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SIGNATURE, TOLEDO AND ETA INVARIANTS FOR SURFACE GROUP REPRESENTATIONS IN THE REAL SYMPLECTIC GROUP

Inkang Kim, Pierre Pansu & Xueyuan Wan

ABSTRACT. — In this paper, by using the Atiyah–Patodi–Singer index theorem, we obtain a formula for the signature of a flat symplectic vector bundle over a surface with boundary, which is related to the Toledo invariant of a surface group representation in the real symplectic group and $\boldsymbol{\rho}$ invariant on the boundary. As an application, we obtain a Milnor–Wood type inequality for the signature. In particular, we give a new proof of the Milnor–Wood inequality for the Toledo invariant in the case of closed surfaces and obtain some modified inequalities for surfaces with boundary.

Introduction

Let Σ be a closed surface, and consider a surface group representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(E, \Omega)$ into the real symplectic group $\mathrm{Sp}(E, \Omega)$, where (E, Ω) is a symplectic vector space. Let $(\mathcal{E}, \Omega)$ denote the flat symplectic vector bundle over Σ associated to the representation ρ. There is a canonical quadratic form $\int_\Sigma \Omega(\cdot \cup \cdot)$ on the cohomology $\mathrm{H}^1(\Sigma, \mathcal{E})$. Meyer's signature formula [26, § 4.1, p. 19] implies that the signature of the quadratic form is given by $4\int_\Sigma c_1(\mathcal{E}, \Omega)$, where $c_1(\mathcal{E}, \Omega)$ denotes the first Chern class of the symplectic vector bundle $(\mathcal{E}, \Omega)$, which can be expressed in terms of Toledo invariant, see e.g. [16, Appendix A]. For the case of manifolds with

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boundary, Atiyah, Patodi and Singer introduced η invariant and obtained an index theorem, which is known as Atiyah–Patodi–Singer index theorem, see [2, Theorem 3.10]. For a surface Σ with boundary $\partial\Sigma$, one can consider a unitary representation of the fundamental group $\rho : \pi_1(\Sigma) \to \mathrm{U}(n)$, which gives a flat Hermitian vector bundle over Σ. Similarly, one can also define a quadratic form on the relative cohomology with coefficients in the flat bundle. From [3, Theorem 2.2, 2.4], the signature of the quadratic form is exactly the η invariant (or $\boldsymbol{\rho}$ invariant). Inspired by these results, in this paper, by using the Atiyah–Patodi–Singer index theorem, we will consider the signature of the flat symplectic vector bundle associated to a representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(E, \Omega)$ for the surface Σ with boundary.

Let Σ be a connected oriented surface with smooth boundary $\partial\Sigma$, and g_Σ be a Riemannian metric on Σ. Suppose that on the collar neighborhood $\partial\Sigma \times [0,1] \subset \Sigma$ of $\partial\Sigma$, the metric has a product form. Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(E, \Omega)$ be a surface group representation, which gives a flat symplectic vector bundle $(\mathcal{E}, \Omega)$ over Σ, see Section 1.2. Consider the image of twisted singular cohomology in the absolute cohomology $\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}) = \mathrm{Im}(\mathrm{H}^1(\Sigma, \partial\Sigma, \mathcal{E}) \to \mathrm{H}^1(\Sigma, \mathcal{E}))$, and the canonical quadratic form $Q_{\mathbb{R}}(\cdot, \cdot) = \int_\Sigma \Omega(\cdot \cup \cdot)$ on $\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E})$, which is non-degenerate. We denote by $\mathrm{sign}(\mathcal{E}, \Omega)$ the signature of the quadratic form. For any $\mathbf{J} \in \mathcal{J}(\mathcal{E}, \Omega)$, the set of compatible complex structures, the operator $A_{\mathbf{J}} = \mathbf{J}\frac{d}{dx}$ is a $\mathbb{C}$-linear formally self-adjoint elliptic first order differential operator in the space $A^0(\partial\Sigma, \mathcal{E}_{\mathbb{C}}|_{\partial\Sigma})$, see Proposition 1.1. Hence it has a discrete spectrum with real eigenvalues, and the η invariant $\eta(A_{\mathbf{J}})$ is well-defined, see Section 1.1.3. Let ∇ be any peripheral connection on $(\mathcal{E}, \Omega, \mathbf{J})$, and let $c_1(\mathcal{E}, \Omega, \mathbf{J})$ denote the first Chern class of the flat symplectic vector bundle in the de Rham cohomology with compact support, which is defined as the first Chern class associated to the peripheral connection ∇. By using the Atiyah–Patodi–Singer index theorem [2, Theorem 3.10], we obtain

THEOREM 0.1. — *The signature of the flat symplectic vector bundle* $(\mathcal{E}, \Omega)$ *is*

$$\mathrm{sign}(\mathcal{E}, \Omega) = 4\int_\Sigma c_1(\mathcal{E}, \Omega, \mathbf{J}) + \eta(A_{\mathbf{J}}). \tag{0.1}$$

For the case of closed surfaces, the above theorem was obtained by Meyer [26, § 4.1, p. 19], and by Lusztig [24, § 2] considering the signature of a flat $\mathrm{U}(p, q)$-Hermitian vector bundle. For the case of surfaces with boundary, and a surface group representation in $\mathrm{U}(p, q)$, the above theorem was proved by Atiyah [1, (3.1)] under the assumption that the representation on each component of the boundary is elliptic. By using the formula of signature for elliptic case, Atiyah [1, Theorem 2.13] proved for general case that the signature can be expressed as another kind of formula in terms of the relative Chern class of a certain line bundle, see [1, § 3] for the proof. In our paper [18], we consider the signature, Toledo invariant, $\boldsymbol{\rho}$ invariant and Milnor–Wood type inequality associated with the surface group representations in the $\mathrm{U}(p,q)$-group.

Next, we will show that the first term $4\int_\Sigma c_1(\mathcal{E},\Omega,\mathbf{J})$ is related to the Toledo invariant $\mathrm{T}(\Sigma,\rho)$. For a closed surface Σ, the Toledo invariant was defined in [8, 29] by considering a surface group representation $\rho:\pi_1(\Sigma)\to\mathrm{PSU}(1,n)$ of its fundamental group in the group of motions of complex hyperbolic n-space. In [6], Burger, Iozzi and Wienhard extended the definition of the Toledo invariant to surfaces with boundary and obtained the Milnor–Wood inequality $|\,\mathrm{T}(\Sigma,\rho)|\leqslant\mathrm{rank}(\mathscr{X})|\chi(\Sigma)|$ by using the methods of bounded cohomology. More precisely, consider a representation $\rho:\pi_1(\Sigma)\to G$ into a Lie group G which is of type (RH), so that the associated symmetric space $\mathscr{X}=G/K$ is a Hermitian symmetric space of noncompact type. Let $\omega_{\mathscr{X}}$ denote the (unique) Kähler form such that the minimal holomorphic sectional curvature of the associated Hermitian metric is -1. The Kähler form $\omega_{\mathscr{X}}$ gives a bounded Kähler class $\kappa_G^b\in\mathrm{H}^2_{c,b}(G,\mathbb{R})$. Consider the pullback in bounded cohomology, $\rho_b^*\left(\kappa_G^{\mathrm{b}}\right)\in\mathrm{H}^2_{\mathrm{b}}\left(\pi_1(\Sigma),\mathbb{R}\right)\cong\mathrm{H}^2_{\mathrm{b}}(\Sigma,\mathbb{R})$. The canonical map $j_{\partial\Sigma}:\mathrm{H}^2_{\mathrm{b}}(\Sigma,\partial\Sigma,\mathbb{R})\to\mathrm{H}^2_{\mathrm{b}}(\Sigma,\mathbb{R})$ is an isomorphism, and the Toledo invariant is defined as

$$\mathrm{T}(\Sigma,\rho)=\left\langle j_{\partial\Sigma}^{-1}\rho_b^*\left(\kappa_G^{\mathrm{b}}\right),[\Sigma,\partial\Sigma]\right\rangle,$$

where $j_{\partial\Sigma}^{-1}\rho_b^*\left(\kappa_G^{\mathrm{b}}\right)$ is considered as an ordinary relative class and $[\Sigma,\partial\Sigma]$ is the relative fundamental class. If $\partial\Sigma=\emptyset$, then the Toledo invariant is given by $\mathrm{T}(\Sigma,\rho)=\int_\Sigma f^*\omega_{\mathscr{X}}$, where $f:\widetilde{\Sigma}\to\mathscr{X}$ denotes any ρ-equivariant smooth map. For a manifold with cusps, there are also several equivalent definitions of the volume invariant, which is a natural generalization of Toledo invariant for higher dimensional manifolds, see e.g. [9, 11, 19, 20, 22].

In our case, $G=\mathrm{Sp}(E,\Omega)$ and K is the maximal compact subgroup of G, which is isomorphic to the unitary group. Then the associated symmetric space $\mathscr{X}=G/K$ can be identified with the bounded symmetric domain $\mathrm{D}_n^{\mathrm{III}}$ of type III, where $\dim E=2n$. It is also isomorphic to the space $\mathcal{J}(E,\Omega)$ of all compatible complex structures on (E,Ω). Let $\omega_{\mathrm{D}_n^{\mathrm{III}}}$ denote the Kähler form with holomorphic sectional curvature in $[-1,-1/n]$, see Appendix A. For any $\mathbf{J}\in\mathcal{J}(\mathcal{E},\Omega)$, $\mathbf{J}$ is equivalent to a ρ-equivariant map $\widetilde{\mathbf{J}}:\widetilde{\Sigma}\to\mathrm{D}_n^{\mathrm{III}}$. The form $\widetilde{\mathbf{J}}^*\omega_{\mathrm{D}_n^{\mathrm{III}}}$ is a ρ-equivariant form on $\widetilde{\Sigma}$ and descends to a form on Σ. For each $L\in G$, there exist a L-invariant 1-form α with $d\alpha=\omega_{\mathrm{D}_n^{\mathrm{III}}}$. Let $\chi_i\in[0,1]$ be a smooth cut-off function which is equal to 1 near the boundary component c_i and vanishes outside a small collar neighborhood of c_i. Then one can define the following de Rham cohomology class with compact support

$$\left[\rho^*\omega_{\mathrm{D}_n^{\mathrm{III}}}\right]_c = \left[\widetilde{\mathbf{J}}^*\omega_{\mathrm{D}_n^{\mathrm{III}}} - d\left(\sum_{i=1}^q \chi_i \widetilde{\mathbf{J}}^*\alpha_i\right)\right]_c,$$

where α_i is a $\rho(c_i)$-invariant 1-form with $d\alpha_i = \omega_{\mathrm{D}_n^{\mathrm{III}}}$, q is the number of components of $\partial\Sigma$. Following [22, Proposition-definition 4.1], the class $[\rho^*\omega_{\mathrm{D}_n^{\mathrm{III}}}]_c$ is independent of $\mathbf{J}$ and depends only on the conjugacy class of ρ. The $\boldsymbol{\rho}$ invariant is defined by

$$\boldsymbol{\rho}(\partial\Sigma) = \frac{1}{\pi}\sum_{i=1}^q \int_{c_i} \widetilde{\mathbf{J}}^*\alpha_i + \eta(A_{\mathbf{J}}),$$

which is a natural generalization of Atiyah–Patodi–Singer $\boldsymbol{\rho}$ invariant for unitary representations. Let $\mathcal{J}_o(\mathcal{E},\Omega)$ denote the space of compatible complex structures $\mathbf{J}$, which is the pullback of a compatible complex structure on $\mathcal{E}|_{\partial\Sigma}$ when restricted to a small collar neighborhood of $\partial\Sigma$. If $\mathbf{J}\in\mathcal{J}_o(\mathcal{E},\Omega)$, then the form $\widetilde{\mathbf{J}}^*\omega_{\mathrm{D}_n^{\mathrm{III}}}$ has compact support on $\Sigma_o := \Sigma\backslash\partial\Sigma$.

Theorem 0.2. — *For any $\mathbf{J}\in\mathcal{J}_o(\mathcal{E},\Omega)$, the Toledo invariant $\mathrm{T}(\Sigma,\rho)$ satisfies*

$$\mathrm{T}(\Sigma,\rho) = \frac{1}{2\pi}\int_\Sigma \left[\rho^*\omega_{\mathrm{D}_n^{\mathrm{III}}}\right]_c = 2\int_\Sigma c_1(\mathcal{E},\Omega,\mathbf{J}) - \frac{1}{2\pi}\sum_{i=1}^q\int_{c_i}\widetilde{\mathbf{J}}^*\alpha_i. \tag{0.2}$$

Hence the signature can be given by the following formula:

$$\mathrm{sign}(\mathcal{E},\Omega) = 2\,\mathrm{T}(\Sigma,\rho) + \boldsymbol{\rho}(\partial\Sigma). \tag{0.3}$$

There is a bound for the Toledo invariant, which is known as Milnor–Wood inequality [27, 31]. It can be thought of as an obstruction for a circle bundle to admit a flat structure, see also [12, 14]. This inequality and the maximal representations were widely studied, see e.g. [5, 6, 8, 10, 13, 22, 23, 29, 30]. Here we will deduce a Milnor–Wood type inequality for the signature by using the formula (0.3).

Theorem 0.3. — *The signature satisfies the following Milnor–Wood type inequality:*

$$|\,\mathrm{sign}(\mathcal{E},\Omega)| \leqslant \dim E\cdot|\chi(\Sigma)|.$$

In particular, if Σ is closed, then $\mathrm{sign}(\mathcal{E},\Omega) = 2\,\mathrm{T}(\Sigma,\rho)$ and we obtain the Milnor–Wood inequality for the Toledo invariant in the case of the surface group representations in the real symplectic group.

COROLLARY 0.4 (Turaev [30]). — *The Toledo invariant satisfies*

$$|\operatorname{T}(\Sigma,\rho)| \leqslant \frac{\dim E}{2}|\chi(\Sigma)|.$$

For the case of $\dim E = 2$, $\mathrm{Sp}(E,\Omega) \cong \mathrm{SL}(2,\mathbb{R})$, we obtain the following modified Milnor–Wood inequalities (0.4).

PROPOSITION 0.5. — *For any representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$, one has*

$$(0.4) \quad -|\chi(\Sigma)| - 1 + \sum_{\rho(c_k) \text{ is elliptic}} \frac{\theta_k}{\pi} \leqslant \operatorname{T}(\Sigma,\rho) \leqslant |\chi(\Sigma)| + 1 - \sum_{\rho(c_k) \text{ is elliptic}} \left(1 - \frac{\theta_k}{\pi}\right),$$

where $\theta_k \in (0,\pi)$ such that $[R(\theta_k)]$ is conjugate to $[\rho(c_k)] \in \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$, and $[\bullet]$ denotes the class in

$$\mathrm{PSL}(2,\mathbb{R}), R(\theta_k) = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}.$$

1. Some backgrounds

1.1. Symplectic vector bundle and η invariant over a circle

In this section, we will consider the flat symplectic vector bundle $\mathcal{E}$ associated with a representation ρ of the fundamental group of the circle S^1 into the real symplectic group $\mathrm{Sp}(E,\Omega)$, and define a first order elliptic self-adjoint differential operator $A_{\mathbf{J}}$, and we will recall the definition of η invariant $\eta(A_{\mathbf{J}})$ for the operator, one can refer to [2, 28] for the η invariant.

1.1.1. Symplectic vector bundle

Let (E,Ω) be a real symplectic vector space, where Ω is a symplectic form. For any symplectic linear transformation $L \in \mathrm{Sp}(E,\Omega)$, consider a representation $\rho(\gamma_0) = L$, where γ_0 denotes the generator of $\pi_1(S^1)$, which is given by $\gamma_0(x) = e^{ix}, 0 \leqslant x \leqslant 2\pi$. Then the representation defines a flat vector bundle

$$\mathcal{E} = \mathbb{R} \times_\rho E = (\mathbb{R} \times E)/\sim$$

over S^1, where $(x_1, e_1) \sim (x_2, e_2)$ if $x_2 = x_1 + 2\pi k, k \in \mathbb{Z}$ and $e_2 = L^{-k}(e_1)$. Each global section of $\mathcal{E}$ is equivalent to a map $s : \mathbb{R} \to E$ satisfies the ρ-equivariant condition $s(x + 2\pi) = L^{-1}s(x)$.

A complex structure on a real vector space E is an automorphism $J : E \to E$ such that $J^2 = -\operatorname{Id}$. A complex structure J on a real symplectic vector space (E, Ω) is called *compatible with* Ω (or Ω-*compatible*) if $\Omega(\cdot, J\cdot)$ defines a positive definite inner product. We denote by $\mathcal{J}(E, \Omega)$ the space of all Ω-compatible complex structures on (E, Ω). In particular, one has $\mathcal{J}(E, \Omega) \subset \operatorname{Sp}(E, \Omega)$.

Denote by $\mathcal{J}(\mathcal{E}, \Omega) = \widetilde{\Sigma} \times_\rho \mathcal{J}(E, \Omega)$ the space of all ρ-equivariant complex structures in $\mathcal{J}(E, \Omega)$. For any symplectic transformation $L \in \operatorname{Sp}(E, \Omega)$ which can be written as $L = \pm\exp(2\pi B)$, where $B \in \mathfrak{sp}(E, \Omega)$, i.e. $B^\top\Omega + \Omega B = 0$, we can find a canonical complex structure $\mathbf{J} \in \mathcal{J}(\mathcal{E}, \Omega)$ for any given $J \in \mathcal{J}(E, \Omega)$. In fact, for any $J \in \mathcal{J}(E, \Omega)$, we define

$$\mathbf{J}(x) = \exp(-xB) J \exp(xB) \in \mathcal{J}(E, \Omega),$$

which satisfies $\mathbf{J}(x + 2\pi) = L^{-1}\mathbf{J}(x)L$ and thus gives a complex structure $\mathbf{J}$ on the flat vector bundle $\mathcal{E}$. Hence $\mathbf{J} \in \mathcal{J}(\mathcal{E}, \Omega)$. But in general, L cannot be written as $L = \pm\exp(2\pi B)$ except for $\operatorname{Sp}(2, \mathbb{R})$, hence one needs to choose another complex structure on $\mathcal{E}$.

There exists a canonical flat connection d on $\mathcal{E}$, which is induced from the trivial vector bundle $\mathbb{R} \times E \to \mathbb{R}$. The holonomy representation of the flat connection d is just the representation ρ. Denote by $A^0(S^1, \mathcal{E})$ the space of all smooth sections of $\mathcal{E}$, which can be identified with the space $A^0(\mathbb{R}, E)^L$ of all ρ-equivariant smooth maps $s : \mathbb{R} \to E$. There is a standard L^2-metric on the space $A^0(S^1, \mathcal{E}) \cong A^0(\mathbb{R}, E)^L$ using the inner product $\Omega(\cdot, \mathbf{J}\cdot)$ and the metric $dx \otimes dx$ on S^1, i.e. $\int_{S^1} \Omega(\cdot, \mathbf{J}\cdot)dx$.

1.1.2. First order differential operator

Let (E, Ω) be a real symplectic vector space. By $\mathbb{C}$-linear extension, Ω is also a symplectic form on the complex vector space $E_\mathbb{C} = E \otimes \mathbb{C}$. For any $L \in \operatorname{Sp}(E, \Omega)$, it can be viewed as an element in $\operatorname{Sp}(E_\mathbb{C}, \Omega)$ by $\mathbb{C}$-linear extension. Since

$$\mathcal{E}_\mathbb{C} = \mathcal{E} \otimes \mathbb{C} = \mathbb{R} \times_\rho E_\mathbb{C},$$

so the space $A^0(S^1, \mathcal{E}_\mathbb{C})$ of all smooth sections of $\mathcal{E}_\mathbb{C}$ can be identified with the space $A^0(\mathbb{R}, E_\mathbb{C})^L$ of all smooth ρ-equivariant maps from $\mathbb{R}$ to $E_\mathbb{C}$. For any $\mathbf{J} \in \mathcal{J}(\mathcal{E}, \Omega)$, we can also extend it to a $\mathbb{C}$-linear transformation of $E_\mathbb{C}$.

Consider the following $\mathbb{C}$-linear first order differential operator

$$A_{\mathbf{J}} := \mathbf{J}\frac{d}{dx} \tag{1.1}$$

which acts on the space $A^0(\mathbb{R}, E_{\mathbb{C}})^L \cong A^0(S^1, \mathcal{E}_{\mathbb{C}})$. Denote

$$H(e_1, e_2) = 2\Omega(e_1, \mathbf{J}e_2). \tag{1.2}$$

for any $e_1, e_2 \in \mathcal{E}_{\mathbb{C}}|_x$, $x \in [0, 2\pi]$. One can check that H is a Hermitian metric on $\mathcal{E}_{\mathbb{C}}$, the Hermitian inner product is denoted by

$$(e_1, e_2) := H(e_1, \overline{e_2}).$$

The global L^2-inner product on $A^0(S^1, \mathcal{E}_{\mathbb{C}})$ is defined as

$$\langle \cdot, \cdot \rangle = \int_{S^1} (\cdot, \cdot) dx = 2\int_{S^1} \Omega(\cdot, \mathbf{J}\bar{\cdot}) dx. \tag{1.3}$$

Proposition 1.1. — *$A_{\mathbf{J}}$ is a $\mathbb{C}$-linear formally self-adjoint elliptic first order differential operator in the space $A^0(S^1, \mathcal{E}_{\mathbb{C}})$.*

Proof. — It is obvious that $A_{\mathbf{J}}$ is $\mathbb{C}$-linear, first order and elliptic, so we just need to prove $A_{\mathbf{J}}$ is formally self-adjoint. For any $s_1, s_2 \in A^0(S^1, \mathcal{E}_{\mathbb{C}})$, one has

$$\begin{aligned}\langle A_{\mathbf{J}} s_1, s_2 \rangle - \langle s_1, A_{\mathbf{J}} s_2 \rangle &= 2\int_{S^1} \left(\Omega\left(\frac{d}{dx} s_1, \overline{s_2}\right) + \Omega\left(s_1, \frac{d}{dx}\overline{s_2}\right)\right) dx \\ &= 2\int_{S^1} d(\Omega(s_1, \overline{s_2})) = 0,\end{aligned}$$

which completes the proof. □

Remark 1.2. — The operator $A_{\mathbf{J}}$ has a natural extension in the Hilbert space $\mathrm{L}^2(S^1, \mathcal{E}_{\mathbb{C}})$, we also denote it by $A_{\mathbf{J}}$, see e.g. [21, Definition 7.1 in Appendix]. From Proposition 1.1, $A_{\mathbf{J}}$ is formally self-adjoint and elliptic, so $A_{\mathbf{J}}$ is self-adjoint in the Hilbert space $\mathrm{L}^2(S^1, \mathcal{E}_{\mathbb{C}})$, see e.g. [21, Theorem 7.2 in Appendix].

1.1.3. η invariant

For every elliptic self-adjoint differential operator A, which acts on a Hermitian vector bundle over a closed manifold, the operator A has a discrete spectrum with real eigenvalues. Let λ_j run over the eigenvalues of A, then the η function of A is defined as

$$\eta_A(s) = \sum_{\lambda_j \neq 0} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^s},$$

where $s \in \mathbb{C}$. The η function admits a meromorphic continuation to the whole complex plane and is holomorphic at $s = 0$. The special value $\eta_A(0)$ is then called the η *invariant* of the operator A, and we denote the η invariant by

$$\eta(A) = \eta_A(0). \tag{1.4}$$

Applying this general notion to the operator $A_{\mathbf{J}}$, $A_{\mathbf{J}}$ has discrete spectrum consisting of real eigenvalues λ of finite multiplicity, and the η invariant $\eta(A_{\mathbf{J}})$ of $A_{\mathbf{J}}$ is defined by (1.4).

Example 1.3. — For any representation

$$\rho : \pi_1(S^1) \to \mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}),$$

and the operator

$$A_{\mathbf{J}} = \mathbf{J}\frac{d}{dx},$$

where $\mathbf{J} := \exp(-xB)J\exp(xB)$, and $L = \pm\exp(2\pi B) \in \mathrm{Sp}(2, \mathbb{R})$ denotes the image of the generator of $\pi_1(S^1)$, the η invariant $\eta(A_{\mathbf{J}})$ can be explicitly calculated. See [18, Appendix].

1.2. Signature of flat symplectic vector bundles

In this section, we will define the signature of a flat symplectic vector bundle, and show it can be expressed as the difference of two L^2-indices.

Let Σ be a connected oriented surface with smooth boundary $\partial\Sigma$, each component of $\partial\Sigma$ is homeomorphic to S^1, let $\iota : \partial\Sigma \to \Sigma$ denote the natural inclusion. Let (E, Ω) be a real symplectic vector space, and $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(E, \Omega)$ be a representation from the fundamental group $\pi_1(\Sigma)$ of Σ into the real symplectic group $\mathrm{Sp}(E, \Omega)$. The representation ρ gives a flat vector bundle $\mathcal{E} = \widetilde{\Sigma} \times_\rho E$ over Σ. Any element of $A^*(\Sigma, \mathcal{E})$ can be viewed as a ρ-equivariant element in $A^*(\widetilde{\Sigma}, \mathbb{R}) \otimes E$, where ρ-equivariant means $(\gamma^{-1})^*\omega \otimes \rho(\gamma)v = \omega \otimes v$ for $\omega \in A^*(\widetilde{\Sigma}, \mathbb{R})$ and $v \in E$. There exists a *canonical flat connection* d on the flat bundle $\mathcal{E}$, which is defined by $d(\omega \otimes v) := d\omega \otimes v$. One can also refer to [4, Section 1.1] for the representations, flat bundles and the canonical flat connection.

1.2.1. Definition of signature

Let $\mathrm{H}^*(\Sigma, \mathcal{E})$ (resp. $\mathrm{H}^*(\Sigma, \partial\Sigma, \mathcal{E})$) denote the (resp. relative) twisted singular cohomology, we refer to [7, Chapter 5] for the definitions. Set

$$\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}) := \mathrm{Im}\left(\mathrm{H}^1(\Sigma, \partial\Sigma, \mathcal{E}) \to \mathrm{H}^1(\Sigma, \mathcal{E})\right).$$

There exists a natural quadratic form

$$Q_{\mathbb{R}} : \widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}) \times \widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}) \to \mathbb{R}$$

$$Q_{\mathbb{R}}([a],[b]) = \int_{\Sigma} \Omega([a] \cup [b]).$$

By the same argument as in [2, p. 65], the form $Q_{\mathbb{R}}$ is non-degenerate due to Poincaré duality. Moreover, $Q_{\mathbb{R}}$ is symmetric, i.e. $Q_{\mathbb{R}}([a],[b]) = Q_{\mathbb{R}}([b],[a])$. If $\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}) = \mathscr{H}^+ \oplus \mathscr{H}^-$ such that $Q_{\mathbb{R}}$ is positive definite on $\mathscr{H}^+$ and negative definite on $\mathscr{H}^-$, then the signature of $(\mathcal{E}, \Omega)$ is defined as the signature of the symmetric bilinear form $Q_{\mathbb{R}}$, that is,

$$\operatorname{sign}(\mathcal{E}, \Omega) := \operatorname{sign}(Q_{\mathbb{R}}) = \dim \mathscr{H}^+ - \dim \mathscr{H}^-.$$

By $\mathbb{C}$-linear extension, Ω can be viewed as a symplectic form on $\mathcal{E}_{\mathbb{C}}$. For this symplectic vector bundle $(\mathcal{E}_{\mathbb{C}}, \Omega)$, we obtain a complex quadratic form

$$Q_{\mathbb{C}} : \widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}_{\mathbb{C}}) \times \widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}_{\mathbb{C}}) \to \mathbb{C}$$

$$Q_{\mathbb{C}}([a],[b]) = \int_{\Sigma} \Omega([a] \cup \overline{[b]}),$$

where $\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}_{\mathbb{C}}) := \operatorname{Im}(\mathrm{H}^1(\Sigma, \partial\Sigma, \mathcal{E}_{\mathbb{C}}) \to \mathrm{H}^1(\Sigma, \mathcal{E}_{\mathbb{C}}))$. For any $[a],[b] \in \widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}_{\mathbb{C}})$, one has $Q_{\mathbb{C}}([a],[b]) = \overline{Q_{\mathbb{C}}([b],[a])}$, which means $Q_{\mathbb{C}}$ is a Hermitian form. Naturally, one can define the signature $\operatorname{sign}(\mathcal{E}_{\mathbb{C}}, \Omega)$ of $(\mathcal{E}_{\mathbb{C}}, \Omega)$. We now prove that the two signatures $\operatorname{sign}(\mathcal{E}, \Omega)$ and $\operatorname{sign}(\mathcal{E}_{\mathbb{C}}, \Omega)$ are equal.

Proposition 1.4. — $\operatorname{sign}(\mathcal{E}_{\mathbb{C}}, \Omega) = \operatorname{sign}(\mathcal{E}_{\mathbb{R}}, \Omega)$.

Proof. — Since $\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}) = \mathscr{H}^+ \oplus \mathscr{H}^-$ and $\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}_{\mathbb{C}}) = \widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}) \otimes \mathbb{C}$, so

$$\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}_{\mathbb{C}}) = \mathscr{H}^+_{\mathbb{C}} \oplus \mathscr{H}^-_{\mathbb{C}},$$

where $\mathscr{H}^{\pm}_{\mathbb{C}} := \mathscr{H}^{\pm} \otimes \mathbb{C}$, and $\dim_{\mathbb{C}} \mathscr{H}^{\pm}_{\mathbb{C}} = \dim_{\mathbb{R}} \mathscr{H}^{\pm}$. Hence

$$\begin{aligned}\operatorname{sign}(\mathcal{E}_{\mathbb{C}}, \Omega) &= \dim_{\mathbb{C}} \mathscr{H}^+_{\mathbb{C}} - \dim_{\mathbb{C}} \mathscr{H}^-_{\mathbb{C}} \\ &= \dim_{\mathbb{R}} \mathscr{H}^+ - \dim_{\mathbb{R}} \mathscr{H}^- \\ &= \operatorname{sign}(\mathcal{E}_{\mathbb{R}}, \Omega).\end{aligned}$$

The proof is complete. □

2. Sketch of proofs

Now we briefly explain the proof of the above results. All the details appear in [17, 18]. By complexification, we consider the complex vector bundle $\mathcal{E}_{\mathbb{C}} = \mathcal{E} \otimes \mathbb{C}$ which is equipped with the symplectic form Ω. The real quadratic from $Q_{\mathbb{R}}$ can be extended to a non-degenerate quadratic form $Q_{\mathbb{C}}(\cdot,\cdot) = \int_\Sigma \Omega(\cdot \cup \cdot)$ on $\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}_{\mathbb{C}})$. We denote by $\operatorname{sign}(\mathcal{E}_{\mathbb{C}}, \Omega)$ the signature of the quadratic form on $\widehat{\mathrm{H}}^1(\Sigma, \mathcal{E}_{\mathbb{C}})$, which equals $\operatorname{sign}(\mathcal{E}, \Omega)$ for the real symplectic vector bundle $(\mathcal{E}, \Omega)$. Let g_Σ be a Riemannian metric on Σ which has the product form $g_\Sigma = du^2 + g_{\partial\Sigma}$ on $\partial\Sigma \times [0,1]$. For any complex structure $\mathbf{J} \in \mathcal{J}(\mathcal{E}, \Omega)$, one can define the Hermitian inner product $(\cdot,\cdot) = 2\Omega(\cdot, \mathbf{J}\bar{\cdot})$ and the global L^2-inner product $\langle \cdot,\cdot \rangle$. The operator $*\mathbf{J}$ satisfies $(*\mathbf{J})^2 = \mathrm{Id}$ when acting on the space $\wedge^1 T^*\Sigma \otimes \mathcal{E}_{\mathbb{C}}$. Let $\wedge^\pm$ denote the ±1-eigenspaces of $*\mathbf{J}$, and denote by $\pi^\pm = \frac{1 \pm *\mathbf{J}}{2} : \wedge^1 T^*\Sigma \otimes \mathcal{E}_{\mathbb{C}} \to \wedge^\pm$ the natural projections. Set $d^\pm = \pi^\pm \circ d$, where d is the canonical flat connection on the space $A^*(\Sigma, \mathcal{E}_{\mathbb{C}})$. The operators $d^\pm$ have the form $d^\pm = \sigma^\pm(\frac{\partial}{\partial u} + A^\pm_{\mathbf{J}})$, where $\sigma^\pm : \mathcal{E}_{\mathbb{C}} \to \wedge^\pm$ are bundle isomorphisms, and $A^\pm_{\mathbf{J}} = \pm A_{\mathbf{J}}$ are the first order elliptic formally self-adjoint operators on the boundary. Let $\widehat{\Sigma} = \Sigma \cup ((-\infty, 0] \times \partial\Sigma)$ be the complete manifold obtained form Σ by gluing the negative half-cylinder $(-\infty, 0] \times \partial\Sigma$ to the boundary of Σ. The vector bundle $\mathcal{E}$, the complex structure $\mathbf{J}$ and the canonical flat connection d can be extended naturally to $\widehat{\Sigma}$. Denote by $\mathscr{H}^*(\widehat{\Sigma}, \mathcal{E}_{\mathbb{C}})$ the space of harmonic L^2-forms on $\widehat{\Sigma}$, which is isomorphic to $\widehat{\mathrm{H}}^*(\Sigma, \mathcal{E}_{\mathbb{C}})$. Moreover, it is the direct sum of the two subspaces $\operatorname{Ker}(d^+)^* \cap \mathrm{L}^2(\widehat{\Sigma}, \wedge^+)$ and $\operatorname{Ker}(d^-)^* \cap \mathrm{L}^2(\widehat{\Sigma}, \wedge^-)$, which correspond to the positive and negative definite subspaces of quadratic form $Q_{\mathbb{C}}$ respectively. The signature is then given by

$$\operatorname{sign}(\mathcal{E}, \Omega) = \mathrm{L}^2 \operatorname{Index}(d^-) - \mathrm{L}^2 \operatorname{Index}(d^+),$$

where $\mathrm{L}^2 \operatorname{Index}(d^\pm)$ denote the L^2-indices of the operators $d^\pm$. Let $P_\pm$ denote the orthogonal projections of $\mathrm{L}^2(\partial\Sigma, \mathcal{E}_{\mathbb{C}})$ onto the subspace spanned by all eigenfunctions of $A^\pm_{\mathbf{J}}$ with eigenvalues $\lambda > 0$, and $A^0(\Sigma, \mathcal{E}_{\mathbb{C}}; P_\pm)$ be the subspace of $A^0(\Sigma, \mathcal{E}_{\mathbb{C}})$ consisting of all sections φ which satisfy the boundary conditions $P_\pm(\varphi|_{\partial\Sigma}) = 0$. Denote by $d^\pm_P : A^0(\Sigma, \mathcal{E}_{\mathbb{C}}; P_\pm) \to A^0(\Sigma, \wedge^\pm)$ the restriction of $d^\pm$. The L^2-index of $d^\pm$ can be expressed as the sum of $\operatorname{Index}(d^-_P)$ and $h_\infty(\wedge^\pm)$, where $h_\infty(\wedge^\pm)$ denote the dimension of the subspace of $\operatorname{Ker} \sigma^\pm A^\pm_{\mathbf{J}} (\sigma^\pm)^{-1}$ consisting of limiting values of extended L^2-sections a of $\wedge^\pm$ satisfying $(d^\pm)^* a = 0$. Hence

$$\operatorname{sign}(\mathcal{E}, \Omega) = \operatorname{Index}\left(d^-_P\right) - \operatorname{Index}\left(d^+_P\right) + h_\infty(\wedge^-) - h_\infty(\wedge^+).$$

By the Atiyah–Patodi–Singer index theorem [2, Theorem 3.10], $d_P^{\pm}$ are Fredholm operators and

$$\operatorname{Index}\left(d_P^{\pm}\right) = \int_\Sigma \alpha_\pm(z) d\mu_g - \frac{\eta\left(A_{\mathbf{J}}^{\pm}\right) + \dim \operatorname{Ker} A_{\mathbf{J}}^{\pm}}{2},$$

where $\int_\Sigma \alpha_\pm(z) d\mu_g$ are the Atiyah–Singer integral. Hence

$$\operatorname{sign}(\mathcal{E}, \Omega) = \int_\Sigma \alpha_-(z) d\mu_g - \int_\Sigma \alpha_+(z) d\mu_g + h_\infty(\wedge^-) - h_\infty\left(\wedge^+\right) + \eta(A_{\mathbf{J}}).$$

Following [2] we consider the double $\Sigma \cup_{\partial\Sigma} \Sigma$ of Σ, and consider the $\mathbb{Z}_2$-graded vector bundle $\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^-$ over the double $\Sigma \cup_{\partial\Sigma} \Sigma$, where $\mathcal{F}^+ = \mathcal{E}_{\mathbb{C}}$ and $\mathcal{F}^- = \wedge^-$. Then the Atiyah–Singer integral can be given by

$$\int_\Sigma \alpha_-(z) d\mu_g = \lim_{t\to 0} \int_\Sigma \operatorname{Str}\left\langle z \left| e^{-tD^2} \right| z \right\rangle d\mu_g,$$

where D is a Dirac operator on $\mathcal{F}$. For any flat symplectic vector bundle $(\mathcal{E}, \Omega)$ with $\mathbf{J} \in \mathcal{J}(\mathcal{E}, \Omega)$, we define the *peripheral connection* to be a connection which commutes with $\mathbf{J}$, preserves the symplectic form, and depends only x on a small collar neighborhood of $\partial\Sigma$. For each peripheral connection, there is a natural Dirac operator $D^{\mathcal{F}}$ on $\mathcal{F}$, which is associated with a Clifford connection $\nabla^{\mathcal{F}}$. The Duhamel's formula gives

$$\lim_{t\to 0} \int_\Sigma \operatorname{Str}\left\langle z \left| e^{-tD^2} \right| z \right\rangle d\mu_g = \lim_{t\to 0} \int_\Sigma \operatorname{Str}\left\langle z \left| e^{-t\left(D^{\mathcal{F}}\right)^2} \right| z \right\rangle d\mu_g.$$

By the local index theorem, see e.g. [25, Theorem 8.34], we obtain

$$\lim_{t\to 0} \int_\Sigma \operatorname{Str}\left\langle z \left| e^{-t\left(D^{\mathcal{F}}\right)^2} \right| z \right\rangle d\mu_g = 2\int_\Sigma c_1(\mathcal{E}, \Omega, \mathbf{J}) + \frac{\dim E}{2}\chi(\Sigma).$$

Similarly, we can calculate the term $\int_\Sigma \alpha_+(z) d\mu_g$. Following a similar argument in [2], we obtain $h_\infty(\wedge^-) = h_\infty(\wedge^+)$. Combining with the above equalities, Theorem 0.1 is proved.

For the trivial symplectic vector bundle $(F, \Omega) = \mathrm{D}_n^{\mathrm{III}} \times (\mathbb{R}^{2n}, \Omega)$ over the bounded symmetric domain $\mathrm{D}_n^{\mathrm{III}}$ of type III, there is a canonical complex structure $\mathbf{J}_F$ on F. Moreover, we can define a complex connection ∇^F on $(F, \mathbf{J}_F)$ such that the first Chern form of the connection is $\frac{1}{4\pi}\omega_{\mathrm{D}_n^{\mathrm{III}}}$. For any representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(E, \Omega)$ and any $\mathbf{J} \in \mathcal{J}_o(\mathcal{E}, \Omega)$, we get a ρ-equivariant map $\widetilde{\mathbf{J}} : \widetilde{\Sigma} \to \mathrm{D}_n^{\mathrm{III}}$ by using the identification $\mathcal{J}(E, \Omega) \cong \mathrm{D}_n^{\mathrm{III}}$, which is also equivalent to the smooth map $\mathbf{J} : \Sigma \to \widetilde{\Sigma} \times_\rho \mathrm{D}_n^{\mathrm{III}}$. For the vector bundle $F_\rho = \widetilde{\Sigma} \times_\rho F$ over $\widetilde{\Sigma} \times_\rho \mathrm{D}_n^{\mathrm{III}}$, the identification $(E, \Omega) \cong (\mathbb{R}^{2n}, \Omega)$ induces a complex linear symplectic isomorphism between $(\mathbf{J}^* F_\rho, \mathbf{J}^* \mathbf{J}_{F_\rho}, \mathbf{J}^* \Omega)$ and $(\mathcal{E}, \mathbf{J}, \Omega)$, where $\mathbf{J}^*\Omega$ denotes the induced symplectic form on F_ρ. The connection ∇^F induces a natural connection

∇^{F_ρ} on F_ρ, and by pullback, the connection $\mathbf{J}^*\nabla^{F_\rho}$ can be proved to be a peripheral connection. On the other hand, the invariant Kähler form $\omega_{\mathrm{D}_n^{\mathrm{III}}}$ is also well-defined on $\widetilde{\Sigma}\times_\rho \mathrm{D}_n^{\mathrm{III}}$, which is just the curvature of the connection ∇^{F_ρ} up to a factor. The pullback 2-form $\widetilde{\mathbf{J}}^*\omega_{\mathrm{D}_n^{\mathrm{III}}}$ is a ρ-equivariant 2-form on $\widetilde{\Sigma}$, so it descends to a 2-form on Σ, which is just $\mathbf{J}^*\omega_{\mathrm{D}_n^{\mathrm{III}}}$. Note that $c_1(\mathcal{E},\Omega,\mathbf{J})$ is independent of the peripheral connection, thus

$$2\int_\Sigma c_1(\mathcal{E},\Omega,\mathbf{J})=\frac{1}{2\pi}\int_\Sigma \widetilde{\mathbf{J}}^*\omega_{\mathrm{D}_n^{\mathrm{III}}}.$$

By considering the specific correspondence between the bounded group cohomology and de Rham cohomolgy, we obtain

$$\mathrm{T}(\Sigma,\rho)=\frac{1}{2\pi}\int_\Sigma \widetilde{\mathbf{J}}^*\omega_{\mathrm{D}_n^{\mathrm{III}}}-\frac{1}{2\pi}\sum_{i=1}^q\int_{c_i}\widetilde{\mathbf{J}}^*\alpha_i,$$

and Theorem 0.2 is proved.

The signature also can be given by

$$\begin{aligned}\pm\operatorname{sign}(\mathcal{E},\Omega)=&-\dim E\cdot\chi(\Sigma)-\dim\mathrm{H}^0(\partial\Sigma,\mathcal{E})\\&+2\dim\mathrm{H}^0(\Sigma,\mathcal{E})-2\dim_{\mathbb{C}}\operatorname{Ker}\left(d^{\mp}\right)^*\cap\mathrm{L}^2\left(\widehat{\Sigma},\wedge^{\mp}\right).\end{aligned}$$

Since $\dim\mathrm{H}^0(\partial\Sigma,\mathcal{E})\geqslant q\dim\mathrm{H}^0(\Sigma,\mathcal{E})$ and so if $q\geqslant 2$, we conclude

$$|\operatorname{sign}(\mathcal{E},\Omega)|\leqslant\dim E|\chi(\Sigma)|.$$

For a general $q\geqslant 1$, and for any representation $\rho:\pi_1(\Sigma)\to\mathrm{Sp}(E,\Omega)$, by a small perturbation with the representation being fixed on the boundary, we obtain a smooth family $\{\rho_\epsilon\}_{\epsilon>0}$ such that the associated signature is invariant and $\rho_\epsilon\to\rho$ as $\epsilon\to 0$. Moreover $\dim\mathrm{H}^0(\Sigma,\mathcal{E}_\epsilon)=0$ for any $\epsilon>0$. Therefore, Theorem 0.3 is proved.

We also consider the case of $n=1$, i.e. $\rho:\pi_1(\Sigma)\to\mathrm{SL}(2,\mathbb{R})$. In this case, each element L has the form $L=\pm\exp(2\pi B)$ for some $B\in\mathfrak{sl}(2,\mathbb{R})$, we can define a canonical compatible complex structure by $\mathbf{J}(x)=\exp(-xB)J\exp(xB)$ for each boundary. Then η invariant of $A_{\mathbf{J}}$ and $\boldsymbol{\rho}$ invariant can be calculated explicitly, see the [18, Appendix]. By reversing the maximal even representations on boundary which are conjugate to the rotation $R(\theta),\theta\in(\pi,2\pi)$ or are parabolic with eigenvalues only 1, we obtain a new representation ρ_1 with the same Toledo invariant, i.e. $\mathrm{T}(\Sigma,\rho)=\mathrm{T}(\Sigma,\rho_1)$. By using (0.3) and Theorem 0.3 to ρ_1, Proposition 0.5 is proved.

Appendix A.

In this section, we will calculate the curvature of the bounded symmetric domain of type III.

Denote by

$$\mathrm{D}_n^{\mathrm{III}} := \left\{ W \in \mathfrak{gl}(n, \mathbb{C}) : W = W^\top, I_n - \overline{W} W > 0 \right\}$$

the Siegel's generalized upper half-plane, see [15, Chapter VIII, § 7]. Then

$$\omega_{\mathrm{D}_n^{\mathrm{III}}} := -2i\partial\bar{\partial} \log \det \left(I - \overline{W} W \right) = \frac{i}{2} h_{a\bar{b}} dw^a \wedge d\overline{w}^b$$

is a Kähler form on $\mathrm{D}_n^{\mathrm{III}}$. The Hermitian metric is denoted by $h = h_{a\bar{b}} dw^a \otimes d\bar{w}^b$, and the holomorphic sectional curvature is

$$\mathrm{K}(V) = \frac{2R\left(V, \overline{V}, V, \overline{V}\right)}{\|V\|^4} = \frac{2R_{a\bar{b}c\bar{d}} V^a \overline{V^b} V^c \overline{V^d}}{\left(h_{a\bar{b}} V^a \overline{V^b} \right)^2}$$

for any $V = V^a \frac{\partial}{\partial w^a}$ and $R_{a\bar{b}c\bar{d}} = -\partial_c \partial_{\bar{d}} h_{a\bar{b}} + \mathrm{H}^{p\bar{q}} \partial_c h_{a\bar{q}} \partial_{\bar{d}} h_{p\bar{b}}$. Since $\omega_{\mathrm{D}_n^{\mathrm{III}}}$ is invariant under the group $\mathrm{Aut}(\mathrm{D}_n^{\mathrm{III}})$ of holomorphic automorphisms, so we just need to calculate the holomorphic sectional curvature of the Bergman metric $\omega_{\mathrm{D}_n^{\mathrm{III}}}$ at $W = 0$. If H is a Hermitian matrix, then

$$\partial\bar{\partial} \log \det H = \mathrm{Tr}\left(H^{-1} \partial\bar{\partial} H \right) - \mathrm{Tr}\left(H^{-1} \partial H \wedge H^{-1} \bar{\partial} H \right).$$

Denote $\frac{1}{4} h_{ab\overline{cd}} = (I - \overline{W} W)^{-1}_{ac} \delta_{bd} + (I - \overline{W} W)^{-1}_{nm} (I - \overline{W} W)^{-1}_{bc} \overline{W_{am}} W_{dn}$. Then

$$\begin{aligned}
\omega_{\mathrm{D}_n^{\mathrm{III}}} = &\frac{i}{2} \sum_{a<b,\, c<d} \left(h_{ab\overline{cd}} + h_{ba\overline{cd}} + h_{ab\overline{dc}} + h_{ba\overline{dc}} \right) dW_{ab} \wedge d\overline{W_{cd}} \\
&+ \frac{i}{2} \sum_{a<b,\, c} \left(h_{ab\overline{cc}} + h_{ba\overline{cc}} \right) dW_{ab} \wedge d\overline{W_{cc}} \\
&+ \frac{i}{2} \sum_{c<d,\, a} \left(h_{aa\overline{cd}} + h_{aa\overline{dc}} \right) dW_{aa} \wedge d\overline{W_{cd}} \\
&+ \frac{i}{2} h_{aa\overline{cc}} dW_{aa} \wedge d\overline{W_{cc}} = \frac{i}{2} \sum_{a \leqslant b,\, c \leqslant d} H_{ab\overline{cd}} dW_{ab} \wedge d\overline{W_{cd}},
\end{aligned}$$

where $H_{ab\overline{cd}} = (h_{ab\overline{cd}} + h_{ba\overline{cd}} + h_{ab\overline{dc}} + h_{ba\overline{dc}})(1 - \frac{1}{2}\delta_{ab})(1 - \frac{1}{2}\delta_{cd})$. At the point $W = 0$, one has

$$\frac{1}{4} h_{ab\overline{cd}} = \delta_{ac}\delta_{bd}, \quad \partial h_{abcd} = 0,$$

and

$$\begin{aligned}
&\frac{1}{4}\frac{\partial^2 h_{ab\overline{cd}}}{\partial W_{kl}\partial\overline{W_{pq}}}\\
&\quad=(\delta_{ap}\delta_{mq}+\delta_{aq}\delta_{mp})(\delta_{mk}\delta_{cl}+\delta_{ml}\delta_{ck})\delta_{bd}\left(1-\frac{1}{2}\delta_{pq}\right)\left(1-\frac{1}{2}\delta_{kl}\right)\\
&\qquad+(\delta_{ap}\delta_{mq}+\delta_{aq}\delta_{mp})(\delta_{dk}\delta_{ml}+\delta_{dl}\delta_{mk})\delta_{bc}\left(1-\frac{1}{2}\delta_{pq})(1-\frac{1}{2}\delta_{kl}\right).
\end{aligned}$$

The curvature is

$$\begin{aligned}
R_{ab\overline{cd}kl\overline{pq}}&=-\frac{\partial^2 H_{ab\overline{cd}}}{\partial W_{kl}\partial\overline{W_{pq}}}\\
&=-8\Big((\delta_{ap}\delta_{mq}+\delta_{aq}\delta_{mp})(\delta_{mk}\delta_{cl}+\delta_{ml}\delta_{ck})\delta_{bd}\\
&\quad+(\delta_{ap}\delta_{mq}+\delta_{aq}\delta_{mp})(\delta_{mk}\delta_{dl}+\delta_{ml}\delta_{dk})\delta_{bc}\\
&\quad+(\delta_{bp}\delta_{mq}+\delta_{bq}\delta_{mp})(\delta_{mk}\delta_{cl}+\delta_{ml}\delta_{ck})\delta_{ad}\\
&\quad+(\delta_{bp}\delta_{mq}+\delta_{bq}\delta_{mp})(\delta_{mk}\delta_{dl}+\delta_{ml}\delta_{dk})\delta_{ac}\Big)\\
&\quad\cdot\left(1-\frac{1}{2}\delta_{ab}\right)\left(1-\frac{1}{2}\delta_{cd}\right)\left(1-\frac{1}{2}\delta_{pq}\right)\left(1-\frac{1}{2}\delta_{kl}\right),
\end{aligned}$$

where $a\leqslant b, c\leqslant d, k\leqslant l, p\leqslant q$. Since

$$\begin{aligned}
H(0)&=\sum_{a\leqslant b,\,c\leqslant d}H_{ab\overline{cd}}(0)dW_{ab}\otimes d\overline{W_{cd}}\\
&=4\left(\sum_{a<b}2dW_{ab}\otimes d\overline{W_{ab}}+\sum_{a}dW_{aa}\otimes d\overline{W_{aa}}\right)
\end{aligned}$$

where $H_{ab\overline{cd}}(0)=8(\delta_{ac}\delta_{bd}+\delta_{bc}\delta_{ad})(1-\frac{1}{2}\delta_{ab})(1-\frac{1}{2}\delta_{cd})$. So the inverse matrix is

$$H^{\overline{cd}ab}(0)=\frac{1}{8}(\delta_{ac}\delta_{bd}+\delta_{bc}\delta_{ad}).$$

Then the Ricci curvature is

$$\begin{aligned}
R_{kl\overline{pq}}&=H^{\overline{cd}ab}R_{ab\overline{cd}kl\overline{pq}}\\
&=\frac{1}{8}\sum_{a<b}R_{ab\overline{ab}kl\overline{pq}}+\frac{1}{4}\sum_{a}R_{aa\overline{aa}kl\overline{pq}}\\
&=-(n+1)\sum_{a=1}^{n}(\delta_{ap}\delta_{mq}+\delta_{aq}\delta_{mp})\\
&\quad(\delta_{mk}\delta_{al}+\delta_{ml}\delta_{ak})\left(1-\frac{1}{2}\delta_{pq}\right)\left(1-\frac{1}{2}\delta_{kl}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -2(n+1)\left(\delta_{pl}\delta_{qk}+\delta_{pk}\delta_{ql}\right)\left(1-\frac{1}{2}\delta_{pq}\right)\left(1-\frac{1}{2}\delta_{kl}\right)\\
&= -\frac{n+1}{4}H_{kl\overline{pq}}.
\end{aligned}
$$

Thus the first Chern form satisfies

$$
\frac{i}{2\pi}R_{kl\overline{pq}}dW_{kl}\wedge d\overline{W_{pq}} = -\frac{i}{2\pi}\frac{n+1}{4}H_{kl\overline{pq}}dW_{kl}\wedge d\overline{W_{pq}} = -\frac{n+1}{2}\frac{1}{2\pi}\omega_{\mathrm{D}_n^{\mathrm{III}}}.
$$

Now we calculate the holomorphic sectional curvature. Note that

$$
\begin{aligned}
&\sum_{\substack{a\leqslant b,\, c\leqslant d\\ k\leqslant l,\, p\leqslant q}} R_{ab\overline{cd}kl\overline{pq}}dW_{kl}\wedge d\overline{W_{pq}}\otimes dW_{ab}\wedge d\overline{W_{cd}}\\
&\qquad = -\sum_{a\leqslant b,\, c\leqslant d}\partial\bar\partial H_{ab\overline{cd}}\otimes dW_{ab}\wedge d\overline{W_{cd}}\\
&\qquad = -\sum_{a,b,c,d}\partial\bar\partial h_{ab\overline{cd}}\otimes dW_{ab}\wedge d\overline{W_{cd}}\\
&\qquad = \sum_{a,b,c,d,k,l,p,q}\widetilde{R}_{ab\overline{cd}kl\overline{pq}}dW_{kl}\wedge d\overline{W_{pq}}\otimes dW_{ab}\wedge d\overline{W_{cd}},
\end{aligned}
$$

where $\widetilde{R}_{ab\overline{cd}kl\overline{pq}} = -4(\delta_{bd}\delta_{ck}\delta_{ap}\delta_{ql}+\delta_{bq}\delta_{ca}\delta_{pk}\delta_{dl})$. For any $(1,0)$-type tangent vector $V=(V^{ab})$, $a\leqslant b$, at 0. We also denote by $\mathbf{V}$ the matrix associated with the vector (V^{ab}) by setting $V^{ab}=V^{ba}$. Then

$$
\begin{aligned}
&R_{ab\overline{cd}kl\overline{pq}}V^{ab}\overline{V^{cd}}V^{kl}\overline{V^{pq}}\\
&= \left(\sum_{\substack{a\leqslant b,\, c\leqslant d\\ k\leqslant l,\, p\leqslant q}} R_{ab\overline{cd}kl\overline{pq}}dW_{kl}\wedge d\overline{W_{pq}}\otimes dW_{ab}\wedge d\overline{W_{cd}}\right)\left(V\wedge\overline{V}\otimes V\wedge\overline{V}\right)\\
&= \left(\sum_{a,b,c,d,k,l,p,q}\widetilde{R}_{ab\overline{cd}kl\overline{pq}}dW_{kl}\wedge d\overline{W_{pq}}\otimes dW_{ab}\wedge d\overline{W_{cd}}\right)\left(\mathbf{V}\wedge\overline{\mathbf{V}}\otimes\mathbf{V}\wedge\overline{\mathbf{V}}\right)\\
&= \widetilde{R}_{ab\overline{cd}kl\overline{pq}}\mathbf{V}^{ab}\overline{\mathbf{V}^{cd}}\mathbf{V}^{kl}\overline{\mathbf{V}^{pq}} = -8\,\mathrm{Tr}\left(\left(\overline{\mathbf{V}}\mathbf{V}\right)^2\right),
\end{aligned}
$$

and $\|V\|^2 = 4(2\sum_{a<b}|V^{ab}|^2+\sum_a|V^{aa}|^2) = 4\,\mathrm{Tr}(\overline{\mathbf{V}}\mathbf{V})$. Thus the holomorphic sectional curvature is

$$
\mathrm{K}(V) = -\frac{\mathrm{Tr}\left(\left(\overline{\mathbf{V}}\mathbf{V}\right)^2\right)}{\left(\mathrm{Tr}\left(\overline{\mathbf{V}}\mathbf{V}\right)\right)^2}\in[-1,-1/n],
$$

and $\mathrm{K}(V)=-1/n$ iff $\mathbf{V}(\det\mathbf{V})^{-1/n}$ is a unitary group.

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Inkang Kim
School of Mathematics,
KIAS, Heogiro 85,
Dongdaemun-gu Seoul,
02455 (Republic of Korea)
inkang@kias.re.kr

Pierre Pansu
Universit Paris-Saclay, CNRS,
Laboratoire de Mathématiques d'Orsay
91405 Orsay Cédex (France)
pierre.pansu@universite-paris-saclay.fr

Xueyuan Wan
Mathematical Science Research Center
Chongqing University of Technology,
Chongqing 400054 (China)
xwan@cqut.edu.cn