Institut Fourier — Université de Grenoble I

Actes du séminaire de

Théorie spectrale et géométrie

Tommaso Rossi

The Relative Heat Content for Submanifolds in Sub-Riemannian Geometry

Volume 36 (2019-2021), p. 191-212.

https://doi.org/10.5802/tsg.376

e-ISSN : 2118-9242

Séminaire de théorie spectrale et géométrie
GRENOBLE
Volume 36 (2019-2021) 191-212

THE RELATIVE HEAT CONTENT FOR SUBMANIFOLDS IN SUB-RIEMANNIAN GEOMETRY

Tommaso Rossi

ABSTRACT. — We study the small-time asymptotics of the relative heat content for submanifolds in sub-Riemannian geometry. First, we prove the existence of a smooth tubular neighborhood for submanifolds of any codimension, assuming they do not have characteristic points. Next, we propose a definition of relative heat content for submanifolds of codimension $k \geqslant 1$ and we build an approximation of this quantity, via smooth tubular neighborhoods. Finally, we show that this approximation fails to recover the asymptotic expansion of the relative heat content of the submanifold, by studying an explicit example.

1. Introduction

In this paper, we study the small-time behavior of the relative heat content for submanifolds in sub-Riemannian geometry.

Loosely speaking, the relative heat content for a submanifold $S \subset M$ can be regarded as the total amount of heat contained in S at time t, corresponding to a uniform initial temperature distribution concentrated on S. When $S = \Omega$ is an open and bounded domain, the problem of finding an asymptotic expansion for the heat content has been extensively studied, see for example [5, 6, 7, 8, 9, 17] for the Euclidean and Riemannian case, and [2, 16, 19] for the sub-Riemannian case. In particular, in this situation, in both the Riemannian and sub-Riemannian setting, the small-time behavior of the heat content associated with Ω encodes geometrical information of $\partial\Omega$, such as its perimeter or its mean curvature (c.f. Theorem 2.3). Thus, we may expect that, in an analogous way, the asymptotics of the relative heat content for a submanifold of higher codimension detects its geometrical invariants.

Acknowledgements: This work was supported by the Grant ANR-18-CE40-0012 of the ANR, by the Project VINCI 2019 ref. c2-1212 and by ERC Starting Grant 2020, project GEOSUB (grant agreement No. 945655).

Up to our knowledge, the relative heat content for submanifolds has never been systematically studied, not even in Riemannian geometry, so we propose the following definition. Let M be a sub-Riemannian manifold and let $S \subset M$ be a smooth, compact submanifold of codimension $k \geqslant 0$. Let ω be a smooth measure on M, and let μ be a probability measure on S. Then, we consider μ as the initial datum for the heat equation in the sense of distributions, and study the associated Cauchy problem:

$$\begin{aligned} (\partial_t - \Delta)u(t,x) &= 0, && \forall\ (t,x) \in (0,\infty) \times M, \\ u(t,\cdot) &\xrightarrow{t\to 0} \mu, && \text{in } \mathcal{D}'(M), \end{aligned} \tag{1.1}$$

where $\Delta = \operatorname{div}_\omega \circ \nabla$ is the usual sub-Laplacian associated with ω. A solution to this problem, in the sense of distribution, is given by

$$u(t,x) = \int_S p_t(x,y)d\mu(y), \qquad \forall\ (t,x) \in (0,\infty) \times M, \tag{1.2}$$

where $p_t(x,y)$ is the usual heat kernel associated with Δ and ω. We define the *relative heat content for a submanifold* S as:

$$H_S(t) = \int_S \int_S p_t(x,y)d\mu(x)d\mu(y), \qquad \forall\ t > 0. \tag{1.3}$$

Notice that, on the one hand if $S = \{x_0\}$ and $\mu = \delta_{x_0}$, we obtain the trace heat kernel $p_t(x_0,x_0)$. On the other hand, if S is a 0-codimensional submanifold, i.e. S is an open, relatively compact and smooth set, and we choose $\mu = \mathbb{1}_S\omega$, then (1.3) coincides with the usual relative heat content as defined in [2]. Therefore, a small-time asymptotic expansion of (1.3) would include many cases of interest, ranging from the trace heat kernel asymptotics, see for example [4, 20], to the results of contained in [2].

Our attempt to compute the asymptotics of (1.3) consists in building a suitable approximation of it, using smooth tubular neighborhoods of S. When S is a smooth hypersurface, the existence of a tubular neighborhood is guaranteed assuming that there are no characteristic points, cf. [2, Def. 2.4]. Therefore, as a first step, we introduce a definition of a non-characteristic submanifold, cf. Definition 3.1, generalizing the classical one of non-characteristic hypersurface. Then, denoting by $\delta_S \colon M \to [0,+\infty)$ the distance function from S, see (2.9) for the precise definition, we prove the following.

Theorem 1.1. — *Let M be a sub-Riemannian manifold and $S \subset M$ be a compact smooth non-characteristic submanifold of codimension $k \geqslant 1$. Then, there exists $r_0 > 0$ such that, denoting by $S_{r_0} = \{p \in M \mid 0 < \delta_{\mathrm{S}} < r_0\}$, the following conditions hold:*

(i) $\delta_{\mathrm{S}}\colon S_{r_0} \to [0,\infty)$ *is smooth and such that* $\|\nabla\delta_{\mathrm{S}}\|_g = 1$*;*
(ii) *there exists a diffeomorphism* $G\colon (0,r_0)\times\{\delta_{\mathrm{S}} = r_0\} \to S_{r_0}$ *such that:*

$$\delta_{\mathrm{S}}(G(r,p)) = r \qquad \textit{and} \qquad G_*\partial_r = \nabla\delta_{\mathrm{S}}. \tag{1.4}$$

Theorem 1.1 generalizes the analogous result in [10, Prop. 3.1], and [14, Lem. 23], to submanifolds of any codimension and is a key tool to build both a canonical probability measure on S, cf. Lemma 4.1, and the approximation of the relative heat content. Indeed, for any $\varepsilon \leqslant r_0$, denoting by S_ε the tubular neighborhood of S, of radius ε, we consider the (rescaled) relative heat content associated with S_ε, namely

$$H_S^\varepsilon(t) = \frac{1}{\omega(S_\varepsilon)^2}\int_{S_\varepsilon}\int_{S_\varepsilon} p_t(x,y)d\omega(x)d\omega(y), \qquad \forall\ t>0. \tag{1.5}$$

We show that H_S^ε converges point-wise to H_S, as $\varepsilon \to 0$, cf. Proposition 5.1. Since now H_S^ε is the (rescaled) relative heat content associated with a non-characteristic open and bounded set, for any $\varepsilon > 0$, we can apply [2, Thm. 1.1], see also Theorem 2.3 for the statement, and try to deduce the asymptotics for the limit as $\varepsilon \to 0$.

Unfortunately, the point-wise convergence of H_S^ε is too weak to recover any information regarding the small-time asymptotic expansion of the limit. Indeed, by studying an explicit example, we show that the approximation procedure using (1.5) fails to recover the asymptotic expansion of H_S. More precisely, we consider a closed simple curve in $\mathbb{R}^3$, equipped with the Euclidean metric and the Lebesgue measure. In this case, on the one hand, by a standard application of the Laplace method, it is possible to compute the asymptotic expansion of $H_S(t)$ as $t \to 0$ at any order: the coefficients appearing in the expansion depend on the curvature of the curve and its derivatives of *any order*. On the other hand, following the approximation strategy described above, we obtain an expansion in the limit which can't possibly agree with the correct one since

- the only geometrical invariant appearing in the coefficients is the curvature of the curve *without* its derivatives;
- the orders of the expansion don't agree.

Structure of the paper

In Section 2, we recall the basic definitions of sub-Riemannian geometry. In Section 3, we give the definition of a non-characteristic submanifold and we prove Theorem 1.1. In Section 4, we build the canonical probability measure on S, which is induced by the outer measure ω. In Section 5, we introduce the definition of relative heat content for a submanifold S, we build its approximation and we prove its point-wise convergence. Finally, in Section 6, we show how the approximation fails to recover the asymptotic expansion of the relative heat content of S.

2. Preliminaries

We recall some essential facts in sub-Riemannian geometry, following [1].

Sub-Riemannian geometry

Let M be a smooth, connected finite-dimensional manifold. A sub-Riemannian structure on M is defined by a set of N global smooth vector fields $X_1, \dots, X_N$, called a *generating frame*. The generating frame defines a *distribution* of subspaces of the tangent spaces at each point $x \in M$, given by

$$\mathcal{D}_x = \operatorname{span}\{X_1(x), \dots, X_N(x)\} \subseteq T_xM. \tag{2.1}$$

We assume that the distribution is *bracket-generating*, i.e. the Lie algebra of smooth vector fields generated by $X_1, \dots, X_N$, evaluated at the point x, coincides with T_xM, for all $x \in M$. The generating frame induces a norm on the distribution at x, namely

$$g_x(v, v) = \inf\left\{ \sum_{i=1}^N u_i^2 \;\middle|\; \sum_{i=1}^N u_i X_i(x) = v \right\}, \qquad \forall\ v \in \mathcal{D}_x, \tag{2.2}$$

which, in turn, defines an inner product on $\mathcal{D}_x$ by polarization. We use the shorthand $\|\cdot\|_g$ for the corresponding norm. We say that $\gamma : [0,T] \to M$ is a *horizontal curve*, if it is absolutely continuous and

$$\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}, \qquad \text{for a.e. } t \in [0,T].$$

This implies that there exists $u : [0,T] \to \mathbb{R}^N$, such that

$$\dot\gamma(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)), \qquad \text{for a.e. } t \in [0,T].$$

Moreover, we require that $u \in L^2([0,T],\mathbb{R}^N)$. If γ is a horizontal curve, then the map $t \mapsto \|\dot\gamma(t)\|_g$ is integrable on $[0,T]$, see [1, Lem. 3.12]. We define the *length* of a horizontal curve as follows:

$$\ell(\gamma) = \int_0^T \|\dot\gamma(t)\|_g \, dt. \tag{2.3}$$

The *sub-Riemannian distance* is defined, for any $x, y \in M$, by

$$d_{\mathrm{SR}}(x,y) = \inf\{\ell(\gamma) \mid \gamma \text{ horizontal curve between } x \text{ and } y\}. \tag{2.4}$$

By Chow–Rashevskii Theorem, the bracket-generating assumption ensures that the distance $d_{\mathrm{SR}}\colon M \times M \to \mathbb{R}$ is finite and continuous. Furthermore it induces the same topology as the manifold one.

Remark 2.1. — The above definition includes all classical constant-rank sub-Riemannian structures as in [12, 15] (where $\mathcal{D}$ is a vector distribution and g a symmetric and positive tensor on $\mathcal{D}$), but also general rank-varying sub-Riemannian structures. The same sub-Riemannian structure can arise from different generating families.

Geodesics and Hamiltonian flow

A *geodesic* is a horizontal curve $\gamma : [0,T] \to M$, parametrized with constant speed, and such that any sufficiently short segment is length-minimizing. The *sub-Riemannian Hamiltonian* is the smooth function $H : T^*M \to \mathbb{R}$,

$$H(\lambda) = \frac{1}{2}\sum_{i=1}^N \langle \lambda, X_i\rangle^2, \qquad \lambda \in T^*M, \tag{2.5}$$

where $X_1, \ldots, X_N$ is a generating frame for the sub-Riemannian structure, and $\langle \lambda, \cdot\rangle$ denotes the action of covectors on vectors. The *Hamiltonian vector field* $\vec{H}$ on T^*M is then defined by $\varsigma(\cdot, \vec{H}) = dH$, where $\varsigma \in \Lambda^2(T^*M)$ is the canonical symplectic form.

Solutions $\lambda : [0,T] \to T^*M$ of the *Hamilton equations*

$$\dot\lambda(t) = \vec{H}(\lambda(t)), \tag{2.6}$$

are called *normal extremals*. Their projections $\gamma(t) = \pi(\lambda(t))$ on M, where $\pi : T^*M \to M$ is the bundle projection, are locally length-minimizing horizontal curves parametrized with constant speed, and are called *normal* *geodesics*. If γ is a normal geodesic with normal extremal λ, then its speed is given by $\|\dot\gamma\|_g = \sqrt{2H(\lambda)}$. In particular

$$\ell\left(\gamma|_{[0,t]}\right) = t\sqrt{2H(\lambda(0))}, \qquad \forall\ t \in [0,T]. \tag{2.7}$$

There is another class of length-minimizing curves in sub-Riemannian geometry, called *abnormal* or *singular*. As for the normal case, to these curves it corresponds an extremal lift $\lambda(t)$ on T^*M, which however may not follow the Hamiltonian dynamics (2.6). Here we only observe that an abnormal extremal lift $\lambda(t) \in T^*M$ satisfies

$$\left\langle \lambda(t), \mathcal{D}_{\pi(\lambda(t))} \right\rangle = 0 \quad \text{and} \quad \lambda(t) \neq 0, \qquad \forall\ t \in [0,T], \tag{2.8}$$

that is $H(\lambda(t)) \equiv 0$. A geodesic may be abnormal and normal at the same time.

Length-minimizers to a submanifold

Let $S \subset M$ be a closed embedded submanifold of codimension $k \geqslant 0$ and define the sub-Riemannian distance from S:

$$\delta_{\mathrm{S}}(p) = \inf\left\{d_{\mathrm{SR}}(q,p), q \in S\right\}, \qquad \forall\, p \in M. \tag{2.9}$$

Let $\gamma : [0,T] \to M$ be a horizontal curve, parametrized with constant speed, such that $\gamma(0) \in S$, $\gamma(T) = p \in M \setminus S$ and assume γ is a minimizer for δ_{S}, that is $\ell(\gamma) = \delta_{\mathrm{S}}(p)$. In particular, γ is a geodesic. Any corresponding normal or abnormal lift, say $\lambda : [0,T] \to T^*M$, must satisfy the transversality conditions, cf. [3, Thm. 12.13],

$$\langle \lambda(0), v \rangle = 0, \qquad \forall\ v \in T_{\gamma(0)}S. \tag{2.10}$$

Equivalently, the initial covector $\lambda(0)$ must belong to the annihilator bundle $\mathcal{A}(S) = \{\lambda \in T^*M \mid \langle \lambda, T_{\pi(\lambda)}S \rangle = 0\}$ of S.

The heat equation and the relative heat content for a domain

Let M be a sub-Riemannian manifold and let ω be a smooth measure on M, i.e. defined by a positive tensor density. The *divergence* of a smooth vector field is defined by

$$\operatorname{div}_\omega(X)\omega = \mathcal{L}_X\omega, \qquad \forall\ X \in \Gamma(TM),$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X. The *horizontal gradient* of a function $f \in C^\infty(M)$, denoted by ∇f, is defined as the horizontal vector field (i.e. tangent to the distribution at each point), such that

$$g_x\left(\nabla f(x), v\right) = v(f)(x), \qquad \forall\ v \in \mathcal{D}_x,$$

where v acts as a derivation on f. In terms of a generating frame as in (2.1), one has

$$\nabla f = \sum_{i=1}^{N} X_i(f)X_i, \qquad \forall\ f \in C^\infty(M).$$

The *sub-Laplacian* is the operator $\Delta = \operatorname{div}_\omega \circ \nabla$, acting on $C^\infty(M)$. Again, we may write its expression with respect to a generating frame (2.1), obtaining

$$\Delta f = \sum_{i=1}^{N} \left\{X_i^2(f) + X_i(f)\operatorname{div}_\omega(X_i)\right\}, \qquad \forall\ f \in C^\infty(M). \tag{2.11}$$

We denote by $L^2(M,\omega)$, or simply by L^2, the space of real functions on M which are square-integrable with respect to the measure ω. Let $\Omega \subset M$ be an open relatively compact set with smooth boundary. This means that the closure $\bar{\Omega}$ is a compact manifold with smooth boundary. We consider the *Cauchy problem for the heat equation* on Ω, that is we look for functions u such that

$$\begin{aligned} (\partial_t - \Delta)\, u(t,x) &= 0, &\qquad& \forall\ (t,x) \in (0,\infty) \times M, \\ u(0,\cdot) &= \mathbb{1}_\Omega, && \text{in } L^2(M,\omega), \end{aligned} \tag{2.12}$$

where $u(0,\cdot)$ is a shorthand notation for the L^2-limit of $u(t,x)$ as $t \to 0$. Notice that Δ is symmetric with respect to the L^2-scalar product and negative, moreover, if (M, d_{SR}) is complete as a metric space, it is essentially self-adjoint, see [18]. Thus, there exists a unique solution to (2.12), and it can be represented as

$$u(t,x) = e^{t\Delta}\mathbb{1}_\Omega(x), \qquad \forall\ x \in M,\ t > 0,$$

where $e^{t\Delta} \colon L^2 \to L^2$ denotes the heat semi-group, associated with Δ. We remark that for all $\varphi \in L^2$, the function $e^{t\Delta}\varphi$ is smooth for all $(t,x) \in (0,\infty) \times M$, by hypoellipticity of the heat operator, see [11]. Furthermore, there exists a heat kernel associated with (2.12), i.e. a positive function $p_t(x,y) \in C^\infty((0,+\infty) \times M \times M)$ such that:

$$u(t,x) = \int_M p_t(x,y)\mathbb{1}_\Omega(y)d\omega(y) = \int_\Omega p_t(x,y)d\omega(y). \tag{2.13}$$

Definition 2.2 (Relative heat content). — *Let $u(t,x)$ be the solution to* (2.12)*. We define the* relative heat content*, associated with Ω, as*

$$H_\Omega(t) = \int_\Omega u(t,x)d\omega(x), \qquad \forall\ t>0. \tag{2.14}$$

In [2], the authors proved the existence of a small-time asymptotic expansion for $H_\Omega(t)$, provided that Ω is a non-characteristic domain. Precisely, denoting by

$$\delta_{\partial\Omega}(p) = \inf\left\{d_{\mathrm{SR}}(q,p), q\in\partial\Omega\right\}, \qquad \forall\ p\in M,$$

the distance from the boundary of Ω and by σ the induced sub-Riemannian measure on $\partial\Omega$ (i.e. the one whose density is $\sigma = |i_\nu\omega|_{\partial\Omega}$, where ν is the outward-pointing normal vector field to Ω), we have the following result. See Definition 3.1 for the notion of a characteristic point.

Theorem 2.3. — *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega\subset M$ be an open subset whose boundary is smooth and has no characteristic points. Then, as $t\to 0$,*

$$\begin{aligned} H_\Omega(t) &= \omega(\Omega) - \frac{1}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2} \\ &\quad - \frac{1}{12\sqrt{\pi}}\int_{\partial\Omega}\left(2g(\nabla\delta_{\partial\Omega},\nabla(\Delta\delta_{\partial\Omega})) - (\Delta\delta_{\partial\Omega})^2\right)d\sigma t^{3/2} + o\left(t^2\right). \end{aligned} \tag{2.15}$$

Remark 2.4. — The compactness assumption in Theorem 2.3 is technical and can be relaxed by requiring, instead, global doubling of the measure and a global Poincaré inequality. We refer to [2] for more details.

3. Tubular neighborhood for submanifolds

Definition 3.1 (Non-characteristic submanifold). — *Let M be a sub-Riemannian manifold and let $S\subset M$ be a smooth submanifold of codimension $k\geqslant 0$. We say that a point $q\in S$ is* non-characteristic *if*

$$\mathcal{D}_q + T_qS = T_qM. \tag{3.1}$$

We say that S is a non-characteristic submanifold *if* (3.1) *holds for any point $q\in S$.*

Remark 3.2. — Notice that Definition 3.1 includes the usual one for hypersurfaces, indeed if $S\subset M$ is a submanifold of codimension 1, it is easy to check that

$$\mathcal{D}_q\subset T_qS \quad\Leftrightarrow\quad \mathcal{D}_q + T_qS \subsetneq T_qM.$$

Under the assumption of non-characteristic submanifold, the distance from S, δ_{S} defined in (2.9), is smooth and it allows to build smooth tubular neighborhoods of S.

THEOREM 3.3. — *Let M be a sub-Riemannian manifold and $S \subset M$ be a compact smooth non-characteristic submanifold of codimension $k \geqslant 1$. Then, there exists $r_0 > 0$ such that, denoting by $S_{r_0} = \{p \in M \mid 0 < \delta_{\mathrm{S}} < r_0\}$, the following conditions hold:*

(i) *$\delta_{\mathrm{S}}\colon S_{r_0} \to [0,\infty)$ is smooth and such that $\|\nabla \delta_{\mathrm{S}}\|_g = 1$;*
(ii) *there exists a diffeomorphism $G\colon (0, r_0) \times \{\delta_{\mathrm{S}} = r_0\} \to S_{r_0}$ such that:*

$$\delta_{\mathrm{S}}(G(r,p)) = r \qquad \textit{and} \qquad G_* \partial_r = \nabla \delta_{\mathrm{S}}.$$

Before giving the proof of the theorem, we need a preliminary lemma, which can be regarded as a partial generalization of [10, Prop. 2.7].

LEMMA 3.4. — *Let M be a sub-Riemannian manifold and $S \subset M$ be a smooth submanifold of codimension $k \geqslant 1$. Let $\gamma\colon [0,1] \to M$ be a minimizing geodesic such that*

$$\gamma(0) \in S, \qquad \gamma(1) = p \in M \setminus S, \qquad \delta_{\mathrm{S}}(p) = \ell(\gamma).$$

If γ is an abnormal geodesic, then $\gamma(0)$ is a characteristic point of S.

Proof. — Let $\lambda\colon [0,1] \to T^*M$ be an abnormal lift of γ: this means in particular that $\pi(\lambda(t)) = \gamma(t)$ and

$$\langle \lambda(0), \mathcal{D}_{\gamma(0)} \rangle = 0, \qquad \text{with } \lambda(0) \neq 0, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual coupling. Moreover, since γ is a minimizing geodesic, any lift must necessarily satisfy the transversality condition (2.10). Thus, since $\lambda(0) \neq 0$, conditions (3.2), (2.10) imply that (3.1) fails at $q = \gamma(0)$. □

Proof of Theorem 3.3. — Let us consider the annihilator bundle of S, $\mathcal{A}S$, namely the vector bundle of rank k, whose fibers are given by

$$\mathcal{A}_q S = \left\{ \lambda \in T_q^* M \,\middle|\, \langle \lambda, T_q S \rangle = 0 \right\}, \qquad \forall\, q \in S.$$

At a point $q \in S$, let us fix a basis of the fiber $\mathcal{A}_q S$, say $\{\lambda_1, \ldots, \lambda_k\}$ and define, for any $j = 1, \ldots, k$, the element $v_j \in \mathcal{D}_q$ dual to λ_i via the Hamiltonian H, i.e.

$$v_j = \pi_* \vec{H}(\lambda_j) = \sum_{i=1}^{N} \langle \lambda_j, X_i(q) \rangle \, X_i(q) \qquad j = 1, \ldots, k. \tag{3.3}$$

Step 1. — If q is non-characteristic, then the set $\{v_1, \ldots, v_k\}$ is linearly independent.
Indeed assume there exists constants α_i for $i = 1, \ldots, k$, such that $\sum_{i=1}^k \alpha_i v_i = 0$. Then,

$$\begin{aligned} 0 = \sum_{j=1}^k \alpha_j v_j &= \sum_{j=1}^k \alpha_j \sum_{i=1}^N \langle \lambda_j, X_i(q)\rangle X_i(q) \\ &= \sum_{i=1}^N \left\langle \sum_{j=1}^k \alpha_j \lambda_j, X_i(q) \right\rangle X_i(q) = \pi_* \vec{H}(\lambda), \end{aligned} \tag{3.4}$$

having set $\lambda = \sum_{j=1}^k \alpha_j \lambda_j \in \mathcal{A}_q S$. Notice that, by the Lagrange multiplier rule, denoting by $v_\lambda = \pi_* \vec{H}(\lambda)$, for any $\lambda \in T_q^* M$, we have

$$\begin{aligned} \|v_\lambda\|_g^2 &= \inf \left\{ \sum_{i=1}^N u_i^2 \,\middle|\, v_\lambda = \sum_{i=1}^N u_i X_i(q) \right\} \\ &= \sum_{i=1}^N \langle \lambda, X_i(q)\rangle^2 = 2H(\lambda). \end{aligned} \tag{3.5}$$

Therefore, (3.4) implies that $\|\pi_* \vec{H}(\lambda)\|_g^2 = 2H(\lambda) = 0$, or equivalently:

$$\langle \lambda, \mathcal{D}_q \rangle = 0. \tag{3.6}$$

Since $\lambda \in \mathcal{A}_q S$ and q is non-characteristic, by (3.6), we deduce that $\lambda = 0$. Thus:

$$0 = \lambda = \sum_{j=1}^k \alpha_j \lambda_j \quad \Rightarrow \quad \alpha_j = 0, \quad \text{for any } j = 1, \ldots, k,$$

since $\{\lambda_1, \ldots, \lambda_k\}$ was a basis of the fiber of $\mathcal{A}S$. This concludes the proof of the first step. Define now the sub-Riemannian exponential map from S, i.e. the map

$$E \colon D \cap \mathcal{A}S \to M; \qquad E(\lambda) = \pi \circ e^{\vec{H}}(\lambda),$$

where $D \subset T^*M$ is the open set where the flow of $\vec{H}$ is defined up to time 1. Consider also the zero section of the annihilator bundle, namely

$$i \colon S \to \mathcal{A}S; \qquad i(q) = (q, 0) \in \mathcal{A}_q S.$$

Step 2. — E is a local diffeomorphism at points of $i(S)$. To prove the claim, we consider a point $(q, 0) \in i(S)$ and verify that $d_{(q,0)}E$ is invertible. Identifying $T_{(q,0)}(D \cap \mathcal{A}S) \cong T_q S \oplus \mathcal{A}_q S$, we have, on the one hand $E \circ i = Id_S$, therefore for a vector $v = (v, 0) \in T_q S \oplus \mathcal{A}_q S$,

$$d_{(q,0)}E(v) = \left.\frac{d}{dt}\right|_{t=0} E(\lambda(t)) = \left.\frac{d}{dt}\right|_{t=0} E \circ i(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} \gamma(t) = v,$$

since $\lambda(t) = (\gamma(t), 0)$, with $\gamma\colon (-\varepsilon, \varepsilon) \to S$, such that $\gamma(0) = q$ and $\dot{\gamma}(0) = v$. On the other hand, take an element $\lambda = (0, \lambda) \in T_qS \oplus \mathcal{A}_qS$, then by definition, we obtain

$$\begin{aligned} d_{(q,0)}E(\lambda) = \left.\frac{d}{dt}\right|_{t=0} E(q, t\lambda) = \left.\frac{d}{dt}\right|_{t=0} E(q, t\lambda) = \left.\frac{d}{dt}\right|_{t=0} \pi \circ e^{t\vec{H}}(\lambda) \\ = \pi_* \vec{H}(\lambda) = v_\lambda. \end{aligned}$$

Thus, choosing any basis for T_qS and the basis $\{\lambda_1, \ldots, \lambda_k\}$ for $\mathcal{A}_qS$, as before, we may write the $n \times n$ matrix representing the differential of E as

$$d_{(q,0)}E = \left(\begin{matrix} Id_{n-k} \\ \mathbf{0} \end{matrix} \middle| \, v_1, \ldots, v_k \right)$$

where the vectors v_j are defined in (3.3). Since, by the previous step, the set $\{v_1, \ldots, v_k\}$ is linearly independent in $\mathcal{D}_q$ and, by construction, $v_j \notin T_qS$ for any $j = 1, \ldots, k$, we conclude that dE is invertible at $i(S)$.

Step 3. — There exists $U \subset D \cap \mathcal{A}S$, such that $E|_U$ is a diffeomorphism on its image. Moreover, U can be chosen of the form:

$$U = \left\{ \lambda \in \mathcal{A}S \,\middle|\, \sqrt{2H(\lambda)} < r_0 \right\}, \qquad \text{for some } r_0 > 0. \tag{3.7}$$

The proof of this step follows verbatim what has been done in [10, Prop. 3.1], cf. also [14, Lem. 23], once we have verified that $\sqrt{2H(\cdot)}$ is a fiber-wise norm on the annihilator bundle. Since H is quadratic on fibers, it immediately follows that $\sqrt{2H(\cdot)}$ is positive, 1-homogeneous and sub-additive. We are left to prove that, for $\lambda \in \mathcal{A}_qS$,

$$\sqrt{2H(\lambda)} = 0 \quad \Leftrightarrow \quad \lambda = 0.$$

As already remarked in (3.6), an element $\lambda \in \mathcal{A}_qS$, such that $\sqrt{2H((q,\lambda))} = 0$, annihilates both the distribution and T_qS, thus, being q non-characteristic, $\lambda = 0$.

Step 4. — $E(U) = \{p \in M \mid \delta_{\mathrm{S}}(p) < r_0\} = S_{r_0} \cup S$ and, for elements $(q, \lambda) \in U$ we have $\delta_{\mathrm{S}}(E(q,\lambda)) = \sqrt{2H(\lambda)}$. In particular, $\delta_{\mathrm{S}} \in C^\infty(S_{r_0})$. Firstly, we recall that, for an element $\lambda \in U$, the length of the curve

$$[0,1] \ni t \mapsto \pi \circ e^{t\vec{H}}(\lambda) \in M$$

is equal to $\sqrt{2H(\lambda)} < r_0$, as one can check using (3.5). Thus, $E(U) \subset S_{r_0} \cup S$. Secondly, we prove the opposite inclusion: up to restricting r_0, we may assume that $S_{r_0} \subset K$, for a compact set $K \subset M$. Therefore, for an element $p \in S_{r_0}$, there exists a minimizing geodesic $\gamma\colon [0,1] \to M$ such that

$$\gamma(0) = q \in S, \quad \gamma(1) = p \quad \text{and} \quad \ell(\gamma) = \delta_{\mathrm{S}}(p).$$

Applying Lemma 3.4, we deduce that γ is not an abnormal geodesic, meaning that there exists a unique normal lift for γ, with initial covector given by $\lambda \in T_q^*M$, which implies

$$\gamma(t) = \pi \circ e^{t\vec{H}}(\lambda), \tag{3.8}$$

and in particular, $E(q,\lambda) = p$. Moreover, $\lambda \in U$ as, by optimality, it satisfies the transversality condition (2.10), and also

$$\ell(\gamma) = \sqrt{2H(\lambda)} < r_0, \tag{3.9}$$

being $p \in S_{r_0}$. Finally, we conclude that $p \in E(U)$ and $\delta_{\mathrm{S}}(E(q,\lambda)) = \sqrt{2H(\lambda)}$, by (3.9). Since $\sqrt{2H(\cdot)}$ is smooth, as long as $H(\lambda) \neq 0$, we also have that δ_{S} is smooth on the set $E(U \setminus i(S)) = S_{r_0}$.

Step 5. — There exists a diffeomorphism $G\colon (0,r_0) \times \{\delta_{\mathrm{S}} = r_0\} \to S_{r_0}$ satisfying item (ii) of the statement. Moreover, $\|\nabla\delta_{\mathrm{S}}\|_g = 1$ in S_{r_0}.

Once again, this part of the proof follows verbatim [10, Prop. 3.1]. □

Remark 3.5. — Consider the set $U = \mathcal{A}S \cap \{\sqrt{2H(\cdot)} < r_0\}$ defined in (3.7). What we proved in the previous Theorem is that E defines a diffeomorphism between U and $S_{r_0} \cup S$. In particular, choosing a local trivialization of the annihilator bundle, this means that

$$S_{r_0} \cup S \cong \mathcal{A}S \cap \left\{\sqrt{2H(\cdot)} < r_0\right\} \underset{locally}{\cong} S \times B^H_{r_0}(0), \tag{3.10}$$

where $B^H_{r_0}(0)$ denotes the ball of radius r_0, centered at the origin of the Euclidean space $(\mathbb{R}^k, \sqrt{2H(\cdot)})$. Of course, in general, the annihilator bundle will not be globally trivializable, however, this is the case when S is the boundary of an open set and we are able to extend (3.10) to the whole submanifold.

Whenever S is a boundary of an open set, we can refine Theorem 3.3 building a double-sided tubular neighborhood of S, in which we are able to distinguish the inside and the outside of the open set. This is done using the signed distance function. We recall here its definition.

Definition 3.6 (Signed distance). — *Let M be a sub-Riemannian manifold and $\Omega \subset M$ be an open subset. Define $\delta\colon M \to \mathbb{R}$ to be the* signed distance function *from $\partial\Omega$, i.e.*

$$\delta(p) = \begin{cases} \delta_{\partial\Omega}(p) & p \in \Omega, \\ -\delta_{\partial\Omega}(p) & p \in M \setminus \Omega, \end{cases}$$

where $\delta_{\partial\Omega}\colon M \to [0, +\infty)$ *denotes the usual distance function from the boundary of* Ω.

Theorem 3.7 (Double-sided tubular neighborhood). — *Let* M *be a sub-Riemannian manifold and* $\Omega \subset M$ *be an open, relatively compact subset, whose boundary is smooth and has no characteristic points. Denote by* $\Omega_{-r_0}^{r_0} = \{p \in M \mid -r_0 < \delta < r_0\}$. *Then, there exists* $r_0 > 0$ *such that, the following conditions hold:*

(i) $\delta\colon \Omega_{-r_0}^{r_0} \to \mathbb{R}$ *is smooth and such that* $\|\nabla\delta\|_g = 1$;

(ii) *there exists a diffeomorphism* $G\colon (-r_0, r_0) \times \partial\Omega \to \Omega_{-r_0}^{r_0}$ *such that:*

$$\delta(G(t,p)) = t \qquad \text{and} \qquad G_*\partial_t = \nabla\delta.$$

Remark 3.8. — The main differences with respect to Theorem 3.3 are that δ is smooth up to the boundary of Ω, and the diffeomorphism is built starting from $\partial\Omega$.

Proof of Theorem 3.7. — By Theorem 3.3, applied with $S = \partial\Omega$, the sub-Riemannian exponential map from $\partial\Omega$ is a diffeomorphism for small covectors, namely there exists $r_0 > 0$, such that:

$$E\colon \mathcal{A}(\partial\Omega) \cap \left\{\sqrt{2H(\lambda)} < r_0\right\} \xrightarrow{\cong} \Omega_{-r_0}^{r_0}$$

and $|\delta(E(q,\lambda))| = \sqrt{2H(\lambda)}$. Now, since Ω is an open set with smooth boundary, $\mathcal{A}(\partial\Omega)$ is trivializable, i.e. there exists a never-vanishing and inward-pointing smooth section

$$\lambda^+\colon \partial\Omega \to \mathcal{A}(\partial\Omega); \qquad q \mapsto \lambda_q^+.$$

Furthermore, by non-characteristic assumption, $\sqrt{2H(\cdot)}$ is a fiber-wise norm on the annihilator bundle, hence we may assume without loss of generality that

$$\sqrt{2H\left(\lambda_q^+\right)} = 1, \qquad \forall\ q \in S.$$

Thus, we find a unique smooth function $\xi(\lambda) \in C^\infty(\mathcal{A}(\partial\Omega))$ such that

$$\lambda = \xi(\lambda)\lambda_q^+, \qquad \lambda \in \mathcal{A}_q(\partial\Omega).$$

Hence, the annihilator bundle is trivializable via the map ξ, i.e.

$$F\colon \mathcal{A}(\partial\Omega) \xrightarrow{\cong} \partial\Omega \times \mathbb{R}; \qquad F(\lambda) = (\pi(\lambda), \xi(\lambda)).$$

Notice that, by definition, $|\xi(\lambda)| = \sqrt{2H(\lambda)}$. Moreover, $\xi(\lambda) > 0$, whenever $E(q,\lambda) \in \Omega$, by definition of λ^+, $\xi(0) = 0$ and negative otherwise. Therefore, having defined the signed distance such that it is positive inside of Ω, we obtain that

$$\delta(E(q,\lambda)) = \xi(\lambda), \qquad \forall\ \lambda \in \left\{\sqrt{2H(\lambda)} < r_0\right\},$$

proving the smoothness of δ on the set $\Omega_{-r_0}^{r_0}$. Finally, define G as the composition of $E \circ F^{-1}$ restricted to the set $(-r_0, r_0) \times \partial\Omega$. Since E and F are diffeomorphisms, also G is and moreover,

$$G(t,q) = E\left(q, t\lambda_q^+\right) \qquad \forall\ (t,q) \in (-r_0, r_0) \times \partial\Omega,$$

therefore $\delta(G(t,q)) = \delta(E(q, t\lambda_q^+)) = \xi(t\lambda_q^+) = t$. This concludes the proof of Theorem 3.7. □

4. Induced measure on S

Let ω be a smooth measure on M. We define a measure on S induced by ω, assigning a tensor density. This construction specializes to the sub-Riemannian perimeter measure, when S is the boundary of an open set. Recall that, by Theorem 3.3, there exists $r_0 > 0$ such that (3.10) holds locally and define vol_H as the Riemannian measure associated with $(\mathbb{R}^k, \|\cdot\|_\perp)$, where $\|\cdot\|_\perp$ is a shorthand notation for $\sqrt{2H(\cdot)}|_{\mathcal{A}_q S}$. In particular, vol_H is well-defined since $\|\cdot\|_\perp$ is induced by the fiber-wise bilinear form

$$(\lambda_1, \lambda_2)_\perp = \sum_{i=1}^{N} \langle \lambda_1, X_i\rangle \langle \lambda_2, X_i\rangle, \qquad \forall\ \lambda_1, \lambda_2 \in \mathcal{A}_q S,\ q \in S, \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual coupling.

Lemma 4.1. — *Let M be a sub-Riemannian manifold, equipped with a smooth measure ω, and let $S \subset M$ be a compact smooth non-characteristic submanifold of codimension $k \geqslant 1$. Then, there exists a unique smooth probability measure μ_{S} on S, such that,*

$$\int_M h(p) d\omega^\varepsilon(p) \xrightarrow{\varepsilon \to 0} \int_S h(q) d\mu_{\mathrm{S}}(q), \tag{4.2}$$

for any $h \in C_c(M)$, where,

$$\omega^\varepsilon = \frac{\mathbb{1}_{S_\varepsilon}}{\omega(S_\varepsilon)} \omega, \qquad \forall\ \varepsilon > 0.$$

Proof. — Proceeding with hindsight, we are going to define explicitly the measure μ_{S} and then prove the convergence. We may define μ_{S} locally, hence, fix an open coordinate chart $V \subset S$ for S and a local trivialization of $\mathcal{A}S$ over V, so that

$$\mathcal{A}S|_V \cong V \times \mathbb{R}^k.$$

By Theorem 3.3, we have that, denoting by $V_{r_0} = E(\mathcal{A}S|_V \cap \{\sqrt{2H(\cdot)} < r_0\})$,

$$V_{r_0} \cong \mathcal{A}S|_V \cap \left\{\sqrt{2H(\cdot)} < r_0\right\} \cong V \times B^H_{r_0}(0).$$

Consider on V_{r_0}, coordinates (x, z) where $(x_1, \ldots, x_{n-k})$ are coordinates on V and $S \cap V_{r_0} = \{(x, z) \mid z = 0\}$. Thus, since ω is smooth, we have

$$d\omega(x, z) = \omega(x, z)dxdz, \quad \text{with } \omega(\cdot) \in C^\infty(V_{r_0}),$$

where dx and dz are the Lebesgue measures in coordinates. Moreover, since (4.1) is a metric along the fibers, we can define canonically a volume associated with H, which in coordinates is given by

$$d\mathrm{vol}_H(z) = \sqrt{\det H_q(z)}dz, \qquad \forall\ q \in S, \tag{4.3}$$

with never-vanishing density. Therefore, we may rewrite ω in terms of vol_H, obtaining

$$d\omega(x, z) = \omega(x, z)dxdz = \frac{\omega(x, z)}{\sqrt{\det H_q(z)}}dxd\mathrm{vol}_H(z) \tag{4.4}$$

Finally, on the fiber, we can choose an orthonormal (w.r.t. $\sqrt{2H(\cdot)}$) basis of smooth local sections $\{\lambda_1, \ldots, \lambda_k\}$, so that $\mathrm{vol}_H(\lambda_1, \ldots, \lambda_k) = 1$, and define $\widetilde{\mu}_S$ in coordinates (x, z), to be the contraction of (4.4) along these covectors, restricted to S, namely

$$d\widetilde{\mu}_S(x) = \frac{\omega(x, 0)}{\sqrt{\det H_q(0)}}dxd\mathrm{vol}_H(\lambda_1, \ldots, \lambda_k) = \frac{\omega(x, 0)}{\sqrt{\det H_q(0)}}dx.$$

One can check that this procedure defines a smooth measure on S, independently on the choice of the coordinates. We can now verify the convergence, using a partition of the unity argument. Fix a covering of S with a finite number of open charts $\{V_i\}_{i=1}^L$ and consider the associated covering $\{V^i_{r_0}\}$ of $S \cup S_{r_0}$, defined by

$$V^i_{r_0} = E\left(\mathcal{A}S|_{V_i} \cap \left\{\sqrt{2H(\cdot)} < r_0\right\}\right), \qquad \forall\ i = 1, \ldots, L.$$

Then, consider $\{\rho_i\}_{i=1}^L$ to be a partition of unity subordinate to the covering $\{V^i_{r_0}\}$ of $S \cup S_{r_0}$. Exploiting the coordinate expression of $\widetilde{\mu}_S$, we have, for any $\varepsilon \leqslant r_0$:

$$\begin{aligned}
\omega(S_\varepsilon) &= \int_{S_\varepsilon} \sum_{i=1}^{L} \rho_i(q) d\omega(q) \\
&= \sum_{i=1}^{L} \int_{V_i} \int_{B^H_\varepsilon(x)} \rho_i(x,z) \frac{\omega(x,z)}{\sqrt{\det H_q(z)}} d\mathrm{vol}_H(z) dx \\
&= \varepsilon^k \sum_{i=1}^{L} \int_{V_i} \int_0^1 \int_{\mathbb{S}^{k-1}} \rho_i(x,\varepsilon r,\theta) \frac{\omega(x,\varepsilon r,\theta)}{\sqrt{\det H_q(\varepsilon r,\theta)}} r^{k-1} d\theta dr dx,
\end{aligned}$$

having expressed the volume vol_H in polar coordinates $r^{k-1}d\theta dr$. Therefore, up to a factor ε^k, we see that

$$\frac{\omega(S_\varepsilon)}{\varepsilon^k} \xrightarrow{\varepsilon\to 0} \varpi_k \sum_{i=1}^{L} \int_{V_i} \rho_i(x,0) d\widetilde{\mu}_S(x) = \varpi_k \int_S d\widetilde{\mu}_S$$

where ϖ_k is the volume of the standard unit ball in $\mathbb{R}^k$. Finally, reasoning as above, since for any $h \in C_c(M)$, we are able to extract a factor ε^k from the integral of h over S_ε, we obtain the convergence in the weak-star topology (4.2), having normalized $\widetilde{\mu}_S$ to obtain a probability measure μ_{S}. □

Remark 4.2. — Since ω^ε, for any $\varepsilon \leqslant r_0$, has compact support which is contained in S_{r_0}, we can extend the convergence (4.2) to any continuous function on M.

5. Heat content for submanifolds

Let M be a sub-Riemannian manifold, equipped with a smooth measure ω, and let $S \subset M$ be a smooth, compact submanifold of codimension $k \geqslant 1$. We may consider μ a smooth probability measure on S as initial datum for the heat equation, in the sense of distributions, and study the associated Cauchy problem:

$$\begin{aligned}
(\partial_t - \Delta) u(t,x) &= 0, && \forall\ (t,x) \in (0,\infty) \times M, \\
u(t,\cdot) &\xrightarrow{t\to 0} \mu, && \text{in } \mathcal{D}'(M).
\end{aligned} \tag{5.1}$$

A solution to this problem, in the sense of distribution, is given by

$$u(t,x) = \int_S p_t(x,y) d\mu(x), \qquad \forall\ (t,x) \in (0,\infty) \times M,$$

which, by hypoellipticity, is a smooth function for positive times. Recall that, by definition of the relative heat content associated with an open set $\Omega \subset M$, we have

$$H_\Omega(t) = \int_\Omega \int_\Omega p_t(x,y)d\omega(x)d\omega(y), \qquad \forall\ t > 0. \tag{5.2}$$

Thus, a suitable generalization of (5.2) for a submanifold S seems to be:

$$H_S(t) = \int_S \int_S p_t(x,y)d\mu(x)d\mu(y), \qquad \forall\ t > 0. \tag{5.3}$$

Moreover, when S is non-characteristic, Lemma 4.1 provides with a canonical probability measure on S, induced by ω, i.e. μ_{S}. Henceforth, we assume S non-characteristic and fix $\mu = \mu_{\mathrm{S}}$. In this setting, we can hope to obtain an asymptotic expansion of (5.3).

Proposition 5.1. — *Let M be a sub-Riemannian manifold, equipped with a smooth measure ω, let $S \subset M$ be a smooth, compact and non-characteristic submanifold of codimension $k \geqslant 1$ and fix the probability measure μ_{S} on S. Define, for any $\varepsilon \leqslant r_0$,*

$$H_S^\varepsilon(t) = \int_M \int_M p_t(x,y)d\omega^\varepsilon(x)d\omega^\varepsilon(y), \qquad \forall\ t > 0. \tag{5.4}$$

Then, for any $t > 0$,

$$H_S^\varepsilon(t) \xrightarrow{\varepsilon \to 0} H_S(t). \tag{5.5}$$

Proof. — Firstly, notice that, applying Lemma 4.1, we have that

$$u^\varepsilon(t,x) = \int_M p_t(x,y)d\omega^\varepsilon(y) = \langle \omega^\varepsilon, p_t(x,\cdot)\rangle \xrightarrow{\varepsilon \to 0} \langle \mu_{\mathrm{S}}, p_t(x,\cdot)\rangle, \tag{5.6}$$

for any $t > 0$ and $y \in M$. Secondly, since the heat kernel p_t is smooth on $M \times M$, there exists a constant $C(t) > 0$ depending on t, such that:

$$\|p_t(\cdot,\cdot)\|_{L^\infty_{\mathrm{loc}}(M\times M)} \leqslant C(t), \tag{5.7}$$

and we remark that the constant $C(t)$ explodes as $t \to 0$. Therefore, the convergence (5.6) is locally uniform with respect to $x \in M$. In conclusion,

$$\begin{aligned}
|H_S^\varepsilon(t) - H_S(t)| &= |\langle \omega^\varepsilon, u^\varepsilon(t,\cdot)\rangle - \langle \mu_{\mathrm{S}}, u(t,\cdot)\rangle| \\
&\leqslant |\langle \omega^\varepsilon, u^\varepsilon(t,\cdot) - u(t,\cdot)\rangle| + |\langle \omega^\varepsilon, u(t,\cdot)\rangle - \langle \mu_{\mathrm{S}}, u(t,\cdot)\rangle| \\
&\leqslant \|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^\infty(S_{r_0})} |\langle \omega^\varepsilon, 1\rangle| + |\langle \omega^\varepsilon - \mu_{\mathrm{S}}, u(t,\cdot)\rangle|,
\end{aligned}$$

and taking the limit as $\varepsilon \to 0$ in the last line proves the desired result. □

Remark 5.2. — The convergence (5.5) is never uniform as $t \to 0$, being the constant $C(t)$ in (5.7) not bounded as $t \to 0$. This suggests that, while H_S^ε seems to be the best possible approximation of the heat content associated with S, using such a strategy to deduce the asymptotics of $H_S(t)$ is not correct. Indeed, we can show that the coefficients of the expansion of H_S^ε can not approximate those of $H_S(t)$, in general.

6. An example: closed simple curve in $\mathbb{R}^3$

In $\mathbb{R}^3$ equipped with the Euclidean scalar product and the Lebesgue measure, let us consider a biregular closed simple curve, parametrized by arc-length, $\gamma\colon [0,\ell] \to \mathbb{R}^3$, where ℓ denotes the length of γ. Recall that a smooth curve $\gamma\colon I \to \mathbb{R}^3$ is biregular if

$$\dot\gamma(s) \wedge \ddot\gamma(s) \neq 0, \qquad \forall\ s \in I,$$

where $\wedge$ denotes the cross product in $\mathbb{R}^3$. In this setting, define $S = \gamma([0,\ell]) \subset \mathbb{R}^3$, submanifold of codimension 2. The tubular neighborhood of S given by Theorem 3.3 coincides with the usual Euclidean tubular neighborhood, which can be conveniently described by the Frenet-Serret moving frame along γ, it being biregular. In particular, denoting by $\{T(s), N(s), B(s)\}$ the Frenet–Serret frame for $s \in [0,\ell]$, we have,

$$S_\varepsilon = \{\gamma(s) + r(\cos\theta N(s) + \sin\theta B(s)) \mid s \in [0,\ell],\ \theta \in (0,2\pi],\ r \in (0,\varepsilon)\}, \qquad \forall\, \varepsilon \leqslant r_0.$$

Thus, in coordinates (s, r, θ), the Lebesgue measure is

$$dxdydz = r(1 - rk(s)\cos\theta)dsdrd\theta, \tag{6.1}$$

where $k(s) = \|\ddot\gamma(s)\|$ is the curvature of γ, and the procedure of Lemma 4.1 gives the probability measure $\mu_{\mathrm{S}} = ds/\ell$. Following the discussion of Section 5, we define the heat content associated with S as

$$H_S(t) = \frac{1}{\ell^2}\frac{1}{(4\pi t)^{3/2}}\int_0^\ell\int_0^\ell e^{-\frac{|\gamma(s)-\gamma(\tau)|^2}{4t}}dsd\tau, \qquad \forall\, t > 0.$$

Asymptotic expansion of $H_S(t)$

Denote by $\phi_\tau(s) = |\gamma(s) - \gamma(\tau)|^2$. We can explicitly compute the asymptotic expansion of H_S applying the Laplace method to the integral

$$I_\tau(\lambda) = \int_0^\ell e^{-\phi_\tau(s)\lambda}ds, \qquad \text{as } \lambda \to +\infty. \tag{6.2}$$

In particular, since γ is a simple curve, the phase $\phi_\tau(s)$ has a strict minimum at $s=\tau$, thus there exists $\varepsilon=\varepsilon(\tau)$ such that $\phi'_\tau(s)>0$ for any $s\in[\tau-\varepsilon,\tau+\varepsilon]\setminus\{\tau\}$. A direct computation, building upon $\|\dot\gamma\|=1$, yields:

$$\phi'_\tau(\tau)=0,\qquad \phi''_\tau(\tau)=2,\qquad \phi^{(5)}_\tau(\tau)=-5\partial_\tau\left(k(\tau)^2\right),$$
$$\phi'''_\tau(\tau)=0,\qquad \phi^{(4)}_\tau(\tau)=-2k(\tau)^2,\qquad \phi^{(6)}_\tau(\tau)=-9\partial^2_\tau\left(k(\tau)^2\right)+2\|\dddot\gamma(\tau)\|^2.$$

Therefore the phase has a Taylor expansion at its minimum and we can apply the Laplace method, which gives a full asymptotic expansion, cf. [13, Thm. 8.1],

$$I_\tau(\lambda)\sim e^{-\lambda\phi_\tau(\tau)}\sum_{i=0}^\infty\Gamma\left(\frac{i+1}{2}\right)\frac{a_i(\tau)}{\lambda^{\frac{i+1}{2}}}\qquad\text{as }\lambda\to\infty,$$

where Γ is the Euler Gamma function, and the $a_i(\tau)$ are given by explicit formulas in terms of the derivatives of ϕ_τ at its minimum. Moreover, since, for any $\tau\in[0,\ell]$, the phase has an interior minimum, the odd coefficients of the expansion vanish. For the even-order coefficients, we have

$$\begin{aligned}&a_0=1,\qquad a_2=\frac{1}{8}k(\tau)^2,\\ &a_4=\frac{1}{1152}\left(36\partial^2_\tau\left(k(\tau)^2\right)+35k(\tau)^2-8\|\dddot\gamma(\tau)\|^2\right).\end{aligned}\tag{6.3}$$

To conclude, we have to integrate with respect to τ the asymptotic expansion of (6.2). In general, the expansion may not be uniform in $\tau\in(0,\ell)$, however, since γ is uniformly continuous on $[0,\ell]$ and can be extended by periodicity on the whole real line, the choice of $\varepsilon>0$ such that $\phi'_\tau(s)>0$ for any $s\in[\tau-\varepsilon,\tau+\varepsilon]\setminus\{\tau\}$ can be made uniform, providing uniform estimates of the remainder. In particular, we have

$$H_S(t)\sim\frac{1}{\ell^2}\frac{1}{(4\pi t)^{3/2}}\int_0^\ell\int_{\tau-\varepsilon}^{\tau+\varepsilon}e^{-\frac{\phi_\tau(s)}{4t}}dsd\tau,\qquad\text{as }t\to0,$$

where $\varepsilon>0$ is chosen uniformly with respect to τ. Hence, we conclude that:

$$\begin{aligned}H_S(t)&\sim\frac{1}{\ell^2}\frac{1}{(4\pi t)^{3/2}}\int_0^\ell\sum_{i=0}^\infty\Gamma\left(\frac{2i+1}{2}\right)a_{2i}(\tau)(4t)^{\frac{2i+1}{2}}d\tau\\ &=\frac{1}{\ell^2}\frac{1}{4\pi t}\sum_{i=0}^\infty\alpha_it^i,\end{aligned}\tag{6.4}$$

as $t\to0$, where the coefficients α_i's are defined by

$$\alpha_i=2^{2i-1}(2i+1)\int_0^\ell a_{2i}(\tau)d\tau,\qquad\forall\ i\geqslant0,$$

and are given explicitly by (6.3), up to $i = 2$.

Approximation via tubular neighborhoods

We compute the asymptotic expansion of the approximation defined in Proposition 5.1, when $S = \gamma([0,\ell])$, and we compare its coefficients with those obtained in (6.4). Recall that, by (5.4), we set

$$H_S^\varepsilon(t) = \frac{1}{|S_\varepsilon|^2}\int_{S_\varepsilon}\int_{S_\varepsilon}\frac{1}{(4\pi t)^{3/2}}e^{-\frac{|x-y|^2}{4t}}dxdy, \qquad \forall\ t>0.$$

By Theorem 2.3, there exists an asymptotic expansion up to order 4 in $\sqrt{t}$, of the form

$$H_S^\varepsilon(t) = \frac{1}{|S_\varepsilon|^2}\left(\alpha_0^\varepsilon + \alpha_1^\varepsilon t^{1/2} + \alpha_3^\varepsilon t^{3/2} + o(t^2)\right), \tag{6.5}$$

where $\alpha_0^\varepsilon = |S_\varepsilon|$ and

$$\alpha_1^\varepsilon = -\frac{1}{\sqrt{\pi}}\sigma^\varepsilon(\partial S_\varepsilon), \qquad \alpha_3^\varepsilon = -\frac{1}{12\sqrt{\pi}}\int_{\partial S_\varepsilon}\left(2\nabla\delta_{\mathrm{S}}\cdot\nabla(\Delta\delta_{\mathrm{S}}) - (\Delta\delta_{\mathrm{S}})^2\right)d\sigma^\varepsilon. \tag{6.6}$$

Notice that, in tubular coordinates (s,r,θ), $\nabla\delta_{\mathrm{S}} = \partial_r$, and, since (6.1) holds,

$$d\sigma^r = r(1-rk(s)\cos\theta)dsd\theta, \qquad \Delta\delta_{\mathrm{S}} = \frac{1}{r} + \partial_r\left(\log(1-rk(s)\cos\theta)\right).$$

Thus, we can explicitly compute the coefficients (6.6):

$$\begin{aligned}\alpha_1^\varepsilon &= -\frac{\varepsilon}{\sqrt{\pi}}\int_0^\ell\int_0^{2\pi}(1-\varepsilon k(s)\cos\theta)dsd\theta = -2\varepsilon\ell\sqrt{\pi}\\ \alpha_3^\varepsilon &= -\frac{\varepsilon}{12\sqrt{\pi}}\int_0^\ell\int_0^{2\pi}\left(-\frac{3}{\varepsilon^2} + \frac{A_1(s,\varepsilon,\theta)}{\varepsilon} + A_0(s,\varepsilon,\theta)\right)\\ &\qquad\qquad (1-\varepsilon k(s)\cos\theta)dsd\theta,\end{aligned} \tag{6.7}$$

where A_0, A_1 are smooth functions defined by

$$\begin{aligned}A_0(s,r,\theta) &= 2\partial_r^2\left(\log(1-rk(s)\cos\theta)\right) - \left(\partial_r\left(\log(1-rk(s)\cos\theta)\right)\right)^2,\\ A_1(s,r,\theta) &= -2\partial_r\left(\log(1-rk(s)\cos\theta)\right).\end{aligned}$$

Comparison between the two approaches

Let us compare the asymptotics of the two quantities in exam: for a fixed $\varepsilon > 0$ and as $t \to 0$, we have from (6.4) and (6.5)

$$H_S^\varepsilon(t) = \frac{1}{|S_\varepsilon|^2}\left(\alpha_0^\varepsilon + \alpha_1^\varepsilon t^{1/2} + \alpha_3^\varepsilon t^{3/2} + o(t^2)\right),$$
$$H_S(t) = \frac{1}{\ell^2}\frac{1}{4\pi t}\left(\alpha_0 + \alpha_1 t + \alpha_3 t^2 + o(t^2)\right),$$

where the coefficients are given by (6.7) and (6.3), respectively. At this stage, on the one hand, we notice that the order in t of the expansions doesn't agree. On the other hand, the coefficients $\alpha_1^\varepsilon, \alpha_3^\varepsilon$ do not contain fine geometrical information of S, indeed, the functions A_0, A_1 depend only the curvature of γ, as opposed to (6.3), where derivatives of $k(s)$ appear. Moreover, at a formal level, $\alpha_1^\varepsilon \to 0$ as $\varepsilon \to 0$, whereas α_3^ε explodes: it is possible to give meaning to these limits, taking into account the parabolic scaling between the space and time variables of the heat equation and formally replacing $t \mapsto \varepsilon^2 t$, in (6.5), however, we do not recover any geometrical meaning.

Remark 6.1. — The approximating relative heat content is too coarse a tool to detect the geometry of a submanifold of high codimension and a different strategy is needed. Inspired by the construction of the tubular neighborhood for S, we may study the asymptotic expansion of $H_S(t)$ using a perturbative approach similarly to what has been done in [4], presenting the sub-Laplacian of S as a perturbation of a simpler operator. This will be object of future research.

BIBLIOGRAPHY

[1] A. Agrachev, D. Barilari & U. Boscain, *A comprehensive introduction to sub-Riemannian geometry. From the Hamiltonian viewpoint*, Cambridge Studies in Advanced Mathematics, vol. 181, Cambridge University Press, 2020.

[2] A. Agrachev, L. Rizzi & T. Rossi, "Relative heat content asymptotics for sub-Riemannian manifolds", arXiv preprint arXiv:2110.03926, 2021.

[3] A. Agrachev & Y. L. Sachkov, *Control theory from the geometric viewpoint*, Encyclopaedia of Mathematical Sciences, vol. 87, Springer, 2004, Control Theory and Optimization, II.

[4] D. Barilari, "Trace heat kernel asymptotics in 3D contact sub-Riemannian geometry", *J. Math. Sci., New York* **195** (2013), no. 3, p. 391-411, Translation of Sovrem. Mat. Prilozh. No. 82 (2012).

[5] M. van den Berg, S. Desjardins & P. B. Gilkey, "Functorality and heat content asymptotics for operators of Laplace type", *Topol. Methods Nonlinear Anal.* **2** (1993), no. 1, p. 147-162.

[6] M. van den Berg & P. B. Gilkey, “Heat content asymptotics of a Riemannian manifold with boundary”, *J. Funct. Anal.* **120** (1994), no. 1, p. 48-71.

[7] ———, “Heat flow out of a compact manifold”, *J. Geom. Anal.* **25** (2015), no. 3, p. 1576-1601.

[8] M. van den Berg & J.-F. Le Gall, “Mean curvature and the heat equation”, *Math. Z.* **215** (1994), no. 3, p. 437-464.

[9] S. Desjardins & P. B. Gilkey, “Heat content asymptotics for operators of Laplace type with Neumann boundary conditions”, *Math. Z.* **215** (1994), no. 2, p. 251-268.

[10] V. Franceschi, D. Prandi & L. Rizzi, “On the essential self-adjointness of singular sub-Laplacians”, *Potential Anal.* **53** (2020), no. 1, p. 89-112.

[11] L. Hörmander, “Hypoelliptic second order differential equations”, *Acta Math.* **119** (1967), p. 147-171.

[12] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, vol. 91, American Mathematical Society, 2002.

[13] F. W. J. Olver, *Asymptotics and special functions*, AKP Classics, A. K. Peters, 1997, Reprint of the 1974 original [Academic Press, New York; MR0435697 (55 #8655)].

[14] D. Prandi, L. Rizzi & M. Seri, “Quantum confinement on non-complete Riemannian manifolds”, *J. Spectr. Theory* **8** (2018), no. 4, p. 1221-1280.

[15] L. Rifford, *Sub-Riemannian geometry and optimal transport*, SpringerBriefs in Mathematics, Springer, 2014.

[16] L. Rizzi & T. Rossi, “Heat content asymptotics for sub-Riemannian manifolds”, *J. Math. Pures Appl.* **148** (2021), p. 267-307.

[17] A. Savo, “Uniform estimates and the whole asymptotic series of the heat content on manifolds”, *Geom. Dedicata* **73** (1998), no. 2, p. 181-214.

[18] R. S. Strichartz, “Sub-Riemannian geometry”, *J. Differ. Geom.* **24** (1986), no. 2, p. 221-263.

[19] J. Tyson & J. Wang, “Heat content and horizontal mean curvature on the Heisenberg group”, *Commun. Partial Differ. Equations* **43** (2018), no. 3, p. 467-505.

[20] Y. Colin de Verdière, L. Hillairet & E. Trélat, “Small-time asymptotics of hypoelliptic heat kernels near the diagonal, nilpotentization and related results”, arXiv preprint, arXiv:2004.06461, 2020.

Tommaso Rossi
Institut für Angewandte Mathematik,
Universität Bonn, Bonn, Germany
rossi@iam.uni-bonn.de