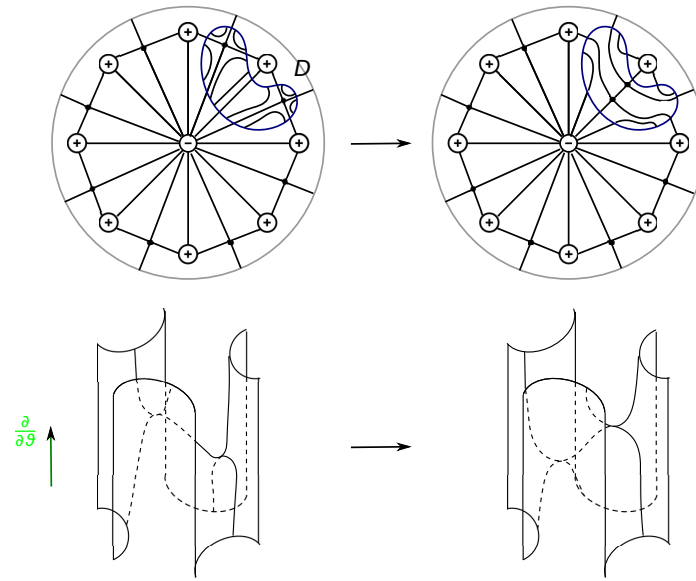
Figure 3.12: top left: the neighborhood of D_s with two saddle points of the same sign enclosed in the blue circle. top right: $\mathcal{F}_b$ after exchanging the order of the saddle points. bottom: the embedded picture for exchanging the order of the saddle points.

of Figure 3.13, and if we slightly push up the middle of D then it modifies to the left hand side of 3.13. The
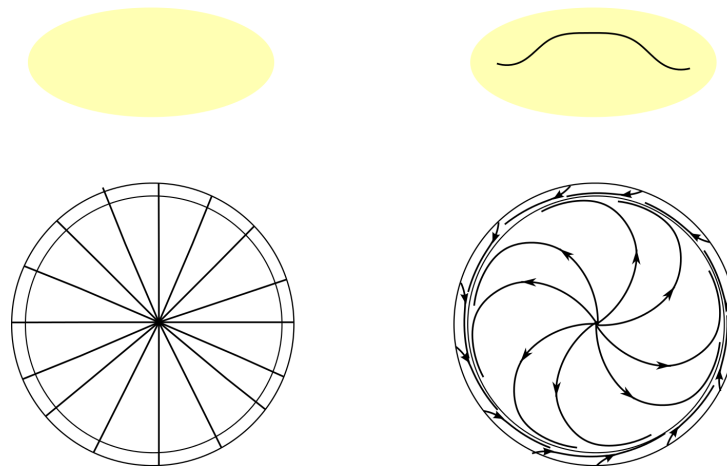

Figure 3.13: Characteristic foliation on the disc D, and on a slight push up of D.

characteristic foliation $\mathcal{F}_{\xi_{\mathrm{OT}}}$ on this pushed up disc has one positive elliptic point and no other singularities, thus

$$sl(U) = -e_- = 1 > -\mathcal{X}(D) = -1,$$

Vera Vértesi

as needed. □

4. Open books and Generalisation of braids

4.1. Open books

The key in the proof of Bennequin's Theorem was that there was a foliation of $\mathbb{R}^3 \setminus \{z - \text{axis}\}$ given by the half-planes H_ϑ which was "close" to ξ_{sym}. By compactifying $\mathbb{R}^3$ to S^3 the z-axis turns into a circle, and planes become discs D_ϑ. In the following we will generalise this concept to any closed oriented 3–manifold M, and we will also relate it to contact structures ξ on M.

Definition 4.1. An *open book* (L, π) of a closed oriented 3–manifold M consists of a link $L \hookrightarrow M$ and a fibration $\pi \colon M \setminus L \to S^1$ such that for each $\vartheta \in S^1$ the closure S_ϑ of the surface $\mathring{S}_\vartheta = \pi^{-1}(\vartheta)$ is a Seifert surface for L. The link L is called the *binding*, while the surfaces S_ϑ are the *pages* of the open book.

Note that the naming "open book" comes from the structure of the fibration near the bindings. See Figure 3.4. For a more complete treatment of open books see [OS04, Etn06]. The first example of an open book comes from the compactification of S^3 as described above. In this case the binding is the unknot coming from the compactification of the z-axis and the pages are the discs D_ϑ.

Example 4.2. Let $H_+ = U \cup U'$ be the positive Hopf link of Figure 4.1. Then $S^3 - H_+$ is a thickened torus
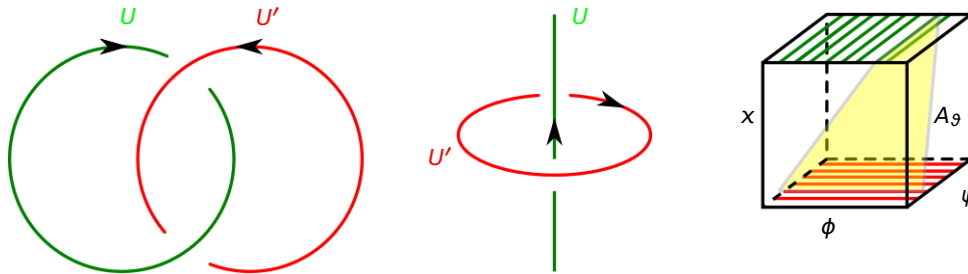

Figure 4.1: left and middle: positive Hopf link H_+, right:, the complement of $H_+ \cong T^2 \times I$, the back and front, and the right and left of the cube are glued together.

$T^2 \times I$. Let us introduce coordinates $(\psi, \phi) \times x$ on $T^2 \times I$, and identify it with $S^3 - L$ so that $(\psi, \phi_0)_{\psi \in S^1} \times \{1\}$ parametrises U (for any fixed $\phi_0 \in S^1$), while $(\psi_0, \phi)_{\phi \in S^1} \times \{0\}$ parametrises U' (for any fixed $\psi_0 \in S^1$). Consider the fibration $\pi \colon T^2 \times I \to S^1$ given by $\pi = \psi + \phi - x$. Then (H_+, π) gives an open book decomposition for S^3 with annuli pages $A_\vartheta = \{\psi + \phi - x = \vartheta\}$.

The same example can be described if we consider S^3 as the unit sphere $\{|z_1|^2 + |z_2|^2 = 1\}$ in $\mathbb{C}^2$. Then the positive Hopf link $H_+ = \{z_1 z_2 = 0\}$, and the fibration $\pi \colon S^3 \setminus H_+ \to S^1$ is given by the equation $\pi = \frac{z_1 z_2}{|z_1 z_2|}$. One can generalise this example for other polynomials:

Example 4.3. Let $p(z_1, z_2) \colon \mathbb{C}^2 \to \mathbb{C}$ be a polinomial with a unique zero at $(0, 0)$. Then for a sphere S_ε^3 with sufficiently small radius ε the pair $(K_p = p^{-1}(0) \cap S_\varepsilon^3, \pi = \frac{p(z_1, z_2)}{|p(z_1, z_2)|})$ gives an open book decomposition for S_ε^3.

Theorem 4.4 (Alexander)**.** *Any closed oriented 3–manifold has an open book decomposition.*

Over the years several proofs were given to this statement, in this note we will outline one using branched covers.

IV–14

Sketch of Proof. We use the fact [Ale20], that every 3–manifold can be obtained as a 3–fold branched cover over a link K in S^3. This means that there is a map $\rho\colon M \to S^3$ that is 3 to 1 in the complement of the link K. Put K in braid position, and "pull back" the trivial open book (U, π_U) of S^3. In other words we consider the open book $(L = \rho^{-1}(U), \pi = \pi_u \circ \rho)$. This is indeed an open book for M where L is a triple cover of the unknot, and the pages are branched covers of D_ϑ. □

An open book decomposition (L, π) for M can be described using the fact that π restricted to the complement of a page S_0 gives a fibration over $I = S^1 \setminus \{0\}$ which is diffeomorphic to $S \times I$, and thus the important information is carried in the way the two ends are glued together.

Definition 4.5. An *abstract open book* is the pair (S, h), where S is a surface with nonempty boundary, and $h\colon S \to S$ is a diffeomorphism of S that is the identity near ∂S. Here, again, S is the *page* of the abstract open book, and h is its *monodromy*.

We have already seen that open book decompositions give rise to abstract open books. For the converse for any abstract open book (S, h) we construct a 3–manifold M with an open book decomposition corresponding to (S, h). Consider the mapping torus of h:

$$M_h = S \times I/(x, 0) \sim (h(x), 1)$$

and identify the points on its boundary that come from the same points of ∂S:

$$M = \frac{S \times I}{(x, 0) \sim (h(x), 1) \text{ for } x \in S \text{ and } (x, t) \sim (x, t') \text{ for } x \in \partial S \text{ and } t, t' \in I}.$$

Then the equivalence class of ∂S gives the binding L and π is given by the projection map to the I-coordinate. With this equivalence open book decompositions up to isotopies of M are in one to one correspondence with abstract open books up to conjugation of h and composing h with an isotopy of S.

Abstract open books give a very efficient ways of describing open books, as one just needs to give a surface with boundary, and then the isotopy class of the monodromy can be described as compositions of *Dehn twists* [Lic62]:

Definition 4.6. Let γ be an embedded closed curve in S then the *Dehn twist*, $D_\gamma\colon S \to S$ along γ is a diffeomorphism $D_\gamma\colon S \to S$ that is the identity in the complement of a neighborhood $N(\gamma)$ of γ, and it is given by $(\psi, t) \mapsto (\psi + f(t), t)$ in $N(\gamma) \cong S^1 \times I$, where $f(t)$ is a smooth function that is 0 near 0 and 1, monotonely increasing up to 2π in $[0, \frac{1}{2}]$ and monotonely decreasing in $[\frac{1}{2}, 1]$. See Figure 4.2.
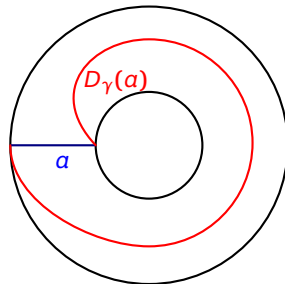

Figure 4.2: Image of an arc $a = I \times \{0\}$ under the Dehn twist D_γ.

Example 4.7. The trivial open book decomposition of S^3 gives the abstract open book (D^2, id).

Example 4.8. The open book decomposition corresponding to the Hopf fibration, of Example 4.2, gives (A, D_γ), where A is an annulus, and D_γ is the right handed Dehn twist along the core of A. See Figure 4.3.
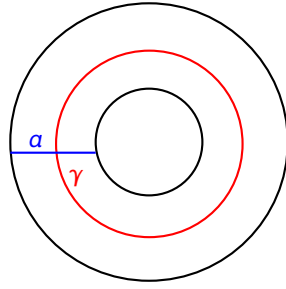
Figure 4.3: Open book corresponding to the Hopf fibration.

Example 4.9. The negative Hopf fibration H_- with binding $U \cup \overleftarrow{U'}$ (here $\overleftarrow{U'}$ is U' with opposite orientation) and annular pages gives the abstract open book is (A, D_γ^{-1}).

4.2. Open books and Contact structures

As it turns out, open books are also an efficient tool in describing contact structures:

Definition 4.10. [Gir02] An open book (L, π) *supports* a contact structure ξ if ξ it has a contact form α (i.e. $\xi = \ker \alpha$) such that $d\alpha$ gives an area form on the pages S_ϑ, and the binding, L is transverse. In this case we say that the contact structure is *compatible* with ξ.

We have already seen that the trivial open book (U, π_U) supports ξ_{sym}.

Example 4.11. (H_+, π) also supports ξ_{sym}.

Thurston-Winkelnkemper proved the existence of compatible contact structures before the concept was generally defined:

Theorem 4.12. [TW75] *Every open book supports some contact structure.*

Proof. Let (S, h) be an abstract open book, and remember that we constructed M by collapsing the circles corresponding to points of ∂S on the boundary of the mapping torus M_h. Take a neighborhood $N(\partial S)$ of ∂S so that h is the identity restricted to $N(\partial S)$. Let $S' = S \setminus N(\partial S) \subset S$, then M is the union of some solid tori

$$N = N(\partial S) \times S^1 / (x, t) \sim (x, t'),$$

and the mapping torus of $h|_{S'}$:

$$M' = M_{h|_{S'}} = S' \times I / (x, 0) \sim (h(x), 1).$$

Our strategy is to construct matching contact structures on both separately, and then glue them together.

To construct ξ on M' first take a 1-form η on S' such that $d\eta$ is an area form on S' and in the local coordinates (r, ψ) on $N(\partial S') \cong I \times \partial S'$ (with $\partial S' = \{r = 1\}$) the 1-form can be written as $\eta = r d\psi$. See Figure 4.4. Then on $S' \times I$ take the 1-form

$$\vartheta \eta + (1 - \vartheta) h^* \eta,$$

where ϑ parametrises I in the mapping torus. After smoothing this 1-form glues to a 1-form α' on M' such that $d\alpha'$ is an area form on each $S'_\vartheta = S' \times \{\vartheta\}$, and for K sufficiently large the 1-form

$$\alpha_K = \alpha' + K d\vartheta$$

is a contact form.

The construction extends to the solid tori $N' = \partial S \times D^2$ (with coordinates $\psi \times (\vartheta, R)$) as

$$d\psi + f(R)d\vartheta,$$

where $f : I \to \mathbb{R}$ is R^2 near 0 and K near 1. See Figure 4.4. This indeed gives a contact form on N with
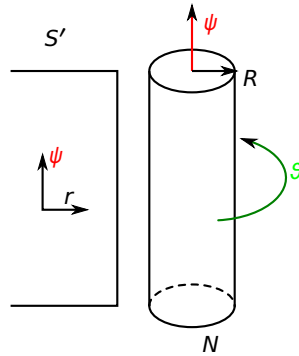

Figure 4.4: Coordinates near the binding of an open book.

transverse binding. □

Moreover we have:

Theorem 4.13. [Gir02] *Contact structures supported by the same open book are isotopic.*

Which gives a map from the set of isotopy classes of open books to the set of isotopy classes of contact structures:

$$\frac{\{\text{open books of } M\}}{\text{isotopy}} \to \frac{\{\text{contact structures on } M\}}{\text{isotopy}}$$

Giroux proved, that the above map is surjective. Moreover he gave a description for when two open books give isotopic contact structures.

Definition 4.14. Let (S, h) be an abstract open book, and let c be a properly embedded arc on the surface S. Add a 1–handle to S along the endpoints of c to obtain, after smoothing a new surface S'. The arc c union the core of the 1–handle gives a closed curve γ in S'. Also h can be extended to S' to $\tilde{h}$ by defining it to be the identity on the 1–handle. The open book (S', h'), where $h' = \tilde{h} \circ D_\gamma$ is the *stabilisation* of (S, h) along γ. Similarly (S, h) is the *destabilisation* of (S', h'). See Figure 4.5.
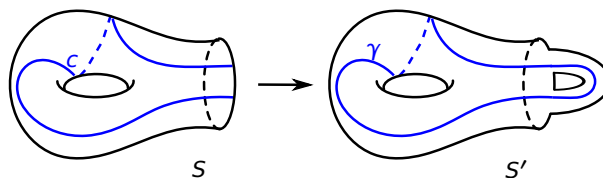

Figure 4.5: Stabilisation of an open book.

The easiest way to see the corresponding embedded picture is through *Hopf-plumbing*. Let (A, D_γ) be the abstract open book corresponding to the Hopf-fibration. Choose an arc α connecting the two boundary

components of the annulus A with neighborhood $N(a) \cong a \times I$. See Figure 4.3. Similarly, for the arc c in S take its neighborhood $N(c) \cong c \times I$. Note that the above 1–handle addition can be described as:

$$S' = S \cup_{N(a) \to N(c)} A,$$

where the map $N(a) \to N(c)$ identifies the a-direction of $N(a)$ with the I-direction of $N(c)$ and vice versa. The mondodromy h' is the composition $h \circ D_\gamma$. Note that this construction works for any second open book instead of (A, D_γ). In this case we take two open books (S_1, h_1) and (S_2, h_2) and glue them along the neighborhoods of two properly embedded arcs $c_1 \subset S_1$ and $c_2 \subset S_2$. The resulting surface

$$S = S_1 \cup_{N(c_1) \to N(c_2)} S_2$$

with monodromy $h_1 \circ h_2$ is called the *Murasugi sum* of (S_1, h_1) and (S_2, h_2).

Theorem 4.15. [Gab83] *If (S_1, h_1) and (S_2, h_2) are open books compatible with the contact manifolds (M_1, ξ_1) and (M_2, ξ_2), then their Murasugi sum (S, h) is compatible with $(M_1 \# M_2, \xi_1 \# \xi_2)$.*

Sketch of Proof. Take (the equivalence class of) $B_1 \cong N(c_1) \times [\frac{1}{2}, 1]/ \sim$ in the 3–manifold M_1 corresponding to the abstract open book (S_1, h_1), and similarly take $B_2 \cong N(c_1) \times [0, \frac{1}{2}]/ \sim \subset M_2$. Then the connected sum of M_1 and M_2 can be formed by glueing $M_1 \setminus B_1$ and $M_2 \setminus B_2$ along their boundaries. We choose the identification $\psi \colon -\partial B_1 \to -\partial B_2$ that sends $\partial N(c_1) \times \{\frac{1}{2} + t\}$ to $\partial N(c_2) \times \{t\}$, where the identification of $\partial N(c_1)$ and $\partial N(c_2)$ matches that in the definition of the Murasugi sum. See Figure 4.6. This proves that the abstract open book (S, h) gives $M_1 \# M_2$, and by the definition of compatibility of contact structure, the compatible contact structure on (S, h) is indeed $\xi_1 \# \xi_2$. □
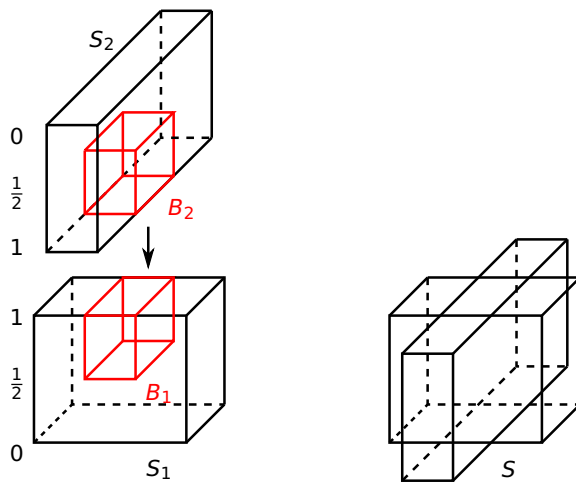

Figure 4.6: Murasugi sums of open books.

As (A, D_γ) describes the standard contact structure on S^3, Hopf stabilisation does not change (M, ξ).

Corollary 4.16. *The stabilisation of an open book (S, h) is compatible with the same contact structure.* □

Moreover the converse of this statement is also true:

Theorem 4.17. [Gir02] *The above map is surjective i.e. any contact structure ξ has a compatible open book. Moreover two open books compatible with a contact structure ξ are related by a sequence of stabilisations and destabilisations.*

4.3. Generalised braids

Open books give a new way to think about knots as "braids" in any 3–manifold:

Definition 4.18. A knot K is *braided* with respect to an open book decomposition (L, π) if K is disjoint from L and positively transverse to all the pages S_ϑ).

Note, this is equivalent to say that $\pi|_K \colon K \to S^1$ is a covering map. This definition generalises the fact, that braids in $\mathbb{R}^3$ (or S^3) naturally give transverse knots for ξ_{sym}. The Alexander Theorem also generalises to this setting:

Theorem 4.19. [Sko92] *Given a knot K and an open book (L, π), then K can be isotoped to be braided with respect to (L, π).*

This means that knots can be described using mapping class groups of a punctured surface. Let (S, h) be the abstract open book corresponding to (L, π). A braided knot K intersects S_0 in n points $P = \{p_0, \dots, p_{n-1}\}$, and K can be described as the mapping torus of a homeomorphism $\widetilde{h} \colon (S, P) \to (S, P)$ fixing the points P as a set, and being isotopic to $h \colon S \to S$ if we forget the points in P.

We can always assume, that the points of P are inside a small disc D, which is identified with a standard disc (D, P) with n punctures. This identification gives an embedding of the usual braid group B_n into the mapping class group of (S, P), and as an abuse of notation we still denote the generators (exchanging p_i and p_{i+1}) by σ_i. The image of B_n does not give all of the mapping class group. For example we can do a "finger move" along any curve starting and ending at a point of P. See Figure 4.7. To get a generating set of the mapping class group we need to take one finger move for each generator of $\pi_1(S, p_0)$.
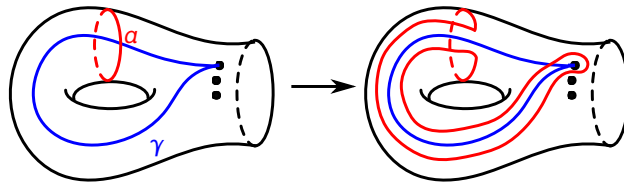

Figure 4.7: The image (right) of the curve α (left) under a fingermove along the curve γ (left).

Choosing a different ϑ (instead of 0) *conjugates* the element of the mapping class group with another mapping class group element, and just as in the classical case we can also change the number of points by *stabilisation* of the braid. Braid stabilisation happens inside the small disc D, and is defined exactly in the same way: adding an extra strand (point to P) and composing with the group element with $\sigma_n^\pm$. Note that braid stabilisation keeps the open book unchanged. The above two moves are called *Markov moves*. Now the Markov theorem can be stated for generalised braids:

Theorem 4.20. [Sun93] *Two knots braided with respect to an open book (L, π) are isotopic if and only if they are related by braid isotopies and Markov moves.*

Note that braided knots with respect to an open book are automatically transverse in the supported contact structure. These theorems can be reformulated for transverse braids as follows:

Theorem 4.21. [Pav11] *Suppose that the contact structure (M, ξ) is supported by the open book (L, π). Then any transverse knot K is transverse isotopic to a knot braided with respect to (L, π). Moreover two braided knots with respect to (L, π) are transverse isotopic if and only if they are related by braid isotopies and positive Markov moves.*

The above new description of knots and transverse knots gives rise to new invariants of knots and transverse knots: one can define the *braid index* of a (transverse) knot K with respect to a given open book (L, π) as the minimal n such that K is (transverse) isotopic to an n-braid with respect to (L, π). Or for any (transverse) knot K one can easily construct an open book (L, π) such that K is a 1-braid with respect to that open book. Then the minimal genus of those open books give another invariant for K. Although these invariants are very natural, little is known about them.

4.4. Open book foliations

Generalising braid foliations of a surface Σ, *open book foliations* [IK14] were defined as "generic" intersections of Σ and the pages of the open book. These objects are very similar to braid foliations; for instance one can understand "moves" for these foliations induced by isotopies of Σ. Moreover one can describe how stabilisation of the open book effects the open book foliation. One can also define a general formula for the self linking number in terms of open book foliations, and this formula helps to give examples of transverse knots that satisfy the Bennequin bound with an equality. Uses of open book foliations are still found, and they have potential to give simple proofs for complicated statements. As an example of how useful it is, I would like to present a proof of a well-known theorem using braid foliations.

4.4.1. Recognising overtwisted discs

Remember, an overtwisted disc was a disc whose boundary was tangent to ξ. A push off of this disc turned out to be a transverse unknot violating Bennequin's inequality. Define a contact structure ξ *overtwisted* if it has a transverse unknot with $sl(K) > -1$. Sometimes one can recognise overtwisted contact structures through their open books.

We say that a properly embedded arc γ' in the surface S is *to the left* of another properly embedded arc γ at their common starting point, n if after putting them in minimally intersecting position the tangents $T_n\gamma$ and $T_n\gamma'$ at n form a positive basis for S. See Figure 4.9. In this case we write $\gamma \prec \gamma'$. If γ γ', then the minimally intersecting representatives are unique, and thus the above notion is well defined.
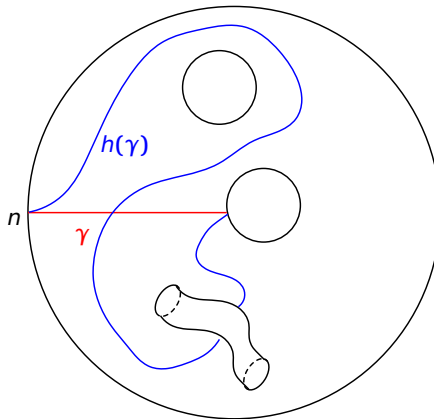

Figure 4.8: An arc γ' and its image to the left at n.

An arc γ in an abstract open book (S, h) is *left veering* if γ is not fixed by h, and $h(\gamma)$ is to the left from γ.

Theorem 4.22. [HKM07] *If an abstract open book* (S, h) *has a left veering arc* γ*, then the compatible contact structure is overtwisted.*

In the following, using open book foliations, we will describe an explicit construction of Ito and Kawamuro for an overtwisted disc for open books satisfying the above conditions. One can actually prove some sort of converse for Theorem 4.22:

Theorem 4.23. [HKM07] *A contact structure is overtwisted if and only if it is compatible with some open book with a left veering arc.* □

Proof of Theorem 4.22. [IK14] Suppose that $h(\gamma)$ is to the left of γ at their common starting point n. In the following we will construct a disc D, whose boundary violates Bennequin inequality.

The intersection of D with the pages S_ϑ will be governed by a sequence of arcs

$$\gamma = \gamma_0 \prec \gamma_1 \cdots \prec \gamma_k = h(\gamma)$$

such that all γ_i starts at n, but the endpoints $p_0, p_1, \ldots, p_{k-1}$ of $\gamma_0, \ldots, \gamma_{k-1}$ are all different and γ_i is disjoint from γ_{i+1}. One can easily find such a sequence. In the following we explain the proof through the example given on Figure 4.8.

Subdivide the circle S^1 into $2k$ equal intervals with dividing points $\{\vartheta_i = \frac{2i\pi}{2k}\}_{i=0}^{2k}$ (here the 0'th and the $2k$'th points agree), and let the disc D intersect $S_{\vartheta_{2i}}$ in $\gamma_i \cup \bigcup_{j \neq i} \overline{\gamma}_j[0, \delta]$, where $\overline{\gamma}_j$ is γ_j with reversed parametrisation and δ is sufficiently small so that $\overline{\gamma}_j[0, \delta]$ is contained in $N(\partial S)$.
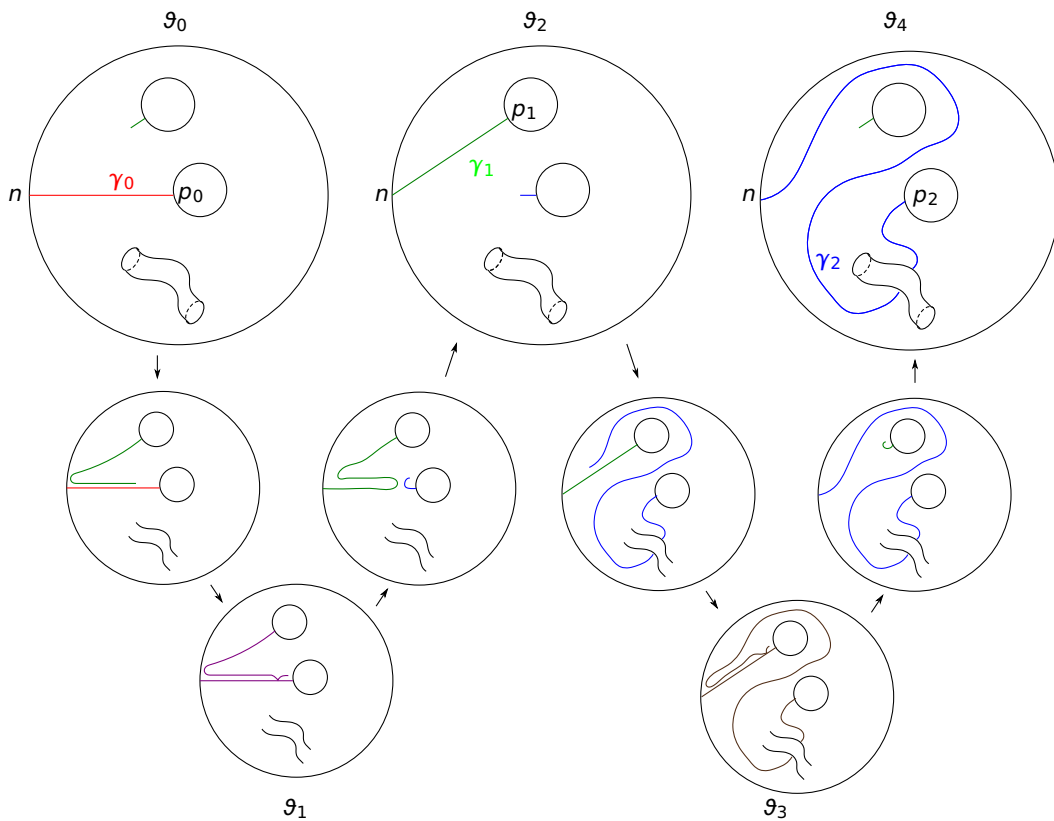

Figure 4.9: The intersection of D with S_ϑ for various ϑs.

In the intervals $[\vartheta_{2i}, \vartheta_{2(i+1)}]$ we change $\gamma_i \cup \bigcup_{j \neq i} \overline{\gamma}_j[0, \delta]$ into $\gamma_{i+1} \cup \bigcup_{j \neq i+1} \overline{\gamma}_j[0, \delta]$ through intersections $S_\vartheta \cap D$, that are the union of $\bigcup_{j \neq i, i+1} \overline{\gamma}_j[0, \delta]$ and two disjoint properly embedded arcs at all times

except at $\vartheta = \vartheta_{2i+1}$. There, the two arcs meet in a single point $q_i = \overline{\gamma}_{i+1}(\delta')$ (See Figure 4.9), which gives rise to a positive hyperbolic point of the open book foliation of D. See Figure 4.10. Now the open book
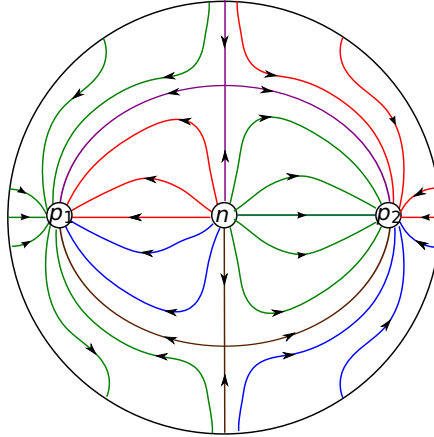

Figure 4.10: The open book foliation on D. The different times are labeled by the same color as on Figure 4.9.

foliation on D is depicted on Figure 4.10, and one can see, that n is a negative elliptic point, the $\{p_i\}_{i=0}^{k-1}$ are positive elliptic points, while the $\{q_i\}_{i=0}^{k-1}$ are positive hyperbolic points (remember that $p_0 = p_k$). Thus $e_+ = 1$, $e_- = k$, $h_+ = k$ and $h_- = 0$. Plugging this data into Equation (3.2) we get:

$$sl(\partial D) = e_- - e_+ + h_+ - h_- = 1 - k + k - 0 = 1 > -1,$$

and thus ∂D indeed violates Bennequin's inequality. □

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