Institut Fourier — Université de Grenoble I

Actes du séminaire de

Théorie spectrale et géométrie

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Volume 37 (2021-2022), p. 137-175.

https://doi.org/10.5802/tsg.385

Publication membre du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2118-9242

Séminaire de théorie spectrale et géométrie
Grenoble
Volume 37 (2021-2022) 137-175

GLOBALLY HYPERBOLIC SPATIALLY COMPACT MAXIMAL CONFORMALLY FLAT SPACETIMES ARISING FROM ANOSOV REPRESENTATIONS

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Abstract. — This paper deals with Anosov representations of a Gromov-hyperbolic group into the semi-simple Lie group $O_0(2,n)$ and their link with conformally flat Lorentzian structures on manifolds. The main result that we discuss states that any P_1-Anosov representation of a Gromov hyperbolic group into $O_0(2,n)$ preserving an acausal subset in the Einstein universe $Ein_{1,n-1}$ is the holonomy of a globally hyperbolic Cauchy-compact maximal conformally flat spacetime. It follows from this result remarkable examples, that we call *black-white holes, conformally flat Misner spacetimes* and *Misner extensions* and that we describe in this paper. Last but not least, we introduce and we discuss the notion of *complete photons* that appears naturally in these examples.

Introduction

In all this paper, we denote by $O_0(1,n)$ and $O_0(2,n)$ the identity components of $O(1,n)$ and $O(2,n)$ respectively.

Anosov representations were introduced by F. Labourie [16] to give a geometrical interpretation to the elements of a particular component, first studied by Hitchin, of the space of representations of the fundamental group of a closed negatively curved surface into $SL_n(\mathbb{R})$. Later, this notion has been extended by Guichard and Wienhard [14] to representations of Gromov-hyperbolic groups into any semi-simple Lie group of non-compact type (e.g. $O(p,q), SL(n,\mathbb{C}), Sp(2n,\mathbb{C}), \ldots$). Actually, any finitely generated group Γ admitting an Anosov representation is Gromov-hyperbolic. This fact comes from a general result of Kapovitch–Leeb–Porti [15] that has been proved again in a more elementary way by Bochi–Potri–Sambarino in [7, Sections 3 and 4].

Anosov representations turn out to be a fundamental notion because of the multitude of examples and because of their close connection with geometric structures on manifolds. The first examples are convex-cocompact subgroups of $O_0(1,n)$: a discrete subgroup Γ of $O_0(1,n)$ is said to be convex-cocompact if it acts properly discontinuously on a convex subset of $\mathbb{H}^n$ with compact quotient. Convex cocompact subgroups of $O_0(1,n)$ are exactly those for which the canonical inclusion into $O_0(1,n)$ is Anosov.

In higher rank, the most naive analogy fails: generically, Anosov representations of a discrete group Γ into a semi-simple Lie group G of rank greater or equal to 2 do not act properly and cocompactly on a convex subset of the associated symmetric space G/K, where K is a maximal compact subgroup of G. However, many authors depicted a geometric picture for Anosov representations into higher rank semi-simple Lie groups that extends naturally the notion of convex-cocompactness to this setting (see e.g. [3, 9, 14]).

This paper deals with Anosov representations into the real semi-simple Lie group $O_0(2,n)$. Since $O_0(2,n)$ is of rank two, there are two different ways for a Gromov hyperbolic group Γ to be Anosov in $O_0(2,n)$. We focus here on representations called P_1-*Anosov* in [13]. These representations act naturally on a Lorentz space: *the Einstein universe*. This is the first relativistic model of the universe proposed by Albert Einstein in 1917 based on his theory of general relativity. The Einstein universe of dimension n, denoted by $Ein_{1,n-1}$, is the space $\mathbb{S}^{n-1} \times \mathbb{S}^1$ equipped with the conformal class[(1)] of the Lorentzian metric $d\sigma^2 - d\theta^2$ where $d\sigma^2$ and $d\theta^2$ are the round metrics on the sphere $\mathbb{S}^{n-1}$ and the circle $\mathbb{S}^1$ respectively. The group of conformal tranformations of $Ein_{1,n-1}$ is $O(2,n)$. For the reader more familiar with Riemannian geometry, let us add that the Einstein universe can be seen as the Lorentzian analogue of the conformal sphere in the following sense: as in Riemannian geometry, the Lorentzian model spaces of constant curvature, namely *Minkowski spacetime* of curvature 0, *de Sitter spacetime* of curvature 1 and *anti-de Sitter spacetime* of curvature -1, embed conformally in the Einstein universe of the same dimension.

In this paper, we propose to study the link between some P_1-Anosov representations of a Gromov hyperbolic group in $O_0(2,n)$ and spaces which are "locally modeled" on the Einstein universe. In dimension at least three, these last ones coincide exactly with *conformally flat spacetimes*[(2)].

A *spacetime* is an oriented Lorentzian manifold (M,g) with a time-orientation given by a timelike vector field, i.e. a vector field X such that $g(X,X) < 0$. The property of spacetimes that turns out to be relevant in

(1) The conformal class of a pseudo-Riemannian metric g on a smooth manifold M is the set of metrics $e^f g$ where f is a smooth function on M.

(2) This is a consequence of the Lorentzian version of Liouville's theorem in conformal geometry (see Section 1.4).

our study is *causality*. Indeed, the tangent vectors to a spacetime (M, g) split into three classes according to the sign of their norm with respect to the metric g: those of negative, null and positive norm, called respectively *timelike*, *lightlike* and *spacelike* tangent vectors. *Causal curves* are the curves whose tangent vectors are either timelike or lightlike. Causality refers to the causal relationships between the points of a spacetime M. More precisely, it deals with the question: which points in M can be joined by a causal curve? Causality of spacetimes have been extensively studied in the 60's by several authors, namely Choquet-Bruhat, Geroch, Kronheimer, Penrose, ... Among the causal properties of a spacetime, a special role is played by *global hyperbolicity*. This notion appeared naturally in the setting of the resolution of the Einstein equations in general relativity. By a classical theorem of Geroch [11], a spacetime M is globally hyperbolic if there exists an embedded Riemannian hypersurface S in M that meets every inextendible causal curve exactly once. Such a hypersurface S is called *a Cauchy hypersurface.* A globally hyperbolic spacetime M with Cauchy hypersurface S is diffeomorphic to $S \times \mathbb{R}$, and so it is never compact. Whereas compactness is a nice property that is usually considered in the Riemannian setting, global hyperbolicity turns out to be the standard assumption to make in the Lorentzian setting in order to get spacetimes with relevant causal properties. However, compactness is a good assumption on a Cauchy hypersurface of a globally hyperbolic spacetime M. Let us point out that since all Cauchy hypersurfaces in M are diffeomorphic, if one of them is compact, they are all compact. In this case, M is said to be *Cauchy-compact.*

Causality is actually a *conformal* notion. Indeed, changing the metric g by another metric in its conformal class consists in multiplying g by a positive smooth function and so the types of tangent vectors stay unchanged. Hence, all the notions defined above still hold in a manifold equipped with a conformal class of Lorentzian metrics. We call *conformal spacetime* an oriented manifold equipped with a conformal class of Lorentzian metric and a time-orientation given by a timelike vector field. For instance, $Ein_{1,n-1}$ is a conformal spacetime: it is oriented and time-oriented by the vector field ∂_θ. The group of orientation and time-orientation preserving conformal transformations is $O_0(2, n)$. A particular class of conformal spacetimes is that of *conformally flat* spacetimes. A spacetime M of dimension n is said to be *conformally flat* if it is locally conformal to Minkowski spacetime $\mathbb{R}^{1,n-1}$, which is the affine space $\mathbb{R}^n$ equipped with the Lorentzian metric $-dt^2 + dx_1^2 + \ldots + dx_{n-1}^2$. The Einstein universe is an example of conformally flat spacetime since it is homogenous and $\mathbb{R}^{1,n-1}$ embeds conformally in $Ein_{1,n-1}$.

Our result concerns globally hyperbolic Cauchy-compact *maximal* conformally flat spacetimes. Maximality is a natural assumption on globally hyperbolic spacetimes that can be seen as a weak notion of completeness in the Lorentzian setting. A globally hyperbolic spacetime M is maximal if any isometric embedding of M into any other globally hyperbolic Lorentzian manifold N that sends a Cauchy hypersurface S of M on a Cauchy hypersurface of N is surjective. Actually, the definition does not depend on the choice of the Cauchy hypersurface S in M. By a classical result of Choquet–Bruhat and Geroch [8] that has been generalized in different settings, globally hyperbolic spacetimes usually admit a maximal extension. Notice that this notion of maximality depends on the metric g on M since the definition involves *isometric* embeddings. In [23], Rossi extends the definition to conformal spacetimes by no longer considering isometric but conformal embeddings. A spacetime which is maximal in this sense is said to be C-*maximal* (C for conformal). Rossi proves then that any conformally flat spacetime admits a conformally flat C-maximal extension. Since our work concerns conformal spacetimes, we will simply say maximal instead of C-maximal while keeping in mind that it is in the conformal sense.

A P_1-Anosov representation ρ of a Gromov hyperbolic group Γ in $O_0(2,n)$ acts on $Ein_{1,n-1}$ and preserves a closed subset in $Ein_{1,n-1}$ called *the limit set of* ρ. We will be interested in P_1-Anosov representations such that the limit set is *acausal*, which in some sense means that for every pair p, q of distinct points of the limit set, there is no causal curve of $Ein_{1,n-1}$ connecting them; in short, the points of the limit set are not causally related one to another. Such a P_1-Anosov representation is called *negative* in [9]. Now, we can state our theorem proved in [24]:

Theorem 0.1. — *Let $n \geqslant 3$ be an integer. A negative P_1-Anosov representation ρ of a Gromov hyperbolic group Γ in $O_0(2,n)$ is the holonomy of a globally hyperbolic maximal Cauchy-compact conformally flat spacetime $M_\rho(\Gamma)$.*

The case where the limit set is a topological $(n-1)$-sphere is due to Barbot-Mérigot's work in [3] (see Section 2.2 for a detailed discussion). In this case, $M_\rho(\Gamma)$ is a spacetime of dimension $(n+1)$ which admits in its conformal class a metric of negative constant curvature. The novelty of our result concerns the general case where the limit set is not a topological $(n-1)$-sphere.

We can see Theorem 0.1 as a manifestation of the "convex-cocompactness" property of Anosov representations. Indeed, the proof consists in finding a domain of discontinuity in the Einstein universe $Ein_{1,n-1}$, i.e. an open subset on which the action of the representation is free and properly discontinuous, such that the quotient is a conformally flat spacetime of dimension n with all the good causal properties, namely globally hyperbolic Cauchy-compact maximal, the most important one being the Cauchy-compactness. Notice that here this is not the quotient of the domain of discontinuity which is compact, as it is usually the case, but a Cauchy hypersurface of the quotient. Actually, the Cauchy-compactness is obtained as a consequence of a general result of Guichard–Kassel–Wienhard in [13, Theorem 4.1] on P_1-Anosov representations in $O(p,q)$ that we rephrase in the Lorentzian setting. The key idea that allows to use Guichard–Kassel–Wienhard result is that in a globally hyperbolic spacetime, the space of lightlike geodesics is homeomorphic to the unit tangent bundle of a Cauchy hypersurface (see Section 2.2).

One application of Theorem 0.1 is that it gives interesting examples of globally hyperbolic Cauchy-compact maximal conformally flat spacetimes. In this paper, we point out families of examples coming from the data of a negative P_1-Anosov representation ρ from a Gromov hyperbolic group Γ in $O_0(2,n)$ that preserves a point p in the Einstein universe $Ein_{1,n-1}$ and more generally a conformal sphere of dimension $1 \leqslant k \leqslant n-2$ in $Ein_{1,n-1}$. We distinguish three families of GHMC conformally flat spacetimes associated to such representations that we call *black-white holes*, *conformally flat Misner spacetimes* and *Misner extensions*. These spacetimes contain photons, i.e. inextendible lightlike geodesics, which have the remarkable property of being *complete* (see Section 4). This gives a strong motivation for the study of globally hyperbolic maximal conformally flat spacetimes with complete photons.

Overview of the paper

Section 1 deals with the basic notions of Lorentzian geometry: we define the causal structure of spacetimes, in particular the global hyperbolicity, before focusing on *conformally flat* structures on spacetimes. In Section 2, we introduce Anosov representations in $O(2,n)$ and we discuss our main result. Section 3 is devoted to the description of black-white holes, conformally flat Misner spacetimes and Misner extensions. In Section 4, we introduce the notion of *complete photons* and we discuss the relevance of the study of GHM conformally flat spacetimes with complete photons.

Acknowledgments

I would like to thank my PhD advisor Thierry Barbot for introducing me to Lorentzian geometry and its surprising link with Anosov representations. I am grateful for all the fruitful discussions we had that made this work possible. I would also like to thank Charles Frances for his interests in my work, his rereading of this survey, his comments and encouragements.

1. Preliminaries

In this section, we briefly recall the background material on (conformal) spacetimes with a special focus on conformally flat spacetimes.

We denote by (p, q) the signature of a nondegenerate quadratic form Q where p and q are respectively the negative and the positive signs in the polar decomposition of Q.

1.1. Spacetimes

A *Lorentzian manifold* is a smooth manifold M of dimension $(n+1)$ equipped with a nondegenerate symmetric 2-form g of signature $(1, n)$ called Lorentzian metric. A non-zero tangent vector v is *timelike* (resp. *lightlike*, *spacelike*) if $g(v, v)$ is negative (resp. null, positive). We say that v is *causal* if it is non-spacelike. In each tangent space, the set of lightlike vectors is a cone, called *the lightcone*, with two connected components (see Figure 1.1).

A Lorentzian manifold M is *time-orientable* if it is possible to make a continuous choice, in each tangent space, of one of the connected components of the lightcone. In this case, a causal tangent vector is said to be *future* if it is in the chosen component, *past* otherwise. More formally, a time-orientation of M is the data of a timelike vector field X on M, i.e. such that $g(X, X) < 0$. A causal vector v in T_pM is future if $g_p(X(p), v) < 0$, past otherwise. Up to a finite covering, a Lorentzian manifold is always time-orientable.

Throughout this paper, we consider only orientable and time-orientable Lorentzian manifolds.

Definition 1.1. — *A* spacetime *is a connected, oriented and time-oriented Lorentzian manifold.*

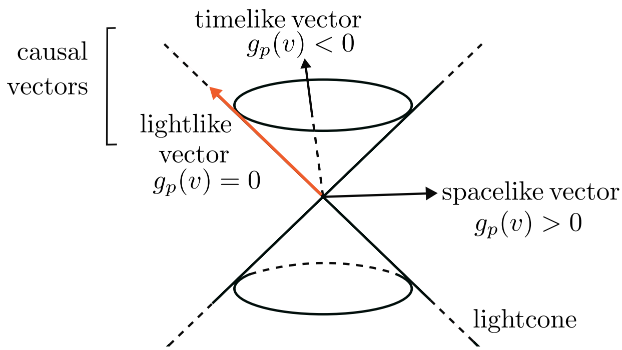

Figure 1.1. Timelike, lightlike and spacelike tangent vectors.

The basic examples of spacetimes are *Minkowski spacetime, de Sitter spacetime* and *anti-de Sitter spacetime* which are the Lorentzian models of constant sectional curvature respectively equal to 0, 1, and −1. They are the Lorentzian analogues of the Riemannian manifolds of constant sectional curvature : the *Euclidean space, the sphere* and *the hyperbolic space* respectively.

For the reader who is not familiar with these spaces, we give brief description of these spaces.

Minkowski spacetime. It is the affine space $\mathbb{R}^{n+1}$ equipped with the Lorentzian metric $-dt^2+dx_1^2+\ldots+dx_n^2$ in the coordinate system associated to the canonical basis of $\mathbb{R}^{n+1}$, and is denoted $\mathbb{R}^{1,n}$. A time-orientation is given by the timelike vector field $\frac{\partial}{\partial_t}$.

De Sitter spacetime. Let $q_{1,n+1}$ be the quadratic form on $\mathbb{R}^{1,n+1}$ given by

$$q_{1,n}(t, x_1, \ldots, x_n) := -t^2 + x_1^2 + \ldots + x_{n+1}^2$$

in the coordinate system $(t, x_1, \ldots, x_{n+1})$ associated to the canonical basis.

The de Sitter space $dS_{1,n}$ is the quadric $\{x \in \mathbb{R}^{1,n} : q_{1,n+1}(x) = 1\}$ equipped with the metric g obtained by restriction of $q_{1,n+1}$. Notice the analogy with the definition of the round sphere.

It is a spacetime. Indeed, the tangent space to $dS_{1,n}$ at a point $x \in dS_{1,n}$ is the orthogonal of x with respect to $q_{1,n+1}$. Since $q_{1,n+1}(x) = 1 > 0$, it is easy to see that the restriction of $q_{1,n}$ to the orthogonal of x is Lorentzian. Besides, the map $f : \mathbb{S}^n \times \mathbb{R} \to dS_{1,n}$ defined by $f(x,t) = (\sinh t, (\cosh t)x)$

is a diffeomorphism. This shows that $dS_{1,n}$ is oriented and time-oriented by the timelike vector field ∂_t.

Anti-de Sitter spacetime. Let $\mathbb{R}^{2,n}$ be the vector space $\mathbb{R}^{n+2}$, with coordinates $(u, v, x_1, \ldots, x_n)$ in the canonical basis of $\mathbb{R}^{n+2}$, equipped with the nondegenerate quadratic form of signature $(2, n)$

$$q_{2,n}(u, v, x_1, \ldots, x_n) = -u^2 - v^2 + x_1^2 + \ldots + x_n^2.$$

We denote by $\langle ., . \rangle_{2,n}$ the bilinear form associated to $q_{2,n}$.

Anti-de Sitter space $AdS_{1,n}$ is the hypersurface $\{x \in \mathbb{R}^{2,n} : q_{2,n}(x) = -1\}$ equipped with the metric g obtained by restriction of $q_{2,n}$. Notice the analogy with the definition of the hyperbolic space.

Anti-de Sitter space is a spacetime:

The metric g is Lorentzian. Indeed, the tangent space to $AdS_{1,n}$ at a point $x \in AdS_{1,n}$ coincides with the orthogonal of x with respect to $q_{2,n}$. Besides, in the coordinates $(r, \theta, x_1, \ldots, x_n)$ with

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

one can easily see that $AdS_{1,n}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{H}^n$, where $\mathbb{H}^n$ is the hyperboloid $\{(r, 0, x_1, \ldots, x_n) : \ -r^2 + x_1^2 + \ldots + x_n^2 = -1, \ r > 0\}$, and thus is oriented. Finally, a time-orientation is given by the vector field $\frac{\partial}{\partial\theta}$.

Remark 1.2. — In the $(r, \theta, x_1, \ldots, x_n)$, the AdS-metric is $(-r^2 d\theta^2 + ds^2_{hyp})$, where ds^2_{hyp} is the hyperbolic metric, i.e. the induced metric by $q_{2,n}$ on $\mathbb{H}^n = \{(r, 0, x_1, \ldots, x_n) : \ -r^2 + x_1^2 + \ldots + x_n^2 = -1, \ r > 0\}$, and $d\theta^2$ is the round metric on $\mathbb{S}^1$.

Remark 1.3. — Anti-de Sitter spacetime has constant sectional curvature -1.

Let $\mathbb{D}^n$ be the upper hemisphere of the round sphere $\mathbb{S}^n$. We denote by $d\sigma^2$ the round metric on $\mathbb{S}^n$.

PROPOSITION 1.4. — *Anti-de Sitter spacetime $AdS_{1,n}$ is conformally isometric to the product $\mathbb{S}^1 \times \mathbb{D}^n$ equipped with the Lorentzian metric $(-d\theta^2 + d\sigma^2)$.*

Proof. — See e.g. [3, Section 2, Proposition 2.4]. □

As for the hyperbolic space, anti-de Sitter spacetime admits *a Klein model*.

Let $\mathbb{S}(\mathbb{R}^{2,n})$ be the quotient of $\mathbb{R}^{2,n}\backslash\{0\}$ by positive homotheties [(3)]. Let $\pi : \mathbb{R}^{2,n}\backslash\{0\} \to \mathbb{S}(\mathbb{R}^{2,n})$ be the radial projection. The restriction of the projection π to $AdS_{1,n}$ is one-to-one. Hence, one can define:

Definition 1.5. — *The* Klein model $\mathbb{ADS}_{1,n}$ *of anti-de Sitter space is the image by π of $AdS_{1,n}$ in $\mathbb{S}(\mathbb{R}^{2,n})$ equipped with the push forward of the AdS Lorentzian metric by the restriction of π to $AdS_{1,n}$.*

Remark 1.6. — Unlike the hyperbolic space, the Klein model of anti-de Sitter space is not contained in an affine chart (see [3, Definition Section 2.7]).

Remark 1.7. — One can also consider the projection of $AdS_{1,n}$ in the projective space $\mathbb{P}(\mathbb{R}^{2,n})$, denoted by $\mathsf{AdS}_{1,n}$. The Klein model $\mathbb{ADS}_{1,n}$ is a double covering of $\mathsf{AdS}_{1,n}$.

1.2. Causality

Definition 1.8. — *A* causal curve *(resp.* timelike, lightlike, spacelike*) of a spacetime M is a $\mathcal{C}^1$ curve on M such that at every point, the tangent vector to the curve is causal (resp. timelike, lightlike, spacelike). A causal curve is said to be* future *(resp.* past*) if all tangent vectors to the curve are future (resp. past) oriented* [(4)].

Causality refers to the general question on which points in a spacetime can be joined by a causal curve, in short causally related. In this setting, the notions of *future* and *past* of a subset of a spacetime have been defined.

Future, past, achronality. Let M be a spacetime and let $A \subset M$. The *causal future* $J^+(A)$ (resp. *chronological future* $I^+(A)$) of A in M is the set of future-ends of causal (resp. timelike) curves starting from a point $p \in A$. Similarly, we define the *causal past* $J^-(A)$ (resp. *chronological past* $I^-(A)$) of A in M by switching future by *past* in the definition.

A spacetime M could contain subsets where no point is causally related to another. A subset A of M is *achronal* (resp. *acausal*) if no timelike (resp. causal) curve meets A more than once.

A subset A of M is said to be *future* (resp. *past*) if $I^+(A) \subset A$ (resp. $I^-(A) \subset A$). The boundary of a future (past) subset of M is a closed achronal topological hypersurface.

(3) Notice that $\mathbb{S}(\mathbb{R}^{2,n})$ is a double covering of the projective space $\mathbb{P}(\mathbb{R}^{2,n})$.

(4) It is possible to generalize the definition of future (resp. past) causal curves to piecewise differential curves (see e.g. [22, Section 3, Chapter 1]).

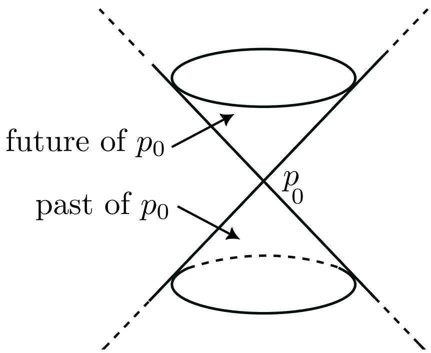

Figure 1.2. Future and past of a point p_0 in $\mathbb{R}^{1,2}$.

Geodesics. As in Riemannian geometry, there is a Levi Civita connection on a spacetime (M, g): this is the unique torsion-free connection on the tangent bundle of M preserving the Lorentzian metric g. A geodesic of M is a curve γ such that parallel transport along γ preserves tangent vectors to the curve. Since the norm of a tangent vector to a geodesic is preserved, the type of a geodesic in a spacetime M is always well-defined. Let us point out that in this setting, geodesics are not considered as minimizing curves anymore since a Lorentzian metric does not provide a distance.

Definition 1.9. — *We call* photon *any inextendible*[(5)] *lightlike geodesic in a spacetime M.*

Example 1.10. — Geodesics of Minkowski spacetime $\mathbb{R}^{1,n}$ are the straight lines. Indeed, the flat connection of the affine space $\mathbb{R}^{n+1}$ is compatible with the Lorentzian metric of $\mathbb{R}^{1,n}$.

Example 1.11. — Geodesics of anti-de Sitter spacetime $AdS_{1,n}$ are intersections with 2-planes (see e.g. [3, Section 2]). The type of a geodesic γ defined by a 2-plane P depends on the signature of the restriction of the quadratic form $q_{2,n}$ to P. If P is

- negative, i.e. the restriction of $q_{2,n}$ to P is negative definite, then γ is timelike;
- degenerate, i.e. the restriction of $q_{2,n}$ to P is of signature $(1, 0)$, then γ is lightlike;
- Lorentzian, i.e. the restriction of $q_{2,n}$ to P is of signature $(1, 1)$, then γ is spacelike.

(5) A causal curve $\gamma : I \to M$ in a spacetime M is said to be *inextendible* if there is no causal curve $\widetilde{\gamma} : J \to M$ that extends γ, i.e. such that $I \subset J$ and $\widetilde{\gamma}_{|I} = \gamma$.

Definition 1.12. — *Let M be a spacetime and let $p \in M$. We call* lightcone of p *the union of photons going through p.*

Global hyperbolicity. Among the causal properties of a spacetime, an important one is *global hyperbolicity*. This is a natural property of spacetimes that comes from the physical theory of general relativity in the setting of the resolution of the Einstein equations. The first definition, due to Leray [18], involves the notion of *diamond*.

Definition 1.13. — *Let M be a spacetime. A* diamond *of M is an intersection $J^+(p) \cap J^-(q)$ where p and q are two points in M, and is denoted by $J(p,q)$.*

Remark 1.14. — A diamond $J(p,q)$ is non-empty if and only if $q \in J^+(p)$.

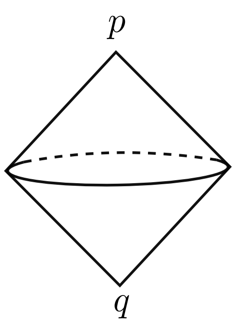

Figure 1.3. Diamond in the Minkowski spacetime $\mathbb{R}^{1,2}$.

Definition 1.15. — *A spacetime M is* globally hyperbolic *if*

(1) *there is no causal loop* [(6)] *in M,*
(2) *all diamonds are compact.*

In what follows, we give a characterization of global hyperbolicity, due to Geroch [11], that uses the notion of *Cauchy hypersurface*.

Definition 1.16. — *Let M be a spacetime. A* Cauchy hypersurface *of M is a topological achronal hypersurface that meets every inextendible causal curve exactly once.*

[(6)] In the definition given by Leray, the first condition was stronger than the one stated here and requires that M is *strongly causal*, i.e. that every point in M admits a *causally convex* (see Definition 1.21) neighorhood, as small as we want. Sanchez [6] weaken this condition to the non-existence of causal loops in M.

Theorem 1.17 ([11]). — *A spacetime M is globally hyperbolic if and only if it admits a Cauchy hypersurface. In this case, M is homeomorphic to $S \times \mathbb{R}$.*

Example 1.18. — Minkowski spacetime $\mathbb{R}^{1,n}$ is globally hyperbolic: the Euclidean hypersurface $\{0\} \times \mathbb{R}^n$ is a Cauchy hypersurface. Anti-de Sitter spacetime $AdS_{1,n}$ is not globally hyperbolic since it contains many causal loops (see e.g. [3, Section 2]).

It follows from Geroch characterization that a globally hyperbolic spacetime M can not be compact. However, M could admit a compact Cauchy hypersurface.

Definition 1.19. — *A globally hyperbolic spacetime is said to be* spatially compact *(or* Cauchy-compact*) if it admits a compact Cauchy hypersurface.*

In a globally hyperbolic spacetime, all Cauchy hypersurfaces are homeomorphic to one another. Consequently, if one of them is compact, they are all compact.

Remark 1.20. — In [5], Bernal and Sanchez state that global hyperbolicity is equivalent to the existence of a *smooth* Cauchy hypersurface. The spacetime is then diffeomorphic to $S \times \mathbb{R}$ where S is a smooth Cauchy hypersurface.

Causal convexity. In Riemannian geometry, it is often useful to consider open neighborhoods which are *geodesically convex*: an open set U is geodesically convex if any geodesic connecting two points in U stays in U. In Lorentzian geometry, we have, in addition, another notion of convexity which is relevant from the causality point of view.

Definition 1.21. — *A subset U of a spacetime M is* causally convex *in M if any causal curve connecting two points in U is contained in U.*

Equivalently, a subset U of M is causally convex in M if every diamond $J(p,q)$, with $p, q \in U$, is contained in U.

Remark 1.22. — The future (past) of a subset of M is causally convex. More generally, future (past) subsets are causally convex.

Example 1.23 (Regular domains of Minkowski spacetime). — In Minkowski spacetime, there is an important class of causally convex open subsets called *regular domains*. *A future (resp. past) regular domain* is the interior of the intersection of future (resp. past) half-spaces bounded by a degenerate hyperplane. It can be described as the strict epigraph (hypograph) of a convex (concave) 1-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$.

Example 1.24 (Misner domains of Minkowski spacetime). — Regular domains of $\mathbb{R}^{1,n-1}$ defined by exactly two degenerate hyperplanes are distinguishable: their quotients by an appropriate discrete subgroup of affine transformations of $\mathbb{R}^{1,n-1}$ define a family of globally hyperbolic Cauchy-compact flat spacetimes, called *Misner flat spacetimes*[(7)], which plays a central role in the description of globally hyperbolic Cauchy-compact flat spacetimes (see [2]). For this reason, these regular domains are called *Misner domains* of $\mathbb{R}^{1,n-1}$ (see Figure 1.4).

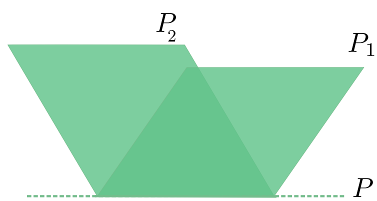

Figure 1.4. The intersection of the strict upper half-spaces bounded by the degenerate hyperplanes P_1 and P_2 of $\mathbb{R}^{1,2}$ is a future Misner domain of $\mathbb{R}^{1,2}$.

LEMMA 1.25. — *Let M be a globally hyperbolic spacetime. Every causally convex open subset U of M is globally hyperbolic.*

Proof. — Since M is globally hyperbolic, it does not contain causal loops and so neither does U. Let $p, q \in U$. We denote by $J_U(p,q)$ the diamond in U, i.e. the union of all causal curves in U connecting p to q. We show that $J_U(p,q)$ is exactly the diamond $J(p,q)$. It is clear that $J_U(p,q) \subset J(p,q)$. Since U is causally convex in M, every causal curve of M connecting p to q is contained in U. This proves that $J(p,q) \subset J_U(p,q)$. The equality follows. Since M is globally hyperbolic, $J(p,q)$ is compact and then so do $J_U(p,q)$. It follows that U is globally hyperbolic. □

Remark 1.26. — In Section 1.5, we define a partial order relation on globally hyperbolic spacetimes. It turns out that regular domains of $\mathbb{R}^{1,n}$

(7) These spacetimes have been called after the mathematician Charles W. Misner since they can be seen as a generalization of the two-dimensional spacetime described by Misner in [20], namely the quotient by a boost of a half space of $\mathbb{R}^{1,1}$ bounded by a lightlike straight line.

are exactly the causally convex open subsets of $\mathbb{R}^{1,n}$ which are maximal for this partial order relation.

Conformal spacetimes. Causality is actually a conformal notion. Recall that two Lorentzian metrics g and g' on a spacetime M are conformally equivalent if there exists a smooth function $f : M \to \mathbb{R}$ such that $g' = e^f g$. The conformal class of g is the set of all Lorenzian metrics on M conformally equivalent to g. Since two conformally equivalent Lorentzian metrics are proportional by a positive function, the types of tangent vectors are preserved. This implies that the causal structure, in particular the time-orientation, of M only depends on the conformal class of g.

Definition 1.27. — *A conformal spacetime is a connected oriented manifold equipped with a conformal class of Lorentzian metrics and a time-orientation.*

While causality is well-defined on a conformal spacetime, geodesics are not well-defined anymore except lightlike geodesics. Indeed, a computation on the Levi–Civita connection (see e.g. [4]) shows that geodesics are not preserved by conformal changes of metrics. However, it has been proved that the lightlike geodesics, seen as unparametrized curves, are preserved.

Theorem 1.28. — *Let (M, g) be a pseudo-Riemannian manifold. Then, lightlike geodesics are the same, up to parametrization, for all metrics conformally equivalent to g.*

Proof. — See e.g. [10, Theorem 3]. □

An important example of conformal spacetime is the *Einstein universe* which can be seen as the Lorentzian analogue of the conformal sphere in Riemannian geometry and that we define now.

1.3. Einstein universe

In Riemannian geometry, the topological boundary of the Klein model $\mathbb{K}^{n+1}$ of the hyperbolic space in $\mathbb{RP}^{n+2}$ is a sphere of dimension n which is naturally equipped with a Riemannian conformal structure. Similarly, in the Lorentzian setting, the topological boundary of the Klein model $\mathbb{ADS}_{1,n}$ of anti-de Sitter spacetime in $\mathbb{S}(\mathbb{R}^{2,n})$ is naturally a conformal spacetime called *the Einstein universe.*

Let C be the isotropic cone of $\mathbb{R}^{2,n}$ with the origin removed

$$C = \left\{x \in \mathbb{R}^{2,n} \backslash \{0\} : \; q_{2,n}(x) = 0\right\}.$$

We denote by $\partial\mathbb{ADS}_{1,n}$ the topological boundary of $\mathbb{ADS}_{1,n}$ in $\mathbb{S}(\mathbb{R}^{2,n})$, i.e. the projection of C in $\mathbb{S}(\mathbb{R}^{2,n})$.

Lemma 1.29. — *The smooth hypersurface $\partial\mathbb{ADS}_{1,n}$ of $\mathbb{S}(\mathbb{R}^{2,n})$ is naturally equipped with a conformal class of Lorentzian metrics.*

Proof. — Let $x \in C$. The restriction of $q_{2,n}$ to the tangent space $T_xC = x^\perp$, that we call $\widehat{q}_{2,n}$, is degenerate. Its kernel is the isotropic line $\mathbb{R}x$. It is easy to see that $\widehat{q}_{2,n}$ induces on the quotient space $x^\perp/\mathbb{R}x$ a quadratic form of signature $(1, n-1)$, that we denote by q_x. A simple computation shows that the kernel of $\widehat{q}_{2,n}$ coincides with the kernel of the tangent map $d_x\pi_{|C}$. Therefore, $d_x\pi_{|C}$ induces an isomorphism between $x^\perp/\mathbb{R}x$ and the tangent space $T_{\pi(x)}\pi(C)$. Thus, the push-forward of q_x by this isomorphism defines a Lorentzian metric on $T_{\pi(x)}\pi(C)$. If $\pi(x) = \pi(y)$, the two Lorentzian metrics on $T_{\pi(x)}\pi(C)$ obtained by pushing forward q_x and q_y are in the same conformal class. It follows that $q_{2,n}$ define a natural conformal class of Lorentzian metrics on $\partial\mathbb{ADS}_{1,n}$. □

Definition 1.30. — *The Einstein universe of dimension n, denoted by $Ein_{1,n-1}$, is the smooth hypersurface $\partial\mathbb{ADS}_{1,n}$ of $\mathbb{S}(\mathbb{R}^{2,n})$ equipped with its natural conformal Lorentzian structure.*

The space $Ein_{1,n-1}$ is also called the *Klein model* of the Einstein universe.

Remark 1.31. — One can also define the projection of the isotropic cone C in the projective space $\mathbb{P}(\mathbb{R}^{2,n})$. Notice that it is the topological boundary of $\mathsf{AdS}_{1,n}$ in $\mathbb{P}(\mathbb{R}^{2,n})$.
The proof of Lemma 1.29 still holds and shows that the image of $\partial\mathsf{AdS}_{1,n}$ is naturally equipped with a conformal Lorentzian structure. We denote by $\mathsf{Ein}_{1,n-1}$ the space $\partial\mathsf{AdS}_{1,n}$ equipped with its natural conformal Lorentzian structure.

Proposition 1.32. — *The Einstein universe of dimension n is conformally isometric to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ equipped with the conformal class of the Lorentzian metric $-d\theta^2 + d\sigma^2$, where $d\theta^2$ and $d\sigma^2$ are the round metrics over $\mathbb{S}^1$ and $\mathbb{S}^{n-1}$.*

Proof. — Let S be the Euclidean sphere of radius $\sqrt{2}$ of $\mathbb{R}^{2,n}$. The projection map π restricted to $S \cap C$ is injective with image $Ein_{1,n-1}$. Let f be the map from $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ to $S \cap C$ defined by $f((u, v), (x_1, \ldots, x_n)) = (u, v, x_1, \ldots, x_n)$. It is easy to check that f is a diffeomorphism and that the pull-back of the conformal class of $Ein_{1,n-1}$ by $\pi \circ f$ coincides with the conformal class of $-d\theta^2 + d\sigma^2$. □

It follows from Proposition 1.32 that $Ein_{1,n-1}$ is oriented and time-oriented by the timelike vector field $\frac{\partial}{\partial_\theta}$. Thus, $Ein_{1,n-1}$ is a *spacetime*.

Proposition 1.33. — *Anti-de Sitter spacetime $AdS_{1,n-1}$ embeds conformally in the Einstein universe $Ein_{1,n-1}$.*

Proof. — It is a direct consequence of Propositions 1.4 and 1.32. □

Causality of the Einstein universe. We briefly recall here some causality properties of the Einstein universe that will be useful for us. We direct the reader to [22, Chapter 2] for more information and more details.

Lemma 1.34. — *Every causal (timelike) curve of $Ein_{1,n-1} \simeq \mathbb{S}^1 \times \mathbb{S}^{n-1}$ can be locally parametrized as $(e^{2i\pi t}, x(t))$ where x is a (strictly) 1-Lipschitz map from an interval I of $\mathbb{R}$ to the sphere $\mathbb{S}^{n-1}$. The photons of $Ein_{1,n-1}$ are the causal curves such that in the previous parametrization, $x : I \to \mathbb{S}^{n-1}$ is a geodesic of $\mathbb{S}^{n-1}$ parametrized by its arc length.*

Proof. — See e.g. [23, Lemma 5]. □

Lemma 1.35. — *A photon of the Einstein universe $Ein_{1,n-1}$ is the projectivization of a totally isotropic 2-plane of $\mathbb{R}^{2,n}$.*

Proof. — See e.g. [22, Lemma 2.12, Chapter 2]. □

It follows easily from Lemma 1.35 the following statement.

Corollary 1.36. — *The lightcone of a point $[x]$ in $Ein_{1,n-1}$ is the intersection of $Ein_{1,n}$ with the projectivization of the orthogonal of x with respect to $q_{2,n}$.*

Proof. — The proof is left to the reader. □

Remark 1.37. — Since photons of $\mathsf{Ein}_{1,n-1}$ are projections of photons of the double cover $\mathsf{Ein}_{1,n-1}$, Lemma 1.35 and Corollary 1.36 still hold in $\mathsf{Ein}_{1,n-1}$.

Remark 1.38. — The lightcone of a point ξ in $Ein_{1,n-1}$ minus its vertices ξ and $-\xi$ is the disjoint union of two topological cylinders $\mathbb{S}^{n-2} \times \mathbb{R}$ (see Figure 1.5). In $\mathsf{Ein}_{1,n-1} \subset \mathbb{P}(\mathbb{R}^{2,n})$, the lightcone of a point ξ is a pinched torus (see Figure 1.6). A detailed description is given in [22, Chapter 2].

Lemma 1.39. — *The Einstein universe $Ein_{1,n-1}$ is totally vicious, i.e. the future and the past of every point is the entire spacetime.*

Proof. — See e.g. [23, Corollary 2]. □

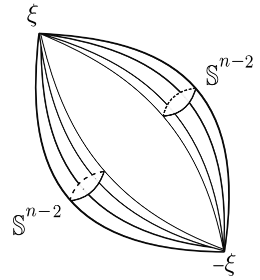

Figure 1.5. Lightcone of a point ξ in the double cover $Ein_{1,n-1}$.

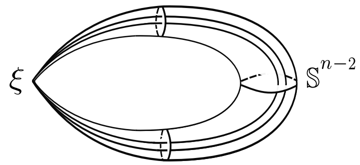

Figure 1.6. Lightcone of a point ξ in $\mathsf{Ein}_{1,n-1}$.

Lemma 1.39 says that causality is trivial in the Einstein universe. However, it turns out that the universal covering of the Einstein universe has a rich causal structure, and is, in particular, globally hyperbolic (see Section 1.3.1).

Remark 1.40. — It is clear that Lemma 1.39 implies that $\mathsf{Ein}_{1,n-1} \subset \mathbb{P}(\mathbb{R}^{2,n})$ is also totally vicious.

Conformal compactification of Minkowski spacetime. In Riemannian geometry, it is known that the round sphere minus a point is conformally isometric to the Euclidean space by the stereographic projection. The situation is similar in the Lorentzian setting. A naive analogy would be that the complement of a point in the Einstein universe is conformally isometric to Minkowski spacetime. However, in the Lorentzian context, one should take into account causality which was latent in the Riemannian setting. Therefore, instead of removing a point, one should remove a lightcone from the Einstein universe to get a conformal copy of Minkowski spacetime (by Lemma 1.39, it would make no sense to remove all the causal curves going through $p \ldots$).

LEMMA 1.41. — *The complement of a lightcone in $\mathsf{Ein}_{1,n-1}$ is conformally isometric to Minkowski spacetime $\mathbb{R}^{1,n-1}$.*

Proof. — Let $(u, v, x_1, \dots, x_n)$ be the canonical coordinate system of $\mathbb{R}^{n+2}$. We make the change of coordinates $u = 1/2(a+b), x_1 = 1/2(a-b)$, $v = t, y_i = x_{i+1}$ for $i = 1, \dots, n-1$. In the coordinate system $(a, b, t, y_1, \dots, y_{n-1})$, we have

$$q_{2,n}(a, b, t, y_1, \dots, y_{n-1}) = -ab - t^2 + y_1^2 + \dots + y_{n-1}^2.$$

We define the conformal map f from $\mathbb{R}^{1,n-1}$ to $\mathbb{P}(\mathbb{R}^{2,n})$ which associates to every point $y = (t, y_1, \dots, y_{n-1})$ in $\mathbb{R}^{1,n-1}$ the point $[q_{1,n-1}(y) : 1 : y]$ where $q_{1,n-1}(y) = -t^2 + y_1^2 + \dots + y_{n-1}^2$. It is clear that the image of f is contained in $\mathsf{Ein}_{1,n-1}$ and that f is injective. An easy computation shows that the image of f is the complement in $\mathsf{Ein}_{1,n-1}$ of the lightcone of $[1:0:0] \in \mathsf{Ein}_{1,n-1}$. □

By Lemma 1.41, the Einstein universe can be seen as *the conformal compactification* of Minkowski spacetime. This corresponds exactly to the compactification defined by Penrose in [21]. We reproduce below the Penrose diagram in $\mathsf{Ein}_{1,n-1}$ (see Figure 1.7) and in $Ein_{1,n-1}$ (see Figure 1.8).

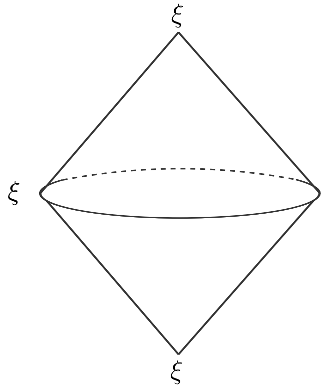

Figure 1.7. Conformal compactification of $\mathbb{R}^{1,2}$ in $\mathsf{Ein}_{1,2}$. The interior of the diamond corresponds to Minkowski spacetime $\mathbb{R}^{1,2}$. The equatorial circle and the vertices of the diamond are identified to the same point ξ in $\mathsf{Ein}_{1,2}$. The boundary at infinity of $\mathbb{R}^{1,2}$ corresponds to the lightcone of ξ in $\mathsf{Ein}_{1,2}$.

Remark 1.42. — In the double cover $Ein_{1,n-1}$, the complement of the lightcone of a point ξ is the disjoint union of two conformal copies of Minkowski spacetime $M(\xi)$ and $M(-\xi)$ where $M(\xi)$ is the open subset of $Ein_{1,n-1}$ given by

$$M(\xi) = \{\xi' \in Ein_{1,n-1} : \langle \xi, \xi' \rangle_{2,n+1} < 0\}$$

where $\langle \xi, \xi' \rangle_{2,n}$ denotes the *sign* of the scalar product $\langle x, x' \rangle_{2,n}$ where $x, x' \in \mathbb{R}^{2,n}$ are representatives of ξ, ξ' (see Figure 1.8).

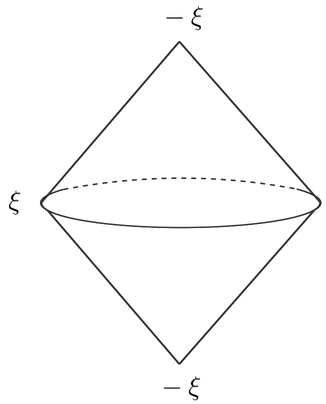

Figure 1.8. Conformal compactification of $\mathbb{R}^{1,2}$ in the double cover $Ein_{1,2}$. The interior of the diamond corresponds to Minkowski spacetime $\mathbb{R}^{1,2}$. The equatorial circle is identified to a point ξ in $Ein_{1,2}$ and the vertices of the diamond are identified to the antipodal point $-\xi$ in $Ein_{1,2}$. The boundary at infinity of $\mathbb{R}^{1,2}$ corresponds to the lightcone of ξ in $Ein_{1,2}$.

1.3.1. Universal Einstein universe

Let $\pi : \widetilde{Ein}_{1,n-1} \to Ein_{1,n-1}$ be a universal covering of the Einstein universe.

The *universal Einstein universe* is the space $\widetilde{Ein}_{1,n-1}$ equipped with the pull back of the natural conformal class of Lorentzian metrics on $Ein_{1,n-1}$. It is conformally isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$ equipped with the conformal class of the Lorentzian metric $-dt^2 + d\sigma^2$, where dt^2 is the usual metric on $\mathbb{R}$ and $d\sigma^2$ the round metric on $\mathbb{S}^{n-1}$.

The fundamental group of $Ein_{1,n-1}$ is isomorphic to $\mathbb{Z}$: it is the cyclic group generated by the conformal diffeomorphism $\delta : \widetilde{Ein}_{1,n-1} \to \widetilde{Ein}_{1,n-1}$ which associates to (t, x) the point $(t + 2\pi, x)$.

The antipodal map $x \in \mathbb{R}^{2,n} \mapsto -x \in \mathbb{R}^{2,n}$ induces a map $\bar{\sigma} : Ein_{1,n-1} \to Ein_{1,n-1}$ such that, in a decomposition $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, $\bar{\sigma}$ is the product of the two antipodal maps of $\mathbb{S}^1$ and $\mathbb{S}^{n-1}$. The map $\bar{\sigma}$ lifts to $\widetilde{Ein}_{1,n-1}$ giving a map $\sigma : \widetilde{Ein}_{1,n-1} \to \widetilde{Ein}_{1,n-1}$ which associates to (t, x) the point $(t + \pi, -x)$. Notice that $\sigma^2 = \delta$. The fundamental group of $\mathsf{Ein}_{1,n-1} \subset \mathbb{P}(\mathbb{R}^{2,n})$ is the group generated by σ.

Causal structure of the universal Einstein universe. We give a quick description of the causal structure of $\widetilde{Ein}_{1,n-1}$ and we direct the reader to [22, Chapter 2] for more details.

Since the projection $\pi : \widetilde{Ein}_{1,n-1} \to Ein_{1,n-1}$ is conformal, the projection of a causal curve (resp. a photon) of $\widetilde{Ein}_{1,n-1}$ is a causal curve (resp. a photon) of $Ein_{1,n-1}$. Therefore, we deduce easily from Lemma 1.34 the following characterization of the causal curves in $\widetilde{Ein}_{1,n-1}$.

Lemma 1.43. — *Every causal (timelike) curve of $\widetilde{Ein}_{1,n-1}$ can be parametrized $(t, x(t))$ where x is a 1-Lipschitz map from an interval of $\mathbb{R}$ to $\mathbb{S}^{n-1}$. The lightlike geodesics are the causal curves such that in the previous parametrization x is a geodesic of $\mathbb{S}^{n-1}$ parametrized by its arc length.*

Remark 1.44. — A causal curve of $\widetilde{Ein}_{1,n-1}$ is inextendible if the parametrization given by Lemma 1.43 is defined for every $t \in \mathbb{R}$.

Corollary 1.45. — *The universal Einstein universe $\widetilde{Ein}_{1,n-1}$ is globally hyperbolic with Cauchy hypersurfaces homeomorphic to $\mathbb{S}^{n-1}$. In particular, $\widetilde{Ein}_{1,n-1}$ is Cauchy-compact.*

Proof. — Set $S := \{0\} \times \mathbb{S}^{n-1} \subset \widetilde{Ein}_{1,n-1} \simeq \mathbb{R} \times \mathbb{S}^{n-1}$. Let $t \in \mathbb{R} \mapsto c(t) = (t, x(t))$ be an inextendible causal curve in $\widetilde{Ein}_{1,n-1}$. The curve c meets S in the unique point $(0, x(0))$. This proves that S is a Cauchy hypersurface of $\widetilde{Ein}_{1,n-1}$ which is compact. The corollary follows. □

The description of photons of $\widetilde{Ein}_{1,n-1}$ shows that:

Lemma 1.46. — *All photons of $\widetilde{Ein}_{1,n-1}$ starting from a point p meet at the points $\sigma^k(p)$.*

Definition 1.47. — *The points $\sigma^k(p)$ in $\widetilde{Ein}_{1,n-1}$, with $k \in \mathbb{Z}$, are said to be* conjugate.

It follows from Lemma 1.43 a simple description of the future (past) of a point p in $\widetilde{Ein}_{1,n-1}$ (see Figure 1.10). We denote by d_0 the distance on the sphere $\mathbb{S}^{n-1}$ induced by the round metric.

Lemma 1.48. — *The causal future (resp. past) of a point (t_0, x_0) in $\widetilde{Ein}_{1,n-1}$ is the set of points (t, x) such that $d_0(x, x_0) \leqslant t - t_0$ (resp. $d_0(x, x_0) \leqslant t_0 - t$).*

Proof. — See e.g. [22, Lemme 2.18, Chapter 2]. □

Remark 1.49. — The chronological future (resp. past) admits a similar description by replacing the large inequalities by strict inequalities.

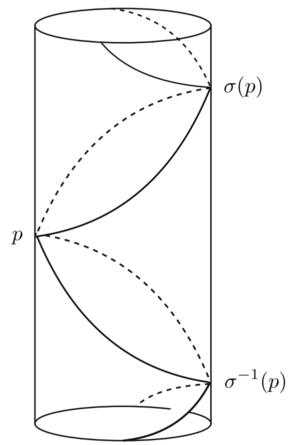

Figure 1.9. Conjugate points in $\widetilde{Ein}_{1,n-1}$ (in the picture $n = 2$).

Affine charts. Let $p \in \widetilde{Ein}_{1,n-1}$. Let $\mathrm{Mink}_0(p)$ be the set of points in $\widetilde{Ein}_{1,n-1}$ which are not causally related to p, namely

$$\mathrm{Mink}_0(p) = \widetilde{Ein}_{1,n-1} \setminus \left(J^+(p) \cup J^-(p)\right).$$

LEMMA 1.50. — *The restriction of the projection $\pi : \widetilde{Ein}_{1,n-1} \to Ein_{1,n-1}$ to the open subset $\mathrm{Mink}_0(p)$ is injective. Besides, its image is exactly the conformal copy of Minkowski spacetime*[8]

$$M(\xi) = \{\xi' \in Ein_{1,n-1} : \langle \xi, \xi' \rangle_{2,n} < 0\}$$

with $\xi = \pi(p)$.

Proof. — See e.g. [22, Lemma 2.21, Chapter 2]. □

Lemma 1.50 motivates the following definition.

DEFINITION 1.51. — *We call* affine chart *of $\widetilde{Ein}_{1,n-1}$ any open subset $\mathrm{Mink}_0(p)$ with $p \in \widetilde{Ein}_{1,n-1}$.*

Remark 1.52. — An affine chart $\mathrm{Mink}_0(p)$ of $\widetilde{Ein}_{1,n-1}$ is causally convex in $\widetilde{Ein}_{1,n-1}$. Indeed, suppose there is a causal curve γ of $\widetilde{Ein}_{1,n-1}$ connecting two points q, q' of $\mathrm{Mink}_0(p)$ which is not contained in $\mathrm{Mink}_0(p)$. This means that there is a point q_0 of γ causally related to p. Since the causality relation is transitive, it follows that q or q' is causally related to p. Contradiction.

(8) See Remark 1.42.

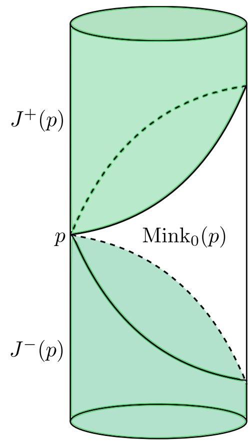

Figure 1.10. Affine chart in $\widetilde{Ein}_{1,n-1}$ (in the picture $n = 2$).

Invisible domains. Let Λ be a closed subset of $\widetilde{Ein}_{1,n-1}$ and let $\Omega(\Lambda)$ be the set of points of $\widetilde{Ein}_{1,n-1}$ non-causally related to any point in Λ:

$$\Omega(\Lambda) := \widetilde{Ein}_{1,n-1} \backslash \left(J^+(\Lambda) \cup J^-(\Lambda)\right).$$

The open subset $\Omega(\Lambda)$ is usually called *the invisible domain from* Λ (see e.g. [1, 3]). Notice that affine charts are invisible domains where Λ is reduced to a single point. As for affine charts, invisible domains are causally convex in $\widetilde{Ein}_{1,n-1}$ (see Remark 1.52).

Lemma 1.53. — *Let Λ be a closed subset of $\widetilde{Ein}_{1,n-1}$. The restriction of the projection $\pi : \widetilde{Ein}_{1,n-1} \to Ein_{1,n-1}$ to the invisible domain $\Omega(\Lambda)$ is injective. Besides, its image is exactly*

$$\{\xi' \in Ein_{1,n-1} : \langle \xi, \xi' \rangle_{2,n} < 0, \ \forall\, \xi \in \Lambda\}$$

Proof. — The invisible domain $\Omega(\Lambda)$ is contained in the affine chart $\mathrm{Mink}_0(p)$ defined by a point p of Λ. Then, by Lemma 1.50, the restriction of π to $\Omega(\Lambda)$ is injective. The description of the image $\pi(\Omega(\Lambda))$ follows immediately from Lemma 1.58. □

1.3.2. Achronality

Let $\pi : \widetilde{Ein}_{1,n-1} \to Ein_{1,n-1}$ be a universal covering of $Ein_{1,n-1}$. Although there is no achronal subset in $Ein_{1,n-1}$ (see Lemma 1.39), the Einstein universe $Ein_{1,n-1}$ inherits from $\widetilde{Ein}_{1,n-1}$ a notion of achronality that we define here.

DEFINITION 1.54. — *A subset A of $Ein_{1,n-1}$ is called* achronal *(resp.* acausal*) if it is the projection of an achronal (resp. acausal) subset of $\widetilde{Ein}_{1,n-1}$.*

This definition is motivated by the two following lemmas.

LEMMA 1.55. — *The restriction of the projection π to any achronal subset is injective.*

Proof. — See e.g. [19, Lemma 2.4]. ☐

LEMMA 1.56. — *Let $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2$ be two achronal subsets of $\widetilde{Ein}_{1,n-1}$ admitting the same projection in $Ein_{1,n-1}$. Then, there exists an integer k such that*

$$\widetilde{\Lambda}_1 = \delta^k \widetilde{\Lambda}_2$$

where δ is the generator of the fundamental group of $Ein_{1,n-1}$ introduced above.

Proof. — See e.g. [19, Corollary 2.5]. ☐

In what follows, we focus on acausal subsets of $Ein_{1,n-1}$. It turns out that the notion of acausal subsets coincide with that of *negative subsets* [(9)] of $Ein_{1,n-1}$. It has been proved for loops by Labourie–Toulisse–Wolf in [17, Proposition 2.11] (a negative loop is called positive in their terminology). We generalized their statement for any subset of $Ein_{1,n-1}$ in [24].

DEFINITION 1.57. — *A subset Λ of $Ein_{1,n-1}$ is called* negative *if for every points $[x], [y]$ in Λ, the sign of the product $\langle x, y\rangle_{2,n}$ is negative.*

LEMMA 1.58. — *Two distinct points p, q of $Ein_{1,n-1}$ can be lifted to points $\widetilde{p}, \widetilde{q}$ of $\widetilde{Ein}_{1,n-1}$ which are not extremities of a causal curve if and only if the sign of product $\langle x, y\rangle_{2,n}$ is negative, where x and y are representatives of p and q.*

Proof. — See e.g. [1, Lemma 10.13]. ☐

PROPOSITION 1.59 ([24, Proposition 2.47]). — *Negative subsets of $Ein_{1,n-1}$ are exactly the acausal ones.*

(9) This notion has been defined in [9] (see Definition 1.9) in the broader context of pseudo-Riemannian geometry. We give here the definition in the Lorentzian setting.

1.4. Conformally flat spacetimes

Definition 1.60. — *A spacetime M is* conformally flat *if it is locally conformal to Minkowski spacetime.*

By Lemma 1.41, the Einstein universe is conformally flat. It follows that any spacetime locally modeled on the Einstein universe, i.e. equipped with a (G, X)-structure with $G = O_0(2, n)$ and $X = Ein_{1,n-1}$, is conformally flat. It turns out that in dimension greater or equal to 3, Lorentzian conformally flat structures are rigid, in other words any conformally flat structure is actually a $(O_0(2, n), Ein_{1,n-1})$-structure:

Proposition 1.61. — *Let M be a smooth manifold of dimension $n \geqslant 3$. A conformally flat Lorentzian structure on M is equivalent to a $(O_0(2, n), Ein_{1,n-1})$-structure.*

This is a easy consequence of the Lorentzian version of Liouville's theorem that we recall here:

Theorem 1.62 (Liouville). — *Let $n \geqslant 3$ be an integer. Any conformal map between two open subsets of $Ein_{1,n-1}$ is the restriction of an element of $O(2, n)$.*

The Lorentzian version of Liouville's theorem implies also that the group of conformal transformations of $Ein_{1,n-1}$ is $O(2, n)$. Indeed, $O(2, n)$ acts naturally on $Ein_{1,n-1}$ and conversely, by Theorem 1.62, every conformal transformation of $Ein_{1,n-1}$ is an element of $O(2, n)$. The group of orientation and time-orientation preserving isometries of $Ein_{1,n-1}$ is the identity component $O_0(2, n)$ of $O(2, n)$.

1.5. Maximality

There is a partial order relation on *globally hyperbolic* spacetimes defined as follow.

Definition 1.63. — *A map $f : (M, g) \to (N, h)$ between two globally hyperbolic spacetimes (M, g) and (N, h) is* a Cauchy-embedding *if*

(1) *f is an isometry, i.e. $f^*h = g$,*
(2) *f sends a Cauchy hypersurface of M on a Cauchy hypersurface of N.*

In this case, we say that N is a Cauchy-extension of M.

Remark 1.64. — This definition does not depend on the choice of the Cauchy hypersurface of M in the second condition. Indeed, Rossi proved that a Cauchy-embedding f from M to N sends *every* Cauchy hypersuface of M on a Cauchy hypersurface of N (see [22, Corollaire 2.3, Chapitre 3]).

Definition 1.65. — *A globally hyperbolic spacetime M is said to be* maximal *if every Cauchy-embedding from M to any other globally hyperbolic spacetime N is surjective.*

A natural question is the existence of a maximal extension and if the answer is positive, is it unique, up to isometric diffeormorphism? The answers to these two questions are yes when we restrict to *a rigid category* of spacetimes, this is the case, for instance, of the category of spacetimes of constant curvature in dimension $\geqslant 3$ (see [22, Chapter 3] for more details).

C-maximality and $\mathcal{C}_0$-maximality. In [23], Rossi extends the notion of maximality to globally hyperbolic *conformal* spacetimes. She defines *conformal* Cauchy-embeddings by requiring that f is *a conformal isometry*, namely that f^*h belongs to the conformal class of g, instead of an isometry.

Definition 1.66. — *A globally hyperbolic conformal spacetime M is* C-maximal *if every conformal Cauchy-embedding from M to any other globally hyperbolic conformal spacetime is surjective.*

Moreover, Rossi proves in [23] that when we restrict to the category of conformally flat spacetimes of dimension greater or equal to 3, the existence and the uniqueness, up to conformal diffeomorphism, of the maximal extension is ensured.

Definition 1.67. — *A globally hyperbolic conformally flat spacetime M is* $\mathcal{C}_0$-maximal *if every conformal Cauchy-embedding from M to any other globally hyperbolic* conformally flat *spacetime is surjective.*

Theorem 1.68 ([23, Theorem 3]). — *Every globally hyperbolic conformally flat spacetime M of dimension $n \geqslant 3$ admits a $\mathcal{C}_0$-maximal extension. Moreover, this extension is unique up to conformal diffeomorphism.*

Notice that if a conformally flat spacetime is C-maximal then it is in particular, maximal among conformally flat spacetimes, in short, it is $\mathcal{C}_0$-maximal. À priori, there is no reason the converse assertion is true. However, Rossi proved in [22, Chapter 7] that it is: a conformally flat spacetime is $\mathcal{C}_0$-maximal if and only if it is C-maximal. For the sake of lightness, from now on, we simply say for a $\mathcal{C}_0$-maximal conformally flat spacetime that it is *maximal* (while keeping in mind that it is in the conformal sense).

In what follows we use the abbreviation GHMC for "globally hyperbolic maximal Cauchy-compact" spacetime.

2. Anosov representations and conformally flat spacetimes

In this section, we establish a link between Anosov representations in $O_0(2,n)$ and GHMC conformally flat spacetimes. Let us first recall briefly the definition of Anosov representation that will be useful for us.

2.1. Anosov representations

Let $\rho : \Gamma \to O_0(2,n)$ be a representation from a Gromov hyperbolic group [(10)] Γ in the semi-simple Lie group $O_0(2,n)$. Since $O_0(2,n)$ is of rank 2, there are two ways for ρ of being Anosov. We focus here on the so-called *P_1-Anosov representations*.

We denote by $\partial\Gamma$ the Gromov boundary of Γ. Let us recall that Γ is a convergence group for its action on $\partial\Gamma$: for any divergent sequence $\{\gamma_i\}$ of Γ, there exist a subsequence $\{\gamma_{i_j}\}$ and points η_+ and η_- in $\partial\Gamma$ such that $\{\gamma_{i_j}\}$ converges uniformly on compact subsets of $\partial\Gamma\backslash\{\eta_-\}$ towards the constant map η_+. This is what we call *a north-south dynamics* (see Figure 2.1).

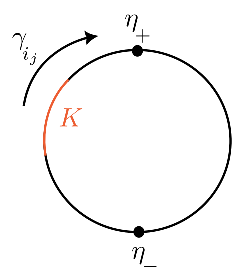

Figure 2.1. North-south dynamics on the Gromov boundary of Γ.

The definition of P_1-Anosov representations involves on the one hand, the north-south dynamics on $\partial\Gamma$ under the action of Γ and on the other hand, a kind of north-south dynamics on the Einstein universe $Ein_{1,n-1}$ under the action of sequences of $O_0(2,n)$ called *P_1-divergent*. A sequence $\{g_i\}$ of $O_0(2,n)$ is P_1-divergent if:

(10) We direct the reader who is not familiar with the notion of Gromov hyperbolic groups to [12, Section 2.1].

- $\{g_i\}$ is divergent, i.e. leaves every compact subset of $O_0(2,n)$,
- there exist a subsequence $\{g_{i_j}\}$, an attracting point $p_+ \in Ein_{1,n-1}$ and a repelling point $p_- \in Ein_{1,n-1}$ such that $\{g_{i_j}\}$ converges uniformly to p_+ on every compact set K of $Ein_{1,n-1}$ in the complement of the lightcone of p_-.

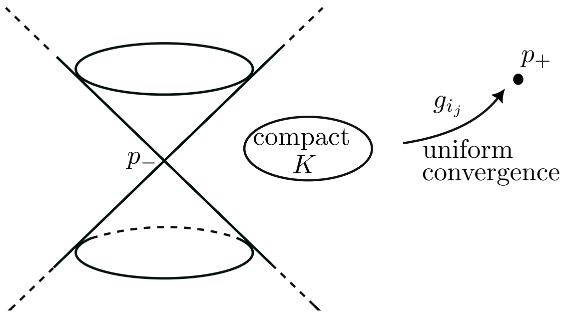

Figure 2.2. Dynamics on $Ein_{1,n-1}$ under the action of a P_1-divergent sequence $\{g_i\}$.

Definition 2.1. — *A representation $\rho : \Gamma \to O_0(2,n)$ of a Gromov hyperbolic group Γ to $O_0(2,n)$ is P_1-Anosov if*

(1) *every sequence of pairwise distinct elements of $\rho(\Gamma)$ is P_1-divergent,*
(2) *there is a continuous ρ-equivariant map $\xi : \partial\Gamma \to Ein_{1,n-1}$ which is*
   (a) tranverse, *meaning that for any pairwise distinct elements η, η' in $\partial\Gamma$, the points $\xi(\eta)$ and $\xi(\eta')$ are not related by a lightlike geodesic;*
   (b) dynamics-preserving, *meaning that if η is the attracting fixed point of some element $\gamma \in \Gamma$ in $\partial\Gamma$, then $\xi(\eta)$ is an attracting fixed point of $\rho(\gamma)$ in $Ein_{1,n-1}$.*

The *limit set* of a P_1-Anosov representation $\rho : \Gamma \to O_0(2,n)$ is the set of all attracting points of P_1-divergent sequences of $\rho(\Gamma)$ and is denoted by Λ_ρ. It turns out that Λ_ρ is exactly the image by ξ of $\partial\Gamma$. In particular, the limit set Λ_ρ is $\rho(\Gamma)$-invariant.

A P_1-Anosov representation $\rho : \Gamma \to O_0(2,n)$ is called *negative* if the limit set Λ_ρ is a negative subset of $Ein_{1,n-1}$ (see Definition 1.57). Notice that in this case, by Proposition 1.59, the limit set admits an *acausal* lift in $\widetilde{Ein}_{1,n-1}$.

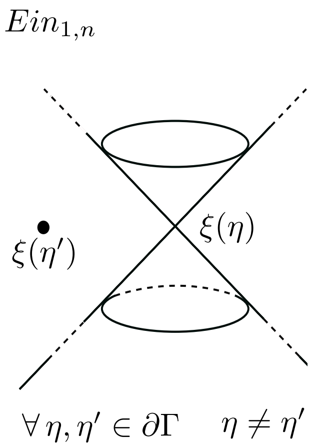

Figure 2.3. Tranversality.

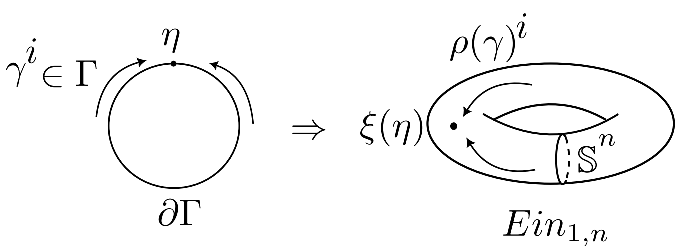

Figure 2.4. The "dynamics-preserving" property.

2.2. Link with conformally flat spacetimes

In [24], we relate negative P_1-Anosov representations to GHCM conformally flat spacetimes. More precisely, we prove the following result.

Theorem 2.2. — *Let Γ be a Gromov hyperbolic group. Every negative P_1-Anosov representation $\rho : \Gamma \to O_0(2,n)$ is the holonomy of a GHMC conformally flat spacetime.*

The particular case where the limit set is a topological $(n-1)$-sphere is due to Barbot–Mérigot [19] (such representations are called *quasi-Fuchsian* in [19]): they proved that ρ is the holonomy of a GHMC AdS-spacetime of dimension $(n+1)$. In their proof, the authors see $Ein_{1,n-1}$ as the conformal boundary of anti-de Sitter spacetime $AdS_{1,n}$ and define the invisible domain $\Omega(\Lambda_\rho)$ from Λ_ρ in $AdS_{1,n}$ as the set of points in $AdS_{1,n}$ which are not causally related to any point in Λ_ρ. They show that the invisible domain is preserved by $\rho(\Gamma)$ and that the action is free and properly discontinous. Then they prove that the quotient space $\rho(\Gamma)\backslash\Omega(\Lambda)$ is GHMC.

Theorem 2.2 extends Barbot–Mérigot result to the case where the limit set is not a topological $(n-1)$-sphere. An application of our result is that it gives remarkable examples of GHCM conformally flat spacetimes that we describe in the next section. Before that, let us give the outline of the proof of Theorem 2.2.

Outline of the proof.

- We mimic Barbot–Mérigot construction by defining *the invisible domain from the limit set* in $Ein_{1,n-1}$:

  $$\Omega(\Lambda_\rho) := \{[x] \in Ein_{1,n-1} : \langle x, x_0\rangle_{2,n} < 0 \ \ \forall\, x_0 \in \Lambda_\rho\}\,.$$

  This is the set of points in $Ein_{1,n-1}$ which are not causally related to any point in Λ_ρ (see Lemma 1.58).

  The fact that Λ_ρ is not a topological $(n-1)$-sphere ensures that the invisible domain $\Omega(\Lambda_\rho)$ is not empty (see [24, Corollary 5.6, Section 5.1]). The invisible domain $\Omega(\Lambda_\rho)$ is a causally convex open subset of $Ein_{1,n-1}$. In particular, it is globally hyperbolic (see Lemma 1.25). Furthermore, it is maximal (see [24, proposition 5.27]). Notice that $\Omega(\Lambda_\rho)$ is preserved by $\rho(\Gamma)$; this is a direct consequence of the fact that the limit set is preserved by $\rho(\Gamma)$.
- The "north-south" dynamics on $Ein_{1,n-1}$ under the action of $\rho(\Gamma)$ allows us to prove that the invisible domain $\Omega(\Lambda_\rho)$ is *a domain of discontinuity*, namely that the action of $\rho(\Gamma)$ on $\Omega(\Lambda_\rho)$ is free and properly discontinuous (see [24, Prop. 5.16]).

  It follows that the quotient $\rho(\Gamma)\backslash\Omega(\Lambda_\rho)$ is a conformally flat spacetime called M_ρ. The dynamical properties of the action shows, without great difficulties, that the global hyperbolicity and the maximality of the invisible domain $\Omega(\Lambda_\rho)$ descends to the quotient M_ρ (see [24, Prop. 5.19 and Prop. 5.27]).
- The tricky point of the proof is *the Cauchy-compactness*. It turns out that it is a consequence of a general result of Guichard–Kassel–Wienhard [13, Theorem 4.1] in pseudo-Riemannian geometry that we state here in the Lorentzian setting (see Proposition 2.4). The proof is based on the following observation.

FACT 2.3 ([24, Proposition 5.20]). — *Let M be a globally hyperbolic spacetime. There is a canonical bijection between the unit tangent bundle of a Cauchy hypersurface of M and the set of photons of M.*

Given a Cauchy hypersurface S of M_ρ, it is sufficient to prove that $T^1 S$ is compact to show that S is compact. By Fact 2.3, this is equivalent to prove that the space of photons[11] of M_ρ, denoted by $\mathcal{P}(M_\rho)$, is compact.

The compactness of $\mathcal{P}(M_\rho)$ is obtained from Guichard-Kassel-Wienhard result [13, Theorem 4.1] applied to the Lorentzian setting. Their result concerns what we call *the space of causal geodesics* defined as the set consisting in timelike geodesics of $AdS_{1,n}$, photons of $AdS_{1,n}$ and photons of $Ein_{1,n-1} = \partial AdS_{1,n}$ (see [24, Section 4]). It is in bijection with the subset of Grassmannian $Gr_2(\mathbb{R}^{2,n})$ defined by[12]

$$\left\{P \in Gr_2\left(\mathbb{R}^{2,n}\right) : \langle x, x\rangle_{2,n} \leqslant 0 \ \ \forall\, x \in P\right\}.$$

Guichard–Kassel–Wienhard result [13, Theorem 4.1] can be rephrased as follow:

Proposition 2.4. — *For every representation P_1-Anosov ρ of a Gromov hyperbolic group Γ into $O_0(2,n)$, the action of $\rho(\Gamma)$ on the space of causal geodesics* which avoid the limit set, *is free, properly discontinuous and cocompact.*

We denote by U the space of causal geodesics which avoid the limit set Λ_ρ. The compactness of $\mathcal{P}(M_\rho)$ is deduced from Proposition 2.4 in two steps:

(1) On the one hand, we prove that $\mathcal{P}(M_\rho)$ is homeomorphic to the quotient of the space $\mathcal{P}(\Omega(\Lambda_\rho))$ of photons of $\Omega(\Lambda_\rho)$ by $\rho(\Gamma)$ (see [24, Proposition 5.25]).
(2) On the other hand, we prove that $\mathcal{P}(\Omega(\Lambda\rho))$ is exactly the set of photons of $Ein_{1,n-1}$ which avoid the limit set, namely the intersection of U with the space $\mathcal{P}(Ein_{1,n-1})$ of photons of $Ein_{1,n-1}$ (see [24, Lemma 5.22 and Corollary 5.23]).

By Proposition 2.4, the quotient $\rho(\Gamma)\backslash U$ is compact. Since

$$\mathcal{P}(Ein_{1,n-1}) \simeq T^1\mathbb{S}^{n-1}$$

is compact, we easily deduce that $\rho(\Gamma)\backslash(U \cap \mathcal{P}(Ein_{1,n-1}))$ is compact (see [24, Prop. 5.24]). Therefore,

[11] The set of photons of M is equipped with the topology for which the canonical bijection with $T^1 S$ is a homeomorphism.
[12] Geodesics of $AdS_{1,n}$ are described in Example 1.11 and photons of $Ein_{1,n-1}$ in Lemma 1.35.

$$T^1 S \simeq \mathcal{P}(M_\rho) \simeq \rho(\Gamma)\backslash\mathcal{P}(\Omega(\Lambda_\rho)) \simeq \rho(\Gamma)\backslash(U \cap \mathcal{P}(Ein_{1,n-1}))$$

is compact. Thus, S is compact.

Remark that the spacetime M_ρ we constructed is of dimension n while, in the case of Barbot–Mérigot, it is of dimension $(n+1)$. However, we can also see $Ein_{1,n-1}$ as a submanifold of codimension 1 in $Ein_{1,n}$ and construct the invisible domain from Λ_ρ in $Ein_{1,n}$, so we get a conformally flat spacetime of dimension $(n+1)$ which is still GHMC.

3. Examples

In this section, we use Theorem 2.2 to construct remarkable examples of GHMC conformally flat spacetimes, namely:

- *black-white holes*, presented in Section 3.1;
- *conformally flat Misner spacetimes*, presented in Section 3.2
- *Misner extensions*, presented in Section 3.3.

3.1. Black-white holes

Let $\rho : \Gamma \to O_0(2,n)$ be a negative P_1-Anosov representation of a Gromov hyperbolic group Γ in $O_0(2,n)$ such that:

(1) $\rho(\Gamma)$ fixes a point ξ in $Ein_{1,n-1}$;
(2) the limit set Λ_ρ is not a topological $(n-2)$-sphere and is contained in the lightcone of ξ.

The first condition can be rephrased by saying that $\rho(\Gamma)$ is a subgroup of the stabilizer of ξ in $O_0(2,n)$. This last one turns out to be isomorphic to the group of conformal transformations of Minkowski spacetime $\mathbb{R}^{1,n-1}$, which is $(\mathbb{R}^*.O_0(1,n-1)) \ltimes \mathbb{R}^{1,n-1}$.

By Theorem 2.2, the representation ρ is the holonomy of a GHMC conformally flat spacetime M_ρ, obtained as the quotient by $\rho(\Gamma)$ of the invisible domain $\Omega(\Lambda_\rho)$ from the limit set Λ_ρ (see Section 2.2).

Notice that the intersection of $\Omega(\Lambda_\rho)$ with the lightcone of ξ is the union of the lightlike geodesics with extremities ξ and $-\xi$ which avoid the limit set Λ_ρ. Every connected component of this union is called *a horizon*[13].

[13] The condition that the limit set Λ_ρ is not a topological $(n-1)$-sphere ensures the existence of horizons. Indeed, otherwise, every point of the lightcone of ξ would be related to a point of Λ_ρ by a lightlike geodesic, so the intersection of $\Omega(\Lambda_\rho)$ with the lightcone of ξ would be empty.

The lightcone of ξ disconnects $Ein_{1,n-1}$ in two affine charts $M(\xi)$ and $M(-\xi)$ conformally diffeomorphic to $\mathbb{R}^{1,n-1}$ (see Section 1.3, Remark 1.42). The fact that the limit set Λ_ρ is contained in the lightcone of ξ ensures that the invisible domain $\Omega(\Lambda_\rho)$ intersects each affine chart in an open subset which is conformally diffeomorphic to *a regular domain* of $\mathbb{R}^{1,n-1}$ (see Example 1.23): one of them is future and will be interpreted as *a white hole* so we denote it W, the other one is past and will be interpreted as *a black hole* so we denote it B (see Figure 3.1).

Indeed, B satisfies the property that no photon of $\Omega(\Lambda_\rho)$ going through a point $\xi_0 \in B$ can escape from B in the future, it can only escape through one of the horizons in the past. This is why B is called *a black hole*.

Similarly, no photon of $\Omega(\Lambda_\rho)$ going through a point $\xi_0 \in W$ can escape from W in the past, it can only escape through one of the horizons in the future. This is why W is called *a white hole*.

The second condition in the definition of ρ, requiring that the limit set is not a topological $(n-2)$-sphere, ensures the existence of horizons and consequently the black-white hole decomposition of the invisible domain. The first condition ensures that this black-white hole decomposition is preserved by $\rho(\Gamma)$ and descends to the quotient. Indeed, since $\rho(\Gamma)$ fixes ξ, it preserves the lightcone of ξ and the affine charts $M(\xi)$ and $M(-\xi)$. Therefore, $\rho(\Gamma)$ preserves the decomposition of $\Omega(\Lambda_\rho)$ as the disjoint union of a black-hole B, horizons H_i and a white-hole W. It follows that M_ρ is the disjoint union of $\mathcal{B} := \rho(\Gamma)\backslash B$, horizons $\mathcal{H}_i := \rho(\Gamma)\backslash H_i$ and $\mathcal{W} := \rho(\Gamma)\backslash W$. Moreover, a photon of M_ρ going through a point of $\mathcal{B}$ (resp. $\mathcal{W}$) can not escape from $\mathcal{B}$ (resp. $\mathcal{W}$) in the future (resp. past), but only in the past (resp. future) through an horizon $\mathcal{H}_i$.

The description above motivates the following definition.

Definition 3.1. — *The GHMC conformally flat spacetime M_ρ is called* a black-white hole.

Remark 3.2. — Black-white holes may be disconnected (we give an example in Remark 3.8, Section 3.3).

3.2. Conformally flat Misner spacetimes

Let $\mathbb{R}^{2,n} = \mathbb{R}^{1,\ell} \oplus^\perp \mathbb{R}^{1,k}$ be an orthogonal splitting of $\mathbb{R}^{2,n}$, with $\ell, k \in \mathbb{N}^*$ such that $n = \ell + k$ and $\ell \leqslant n-2$. Let $O_0(1,\ell) \times O_0(1,k)$ be the subgroup of $O_0(2,n)$ preserving this splitting.

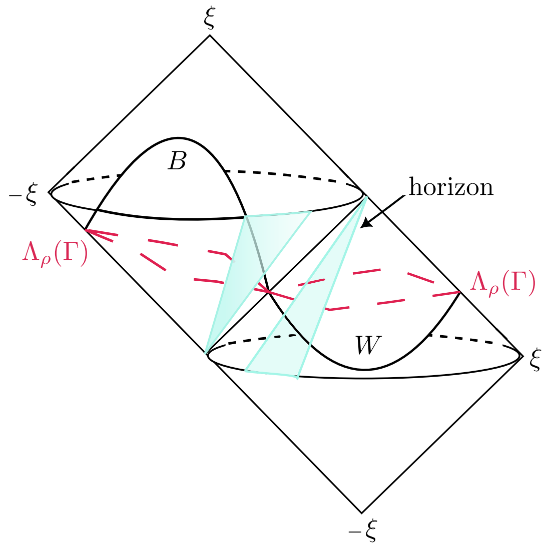

Figure 3.1. The black-white hole decomposition of the invisible domain $\Omega(\Lambda_\rho)$ in $Ein_{1,2}$. The picture represents the affine charts $M(\xi)$ and $M(-\xi)$ glued along the lightcone of ξ. The equatorial circles are identified to ξ and $-\xi$ respectively. The upper (resp. lower) half cone of the conformal boundary of $M(\xi)$ is identified to the lower (resp. upper) half cone of the conformal boundary of $M(-\xi)$.

We denote by $q_{1,\ell}$ and $q_{1,k}$ the restrictions of the quadratic form $q_{2,n}$ to $\mathbb{R}^{1,\ell}$ and $\mathbb{R}^{1,k}$ respectively. We have $q_{2,n} = q_{1,\ell} + q_{1,k}$. Recall that $\mathbb{S}(\mathbb{R}^{2,n})$ is the quotient of $\mathbb{R}^{2,n}\backslash\{0\}$ by the equivalence relation $v \sim \lambda v$ with $\lambda > 0$.

We consider a negative P_1-Anosov representation ρ of a Gromov hyperbolic group Γ in $O_0(2,n)$ defined by a pair (ρ_ℓ, ρ_k) where

(1) $\rho_\ell : \Gamma \to O_0(1,\ell)$ is a convex cocompact representation such that the limit set of ρ_ℓ is the conformal sphere $\mathbb{S}^{\ell-1}$;
(2) $\rho_k : \Gamma \to O_0(1,k)$ is a relatively compact representation, i.e. the image $\rho_k(\Gamma)$ is contained in a compact of $O_0(1,k)$.

The limit set Λ_ρ of the representation $\rho = (\rho_\ell, \rho_k)$ is the conformal sphere $\mathbb{S}^{\ell-1}$. Notice that this last one can be seen as a connected component of the projection in $\mathbb{S}(\mathbb{R}^{2,n})$ of the quadric

$$\left\{(x,0) :\in \mathbb{R}^{1,\ell} \oplus \mathbb{R}^{1,k} :\ q_{1,\ell}(x) = 0\right\}.$$

Moreover, the projection in $\mathbb{S}(\mathbb{R}^{2,n})$ of the quadric

$$\left\{(0,y) :\in \mathbb{R}^{1,\ell} \oplus \mathbb{R}^{1,k} :\ q_{1,k}(y) = 0\right\}$$

is the disjoint union of two antipodal conformal spheres $\mathbb{S}_+^{k-1}$, $\mathbb{S}_-^{k-1}$ of dimension $(k-1)$ contained in $Ein_{1,n-1}$ and preserved by $\rho(\Gamma)$. Since the splitting of $\mathbb{R}^{2,n}$ is orthogonal, it follows from Corollary 1.36 that the limit set $\mathbb{S}^{\ell-1}$ is contained in the lightcone of each point of $\mathbb{S}_+^{k-1} \sqcup \mathbb{S}_-^{k-1}$.

Let $\Omega(\Lambda_\rho)$ be the invisible domain from the limit set Λ_ρ. In what follows, we describe the invisible domain $\Omega(\Lambda_\rho)$. We will see that it presents similarities with that of black-white holes introduced in the previous section.

Homogeneous model of the invisible domain. Let $\mathcal{C}_\ell$ be the connected component of the causal cone of $\mathbb{R}^{1,\ell}$ defining $\mathbb{S}^{\ell-1}$. More precisely, $\mathbb{S}^{\ell-1}$ is the projection in $\mathbb{S}(\mathbb{R}^{2,n})$ of the boundary of $\mathcal{C}_\ell$ in $\mathbb{R}^{1,\ell}$. Let

$$\mathbb{H}^\ell = \{x \in \mathcal{C}_\ell : \ q_{1,\ell}(x) = -1\}$$

be the hyperbolic space of dimension ℓ and let

$$dS_{1,k-1} = \left\{y \in \mathbb{R}^{1,k} : \ q_{1,k}(y) = 1\right\}$$

be the de Sitter space of dimension k.

Proposition 3.3. — *The invisible domain $\Omega(\Lambda_\rho)$ is conformally diffeomorphic to the homogeneous space $\mathbb{H}^\ell \times dS_{1,k-1}$.*

Proof. — We denote by $\langle .,. \rangle_{1,\ell}$ the bilinear form on $\mathbb{R}^{1,\ell}$ associated to the quadratic form $q_{1,\ell}$. By Lemma 1.58, the invisible domain $\Omega(\Lambda_\rho)$ is the set of points $[x:y]$ of $Ein_{1,n-1}$, with $x \in \mathbb{R}^{1,\ell}$ and $y \in \mathbb{R}^{1,k}$, such that $\langle (x;y),(z;0)\rangle_{2,n} < 0$ for every $[z:0] \in \mathbb{S}^{\ell-1}$, equivalently such that $\langle x, z\rangle_{1,\ell} < 0$ for every $z \in \partial\mathcal{C}_\ell$. This last condition means that x is in the intersection of the half spaces bounded by the degenerate hyperplans $\langle z, .\rangle_{1,\ell} = 0$ of $\mathbb{R}^{1,\ell}$. It is easy to see that this intersection is exactly $\mathcal{C}_\ell$. Thus $x \in \mathcal{C}_\ell$. Up to rescaling, one can suppose that $q_{1,\ell}(x) = -1$, i.e. $x \in \mathbb{H}^\ell$. Since $0 = q_{2,n}(x;y) = q_{1,\ell}(x) + q_{1,k}(y)$, we deduce that $q_{1,k}(y) = 1$, i.e. $y \in dS_{1,k-1}$. It follows that

$$\Omega(\Lambda_\rho) = \left\{[x:y] \in Ein_{1,n-1} : \ x \in \mathbb{H}^\ell, \ y \in dS_{1,k-1}\right\}.$$

This proves the proposition. ☐

Black-white hole decomposition. Fix a point ξ in $\mathbb{S}_+^{k-1} \sqcup \mathbb{S}_-^{k-1}$. The limit set Λ_ρ is contained in the lightcone of ξ, so the description presented in Section 3.1 applies here: the invisible domain $\Omega(\Lambda_\rho)$ is the disjoint union of a black hole B, horizons H_i and a white hole W. The fact that Λ_ρ is a conformal $(\ell-1)$-sphere implies that B and W are conformally diffeomorphic to past and future *Misner domains of* $\mathbb{R}^{1,n-1}$ (see Figure 3.2).

Moreover, there is a single horizon when $\ell < n-2$ and there are exactly two horizons when $\ell = n-2$ (see Figure 3.2).

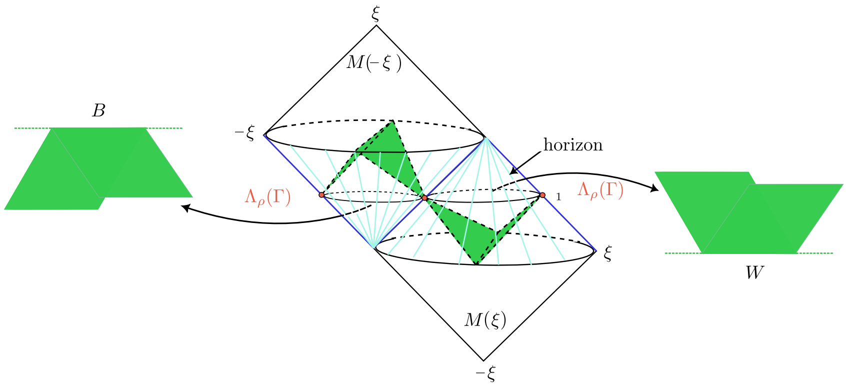

Figure 3.2. A Misner domain of $Ein_{1,2}$. The limit set Λ_ρ (in red) is a conformal sphere of dimension 0*, i.e. the union of two points. The black hole B and the white hole W are conformally isometric to past and future Misner domains of $\mathbb{R}^{1,2}$. The lightlike geodesics going through the points of the limit set (in dark blue) disconnect the future lightcone of ξ in two horizons: the half cone forwards (in light blue) and the half cone backwards.*

In general, this decomposition is not preserved by $\rho(\Gamma)$ unless ξ is stabilized by $\rho(\Gamma)$. However, since the spheres $\mathbb{S}^{k-1}_{\pm}$ are preserved by $\rho(\Gamma)$, the limit set is contained in the lightcone of $\rho(\gamma).\xi$ for every $\gamma \in \Gamma$. Therefore, every $\rho(\gamma).\xi$ defines a black-white hole decomposition of $\Omega(\Lambda_\rho)$ which is conformally diffeomorphic to that defined by ξ.

To sum up, each point ξ of $\mathbb{S}^{k-1}_{\pm}$ defines a decomposition of $\Omega(\Lambda_\rho)$ as the union of two Misner domains of $\mathbb{R}^{1,n-1}$ separated by one or two horizons. For this reason, the invisible domain $\Omega(\Lambda_\rho)$ is called *a Misner domain of the Einstein universe.* This motivates the following definition.

Definition 3.4. — *The GHMC conformally flat spacetime M_ρ is called* a conformally flat Misner spacetime.

Remark 3.5. — Conformally flat Misner spacetimes are connected.

3.3. Misner extensions

Let $\mathbb{R}^{2,n} = \mathbb{R}^{1,\ell} \oplus^{\perp} \mathbb{R}^{1,k}$ be an orthogonal splitting of $\mathbb{R}^{2,n}$, with $\ell, k \in \mathbb{N}^*$ such that $n = \ell + k$ and $\ell \leqslant n-2$. Let $O_0(1,\ell) \times O_0(1,k)$ be the subgroup of $O_0(2,n)$ preserving this splitting.

We consider $\mathbb{S}^{\ell-1}$ the conformal sphere of dimension $(\ell-1)$ realized as a connected component of the projection in $\mathbb{S}(\mathbb{R}^{2,n})$ of the quadric

$$\left\{[x:0] \in Ein_{1,n-1} : \ x \in \mathbb{R}^{1,\ell}\right\}.$$

Let Ω be the set of point in $Ein_{1,n-1}$ which are not causally related to any point in $\mathbb{S}^{\ell-1}$. Remark that $O_0(1,\ell) \times O_0(1,k)$ preserves Ω. We ask the following question:

> *Is there a discrete subgroup Γ of $O_0(1,\ell) \times O_0(1,k)$ preserving a causally convex open subset Ω' of $Ein_{1,n-1}$ containing strictly Ω?*

The answer to this question is yes! Anosov representations give examples of such subgroups Γ. Indeed, let ρ be a negative P_1-Anosov representation of a Gromov hyperbolic group Γ in $O_0(2,n)$ defined by a pair (ρ_ℓ, ρ_k) where

(1) $\rho_\ell : \Gamma \to O_0(1,\ell)$ is a convex cocompact representation such that the limit set Λ_{ρ_ℓ} is *strictly* contained in the conformal sphere $\mathbb{S}^{\ell-1}$;
(2) $\rho_k : \Gamma \to O_0(1,k)$ is a relatively compact representation, i.e. the image $\rho_k(\Gamma)$ is contained in a compact of $O_0(1,k)$.

The limit set Λ_ρ of the representation ρ is exactly Λ_{ρ_ℓ}. Since $\Lambda_\rho \subsetneq \mathbb{S}^{\ell-1}$, the invisible domain $\Omega(\Lambda_\rho)$ contains stricly Ω. Moreover, since $\rho(\Gamma)$ preserves $\mathbb{S}^{\ell-1}$, it preserves Ω. The quotient spacetime $M_\rho = \rho(\Gamma)\backslash\Omega(\Lambda_\rho)$ is then a GHMC conformally flat spacetime containing the conformally flat Misner spacetime $\rho(\Gamma)\backslash\Omega$. This motivates the following definition.

Definition 3.6. — *The GHMC conformally flat spacetime M_ρ is called* a Misner extension.

Remark 3.7. — We denote by $\mathbb{S}^{k-1}_{\pm}$ the conformal spheres defined as the connected components of the projection in $\mathbb{S}(\mathbb{R}^{2,n})$ of the lightcone of $\mathbb{R}^{1,k}$. As for Misner domains, the limit set Λ_ρ is contained in the lightcone of each point of $\mathbb{S}^{k-1}_{\pm}$. Moreover, the choice of a point in $\mathbb{S}^{k-1}_{\pm}$ defines a black-white hole decomposition of the invisible domain $\Omega(\Lambda_\rho)$.

Remark 3.8. — Misner extensions may be disconnected. Indeed, consider a splitting $\mathbb{R}^{2,3} = \mathbb{R}^{1,2} \oplus^{\perp} \mathbb{R}^{1,1}$ and let Γ be a convex-cocompact subgroup of $O_0(1,2)$. Now, let ρ be the representation of Γ in $O_0(2,3)$ that acts trivially on the factor $\mathbb{R}^{1,1}$ of the splitting. The representation ρ fixes two points ξ_0, ξ_1 in $Ein_{1,2}$ defined as the projection of the two isotropic lines of $\mathbb{R}^{1,1}$ and the limit set of $\rho(\Gamma)$ is a Cantor set contained in the circle $\mathbb{S}^1$ obtained as the intersection of the lightcones of ξ_0 and ξ_1. Therefore, it is easy to see that the Misner extension defined by ρ has as many connected components as the complement of the cantor set in $\mathbb{S}^1$. Notice that since ρ

fixes a point in $Ein_{1,2}$, the Misner extension above is actually a black-white hole. So, this gives also an example of disconnected black-white holes.

4. Complete photons

In a globally hyperbolic conformally flat spacetime M, photons are embedded manifolds of dimension 1 diffeomorphic to $\mathbb{R}$, equipped with a natural $(\widetilde{PSL}(2,\mathbb{R}), \widetilde{\mathbb{RP}}^1)$-structure. This projective structure on each photon is a conformal invariant. We say that a photon Δ *contains conjugate points* if there is a non-surjective projective embedding of the affine line $\mathbb{A}$ in Δ. The photon Δ is said to be *complete* if it is projectively equivalent to $\mathbb{A}$. Equivalently, Δ contains conjugate points if it lifts to a photon $\widetilde{\Delta}$ of $\widetilde{M}$ which develops into a lightlike geodesic of $\widetilde{Ein}_{1,n-1}$ containing conjugate points. The photon Δ is complete if $\widetilde{\Delta}$ develops into a lightlike geodesics connecting strictly a point of $\widetilde{Ein}_{1,n-1}$ to one of its first conjugate points (see Figure 4.1).

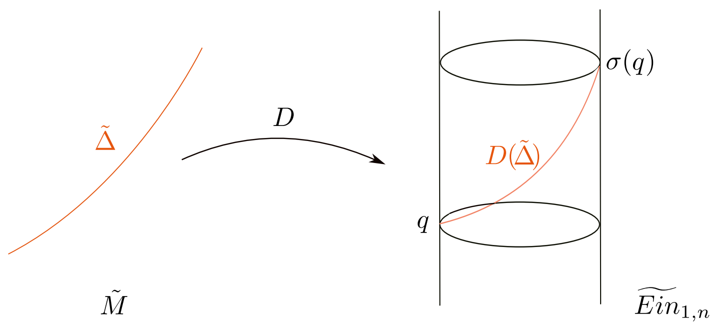

Figure 4.1. Complete photon.

An important result of Rossi [23, Theorem 5] states that if a globally hyperbolic maximal conformally flat spacetime of dimension $n \geqslant 3$ contains a photon with conjugate points then it is a finite quotient of $\widetilde{Ein}_{1,n-1}$. This result completely classifies globally hyperbolic maximal conformally flat spacetimes containing conjugate points. A natural question is then the study of globally hyperbolic maximal conformally flat spacetimes without conjugate points. The globally hyperbolic maximal conformally flat spacetimes presented in Section 3, namely black-white holes, conformally flat Misner spacetimes and Misner extensions, are examples of such spacetimes. In addition, these spacetimes share the remarkable property of containing horizons foliated by *complete photons* (see Figures 3.1 and 3.2). This gives a strong motivation for the study of globally hyperbolic maximal conformally flat spacetimes without conjugate points but containing at least a complete photon. In a forthcoming paper, we present some classification results on these spacetimes.

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