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ON THE NOTIONS OF LOWER RICCI CURVATURE BOUND FOR DISCRETE GRAPHS

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Abstract. — The Ricci curvature plays an important role in Riemannian geometry. The assumption that the manifold has nonnegative Ricci curvature implies some geometric and topological constraints (For instance, the diameter of the manifold is bounded and so the manifold is compact. This is the famous Bonnet–Myers Theorem). In these notes, we present several approaches to extend this kind of results in the setting of discrete graphs, in particular Cayley graphs of finitely generated groups.

1. Introduction

In Riemannian geometry, there are some natural assumptions, for instance:

(i) The manifold has negative sectional curvature (or more generally, the sectional curvature has an upper bound).

(ii) The manifold has nonnegative Ricci curvature (or more generally, the Ricci curvature has a lower bound).

It is possible to extend some of these notions in the case of continuous metric spaces: Hadamard spaces, Alexandrov spaces, CAT-spaces (See for instance [5] or [4]), spaces with bounded synthetic Ricci curvature in the sense of Lott–Villani–Sturm (See [13, 22, 19] and [20]). By a continuous space (Usually called arcwise connected spaces), we mean a metric space (X, d) such that for any pair of points x and y in X, there exists a continuous map $\gamma : [a, b] \to X$ so that $\gamma(a) = x$ and $\gamma(b) = y$. In other words, there is a (continuous) curve joining x and y. Most of the time, we also have to assume that the metric space (X, d) is geodesic, in the sense that given two points x and y in X, there exists a curve $\gamma : [0, l] \to X$ joining x and y so that $d(\gamma(s), \gamma(t)) = |t - s|$ for all s, $t \in [0, l]$. In this case, l is equal to $d(x, y)$.

It is quite natural to try to extend the previous notions of "space with controlled curvature" to the discrete setting, for instance to Cayley graphs of finitely generated groups. This extension is far away to be straightforward, since the existence of (nice) continuous curves plays an important role in these theories. In the case of negative sectional curvature, Gromov (and others) developed the notion of hyperbolic groups which is quite robust. The situation is not so clear in the case of nonnegative Ricci curvature. For instance, one goal is to give discrete versions of these well-known results in Riemannian geometry and geometric analysis.

Theorem 1.1 (Bonnet–Myers). — *Let (M, g) be a complete Riemannian manifold of dimension $n \geqslant 2$ such that there exists a constant $\alpha > 0$ so that for any unit tangent vector ξ of M, $Ric(\xi, \xi) \geqslant (n-1)\alpha > 0$. Then, M is compact with diameter less than $\pi/\sqrt{\alpha}$.*

Note that this theorem concerns manifolds with a positive lower bound for the Ricci curvature. Another way to express this result is to assume that the Ricci curvature of M is at least that of the standard sphere S^n. Then, the diameter of M is at most that of S^n. This is a comparison theorem in the sense of Riemannian geometry (See [6] or [9]).

Theorem 1.2 (Buser). — *Let (M, g) be a complete Riemannian manifold with nonegative Ricci curvature of dimension n. Then, M supports a Poincaré inequality, that is for any smooth function $f : M \to \mathbb{R}$, any ball B in X,*

$$\fint_B |f(x) - f_B| dx \leqslant 2^{n-1} \mathrm{diam}(B) \fint_{2B} |\nabla f|(x) dx.$$

Here, $f_B = \fint_B f(x)dx = \dfrac{1}{Vol_g(B)} \displaystyle\int_B f(x)dx$ where $Vol_g(B)$ denotes the Riemannian volume of the ball B.

The last theorem has important applications in geometric analysis (Estimates of the decay of the heat kernel, boundedness of the Riesz transform, dimension of the space of harmonic functions with polynomial growth, … See [17, Chapter 4]). For a nice proof of Theorem 1.2, see [18].

In the first section, we describe what we call the discrete setting (As opposed to the continuous one). Then, we present three different approaches, all of them are related to optimal transportation. They are inspired by the Riemannian case that we will briefly discuss before going to the discrete case.

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2. The setting: discrete graphs and Cayley graphs

Let G be an infinite graph. By this, we mean that G is given by an infinite set V of vertices and by a set E of edges that join some vertices of V.Thus, an edge is of the form (x,y) where $x,\, y \in V$. We say that the edge (x,y) is oriented if x is the origin of the edge and y is the endpoint (In this case $(x,y) \neq (y,x)$). For convenience, we will assimilate (most of the time) V to G. We will say that x and y in G are neighbours if they are related by an edge of E (Or more precisely the corresponding vertices are related by an edge of E). In this case, we will write $x \sim y$. A path between x and y in G is a subset of G of the form $(x_0,\, \dots,\, x_k)$ where $x_0 = x$, $x_k = y$ and for any $i = 0,\, \dots,\, k-1$, $x_i \sim x_{i+1}$. A graph is connected if for all vertices x and y, there exists a path joining x and y. We will always assume that the graph G is connected. The geodesic distance between x and y is then the smallest k such that there exists a path $(x_0,\, \dots, x_k)$ in G joining x and y. Hence, we can see G as a metric space.

At each adge (x,y), we assign a weight m_{xy} so that $m_{xy} = m_{yx} > 0$ if $x \sim y$ and $m_{xy} = 0$ otherwise. Then, we set $m(x) = \sum_{y \sim x} m_{xy}$ and $p(x,y) = \frac{m_{xy}}{m(x)}$ (if $y \sim x$). Note that, for any $x \in G$, $\sum_{y\,\in\, G} p\ (x,y) = \sum_{y \sim x} p\ (x,y) = 1$. Thus, the weights m_{xy} determine a random walk on G and $p(x,y)$ should be seen as the probability to jump from x to y. We set $V(x,r) = m(B(x,r)) = \sum_{y\,\in\, B(x,r)} m(y)$ where $B(x,r)$ is the ball (For the geodesic distance) of center $x \in G$ and radius $r \in \mathbb{N}^*$.

For $x \in G$, denote by d_x the number of neighbours of x (d_x is the degree of G at x). We say that G is locally finite if for any $x \in G$, d_x is finite, and that G is uniformly locally finite if there exists $D \in \mathbb{N}^*$ so that $d_x \leqslant D$ for any $x \in G$.

Basic (but useful) examples are unweighted graphs. In this case, we set $m_{xy} = 1$ for every $x, y \in G$, $x \sim y$. Since the graph is equipped with weights that are all equal to 1, this terminology of "unweighted graph" is quite ambiguous but is often used. Thus, $m(x)$ is the number of neighbours of x and $p(x,y) = \frac{1}{m(x)}$ is related to the classical random walk on a graph. Note also that $V(x,r) \geqslant card(B(x,r))$ since $m(y) \geqslant 1$ for any $y \in B(x,r)$ and that $V(x,r) \leqslant Dcard(B(x,r))$ if G is uniformly locally finite (with constant D). Hence, the counting measure and the measure m are equivalent, that is for any $x \in G$, any $r > 0$,

$$card(B(x,r)) \leqslant m(B(x,r)) \leqslant Dcard(B(x,r)).$$

Thus, we can also consider that G is equipped with the counting measure. This would not change anything in what follows except some constants.

Typical examples of unweighted (and uniformly locally finite) graphs are given by Cayley graphs. Let $(G,.)$ be a group. We say that G is finitely generated if there exists a finite family, called a generating family, $S = \{s_1, \ldots, s_N\}$ in G so that for any $g \in G$, there exist elements of S denoted by $s_{\sigma(1)}, \ldots, s_{\sigma(M)}$ so that $g = s_{\sigma(1)} \cdot \ldots \cdot s_{\sigma(M)}$. The infimum over the $M \in \mathbb{N}$ such that we get this kind of decomposition is called the length of g and is denoted by $|g|$. We now define the Cayley graph of G related to S by $G = (V, E)$ where V is the set of elements of G and the edges are of the form (x, y) if there exists $s \in S$ so that $y = x.s$ (We then write $y \sim x$). In this case, the geodesic distance on the graph is given by $d(x, y) = |x^{-1}.y|$ (Word metric). Cayley graphs are equipped with the counting measure. We always assume that S does not contain the identity element e of G. A generating family S is said to be symmetric if $s^{-1} \in S$ whereas $s \in S$. This notion will be useful in the sequel. See [8] for more details on geometric group theory.

3. The Poincaré inequality and the Couhlon/Saloff-Coste condition

One motivation of Coulhon and Saloff-Coste in [18] was to prove existence of isoperimetric type inequalities on discrete graphs under suitable geometric conditions. For this, they use a Poincaré inequality which is valid under some curvature assumptions. Thus, they get a discrete analog of Theorem 1.2. Before giving their result, we need a couple of definitions.
Let G be an infinite (But locally finite) connected graph. If $f : G \to \mathbb{R}$ is a function, the length of its gradient is $|\nabla f|(x) = \sum_{y \sim x} |f(y) - f(x)|$.

Remark 3.1. — Another possible definition (Often used in probability to study random walks) is

$$|\nabla f|(x) = \left(\sum_{y \sim x} |f(x) - f(y)|^2 \, p\,(x, y) \right)^{1/2}.$$

A classical assumption on the weights is that there exists some $p_0 > 0$ so that $p(x, y) \geqslant p_0$ whenever x, $y \in G$. If G is uniformly locally finite and under the previous assumption, choosing one of these definitions would change nothing but constants in what concerns Poincaré inequalities.

We say that G satisfies the Poincaré inequality if there exist constants C, $C' > 0$ so that

$$\sum_{y \in B(x,n)} |f(y) - f_n(x)|\, m(y) \leqslant Cn \sum_{y \in B(x,C'n)} |\nabla f|(y) m(y)$$

whenever $x \in G$, $n \in \mathbb{N}^*$ and $f : G \to \mathbb{R}$ is a function (with finite support). Here,

$$f_n(x) = \frac{1}{V(x,n)} \sum_{y \in B(x,n)} f(y)m(y).$$

In the case of uniformly finite and unweighted graphs, we have $1 \leqslant m(y) \leqslant D$ and the previous Poincaré inequality is equivalent to the more classical one:

$$\sum_{y \in B(x,n)} |f(y) - f_n(x)| \leqslant Cn \sum_{y \in B(x,C'n)} |\nabla f|(y)$$

where $f_n(x) = \frac{1}{V(x,n)} \sum_{y \in B(x,n)} f(y)$. As we will see later, it turns out that in a lot of examples, $C' = 2$.

We now will describe what Coulhon and Saloff-Coste (See [18]) call *graphs with positive curvature*. Let G be an unweighted graph (We can also consider weighted graphs. But for simplicty, we first restrict ourself to the unweighted case). Fix $x \in G$ and $n \in \mathbb{N}^*$. For any pair of points $(y, z) \in B(x, n)^2$, assume that a geodesic path γ_{yz} joining y and z has been chosen. Set $\Gamma(x, n) = \{\gamma_{y,z}; y, z \in B(x, n)\}$. Note that if $\gamma \in \Gamma(x, n)$, $\gamma \subset B(x, 2n)$ by the triangle inequality. Then, define the geometric quantity (Which depends on the choice of geodesics between two points):

$$K(x,n) = \frac{1}{V(x,n)} \max_{e \in B(x,2n) \cap E} \operatorname{card} \{\gamma \in \Gamma(x,n); e \in \gamma\}.$$

We say that G is a graph with positive curvature in the sense of Coulhon–Saloff-Coste if there exist a constant $C \geqslant 0$ and a choice of geodesic paths $\gamma_{y,z}$ (joining y and z in G) for all y, $z \in B(x, n)$ so that $K(x, n) \leqslant Cn$ for any $x \in G$ and any $n \in \mathbb{N}^*$.

Example 3.2. — The Cayley graph of a finitely generated group with polynomial growth is a graph with positive curvature in the previous sense. Recall that a group has polynomial growth if there exists $d \in \mathbb{N}$ so that $\limsup_{n\to+\infty} \frac{V(x,n)}{n^d} < +\infty$ for any $x \in G$. This definition does not depend on the set of generators S. By a famous (but difficult) result of Gromov, a group with polynomial growth is almost nilpotent (and the converse is also true). See for instance [21, Chapter 6]. A proof of Kleiner starts by proving a Poincaré inequality in the same spirit as the theorems below (See [10]).

Theorem 3.3. — *Any unweighted graph with positive curvature supports a Poincaré inequality.*

Proof. — If e is an oriented edge of G, we denote by e_+ and e_- its endpoints. Fix $x \in G$ and $n \in \mathbb{N}^*$. Then for any y, $z \in B(x,n)$, by the triangle inequality, we have

$$|f(y) - f(z)| \leqslant \sum_{e \in \gamma_{y,z}} |f(e_+) - f(e_-)|$$

where $\gamma_{y,z}$ is the geodesic joining y and z given by the curvature assumption.
Hence,

$$\sum_{y,z \in B(x,n)} |f(y) - f(z)| \leqslant \sum_{y,z \in B(x,n)} \sum_{e \in \gamma_{y,z}} |f(e_+) - f(e_-)| \,.$$

This implies

$$\begin{aligned}\sum_{y \in B(x,n)} |f(y) - f_n(x)| &= \sum_{y \in B(x,n)} \left| f(y) - \frac{1}{V(x,n)} \sum_{z \in B(x,n)} f(z) \right| \\ &= \sum_{y \in B(x,n)} \left| \frac{1}{V(x,n)} \sum_{z \in B(x,n)} (f(y) - f(z)) \right| \\ &\leqslant \frac{1}{V(x,n)} \sum_{y,z \in B(x,n)} |f(y) - f(z)| \\ &\leqslant \frac{1}{V(x,n)} \sum_{y,z \in B(x,n)} \sum_{e \in \gamma_{y,z}} |f(e_+) - f(e_-)| \,.\end{aligned}$$

Finally, by the curvature assumption, we get

$$\begin{aligned}&\frac{1}{V(x,n)} \sum_{y,z \in B(x,n)} \sum_{e \in \gamma_{y,z}} |f(e_+) - f(e_-)| \\ &\quad \leqslant \frac{1}{V(x,n)} \sum_{e \in B(x,2n) \cap E} \operatorname{card} \{\gamma \in \Gamma(x,n); e \in \gamma\} \, |f(e_+) - f(e_-)| \\ &\quad \leqslant K(x,n) \sum_{y \in B(x,2n)} |\nabla f|(y) \\ &\quad \leqslant Cn \sum_{y \in B(x,2n)} |\nabla f|(y).\end{aligned}$$

The proof of the Poincaré inequality is complete. □

We now consider the case of weighted graphs. For this, we follow the notations of the second section. Fix first $x \in G$ and $n \in \mathbb{N}^*$. For any pair of points $(y,z) \in B(x,n)^2$, we choose a geodesic path γ_{yz} joining y and z. Set $\Gamma_{x,n} = \{\gamma_{yz}, y, z \in B(x,n)\}$. We say that G is a graph with positive curvature in the sense of Coulhon–Saloff-Coste if there exist a constant $C \geqslant 0$ and a choice of geodesic paths $\gamma_{y,z}$ (joining y and z in G) for all y, $z \in B(x,n)$ such that $\frac{1}{V(x,n)} \sum_{\gamma \in \Gamma(x,n),\, e \in \gamma} m(\gamma^+)m(\gamma^-) \leqslant Cnm(e)$ for any $x \in G$, any $n \in \mathbb{N}^*$, any $e \in E \cap B(x,2n)$. Here, γ^+ and γ^- are the endpoints of γ and $m(e) = m_{e_+e_-}$ where e_+ and e_- are the endpoints of the edge e. Note that in the case of unweighted graphs, this definition is equivalent to the previous one.

Theorem 3.4. — *Any weighted graph with positive curvature supports a Poincaré inequality.*

Proof. — The proof is essentially the same as in the unweighted case. For the convenience of the reader, we repeat it. If e is an oriented edge of G, we denote by e_+ and e_- its endpoints. Then for any y, $z \in B(x,n)$, by the triangle inequality, we have

$$|f(y) - f(z)| \leqslant \sum_{e \in \gamma_{y,z}} |f(e_+) - f(e_-)|\,.$$

Hence,

$$\sum_{y,z \in B(x,n)} |f(y) - f(z)|\, m(y)m(z) \leqslant \sum_{y,z \in B(x,n)} \sum_{e \in \gamma_{y,z}} |f(e_+) - f(e_-)|\, m(y)m(z).$$

This implies

$$\begin{aligned}\sum_{y \in B(x,n)} &|f(y) - f_n(x)|\, m(y)\\ &= \sum_{y \in B(x,n)} \left| f(y) - \frac{1}{V(x,n)} \sum_{z \in B(x,n)} f(z)m(z) \right| m(y)\\ &= \sum_{y \in B(x,n)} \left| \frac{1}{V(x,n)} \sum_{z \in B(x,n)} (f(y) - f(z)) \right| m(y)m(z)\\ &\leqslant \frac{1}{V(x,n)} \sum_{y,z \in B(x,n)} |f(y) - f(z)|\, m(y)m(z)\end{aligned}$$

$$\leqslant \frac{1}{V(x,n)} \sum_{y,z\,\in\, B(x,n)} \sum_{e\,\in\,\gamma_{y,z}} |f(e_+)-f(e_-)|\, m(y)m(z)$$

Finally, by using the curvature assumption, we get

$$\begin{aligned}\sum_{y,z\,\in\, B(x,n)} \sum_{e\,\in\,\gamma_{y,z}} & |f(e_+)-f(e_-)| m(y)m(z) \\ &\leqslant \sum_{e\,\in\, B(x,2n)} \sum_{\gamma\,\in\,\Gamma(x,n),\, e\,\in\,\gamma} m(\gamma^+)m(\gamma^-)\, |f(e_+)-f(e_-)| \\ &\leqslant CnV(x,n) \sum_{e\,\in\, B(x,2n)\,\cap\, E} |f(e_+)-f(e_-)|\, m(e) \\ &\leqslant CnV(x,n) \sum_{y\,\in\, B(x,2n)} |\nabla f|(y)m(y).\end{aligned}$$

The proof of the Poincaré inequality is complete. □

We now relate this notion of positive curvature to the so-called democratic condition which was introduced in the continuous case by Lott and Villani in their theory of synthetic Ricci curvature (See [13] or [22]). The democratic condition in a geodesic measure space (X,d,μ) states that, for any pair of points y and z in X, there exists a choice of a geodesic $\gamma_{y,z}$ joining y and z in X such that the number of such geodesics passing through a point $x \in X$ is uniformly controlled (that is, the bound does not depend on x). Roughly speacking, the democratic condition means that if you would like to travel from y to z in X, you are not obliged to pass through a given point $x \in X$. This is a good intuitive idea of how a space with positive curvature should look. Lott and Villani proved that a non-branching geodesic measure space (X,d,μ) with nonnegative synthetic Ricci curvature satisfies the democratic condition which implies the existence of Poincaré inequalities.

We can give a discrete version of the result of Lott and Villani (Unpublished work with Jérôme Bertrand). Given a positive integer D, the set of curves γ of length at most D in G is viewed as a subset of G^{D+1} by repeating the ending point if necessary. We will also use the following notation: If μ is a measure on a space Y and $f : X \to Y$ is a measurable map between the spaces X and Y, we denote by $f_\sharp\mu$ the image measure of μ by f. We say that G satisfies the discrete democratic condition on a ball $B = B(x,n)$ if there exist a constant $C \geqslant 0$ and a probability measure $\pi \in \mathcal{P}(G^{2n})$ supported in the set of geodesics in G such that

(CD1) $(e_0, e_{2k})_\sharp \pi = m_B \otimes m_B$ where $e_i(\gamma) = \gamma(i)$ (Evaluation map) and $m_B = \frac{1}{m(B)} m_{|B}$. Here, $m_B \otimes m_B$ denotes the product measure.

(CD2) If we set $\Theta(e)=\sum_{\gamma\in G^{2k}}\chi_\gamma(e)\pi(\gamma)$ for any $e\in E$ where χ_γ is the characteristic function of γ (namely $\chi_\gamma(e)=1$ if and only if the endpoints of e are of the form $\gamma(i),\gamma(i+1)$), then $\Theta(e)\leqslant Cn\frac{m(e)}{m(B)}$.

Lemma 3.5. — *A graph Γ with positive curvature also satisfies the discrete democratic condition on any ball B of G.*

Proof. — Assume that $B=B(x,n)$ where $x\in G$ and $n\in\mathbb{N}^*$. For any $y,z\in B$, there exists a geodesic path γ_{xy} such that for any $e\in 2B\cap E$ (As previously, we denote by $\Gamma(x,n)$ this set of "good" geodesic paths)

$$\sum_{\gamma\in\Gamma(x,n),\,e\in\gamma} m(\gamma^+)m(\gamma^-)\leqslant Cnm(e)m(B).$$

Let us denote by S the map

$$\begin{aligned} S:B\times B &\longrightarrow G^{2n}\\ (y,z)&\longmapsto \gamma_{y,z}\end{aligned}.$$

Then, we set $\pi=S_\sharp m_B\otimes m_B$. By definition, we have

$$(e_0,e_{2n})_\sharp\pi=m_B\otimes m_B$$

and

$$\pi(\gamma_{y,z})=\frac{m(y)m(z)}{m(B)^2}\text{ but }\pi(\gamma)=0\text{ otherwise.}$$

By using π, the discrete democratic condition is clearly satisfied:

$$\begin{aligned}\sum_{\gamma\in G^{2n}}\chi_\gamma(e)\pi(\gamma)&\leqslant\sum_{\gamma\in\Gamma(x,n),\,e\in\gamma}\frac{m(\gamma^+)m(\gamma^-)}{m(B)^2}\\&\leqslant C\frac{nm(e)}{m(B)}.\end{aligned}$$

□

Lemma 3.6. — *Every graph satisfying the discrete democratic condition on any ball (with a uniform constant) satisfies a Poincaré inequality.*

Proof. — Fix a ball $B=B(x,n)$ in G. Let f be a map on G. We start with the following equality

$$\begin{aligned}&f(y)-f(z)\\&=\frac{1}{\pi(\{\gamma;\gamma(0)=y,\gamma(2n)=z\})}\sum_{\{\gamma;\gamma(0)=y,\gamma(2n)=z\}}\big(f(\gamma(0))-f(\gamma(2n))\big)\pi(\gamma).\end{aligned}$$

Now, by using

$$|f(\gamma(0))-f(\gamma(2n))|\leqslant\sum_{i=0}^{2n-1}|f(\gamma(i)-f(\gamma(i+1))|$$

we get

$$|f(y)-f(z)| \leqslant \frac{1}{\pi\left(\{\gamma;\gamma(0)=y,\gamma(2n)=z;\}\right)} \sum_{\gamma\in(e_0,e_{2n})^{-1}(y,z)} \sum_{e\in E\cap 2B} |f(e+)-f(e-)|\,\chi_\gamma(e)\pi(\gamma).$$

By the democratic condition (CD1), $\pi(\{\gamma;\gamma(0)=y,\gamma(2n)=z\}) = \frac{m(y)m(z)}{m(B)^2}$. So, summing the above inequality over y and z yields

$$\sum_{y,z\in B} |f(y)-f(z)|\,m(y)m(z) \leqslant \sum_{\gamma} \sum_{e\in E\cap 2B} |f(e+)-f(e-)|\,\chi_\gamma(e)\pi(\gamma)m(y)m(z).$$

Hence, by using Fubini theorem and the democratic condition (CD2), we get

$$\begin{aligned}\sum_{y,\,z\in B} |f(y)-f(z)|\,m(y)m(z) &\leqslant Cnm(B) \sum_{e\in E\cap 2B} |f(e+)-f(e-)|\,m(e)\\ &\leqslant Cnm(B) \sum_{y\in 2B} |\nabla f|(y)m(y)\end{aligned}$$

and the proof is complete since

$$\begin{aligned}\sum_{y\in B(x,n)} |f(y)-f_n(x)|\,m(y) &\leqslant \frac{1}{V(x,n)} \sum_{y\in B(x,n)} \sum_{z\in B(x,n)} |f(y)-f(z)|\,m(y)m(z)\\ &= \frac{1}{m(B)} \sum_{y\in B} \sum_{z\in B} |f(y)-f(z)|\,m(y)m(z). \quad\square\end{aligned}$$

4. Optimal transportation and the approach of Ollivier

The starting point of Ollivier's work (See [14]) is the following nice observation which gives an intuitive idea of the meaning of the Ricci curvature. Let (M,g) be a Riemannian manifold. Consider $x\neq y$ in M and $r>0$ small enough (For instance, $r\leqslant d(x,y)/10$). Denote by $\overline{x}$ (Respectively $\overline{y}$) a generic point in $B(x,r)$ (Respectively in $B(y,r)$). If M has nonegative Ricci curvature, the distance $d(\overline{x},\overline{y})$ is less or equal in average than $d(x,y)$. This can be see by using Jacobi fields and parallel transport. The idea of

Ollivier was to formalize this notion in an abstract setting by using optimal transportation, more precisely the Wasserstein distance between probability measures

Let (X, d) be a Polish space (That is, a complete and separable metric space). A random walk m on X is a family of probability measures m_x on X for any $x \in X$ so that

(i) The measure m_x depends continuously on the points $x \in X$ (With respect to the weak topology);
(ii) The measures m_x have finite moment, that is there exists a point $O \in X$ (and hence for any point O in X) so that $\int d(0, y) dm_x(y) < \infty$ for every $x \in X$.

The Ricci curvature $R(x, y)$ of (X, d, m) along the pair of points (x, y) of X is given by

$$R(x, y) = 1 - \frac{W_1(m_x, m_y)}{d(x, y)}$$

where W_1 is the L^1-Wasserstein distance. Recall that if μ and ν are two probabilities measure on X, the L^1-Wasserstein distance between μ and ν is defined by

$$W_1(\mu, \nu) = inf_\sigma \int_{X \times X} d(x, y) d\sigma(x, y)$$

where the infimum is taken over all the probability measures σ on $X \times X$ so that $\sigma(A \times X) = \mu(A)$ and $\sigma(X \times A) = \nu(A)$ whenever $A \subset X$ is a Borel set. By the Kantorovich duality, we also have

$$W_1(\mu, \nu) = \sup_f \left(\int f d\mu - \int f d\nu \right)$$

where the supremum is taken over all the 1-Lipschitz functions $f : X \to \mathbb{R}$ (That is $|f(x) - f(y)| \leqslant d(x, y)$ for every x, $y \in X$). In our setting of graphs, $m_x(y) = p(x, y)$ and thus

$$W_1 (m_x, m_y) = \sup_{f 1-Lip} \left(\sum_{z \sim x} f(z) p(x, z) - \sum_{z \sim y} f(z) p(y, z) \right).$$

Example 4.1. — We come back to the case of Riemannian manifolds. Let (M, g) be a smooth complete Riemannian manifold of dimension n. We denote by d_g the Riemmanian distance and by Vol_g the Riemannian volume. Fix $x \in M$ and consider two tangents vectors v and w at x. Let ε, $\delta > 0$. Set $y = \exp_x(\delta v)$ (where $\exp_x$ is the exponential map at x). If

$\widetilde{u}$ is the tangent vector at y obtained by parallel transport of u along the geodesic $t \to \exp_x(tv)$, then as $(\varepsilon, \delta) \to (0,0)$, we get

$$d_g \left(\exp_x(\varepsilon w), \exp_y(\varepsilon \widetilde{u})\right) = \delta \left(1 - \frac{\varepsilon^2}{2} S(v,w) + O\left(\varepsilon^3 + \delta\varepsilon^2\right)\right),$$

where $S(v,w)$ is the sectional curvature in the tangent plane generated by v anf w. Consider now the Markov chain on M (depending on the parameter $\varepsilon > 0$) given by

$$dm_x(y) = \frac{1}{Vol_g(B(x,\varepsilon))} \chi_{B(x,\varepsilon)} dVol_g(y).$$

Then, if $x \in M$, v is a tangent vector at x and y is a point on the geodesic starting from x in the direction v, we get

$$R(x,y) = \frac{\varepsilon^2 Ric(v,v)}{2(n+2)} + O\left(\varepsilon^3 + \varepsilon^2 d_g(x,y)\right)$$

if $d_g(x,y)$ is small enough and where Ric is the classical Ricci curvature tensor. This example is taken from [14]. Note also that the scaling by ε^2 comes from the fact that Riemannian manifolds are locally Euclidean up to second order.

Example 4.2. — If $G = [0,1]^N$ is the discrete cube equipped with the Haming distance (the length of an edge is one) and the lazy random walk m, that is $m_x(x) = p(x,x) = 1/2$ and $m_x(y) = p(x,y) = \frac{1}{2N}$ if $y \sim x$. Then, $R(x,y) = 1/N$ if $x \sim y$. We now sketch a proof of this statement (See [15] for more details). Without loss of generality, we assume that $x = (0, \ldots, 0)$ and $y = (0,1,0, \ldots, 0)$. For any $i \geqslant 2$, we denote by x^i (Respectively y^i) the neighbour of x (Respectively of y) such that the i-coordinate of x (Respectively of y) is switched. Thus, the measure m_x (Respectively the measure m_y) is supported on the set $\{x, y, x^2, \ldots, x^N\}$ (Respectively on the set $\{y, x, y^2, \ldots, y^N\}$). Consider now the following coupling between m_x and m_y:

- for $i \geqslant 2$, move x^i to y^i. It is possible since $m_x(x^i) = m_y(y^i) = \frac{1}{2N}$.
- Since $m_x(x) = \frac{1}{2}$ and $m_x(y) = \frac{1}{2N}$, move $\frac{1}{2} - \frac{1}{2N}$ of the mass of $m_x(x)$ to y.
- $\frac{1}{2N}$ of the mass of $m_x(x)$ remains at the same place x.

With this coupling, we get $R(x,y) \geqslant 1/N$. It turns out that this coupling is optimal. To see this, consider the function $f : G \to \{0,1\}$ so that the image $f(z)$ of $z \in G$ is the first coordinate of z. Thus, f is 1-Lipschitz and $f(x) = 0$, $f(y) = 1$. By using the function, we get that $W_1(m_x, m_y) \geqslant 1 - 1/N$ and thus $R(x,y) \leqslant 1/N$.

We say that the graph G has nonnegative Ricci curvature in the sense of Ollivier if for every $x \in G$, every $y \in G$, $R(x,y) \geqslant 0$.

For any $x \in X$, we define the jump of the random walk $(m_x)_{x \in X}$ at x by $J(x) = W_1(\delta_x, m_x)$ where δ_x is the Dirac measure at x. We now give a (very weak) discrete version of the Bonnet-Myers theorem (See Theorem 1.1 for a statement in the Riemannian setting).

Theorem 4.3. — *Let X be a Polish space and let $(m_x)_{x \in X}$ be a random walk on X. Assume that there exists $\alpha > 0$ so that $R(x,y) \geqslant \alpha$ for all x, $y \in X$. Then,*

$$d(x,y) \leqslant \frac{J(x)+J(y)}{R(x,y)}$$

In particular, $\operatorname{diam} X \leqslant \frac{2\sup_{x \in X} J(x)}{\alpha}$.

Proof. — We have $W_1(m_x, m_y) \leqslant d(x,y)(1 - R(x,y))$, $W_1(\delta_x, m_x) = J(x)$ and $W_1(\delta_y, m_y) = J(y)$. Hence, by the triangle inequality, $W_1(\delta_x, \delta_y) \leqslant J(x) + J(y) + d(x,y)(1 - R(x,y))$. Since $d(x,y) = W_1(\delta_x, \delta_y)$, we can easily conclude. □

We now consider the case of Cayley graphs (We follow here the presentation of [3]). So, let G be a finitely generated group and let S be a symmetric generating familly of G (That does not contain the identity element e of G). For any $x \in G$ and any $r > 0$, set $\mathcal{S}_r(x) = \{y \in G; |x^{-1}y| = r\}$ and $\mathcal{B}_r(x) = \{y \in G; |x^{-1}y| \leqslant r\}$. Recall that $|x^{-1}y|$ is the word distance between x and y. In the special case $x = e$, we write for simplicity $\mathcal{S}_r = \mathcal{S}_r(e)$ and $\mathcal{B}_r = \mathcal{B}_r(e)$.

If $x \in G$, it is quite natural to consider the random walk m_x given by the uniform law on the neighbours (That is, the probabiliy to jump from x to one of its neighbours is $1/|S|$ where $|S|$ denotes the cardinal of S). With the same notations as above, we have for x, $y \in G$,

$$W_1(x,y) = \sup_{f 1-lip} \left(\frac{1}{|S|} \sum_{s \in S} (f(xs) - f(ys)) \right) \leqslant \frac{1}{|S|} \sum_{s \in S} d(xs, ys).$$

Set for x, $y \in G$ and $r > 0$,

$$\mathcal{S}_r(x,y) = \frac{1}{|\mathcal{S}_r|} \sum_{w \in \mathcal{S}_r} d(xw, yw) \quad \text{and} \quad \mathcal{B}_r(x,y) = \frac{1}{|\mathcal{B}_r|} \sum_{w \in \mathcal{B}_r} d(xw, yw).$$

Therefore, by the previous computation, $W_1(x,y) \leqslant \mathcal{S}_1(x,y)$. By analogy with the definition of Ollivier, define the curvatures by

$$\kappa_r^{\mathcal{S}}(x,y) = 1 - \frac{d(x,y)}{\mathcal{S}_r(x,y)} \quad \text{and} \quad \kappa_r^{\mathcal{B}}(x,y) = 1 - \frac{d(x,y)}{\mathcal{B}_r(x,y)}.$$

By invariance by translation, we get $\kappa_r^{\mathcal{S}}(x,y) = \kappa_r^{\mathcal{S}}(e, x^{-1}y)$ and $\kappa_r^{\mathcal{B}}(x,y) = \kappa_r^{\mathcal{B}}(e, x^{-1}y)$. In the sequel, we will consider only the case $r = 1$. As in the approach of Ollivier, it would be interesting to look also to the case $r > 1$, in particular when r is big enough to understand the geometry of G at large scales. But, this study is quite difficult.

We set, for $g \in G$, $\kappa(g) = \kappa_1^{\mathcal{S}}(g)$ and $Geocon(g) = \frac{1}{|S|}\sum_{s \in S} |s^{-1}gs|$. Therefore, $Geocon(g) = \mathcal{S}_1(e,g)$. We define the Ricci curvature of G (with respect to S) by

$$\kappa(g) = \frac{|g| - Geocon(g)}{|g|} = 1 - \frac{Geocon(g)}{|g|}.$$

Remark 4.4. — For $g \in G$, set $\phi_g(x) = |x^{-1}gx| = d(x, gx)$. So, we have

$$\begin{aligned}\frac{1}{|S|}\sum_{x \sim g} \phi_g(x) &= \frac{1}{|S|}\sum_{x \sim g} \left[x^{-1}gx\right| = \frac{1}{|S|}\sum_{s \in S} \left|(gs)^{-1}g(gs)\right| \\ &= \frac{1}{|S|}\sum_{s \in S} \left|s^{-1}gs\right| = Geocon(g).\end{aligned}$$

Since $\kappa(g) = 0$ if and only if $Geocon(g) = 0$, it follows that $\kappa(g) = 0$ if and only if $\Delta\phi_g(g) = 0$ (That is ϕ_g is harmonic at g). Note also that $\kappa(g) = -\frac{\Delta\phi_g(g)}{\phi_g(g)}$ and that $\kappa(s) \leqslant 0$ if $s \in S$.

It is clear that the curvature of an Abelian group G at each $g \in G$ is zero. The converse is almost true.

Theorem 4.5. — *If for any $x \in S$, $\kappa(s) = 0$, then G is virtually Abelian.*

Proof. — We start with some classical notations. If $x \in G$, the stabilizer of x is given by the equivalent formulas:

$$\begin{aligned} Stab(x) &= \left\{g \in G; g^{-1}xg = x\right\} \\ &= \left\{g \in G; gxg^{-1} = x\right\} \\ &= \{g \in G; xg = gx\} \\ &= Z(x).\end{aligned}$$

Fix $s \in S$ and assume that $\kappa(s) = 0$. Then, by definition, $Geocon(s) = |s| = 1$ and $|S| = \sum_{a \in S} |a^{-1}sa|$. Therefore, for $a \in S$, $|a^{-1}sa| = 1$ and so $a^{-1}sa \in S$. Hence, S is stable as a set under the action of G by conjugation. This implies that the map $x \to x^{-1}sx$ takes a finite set of values. Since x and y are in the same class of equivalence (With respect to $Stab(s)$) if and only if they are conjugate under the action of G, $Stab(s)$ is a subgroup of G of finite index. Hence, for any $s \in S$, $Z(s)$ is a subgroup of finite index of G. Since S is a finite generating family of G, it follows that the center

$Z(G) = \bigcap_{s \in S} Z(s)$ is a subgroup of finite index of G and is Abelian. This completes the proof. □

The main problem of this definition of Ricci curvature for Cayley graphs is its strong dependance with respect to the set of generators. For instance, consider the symmetric group. The curvature at one point can vary from -2 to $1/2$ depending on the set of generators (See [3])!

5. The Bochner formula and the Bakry–Emery condition

We start with some formal definitions. Let M be a smooth complete manifold of dimension n and let L be an operator defined on an algebra $\mathcal{A}$ of functions $f : M \to \mathbb{R}$. Then, we define the operator "carré du champs" for u, $v \in \mathcal{A}$ by

$$\Gamma(u,v) = \frac{1}{2}\left(L(uv) - uLv - vLu\right).$$

Note that $\Gamma(u,v)$ measures how far the operator L is closed to satisfy the chain rule. Then, we set

$$\Gamma_2(u,v) = \frac{1}{2}\left(L\Gamma(u,v) - \Gamma(Lu,v) - \Gamma(u,Lv)\right).$$

To simplify notations, we set $\Gamma(u) = \Gamma(u,u)$ and $\Gamma_2(u) = \Gamma_2(u,u)$.

From now on, we consider only elliptic operators of second order L acting on smooth fonctions on M (We also assume that L has no constant term). In local coordinates, L can be written as

$$Lu(x) = \sum_{i,j} g^{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b^i(x)\frac{\partial u}{\partial x_i},$$

where $g^{i,j}$ and b^i are smooth functions. The ellipticity assumption on L implies that the matrix $(g^{i,j}(x))$ is a (positive and definite) quadratic form for all x on M. Hence, we can introduce the Riemannian metric $g = (g_{i,j}(x))$ which is the inverse of the matrix $(g^{i,j}(x))$. We assume that M is equiped with the Riemannian distance d_g and the Riemannian volume Vol_g associated to g. Then, $\Gamma(u,v) = \nabla u.\nabla v$ (where ∇ is the standard gradient with respect to g). In particular, $\Gamma(u) = |\nabla u|^2$.

We say that the elliptic operator L satisfies the curvature-dimension condition $CD(K,N)$ of Bakry–Emery for some $N \geqslant n$ and a function $K : M \to \mathbb{R}$ if, for any smooth function $u : M \to$ and all $x \in M$, we have

$$\Gamma_2(u)(x) \geqslant \frac{1}{N}(Lu)(x)^2 + K(x)\Gamma(u)(x).$$

Note that if L satisfies $CD(K,N)$, then L satisfies $CD(K',N')$ for every $N' \geqslant N$ and every $K' \leqslant K$. We now investigate two basic examples. For all of them, the function $K(x)$ will be a constant function.

Example 5.1. — Assume that $M = \mathbb{R}^n$ and that $L = \Delta$ is the usual Laplacian:

$$\Delta u(x) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x)$$

for any smooth function $u : \mathbb{R}^n \to \mathbb{R}$. Then, by elementary computations, we get for all u, $v : \mathbb{R}^n \to \mathbb{R}$,

$$\Gamma(u,v) = \langle \nabla u, \nabla v\rangle (\text{Where } \langle .,.\rangle \text{ is the Euclidean scalar product on } \mathbb{R}^n),$$

$$\begin{aligned} L(\Gamma(u)) &= 2\left(\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \frac{\partial u}{\partial x_j}\frac{\partial^3 u}{\partial^2 x_i \partial x_j}\right)\right), \\ \Gamma(u, Lu) &= \sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_j}\frac{\partial^3 u}{\partial^2 x_i \partial x_j}\right), \\ \Gamma_2(u) &= \sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 . \end{aligned}$$

By definition of the Hessian of u, we get that

$$\Gamma_2(u) = \|Hess(u)\|_2^2.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\Delta u|^2 &= \left(\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}\right) \\ &\leqslant n \left(\sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2}\right)^2\right)^{1/2} . \\ &\leqslant n\Gamma_2(u). \end{aligned}$$

Hence, $\Gamma_2(u) \geqslant \dfrac{1}{n}|\Delta u|^2$ and the Laplacian on $\mathbb{R}^n$ satisfies the $CD(0,n)$ condition.

Example 5.2. — We now consider a complete Riemannian manifold (M,g) of dimension n. There is a natural generalization of the Laplacian, on M called the Laplace–Beltrami operator that we will also denote by Δ. If we assume that the Ricci curvature Ric is bounded by K, we get $Ric(\nabla u, \nabla u) \geqslant K\Gamma(u)$.

We recall a well-known result in Riemannian geometry.

THEOREM 5.3 (Bochner formula). — *Let (M, g) be a (smooth complete) Riemannian manifold. Then, if $u : M \to \mathbb{R}$ is smooth, then*

$$\frac{1}{2}\Delta|\nabla u|^2 = g(\nabla\Delta u, \nabla u) + \|Hess(u)\|_2^2 + Ric(\nabla u, \nabla u),$$

where ∇u is the gradient of u with respect to g, $\|Hess(u)\|_2$ is the L^2-norm of the Hessian of u and Ric is the Ricci curvature tensor.

By using the terminology of Bakry–Emery, the Bochner formula can be rewritten as follows:

$$\Gamma_2(u) = Ric(\nabla u, \nabla u) + \|Hess(u)\|_2^2. \tag{5.1}$$

Hence, as in the Euclidean case, we get by the Cauchy–Schwarz inequality that for all smooth function $u : M \to \mathbb{R}$

$$\Gamma_2(u) \geqslant K\Gamma(u) + \frac{1}{n}(\Delta u)^2,$$

if the Ricci curvature of M is bounded above by K. Therefore, if M is a manifold of dimension n so that $Ric \geqslant K$, then the Laplace–Beltrami operator Δ satisfies $CD(0, n)$. It tuns out that the converse is also true for a manifold of dimension n. Note also that if L is a reasonable operator (Namely a diffusion operator) on M which satisfies $CD(K, n)$ for some $K \in \mathbb{R}$, then $L = \Delta$. See [1] and [11] for more details.

Remark 5.4. — If we assume that the function u is harmonic (That is $\Delta u = 0$), then the Bochner formula states that

$$\frac{1}{2}\Delta|\nabla u|^2 = \|Hess(u)\|_2^2 + Ric(\nabla u, \nabla u),$$

Remark 5.5. — The Bochner formula (By applying it to the distance function) provides a very nice proof of the Bishop–Gromov comparison Theorem for volumes of balls. See [9]. Note also that in the continuous case, for most of the possible definitions of spaces with nonnegative Ricci curvature, there exists an analog of the Bishop–Gromov theorem (See [17, Chapter 4]). In the discrete setting, the situation is not so clear.

Remark 5.6. — If we assume that the manifold M satisfies the condition $CD(0, N)$ for some elliptic operator L, then M supports a Poincaré type inequality (Unpublished joint work with Sylvestre Gallot). The proof follows the Riemannian one as in [18] but uses the comparison volume result of [2] instead of the proof of the Bishop–Gromov Theorem (Which is based on the Jacobi fields).

In the two previous examples, the operator Δ satisfies $CD(K, N)$ where K is a lower bound for the Ricci curvature on M and N is the dimension of

M. In general (For instance if $L \neq \Delta$), there is no geometric interpretation of the parameters K and N (See below the discussion in the discrete case).

Consider now a discrete graph G equipped with the usual Laplacian Δ. The key point is to find an analog of (5.1) in this setting. We recall some basic definitions.

$$\Gamma(f,g)(x) = \frac{1}{2}\big(\Delta(fg)(x) - f(x)\Delta g(x) - g(x)\Delta f(x)\big)$$

and then define by iterating Γ:

$$\Gamma_2(f,g)(x) = \frac{1}{2}\left(\Delta\Gamma(f,g)(x) - \Gamma(f,\Delta g)(x) - \Gamma(g,\Delta f)(x)\right.$$

We say that the operator Δ satisifes the curvature-dimension inequality $CD(K,N)$ (in the sense of Bakry–Emery) if for any function $f : G \to \mathbb{R}$ with finite support,

$$\Gamma_2(f) \geqslant \frac{1}{N}(\Delta f)^2 + K\Gamma(f).$$

In this case, $N \in [1,+\infty[$ is an upper bound of the "dimension" of (G,Δ) whereas $K \in \mathbb{R}$ is a lower bound of the Ricci curvature. As in the continuous setting, it is not so easy to give a geometric interpretation of the parameters K and N. For instance, Lin and Yau proved in [12] the following general result (See the comments after the proof).

Theorem 5.7. —

(1) *Let G be a uniformly locally graph (with constant D). Then, $\Gamma_2(f) \geqslant \frac{1}{2}(\Delta f)^2 + (\frac{2}{D}-1)\Gamma(f)$ for any $f : G \to \mathbb{R}$. Hence, the Laplace operator satisfies $CD(\frac{2}{D}-1,2)$. This is the case when G is the Cayley graph of a finitely generated group.*

(2) *Let G be a locally finite graph. Then, for any $f : G \to \mathbb{R}$, $\Gamma_2(f) \geqslant \frac{1}{2}(\Delta f)^2 - \Gamma(f)$. Hence, the Laplace operator satisfies $CD(-1,2)$.*

Proof. — We will use the notation of Lin and Yau. If $f : G \to \mathbb{R}$, we set $\Delta f(x) = \frac{1}{d_x}\sum_{y\sim x}(f(y) - f(x))$ and $|\nabla f|^2(x) = \frac{1}{d_x}\sum_{y\sim x}|f(x) - f(y)|^2$. Recall that d_x is the degree of G at x (See Section 2). This definition of the Laplace operator Δ is not the same as above, but this will not change our discussion. The quantity $|\nabla f|^2$ should be seen as the square of the length of the gradient of f (See Section 3). The proof is a little bit technical and will be divided into several parts. In the first one, we express $\Gamma(f)$ in terms of the operators Δ et $|\nabla f|^2$:

Lemma 5.8. — *If $f : G \to \mathbb{R}$, we have $\Gamma(f) = \frac{1}{2}|\nabla f|^2 = \frac{\Delta f^2}{2} - f\Delta f$.*

Proof. — Let $f : G \to \mathbb{R}$ and let $x \in G$. Then, we have

$$\begin{aligned} 2f\nabla f(x) &= 2f(x)\frac{1}{d_x}\sum_{y\sim x}(f(y)-f(x)) \\ &= -2f(x)^2 + \frac{2f(x)}{d_x}\sum_{y\sim x} f(y). \end{aligned}$$

Hence, we get

$$\begin{aligned} \Delta f^2(x) &= \frac{1}{d_x}\sum_{y\sim x}\left(f(y)^2 - f(x)^2\right) \\ &= \frac{1}{d_x}\sum_{y\sim x}\left[(f(y)-f(x))^2 - 2f(x)^2 + 2f(x)f(y)\right] \\ &= |\nabla f|^2(x) - 2f(x)^2 + \frac{2}{d_x}f(x)\sum_{y\sim x} f(y) \\ &= |\nabla f|^2(x) + 2f(x)\Delta f(x). \end{aligned}$$

So we have, $\Gamma(f) = \frac{1}{2}\Delta f^2 - f\Delta f = \frac{1}{2}|\nabla f|^2$. □

We now estimate the Laplacian of the length of the gradient.

Lemma 5.9. — *For any $f : G \to \mathbb{R}$ and any $x \in G$, we have*

$$\begin{aligned} \Delta|\nabla f|^2(x) = \frac{1}{d_x}\sum_{y\sim x}\frac{1}{d_y}\sum_{z\sim y} \\ \left((f(x)-2f(y)+f(z))^2 - 2(f(x)-2f(y)+f(z))(f(x)-f(y))\right). \end{aligned}$$

The proof of this lemma is straightforward and and is left to the reader. We now can use the two previous lemmas to prove Theorem 5.7. Let $f : G \to \mathbb{R}$ and let $x \in G$. We have to compute $\Gamma_2(f) = \frac{1}{2}(\Delta\Gamma(f) - 2\Gamma(f, \Delta f))$. We first express the Laplacian of $f\Delta f$, and then $\Gamma(f, \Delta f)$:

$$\begin{aligned} \Delta(f\Delta f)(x) &= \frac{1}{d_x}\sum_{y\sim x}\left(f(y)\Delta f(y) - f(x)\Delta f(x)\right) \\ &= \frac{1}{d_x}\sum_{y\sim x}\left[f(y)\Delta f(y) - f(y)\Delta f(x) + f(y)\Delta f(x) - f(x)\Delta f(x)\right] \\ &= \frac{1}{d_x}\sum_{y\sim x}\left[f(y)(\Delta f(y) - \Delta f(x)) + (f(y)-f(x))\Delta f(x)\right] \\ &= \frac{1}{d_x}\sum_{y\sim x}\left[(f(y)-f(x))(\Delta f(y)-\Delta f(x)) + f(x)(\Delta f(y)-\Delta f(x))\right] \\ &\qquad + (\Delta f(x))^2 \end{aligned}$$

$$= \frac{1}{d_x}\sum_{y\sim x}\left[(f(y)-f(x))(\Delta f(y)-\Delta f(x))\right] \quad + f(x)\Delta\Delta f(x) + (\Delta f(x))^2$$

Since $\Gamma(f,\Delta f) = \dfrac{1}{2}(\Delta(f\Delta f) - f\Delta\Delta f - (\Delta f)^2)$, we get

$$\Gamma(f,\Delta f)(x) \quad = \quad \frac{1}{2d_x}\sum_{y\sim x} \quad (f(y)-f(x))(\Delta f(y)-\Delta f(x)). \tag{5.2}$$

We conclude by using (5.2), the definition of Γ_2 and the Lemmas 5.8 et 5.9:

$$\begin{aligned}
\Gamma_2(f)(x) &= \frac{1}{2}\left(\Delta\Gamma(f))(x) - 2\Gamma(f,\Delta f)(x)\right)) \\
&= \frac{1}{2}\left[\frac{1}{2}\Delta|\nabla f|^2(x) - \frac{1}{d_x}\sum_{y\sim x} \quad \big(f(y)-f(x)\big)\big(\Delta f(y)-\Delta f(x)\big)\right] \\
&= \frac{1}{2}\frac{1}{d_x}\sum_{y\sim x}\Bigg[\frac{1}{2d_y}\Big[\sum_{z\sim y} \quad \big(f(x)-2f(y)+f(z)\big)^2 \\
&\qquad - 2\big(f(x)-f(y)\big)\big(f(x)-2f(y)+f(z)\big)\Big] \\
&\qquad - \big(f(y)-f(x)\big)\big(\Delta f(y)-\Delta f(x)\big)\Bigg] \\
&= \frac{1}{4d_x}\sum_{y\sim x}\frac{1}{d_y}\sum_{z\sim y} \quad \big(f(x)-2f(y)+f(z)\big)^2 \\
&\quad + \frac{1}{2d_x}\sum_{y\sim x}\big(f(y)-f(x)\big)\frac{1}{d_y}\sum_{z\sim y}\big(f(z)-f(y)+f(x)-f(y)\big) \\
&\quad - \frac{1}{2d_x}\sum_{y\sim x}\big(f(y)-f(x)\big)\left[\frac{1}{d_y}\sum_{z\sim y}\big(f(z)-f(y)\big) - \Delta f(x)\right] \\
&= \frac{1}{4d_x}\sum_{y\sim x}\frac{1}{d_y}\sum_{z\sim y} \quad \big(f(x)-2f(y)+f(z)\big)^2 \\
&\quad - \frac{1}{2d_x}\sum_{y\sim x}\big(f(y)-f(x)\big)^2 \quad + \frac{\Delta f(x)}{2d_x}\sum_{y\sim x}\big(f(y)-f(x)\big) \\
&\geqslant \frac{1}{4d_x}\sum_{y\sim x}\frac{(2f(x)-2f(y))^2}{d_y} \text{ (Consider only } z=x) \\
&\quad - \frac{1}{2d_x}\sum_{y\sim x}\big(f(y)-f(x)\big)^2 + \frac{1}{2}(\Delta f(x))^2 \\
&= \frac{1}{d_x}\sum_{y\sim x}\frac{1}{d_y}\big(f(x)-f(y)\big)^2 - \frac{1}{2}\frac{1}{d_x}\sum_{y\sim x}\big(f(y)-f(x)\big)^2 + \frac{1}{2}(\Delta f(x))^2
\end{aligned}$$

$$\geqslant \left(\frac{1}{D} - \frac{1}{2}\right) |\nabla f|^2(x) + \frac{1}{2}(\Delta f(x))^2$$
$$\geqslant \left(\frac{2}{D} - 1\right) \Gamma(f)(x) + \frac{1}{2}(\Delta f(x))^2$$

At the end, we used the fact that $d_y \leqslant D$. Hence, G satisfies $CD\left(\frac{2}{D} - 1, 2\right)$. If d_y is not bounded, we showed that G satisfies $CD(-1, 2)$. So, the result of Lin and Yau means that there exist uniform bounds of the "curvature" and of the "dimension" in the case of discrete graphs! □

Remark 5.10. — Another approach of Ricci flat graphs appears in [7]. We say that a graph has local k-frame at $v \in V$ if there exist mappings $\eta_1, \ldots, \eta_k : N(v) \to V$ (Where $N(v)$ is the set of neighbours of v) so that

(i) G is k-regular.
(ii) v is adjacent to $\eta_i(v)$ for any $v \in V$ and any $i = 1, \ldots, k$.
(iii) $\eta_i(v) \neq \eta_j(v)$ if $v \in V$ and $i \neq j$ in $\{1, \ldots k\}$.

A graph G is said to be Ricci flat if G has a local k-frame (At each $v \in V$) so that for any $i \in \{1, \ldots, k\}$ and any vertex $v \in V$,

$$\bigcup_{j=1}^{k} \eta_i \eta_j(v) = \bigcup_{j=1}^{k} \eta_j \eta_i(v).$$

In this case, G satisfies $CD(0, +\infty)$, that is $\Gamma_2(f) \geqslant 0$. For instance, the grid $\mathbb{Z}^n$ is Ricci flat.

We now move to the case of Cayley graphs. Let G be a finitely generated group and lest $S = \{e_1, \ldots, e_N\}$ be a finite generating family (That does not contain the identity element). For the moment, we do not require that S is symmetric. Recall that if $x, y \in G$, $x \sim y$ if and only if there exists $e_i \in S$ so that $y = xe_i$. For $i = 1, \ldots, N$, we set $\partial_i f(x) = f(xe_i) - f(x)$ whenever $x \in G$ and $f : G \to \mathbb{R}$ is a function. The operators Δ, Γ and Γ_2 are defined as above.

Remark 5.11. — The operator ∂_i looks like a partial derivative, but is not a derivation in the usual sense, since it does not satisfy the chain rule. Indeed, simple computations show

$$\partial_i(fg)(x) = \partial_i f(x)g(x) + f(x)\partial_i g(x) + \partial_i f(x)\partial_i g(x).$$

This section is based on [16] where a discrete version of the Bochner formula is given. We start with a simple but useful result (The proof is left to the reader as an easy exercise).

Proposition 5.12. — *For all $x \in G$, all f, $g : G \to \mathbb{R}$, we have*

(i) $\Delta f(x) = \sum_{i=1}^{N} \partial_i f(x)$.
(ii) $\partial_i(fg)(x) = \partial_i f(x) g(xe_i) + f(x)\partial_i g(x)$.
(iii) $\Gamma(f,g)(x) = \frac{1}{2}\sum_{i=1}^{N}(\partial_i f(x)\partial_i g(x))$. *In particular,*

$$\Gamma(f) = \frac{1}{2}\sum_{i=1}^{N}(\partial_i f(x))^2.$$

Remark 5.13. — The formula given in (i) suggests that ∂_i is a second partial derivative. (iii) gives an analog of Lemma 5.8 in the setting of Cayley graphs.

It follows that if the groupe G is Abelian, we have

$$\Gamma_2(f)(x) = \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}(\partial_i\partial_f f(x))^2.$$

Assume now that the generating family S is symmetric (Note that we never used this assumption previously). We set $S = \{e_1^+, \ldots, e_N^+, e_1^{-1}, \ldots, e_N^-\}$ with the convention that $e_i^- = (e_i^+)^{-1}$. For any $i = 1, \ldots, N$, we set $\partial_i^+ f(x) = f(xe_i^+) - f(x)$ and $\partial_i^- f(x) = f(xe_i^-) - f(x)$ whenever $x \in G$ and $f : G \to \mathbb{R}$. Note that

$$\partial_i^+\partial_i^- f(x) = f\left(xe_i^+e_i^-\right) - f\left(xe_i^+\right) - f\left(xe_i^-\right) + f(x) = -\left(\partial_i^+ f(x) + \partial_i^- f(x)\right).$$

For the same reasons, $\partial_i^-\partial_i^+ f(x) = -(\partial_i^+ f(x) + \partial f_i^- f(x)) = \partial_i^+\partial_i^- f(x)$. Hence, we get by the Cauchy-Schwarz inequality

$$\begin{aligned}
\Gamma_2(f) &\geqslant \frac{1}{4}\sum_{i=1}^{N}\left(\partial_i^+\partial_i^- f(x)^2 + \partial i^-\partial_i^+ f(x)^2\right) \\
&\geqslant \frac{1}{2}\sum_{i=1}^{N}\left(\partial_i^+ f(x) + \partial_i^- f(x)\right)^2 \\
&\geqslant \frac{1}{2N}\left(\sum_{i=1}^{N}(\partial_i^+ f(x) + \partial_i^- f(x))\right)^2 \\
&= \frac{1}{2N}\Delta f(x)^2.
\end{aligned}$$

Thus, an Abelian group G with 2N generators satisfies $CD(0, 2N)$.

Remark 5.14. — The case of the symmetric group is still open.

6. What is the conclusion?

We proposed and discussed in these notes several possible notions of lower Ricci curvature bound for discrete spaces, in particular for Cayley graphs of finitely generated groups. A natural question is which definition is more relevant to study for instance geometry of groups (Independently of the generating family). For the moment, the question is still open and I expect that these notes can help to solve the problem…

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