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QUILLEN METRIC THEORY FOR SURFACES WITH CUSPS

Siarhei Finski

Abstract. — In this note, we define the analytic torsion for complete non-compact Riemann surfaces with hyperbolic cusps singularities. We then show that the Quillen metric associated to it satisfies the curvature formula and study the behaviour of it for families of Riemann surfaces when the additional cusps are created by degeneration.

1. Introduction

In this short note we will report on some recent progress which has been done in [16, 17, 15], on Quillen metric theory for complete Riemann surfaces with hyperbolic cusps singularities.

Recall that for complex manifolds X and S, a proper holomorphic map $\pi : X \to S$ and a holomorphic vector bundle ξ over X, the construction of Grothendick–Knudsen–Mumford [22] (cf. also [8, §3]) associates the "determinant of the direct image of ξ, which is the holomorphic line bundle over S, denoted here by $\det(R^\bullet \pi_* \xi)$. In the special case when the cohomology groups $H^\bullet(\pi^{-1}(\cdot), \xi)$ have constant dimension, by Grauert's theorem, they form holomorphic vector bundles, and we have an isomorphism $\det(R^\bullet \pi_* \xi) = \otimes_i (\Lambda^{\max} H^i(\pi^{-1}(\cdot), \xi))^{(-1)^i}$.

Quillen metric has been introduced by Quillen in [26] (in the context of trivial families of compact Riemann surfaces) and later generalized by Bismut–Gillet–Soulé [6, 7, 8] (for locally Kähler submersions in any relative dimension) as a natural Hermitian norm on the determinant line bundle $\det(R^\bullet \pi_* \xi)$ associated to a Hermitian vector bundle (ξ, h^ξ) on a family of complex manifolds endowed with Kähler metrics. It is defined as a product of the Ray–Singer holomorphic analytic torsion of the fiber, [27], and the L^2-norm on the cohomology.

The curvature theorem of Bismut–Gillet–Soulé [8, Theorem 1.9] expresses the curvature of the Chern connection on $\det(R^{\bullet}\pi_*\xi)$, endowed with Quillen metric, as a pushforward along π of the differential form associated by Chern–Weil theory to a cohomological class appearing on the right-hand side of Riemann–Roch–Grothendieck theorem. This formula generalizes the curvature theorem of Quillen [26], obtained for trivial families of Riemann surfaces, and gives a refinement of Riemann–Roch–Grothendieck theorem on the level of differential forms.

In [29], Takhtajan–Zograf generalized the curvature formula in relative dimension 1 for families of stable non-compact Riemann surfaces endowed with a complete metric of finite volume and constant scalar curvature -1. One of the difficulties in their theory is that the spectrum of such surfaces contains a continuous part, and so the usual definition of the holomorphic analytic torsion through spectral "determinants" makes no sense. To overcome this difficulty, instead of the original definition, Takhtajan–Zograf used a version of the analytic torsion, defined using the Selberg zeta function. This was motivated by an earlier result of D'Hoker-Phong [12, (7.30)], [13, (3.6)], stating that for compact surfaces of constant scalar curvature -1, the two definitions coincide. In this setting, Takhtajan–Zograf showed that the singularities of the fibers introduce a new term to the curvature formula (nowadays called Takhtajan–Zograf forms), which can be expressed through Eisenstein–Maass series of the cusps of the fibers. The curvature formula from [29] thus expresses the curvature of the Chern connection on the determinant line bundle, endowed with Takhtajan–Zograf's version of the Quillen metric, on the universal curve on the moduli space of punctured stable surfaces as a linear combination of the Weil–Petersson form and Takhtajan–Zograf forms.

In this note we describe a generalization of cuvature formulas of Bismut–Gillet–Soulé (in relative dimension 1) and Takhtajan–Zograf to non-compact surfaces, endowed with metric with hyperbolic cusps of non necessarily constant scalar curvature, and families, admitting singular fibers. Our approach will be different from [29] and will be based on [8] and Bismut–Bost [5]. The first step of this approach, from [16], is to define the analytic torsion for surfaces with hyperbolic cusps using heat trace and to study the properties of this definition. The second step, from [17], is to study this Quillen metric in the family setting and calculate its curvature, which would give our version of the curvature formula. The third step, from [15], is to study Quillen metric at the singular locus and to obtain the compatibility between our definition of the analytic torsion and Takhtajan–Zograf's version, [29].

2. Surfaces with cusps and associated Quillen metric

Let $\overline{M}$ be a compact Riemann surface, and $D_M = \{P_1^M, \dots, P_m^M\}$ be a finite set of distinct points on $\overline{M}$. Let g^{TM} be a Kähler metric on the punctured Riemann surface $M = \overline{M} \setminus D_M$.

For $\epsilon \in]0,1[$, let $z_i^M : \overline{M} \supset V_i^M(\epsilon) \to D(\epsilon) = \{z \in \mathbb{C} : |z| \leqslant \epsilon\}$, $i = 1, \dots, m$, be a local holomorphic coordinate around P_i^M, and let

$$V_i^M(\epsilon) := \left\{x \in M : \left|z_i^M(x)\right| < \epsilon\right\}. \tag{2.1}$$

We say that g^{TM} is *Poincaré-compatible* with coordinates $z_1^M, \dots, z_m^M$ if for any $i = 1, \dots, m$, there is $\epsilon > 0$ such that $g^{TM}|_{V_i^M(\epsilon)}$ is induced by the Hermitian form

$$\frac{\sqrt{-1}dz_i^M d\overline{z}_i^M}{\left|z_i^M \log \left| z_i^M\right|\right|^2}. \tag{2.2}$$

We say that g^{TM} is a *metric with cusps* if it is Poincaré-compatible with some holomorphic coordinates of D_M. A triple $(\overline{M}, D_M, g^{TM})$ of a Riemann surface $\overline{M}$, a set of punctures D_M and a metric with cusps g^{TM} is called a *surface with cusps* (cf. [23]).

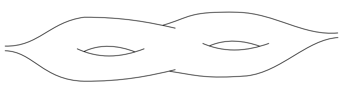

Figure 2.1. A surface with cusps.

The condition (2.2) essentially requires that our surface is complete, has finite volume and constant scalar curvature -1 away from a compact set.

From now on, we fix a surface with cusps $(\overline{M}, D_M, g^{TM})$ and a Hermitian vector bundle (ξ, h^ξ) over it, and by $\omega_{\overline{M}} := T^{*(1,0)}\overline{M}$ the *canonical line bundle* over $\overline{M}$. We denote by $\|\cdot\|_M^\omega$ the norm on $\omega_{\overline{M}}$ induced by g^{TM} over M by the natural identification $TM \ni X \mapsto \frac{1}{2}(X - \sqrt{-1}JX) \in T^{(1,0)}M$, where J is the complex structure of M. Let $\mathscr{O}_{\overline{M}}(D_M)$ be the line bundle associated with the divisor D_M. The *twisted canonical line bundle* is defined as

$$\omega_M(D) := \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M). \tag{2.3}$$

The metric g^{TM} endows the line bundle $\omega_M(D)$ with the induced Hermitian norm $\|\cdot\|_M$ over M, which has logarithmic singularities near D_M.

Assume first $m = 0$, i.e. the surface has no cusps. Then the *holomorphic analytic torsion* was defined by Ray–Singer [27, Definition 1.2] as the regularized determinant of the Kodaira Laplacian $\Box^{\xi\otimes\omega_M(D)^n}$ associated with (M, g^{TM}) and $(\xi \otimes \omega_M(D)^n, h^\xi \otimes \|\cdot\|_M^{2n})$. More precisely, let $\lambda_i, i \in \mathbb{N}$ be the non-zero eigenvalues of $\Box^{\xi\otimes\omega_M(D)^n}$. By Weyl's law, the associated zeta-function

$$\zeta_M(s) := \sum_{i=0}^{\infty} \frac{1}{\lambda_i^s}, \tag{2.4}$$

is well-defined for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, and it is holomorphic in this region. Moreover, we have

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}\left[\exp^\perp\left(-t\Box^{\xi\otimes\omega_M(D)^n}\right)\right] t^s \frac{dt}{t}, \tag{2.5}$$

where $\exp^\perp(-t\Box^{\xi\otimes\omega_M(D)^n})$ is the spectral projection onto the eigenspace corresponding to the non-zero eigenvalues. Also, as it can be seen by the small-time expansion of the heat kernel and the usual properties of the Mellin transform, cf. [3, §9.6], $\zeta_M(s)$ extends meromorphically to the entire s-plane. This extension is holomorphic at 0, and the *analytic torsion* is defined by

$$T\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right) := \exp\left(-\zeta_M'(0)\right). \tag{2.6}$$

By (2.4) and (2.6), we may interpret the analytic torsion as

$$T\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right) = \prod_{i=0}^{\infty} \lambda_i. \tag{2.7}$$

Now, assume $m > 0$. Then M is non-compact, and, classically, the heat operator associated to $\Box^{\xi\otimes\omega_M(D)^n}$ is no longer of trace class. Also the spectrum of $\Box^{\xi\otimes\omega_M(D)^n}$ is not discrete in general. Thus, neither the definitions (2.5), (2.6), nor the interpretation (2.7) are applicable.

In [16, Definition 2.10], for $n \leqslant 0$, we defined the *regularized heat trace* $\mathrm{Tr}^{\mathbf{r}}[\exp^\perp(-t\Box^{\xi\otimes\omega_M(D)^n}]$ as a "difference" of the heat trace of $\Box^{\xi\otimes\omega_M(D)^n}$ and the heat trace of the Kodaira Laplacian $\Box^{\omega_P(D)^n}$, corresponding to the 3-punctured projective plane $P := \overline{P} \setminus \{0, 1, \infty\}$, $\overline{P} := \mathbb{C}P^1$, endowed with the complete metric g^{TP} of constant scalar curvature -1 and the induced metric $\|\cdot\|_P$ on $\omega_P(D) := \omega_{\overline{P}} \otimes \mathscr{O}_{\overline{P}}(0+1+\infty)$. Then in [16, Definition 2.16], we defined the *regularized spectral zeta function* $\zeta_M(s)$ for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, by

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}^{\mathbf{r}} \left[\exp^{\perp} \left(-t \square^{E_M^{\xi,n}} \right) \right] t^s \frac{dt}{t}. \tag{2.8}$$

We proved in [16, p. 17] that similarly to the case $m = 0$, the function $\zeta_M(s)$ extends meromorphically to $\mathbb{C}$ and 0 is a holomorphic point of this extension. This has led in [16, Definition 2.17] to the definition of the regularized analytic torsion by the following formula

$$T\left(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}\right) := \exp\left(-\zeta_M'(0)/2\right) \cdot T_{TZ}\left(g^{TP}, \|\cdot\|_P^{2n}\right)^{m \cdot \mathrm{rk}(\xi)/3}, \tag{2.9}$$

where $T_{TZ}(\cdot,\cdot)$ is the Takhtajan–Zograf version of the analytic torsion from [29] (which will be recalled later, see (4.20)). In other words, our analytic torsion is defined by subtracting the universal contribution of the cusp from the "heat trace" and by normalizing the result in such a way that it coincides with Takhtajan–Zograf's version of the analytic torsion for (P, g^{TP}).

Then for $n \leqslant 0$, in [16, § 2.1], we endowed the complex line

$$\begin{aligned} &\left(\det H^{\bullet}(\overline{M}, \xi \otimes \omega_M(D)^n)\right)^{-1} \\ &\quad := \left(\Lambda^{\max} H^0(\overline{M}, \xi \otimes \omega_M(D)^n)\right)^{-1} \otimes \Lambda^{\max} H^1(\overline{M}, \xi \otimes \omega_M(D)^n), \end{aligned} \tag{2.10}$$

with the L^2-norm $\|\cdot\|_{L^2}(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n})$ induced by the L^2-scalar product, defined by

$$\begin{aligned} &\langle \alpha, \alpha' \rangle_{L^2} \\ &\quad := \int_M \langle \alpha(x), \alpha'(x) \rangle_h \, dv_M(x), \qquad \alpha, \alpha' \in \mathscr{C}_c^{\infty}\left(M, \xi \otimes \omega_M(D)^n\right), \end{aligned} \tag{2.11}$$

and the analogous scalar product for the $(0,1)$-forms, where dv_M is the Riemannian volume form on (M, g^{TM}), and $\langle \cdot, \cdot \rangle_h$ is the pointwise Hermitian product induced by h^{ξ}, g^{TM}, $\|\cdot\|_M$.

The Quillen metric on the complex line (2.10) is then defined as

$$\begin{aligned} &\|\cdot\|_Q\left(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}\right) = \\ &\qquad T\left(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}\right)^{1/2} \cdot \|\cdot\|_{L^2}\left(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}\right). \end{aligned} \tag{2.12}$$

To motivate, when $m = 0$, this coincides with the usual definition of the Quillen metric from Quillen [26], Bismut–Gillet–Soulé [6, (1.64)] and [8, Definition 1.5].

Our first result computes the newly defined Quillen metric in terms of the Quillen metric for compact surfaces. Following [16], we say that a (smooth) metric g_{f}^{TM} over $\overline{M}$ is a *flattening* of g^{TM} if there is $\nu > 0$ such that g^{TM} is induced by (2.2) over $V_i^M(\nu)$, and

$$g_{\mathrm{f}}^{TM}|_{M \setminus (\cup_i V_i^M(\nu))} = g^{TM}|_{M \setminus (\cup_i V_i^M(\nu))}. \tag{2.13}$$

Similarly, we define a flattening $\|\cdot\|_M^{\mathrm{f}}$ of the norm $\|\cdot\|_M$.

Let $(\overline{N}, D_N, g^{TN})$ be another surface with cusps and let g_{f}^{TN} be a *flattening* of g^{TN}. We say that the flattenings g_{f}^{TM} and g_{f}^{TN} are *compatible*, if

$$\left(\left(z_i^N\right)^{-1} \circ z_i^M \right)^* \left(g_{\mathrm{f}}^{TM}|_{V_i^M(\nu)} \right) = g_{\mathrm{f}}^{TN}|_{V_i^N(\nu)}, \quad \text{for any} \quad i = 1, \ldots, m, \tag{2.14}$$

for some $\nu > 0$, satisfying (2.13) and

$$g_{\mathrm{f}}^{TN}|_{N \setminus (\cup_i V_i^N(\nu))} = g^{TN}|_{N \setminus (\cup_i V_i^N(\nu))}. \tag{2.15}$$

Similarly, we define *compatible flattenings* $\|\cdot\|_M^{\mathrm{f}}, \|\cdot\|_N^{\mathrm{f}}$ for Hermitian norms $\|\cdot\|_M, \|\cdot\|_N$.

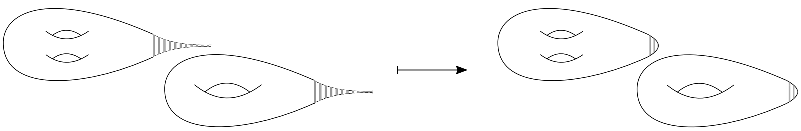

Figure 2.2. An example of compatible flattenings. The striped regions are isometric.

Theorem 2.1 ([16, Theorem A]). — *Let $(\overline{M}, D_M, g^{TM})$, $(\overline{N}, D_N, g^{TN})$ be two surfaces with the same number of cusps. Let (ξ, h^ξ) be a Hermitian vector bundle over $\overline{M}$. Let $\|\cdot\|_M$, $\|\cdot\|_N$ be the norms induced by g^{TM}, g^{TN} on $\omega_M(D)$ and $\omega_N(D)$ over M and N respectively. Let $g_{\mathrm{f}}^{TM}, g_{\mathrm{f}}^{TN}$, $\|\cdot\|_M^{\mathrm{f}}, \|\cdot\|_N^{\mathrm{f}}$ be compatible flattenings of g^{TM}, g^{TN}, $\|\cdot\|_M \|\cdot\|_N$ respectively. For any $n \leqslant 0$, we have*

$$\begin{aligned} 2\log\left(\|\cdot\|_Q \left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right) / \|\cdot\|_Q \left(g_{\mathrm{f}}^{TM}, h^\xi \otimes \left(\|\cdot\|_M^{\mathrm{f}} \right)^{2n} \right) \right) \\ - 2\mathrm{rk}(\xi) \log\left(\|\cdot\|_Q \left(g^{TN}, \|\cdot\|_N^{2n}\right) / \|\cdot\|_Q \left(g_{\mathrm{f}}^{TN}, \left(\|\cdot\|_N^{\mathrm{f}} \right)^{2n} \right) \right) \\ = \int_M c_1\left(\xi, h^\xi\right) \left(2n \log\left(\|\cdot\|_M^{\mathrm{f}} / \|\cdot\|_M \right) + \log\left(g_{\mathrm{f}}^{TM} / g^{TM} \right) \right). \end{aligned} \tag{2.16}$$

In other words, the relative Quillen metric can be computed through a compact perturbation.

To state the next result, explaining how the Quillen metric changes under the change of the metrics g^{TM}, h^{ξ}, we need to recall another natural norm associated with a surface with cusps.

DEFINITION 2.2 ([16, Definition 1.5]). — *For a surface with cusps* $(\overline{M}, D_M, g^{TM})$, *the* Wolpert norms $\|\cdot\|_M^{W,i}$ *on the complex lines* $\omega_{\overline{M}}|_{P_i^M}$, $i = 1, \ldots, m$, *are defined by* $\|dz_i^M\|_M^{W,i} = 1$, *see* (2.2). *They induce the* Wolpert norm $\|\cdot\|_M^W$ *on the complex line* $\otimes_{i=1}^m \omega_{\overline{M}}|_{P_i^M}$.

Remark 2.3. — Wolpert defined this norm in [33] for surfaces of constant scalar curvature to give a geometric interpretation of the Takhtajan–Zograf forms, [29]. The name "Wolpert norm" was coined by Freixas in [18]. For surfaces of non-constant scalar curvature, this metric is still well-defined since Poincaré-compatible coordinates, (2.2), are well-defined up to a unimodular constant.

We recall that the Chern and Todd forms of the Hermitian vector bundle (ξ, h^{ξ}) are defined as

$$\begin{aligned}
&\mathrm{ch}\left(\xi, h^{\xi}\right) = \\
&\mathrm{rk}(\xi) + c_1\left(\xi, h^{\xi}\right) + \tfrac{1}{2} c_1\left(\xi, h^{\xi}\right)^2 - c_2\left(\xi, h^{\xi}\right) + \cdots, \\
&\mathrm{Td}\left(\xi, h^{\xi}\right) = \\
&\mathrm{rk}(\xi) + \tfrac{1}{2} c_1\left(\xi, h^{\xi}\right) + \tfrac{1}{12}\Big(c_1\left(\xi, h^{\xi}\right)^2 + c_2\left(\xi, h^{\xi}\right)\Big) + \cdots,
\end{aligned} \tag{2.17}$$

where the $\cdots$ mean higher degree terms.

Let's recall that by [6, Theorem 1.27], the Bott–Chern classes of a vector bundle ξ with Hermitian metrics h_1^{ξ}, h_2^{ξ} are natural differential forms (strictly speaking, those are classes of differential forms, defined modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}$) satisfying

$$\begin{aligned}
\frac{\partial\overline{\partial}}{2\pi\sqrt{-1}} \widetilde{\mathrm{Td}}\left(\xi, h_1^{\xi}, h_2^{\xi}\right) &= \mathrm{Td}\left(\xi, h_1^{\xi}\right) - \mathrm{Td}\left(\xi, h_2^{\xi}\right), \\
\frac{\partial\overline{\partial}}{2\pi\sqrt{-1}} \widetilde{\mathrm{ch}}\left(\xi, h_1^{\xi}, h_2^{\xi}\right) &= \mathrm{ch}\left(\xi, h_1^{\xi}\right) - \mathrm{ch}\left(\xi, h_2^{\xi}\right).
\end{aligned} \tag{2.18}$$

By [6, Theorem 1.27], we have the following identity

$$\widetilde{\mathrm{ch}}\left(\xi, h_1^{\xi}, h_2^{\xi}\right)^{[0]} = 2\widetilde{\mathrm{Td}}\left(\xi, h_1^{\xi}, h_2^{\xi}\right)^{[0]} = \log\left(\det\left(h_1^{\xi}/h_2^{\xi}\right)\right). \tag{2.19}$$

If, moreover, $\xi := L$ is a line bundle, we have

$$\begin{aligned}
\widetilde{\mathrm{ch}}\left(L, h_1^L, h_2^L\right)^{[2]} &= 6\widetilde{\mathrm{Td}}\left(L, h_1^L, h_2^L\right)^{[2]} \\
&= \log\left(h_1^L/h_2^L\right)\Big(c_1\left(L, h_1^L\right) + c_1\left(L, h_2^L\right)\Big)/2,
\end{aligned} \tag{2.20}$$

where c_1 is the first Chern form.

Theorem 2.4 ([16, Theorem B]). — *Suppose that for the metrics g^{TM}, g_0^{TM}, the triples $(\overline{M}, D_M, g^{TM})$, $(\overline{M}, D_M, g_0^{TM})$ are surfaces with cusps.*

We denote by $\|\cdot\|_M, \|\cdot\|_M^0$ the norms induced by g^{TM}, g_0^{TM} on $\omega_M(D)$, and by $\|\cdot\|_M^W$, $\|\cdot\|_M^{W,0}$ the associated Wolpert norms.

Let h^ξ, h_0^ξ be Hermitian metrics on ξ over $\overline{M}$.

Then the right-hand side of the following equation is finite, and

$$\begin{aligned}(2.21)\quad & 2\log\Big(\|\cdot\|_Q\Big(g_0^{TM}, h_0^\xi\otimes\big(\|\cdot\|_M^0\big)^{2n}\Big)/\|\cdot\|_Q\big(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n}\big)\Big)\\
&= \int_M\Big[\widetilde{\mathrm{Td}}\Big(\omega_M(D)^{-1}, \|\cdot\|_M^{-2}, \big(\|\cdot\|_M^0\big)^{-2}\Big)\,\mathrm{ch}\big(\xi, h^\xi\big)\,\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big)\\
&+\mathrm{Td}\Big(\omega_M(D)^{-1}, \big(\|\cdot\|_M^0\big)^{-2}\Big)\,\widetilde{\mathrm{ch}}\Big(\xi, h^\xi, h_0^\xi\Big)\,\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big)\\
&+\mathrm{Td}\Big(\omega_M(D)^{-1}, \big(\|\cdot\|_M^0\big)^{-2}\Big)\,\mathrm{ch}\Big(\xi, h_0^\xi\Big)\,\widetilde{\mathrm{ch}}\Big(\omega_M(D)^n, \|\cdot\|_M^{2n}, \big(\|\cdot\|_M^0\big)^{2n}\Big)\Big]^{[2]}\\
&-\frac{\mathrm{rk}(\xi)}{6}\log\Big(\|\cdot\|_M^W/\|\cdot\|_M^{W,0}\Big)+\frac{1}{2}\sum\log\Big(\det\Big(h^\xi/h_0^\xi\Big)\Big|_{P_i^M}\Big).\end{aligned}$$

Remark 2.5. — The integral term in the right-hand side of (2.21) is analogous to the term appearing in the anomaly formula of Bismut–Gillet–Soulé [8, Theorem 1.23] (cf. Theorem 2.4 for $m = 0$). Next term is new and it is related to the appearance of the ψ-classes in the Mumford's formula relating λ, κ and δ-classes on the moduli space of pointed curves, cf. [1, Theorem 7.6].

Let's now say a word about the proofs of Theorems 2.1, 2.4. First, Theorem 2.4 is a consequence of Theorem 2.1 and the anomaly formula of Bismut–Gillet–Soulé [8, Theorem 1.23] (cf. Theorem 2.4 for $m = 0$). Indeed, Theorem 2.1 expresses the left-hand side of (2.21) through the ratio of Quillen metrics associated with flattenings of g^{TM}, g_0^{TM}. As the flattenings are compact metrics, this ratio can be computed by the anomaly theorem of Bismut–Gillet–Soulé [8, Theorem 1.23].

Proving Theorem 2.1 is technically more challenging. The main idea can be decomposed in three steps. First, we reduce the problem to the case when (ξ, h^ξ) is trivial around the cusps (then the right-hand side of (2.16) vanishes). To do so, we show in [16, § 3.2] that the Quillen metric behaves continuously under "flattenings" of (ξ, h^ξ) around the cusps. The second step is to prove that the same holds in the relative setting for the flattenings of the metrics g^{TM}, $\|\cdot\|_M$. In other words, we prove in [16, § 3.3] that for a certain family of flattenings $g_{\mathrm{f},\theta}^{TM}$, $\|\cdot\|_M^{\mathrm{f},\theta}$, $\theta\in]0,1]$, approaching g^{TM}, $\|\cdot\|_M$, as $\theta \to 0$, and compatible flattenings $g^{TN}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_N$ of $g^{TN}, \|\cdot\|_N$, we have

$$\lim_{\theta\to 0} \frac{\|\cdot\|_Q\left(g^{TM}_{\mathrm{f},\theta}, h^{\xi}\otimes\left(\|\cdot\|^{\mathrm{f},\theta}_M\right)^{2n}\right)}{\|\cdot\|_Q\left(g^{TN}_{\mathrm{f},\theta}, \left(\|\cdot\|^{\mathrm{f},\theta}_N\right)^{2n}\right)^{\mathrm{rk}(\xi)}} = \frac{\|\cdot\|_Q\left(g^{TM}, h^{\xi}\otimes\|\cdot\|^{2n}_M\right)}{\|\cdot\|_Q\left(g^{TN}, \|\cdot\|^{2n}_N\right)^{\mathrm{rk}(\xi)}}. \tag{2.22}$$

This is done by a very explicit study of the heat kernel associated with $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$ in the neighborhood of the cusps, and it involves, among other things, finding uniform (in θ) elliptic and Sobolev estimates associated with $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$, and proving a uniform spectral gap for the associated Laplacians. In [16, Proposition 3.14], we also prove that the heat kernel associated with $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$ converges to the heat kernel associated with $g^{TM}, \|\cdot\|_M$. To do so we use analytic localization techniques from Bismut–Lebeau [9], based on finite propagation speed of solutions of hyperbolic equations. A specific choice of $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$, described in [16, § 3.5], plays the fundamental role.

The third step consist in noticing that for (ξ, h^{ξ}) trivial around the cusps, the left-hand side of (2.22) is independent on $\theta \in]0,1]$ by [8, Theorem 1.23] (cf. Theorem 2.4 for $m = 0$), which implies that (2.22) holds without taking the limit. This finishes the proof of Theorem 2.1.

Finally, we mention that in [15, Theorem 1.8] we proved a theorem, which generalizes both Theorems 2.1, 2.4. It gives an explicit relation in terms of Bott–Chern classes between the Quillen metric of a metric with cusps and the Quillen metric of a metric on the flattened Riemann surface. The proof of this theorem relies heavily on Theorems 2.1, 2.4 and the results of Section 4.

3. Continuity and curvature theorems

In this section we study the Quillen metric in the family setting. We study the singularities of the Quillen metric for singular families and establish the curvature formula for our version of the Quillen metric.

Let $\pi : X \to S$ be a family of complex curves with at most double point singularities (i.e. those of the form $\{z_1 z_2 = 0 \,|\, z_1, z_2 \in \mathbb{C}\}$). Let $\Sigma_{X/S}$ be the submanifold of singular points of the fibers, and let $\Delta := \pi_*(\Sigma_{X/S})$ be the divisor of singular curves. Assume Δ has normal crossings. Let ξ be a holomorphic vector bundle over X, and let h^{ξ} be a Hermitian metric on ξ over $X \setminus \pi^{-1}(|\Delta|)$.

Let $D_{X/S}$ be a divisor induced by a submanifold $|D_{X/S}| \subset X \setminus \Sigma_{X/S}$, intersecting $\pi^{-1}(|\Delta|)$ transversally and such that $\pi|_{|D_{X/S}|} : |D_{X/S}| \to S$ is locally an isomorphism. Let the norm $\|\cdot\|^{\omega}_{X/S}$ on the canonical line bundle $\omega_{X/S}$ over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$ be such that its restriction over each nonsingular fiber $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ of π induces the Kähler metric with cusps at $|D_{X/S}| \cap X_t$. We endow the twisted relative canonical line bundle $\omega_{X/S}(D) := \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S})$ with the induced Hermitian norm $\|\cdot\|_{X/S}$ over $X \setminus (\pi^{-1}(\Delta) \cup |D_{X/S}|)$.

For $n \leqslant 0$, we endow the line bundle $\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))^{-1}$ with the Quillen metric

$$\|\cdot\|_Q \left(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S}\right), \quad t \in S \setminus |\Delta|, \tag{3.1}$$

over $S \setminus |\Delta|$, defined by gluing the pointwise norms from the previous section.

Let $\det \xi := \Lambda^{\max} \xi$, and let $h^{\det \xi}$ be the metric on $\det \xi$ induced by h^ξ. Denote the norm

$$\begin{aligned} \|\cdot\|_{\mathscr{L}_n} := \left(\|\cdot\|_Q \left(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S}\right)\right)^{12} \otimes \left(\|\cdot\|^{W}_{X/S}\right)^{-\mathrm{rk}(\xi)} \\ \otimes \left(\|\cdot\|^{\mathrm{div}}_{\Delta}\right)^{\mathrm{rk}(\xi)} \otimes \left(\det \left(\pi_* \left(h^{\det \xi}\Big|_{|D_{X/S}|}\right)\right)^3\right) \end{aligned} \tag{3.2}$$

on the line bundle

$$\begin{aligned} \mathscr{L}_n := {} & \\ & \det\left(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n)\right)^{-12} \otimes \left(\det \left(\pi_* \left(\omega_{X/S}\Big|_{|D_{X/S}|}\right)\right)\right)^{-\mathrm{rk}(\xi)} \\ & \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)} \otimes \left(\det \left(\pi_* \left(\det \xi\Big|_{|D_{X/S}|}\right)\right)^6\right). \end{aligned} \tag{3.3}$$

Here $\|\cdot\|^{\mathrm{div}}_{\Delta}$ is the canonical singular norm on $\mathscr{O}_S(\Delta)$. Our first goal is to study the regularity of $\|\cdot\|_{\mathscr{L}_n}$ over $S \setminus |\Delta|$ and its singularities near Δ.

We show that the singularities of $\|\cdot\|_{\mathscr{L}_n}$ are reasonable enough to define the curvature of the Chern connection of $(\mathscr{L}_n, \|\cdot\|_{\mathscr{L}_n})$ as a current on S. Then we compute this current explicitly, which gives us a refinement of Riemann–Roch–Grothendieck theorem on the level of currents.

Note, however, that due to the fact that we work in a non-compact setting, we need to put some additional assumptions on the growth of our metrics near the cusps and singular fibers (at least to ensure that the right-hand side of the curvature formula, see (3.8), makes sense).

For this, recall that for a vector bundle E over a complex manifold Y of dimension q, Burgos Gil–Kramer–Kühn in [11] defined *pre-log-log* Hermitian metric h^E with singularities along a normal crossing divisor $D \subset Y$, as a smooth metric on $Y \setminus D$, verifying the following assumptions. For any local holomorphic frame $e_1, \dots, e_{\mathrm{rk}(E)}$, of E, the functions $h^E(e_i, e_j)$, $(\det(H))^{-1}$, for $H = (h^E(e_i, e_j))_{i,j=1}^{\mathrm{rk}(E)}$, have at most logarithmic singularities along D, and the entries of the matrix $(\partial H)H^{-1}$, as well as its ∂, $\overline{\partial}$ and $\partial\overline{\partial}$ derivatives, have at most log-log singularities along D in the basis $d\zeta_1, \dots, d\zeta_q, d\overline{\zeta_1}, \dots, d\overline{\zeta_q}$, where D in a neighborhood $\{(z_1, \dots, z_q) \in \mathbb{C}^q : |z_i| < 1, \text{ for all } i = 1, \dots, q\}$ can be locally written as $\{z_1 \cdots z_l = 0\}$, and

$$\begin{cases} dz_k/(z_k \log|z_k|), & \text{if} \quad 1 \leqslant k \leqslant l, \\ dz_k, & \text{if} \quad l+1 \leqslant k \leqslant q. \end{cases} \tag{3.4}$$

(3.5) *Suppose that the Hermitian metric h^ξ is pre-log-log with singularities along $\pi^{-1}(\Delta)$, and the Hermitian norm $\|\cdot\|_{X/S}$ is pre-log-logwith singularities along $\pi^{-1}(\Delta) \cup D_{X/S}$.*

This assumption encapsulates interesting geometric phenomena because due to the results of Wolpert, [32] (in the compact case) and Freixas, [18] (in the non-compact case), the main example of degenerating hyperbolic surfaces endowed with constant scalar curvature -1 metric satisfies (3.5) (cf. [17, Proposition 5.7]).

Theorem 3.1 ([17, Theorem C]). — *Under assumption* (3.5)*, the norm $\|\cdot\|_{\mathscr{L}_n}$ is continuous over $S \setminus |\Delta|$, L^1 over S, and for a frame υ of $\mathscr{L}_n$, the ∂, $\overline{\partial}$ and $\partial\overline{\partial}$ derivatives of the L^1-extension of the current* $\log\|\upsilon\|_{\mathscr{L}_n}$*, are given by the integration against continuous forms over $S \setminus |\Delta|$.*

The theorem 3.1 makes it possible to define the first Chern form of $(\mathscr{L}_n, \|\cdot\|_{\mathscr{L}_n})$ by

$$c_1\left(\mathscr{L}_n, \|\cdot\|_{\mathscr{L}_n}^2\right) := \frac{\partial\overline{\partial}\left[\log h^L(\upsilon, \upsilon)\right]_{L^1}}{2\pi\sqrt{-1}}. \tag{3.6}$$

Theorem 3.2 ([17, Theorem D]). — *Under assumption* (3.5)*, the current*

$$\pi_*\left[\mathrm{Td}\left(\omega_{X/S}(D)^{-1}, \|\cdot\|_{X/S}^{-2}\right) \mathrm{ch}\left(\xi, h^\xi\right) \mathrm{ch}\left(\omega_{X/S}(D)^n, \|\cdot\|_{X/S}^{2n}\right)\right]^{(2,2)} \tag{3.7}$$

is $L^1_{loc}(S)$. We denote by the same symbol the L^1-extension of this current over S. This extension is closed. Moreover, the following identity of currents over S holds

$$(3.8)\quad c_1\big(\mathscr{L}_n, \|\cdot\|_{\mathscr{L}_n}^2\big)$$
$$= -12\pi_*\Big[\mathrm{Td}\left(\omega_{X/S}(D)^{-1}, \|\cdot\|_{X/S}^{-2}\right)\mathrm{ch}\left(\xi, h^{\xi}\right)\mathrm{ch}\left(\omega_{X/S}(D)^{n}, \|\cdot\|_{X/S}^{2n}\right)\Big]^{(2,2)}.$$

Remark 3.3. — By Chern–Weil theory, the de Rham class of $c_1(\mathscr{L}_n, \|\cdot\|_{\mathscr{L}_n}^2)$ coincides with $c_1(\mathscr{L}_n)$. So (3.8) refines Riemann–Roch–Grothendieck theorem on the level of currents.

Let's say a word about the way we proved those theorems. As their statements are local over the base, we may suppose that the base is a small ball. Then by Theorem 2.4, we may relate the norm $\|\cdot\|_{\mathscr{L}_n}$ with the analogous norm associated with a metric which is "constant" in the horizontal direction in the neighborhood of the cusps. Then Theorem 2.1 permits us to compare the Quillen metric associated with this metric and the Quillen metric associated with the "flattened" metric.

Now, the anomaly formula of Bismut–Gillet–Soulé [8, Theorem 1.23] (cf. Theorem 2.4 for $m = 0$) describes the *defect* up to which the Quillen metric associated with the "flattened" metric coincides with the Quillen metric associated with some smooth metrics on ξ, $\omega_{X/S}$. Once the singularities of this *defect* are studied, Theorem 3.1 reduces to a result of Bismut–Bost [5, Théorème 2.2], stating that Theorem 3.1 holds for families with no cusps and no degeneration of the metrics h^{ξ} and $\|\cdot\|_{X/S}^{\omega}$ in the neighborhood of $\pi^{-1}(|\Delta|)$. To get further Theorem 3.2, we need to analyze the derivatives of the mentioned *defect*. This is done by developing potential theory for currents with log-log type singularities, which reduces Theorem 3.2 to the curvature theorem of Bismut–Gillet–Soulé [8, Theorem 1.9].

4. Understanding the Quillen metric on the singular fibers

It turns out that under some stronger hypothesizes, the norm $\|\cdot\|_{\mathscr{L}_n}$ on $\mathscr{L}_n$ from Section 3 actually extends continuously over S (including the singular locus $|\Delta|$). The main goal of this section is to give the precise value of this continuous extension, and give a geometric interpretation of the values of this extension over the singular locus of the family as the Quillen metric of the normalization of a singular fiber.

More precisely, as Δ has normal crossings, by shrinking the base S, we may always assume that for any $t \in S$, there is $l \in \mathbb{N}$, so that the divisor Δ decomposes near t as

$$(4.1)\qquad \Delta = k \cdot \Delta_0 + k_1 \cdot \Delta_1 + \cdots + k_l \cdot \Delta_l,$$

where Δ_i, $i = 0, \ldots, l$, are divisors induced by the submanifolds $|\Delta_i|$ and $k, k_j \in \mathbb{N}^*$, $j = 1, \ldots, l$. Let $\Delta_j^0 := \Delta_j \cap \Delta_0$ be the induced divisor on $S' := |\Delta_0|$, and let Δ' be the divisor on S' given by

$$\Delta' := k_1 \cdot \Delta_1^0 + \cdots + k_l \cdot \Delta_l^0. \tag{4.2}$$

Let $\iota : S' \to S$ be the obvious inclusion. We denote $Z := \pi^{-1}(S')$, $Z_t := \pi^{-1}(t)$, $t \in S'$, and let $\rho : Y \to Z$ be the normalization of Z. We denote by $\pi' : Y \to S'$ the family of curves, induced by the following commutative square

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\iota} & S \end{array} \tag{4.3}$$

The restriction of the holomorphic sections $\sigma_1, \ldots, \sigma_m$ on S' induce the holomorphic sections of Y, which we denote by $\sigma'_1, \ldots, \sigma'_m : S' \to Y$.

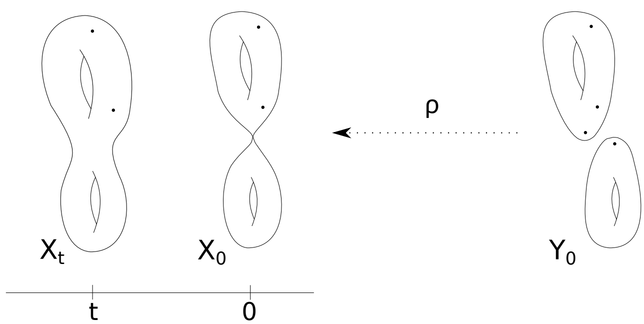

Figure 4.1. Singular family and the normalization of the singular fiber. Notice marked points on Y_0.

Let $\Sigma_{Z/S'}$ be the locus of points, normalized in ρ. The manifold $\Sigma_{Z/S'}$ is a union of connected components of $\Sigma_{X/S}$. In particular, $\Sigma_{Z/S'}$ has codimension 2 in X. Let

$$\kappa : \Sigma_{Z/S'} \hookrightarrow X \tag{4.4}$$

the obvious inclusion.

Then the restriction of π' to $\rho^{-1}(\kappa(\Sigma_{Z/S'}))$ is the covering map of degree $2k$, see (4.1). By shrinking the base, we may suppose that it is a trivial cover, so there are holomoprhic sections $\sigma'_{m+1}, \ldots, \sigma'_{m+2k} : S' \to Y$

such that $\rho^{-1}(\Sigma_{Z/S'}) = \cup_{i=1}^{2k} \operatorname{Im}(\sigma'_{m+i})$ and $\rho \circ \sigma'_{m+2i-1} = \rho \circ \sigma'_{m+2i}$, $i = 1, \ldots, k$. We define the divisor $D_{Y/S'}$ over Y by

$$D_{Y/S'} := \operatorname{Im}(\sigma'_1) + \cdots + \operatorname{Im}\left(\sigma'_{m+2k}\right). \tag{4.5}$$

We also define the *twisted canonical line bundle* of π' as follows

$$\omega_{Y/S'}(D) := \omega_{Y/S'} \otimes \mathscr{O}_Y\left(D_{Y/S'}\right). \tag{4.6}$$

Then, we have the canonical isomorphism (cf. Section 2)

$$\rho^*\left(\omega_{X/S}(D)\right) \simeq \omega_{Y/S'}(D). \tag{4.7}$$

The notion of *good vector bundle* with singularities along $\pi^{-1}(\Delta) \cup D_{X/S}$ was defined by Mumford in [24]. It is slightly stronger than the pre-log-log assumption from (3.5), as it requires the singularities from the definition of pre-log-log line bundle to be *bounded* instead of *log-log*.

(4.8) *We assume that h^{ξ} extends smoothly over X; the norm $\|\cdot\|_{X/S}$ is continuous over $X \setminus (\Sigma_{X/S} \cup |D_{X/S}|)$, has log-log growth with singularities along $\Sigma_{X/S} \cup |D_{X/S}|$, is good in the sense of Mumford on X with singularities along $\pi^{-1}(\Delta) \cup D_{X/S}$, and the coupling of $c_1(\omega_{X/S}(D), \|\cdot\|^2_{X/S})$ with two continuous vertical vector fields over $X \setminus (\Sigma_{X/S} \cup |D_{X/S}|)$ is continuous over $X \setminus (\Sigma_{X/S} \cup |D_{X/S}|)$.*

Under assumptions (4.8), the isomorphism (4.7) induces the Hermitian norm $\|\cdot\|_{Y/S'}$ on $\omega_{Y/S'}(D)$ over $Y \setminus ((\pi')^{-1}(|\Delta'|) \cup |D_{Y/S'}|)$ by

$$\|\cdot\|_{Y/S'} := \rho^*\left(\|\cdot\|_{X/S}\right). \tag{4.9}$$

Let $\|\cdot\|^{\omega}_{Y/S'}$ be the norm on $\omega_{Y/S'}$, induced by $\|\cdot\|_{Y/S'}$ over $Y \setminus ((\pi')^{-1}(|\Delta'|) \cup |D_{Y/S'}|)$.

(4.10) *We suppose that the norm $\|\cdot\|^{\omega}_{Y/S'}$ over $Y \setminus ((\pi')^{-1}(|\Delta'|) \cup |D_{Y/S'}|)$ is such that its restriction over nonsingular fibers $Y_t := (\pi')^{-1}(t)$, $t \in S' \setminus |\Delta'|$, of π' induces Kähler metric g^{TY_t}, for which the triple $(Y_t, \{\sigma'_1(t), \ldots, \sigma'_{m+2k}(t)\}, g^{TY_t})$ is a surface with cusps.*

Essentially the assumption (4.10) says that the cusps on the normalization of the singular fibers are produced either by the extension of the existing cusps or by degeneration.

Again, the assumptions (4.8), (4.10) encapsulate interesting geometric phenomena because by Wolpert, [32] (in the compact case) and Freixas, [18] (in the non-compact case), the example of degenerating hyperbolic surfaces endowed with constant scalar curvature -1 metric satisfies them (cf. [17, Proposition 5.7]).

We denote by $\|\cdot\|_{Y/S'}^{W}$ the Wolpert norm on $\otimes_{i=1}^{m+2k}(\sigma_i')^*\omega_{Y/S'}$, induced by $\|\cdot\|_{Y/S'}^{\omega}$ (it is well-defined by the assumption (4.10)). Now, by (4.10), similarly to (3.2), (3.3), we define the norm

$$\begin{aligned}\|\cdot\|_{\mathscr{L}_n'} := \Big(\|\cdot\|_Q\Big(g^{TY_t}, \rho^*\big(h^{\xi}\big)\otimes\|\cdot\|_{Y/S'}^{2n}\Big)\Big)^{12}\otimes\Big(\|\cdot\|_{Y/S'}^{W}\Big)^{-\mathrm{rk}(\xi)}\\ \otimes\big(\|\cdot\|_{\Delta'}^{\mathrm{div}}\big)^{\mathrm{rk}(\xi)}\otimes\big(\otimes_{i=1}^{m+2k}(\sigma_i'\circ\rho)^*h^{\det\xi}\big)^3\end{aligned} \tag{4.11}$$

on the line bundle

$$\begin{aligned}\mathscr{L}_n' := \lambda\,\big(j^*(\rho^*(\xi)\otimes\omega_{Y/S'}(D)^n)\big)^{12}\otimes\big(\otimes_{i=1}^{m+2k}(\sigma_i')^*\omega_{Y/S'}\big)^{-\mathrm{rk}(\xi)}\\ (\Delta')^{\mathrm{rk}(\xi)}\otimes\big(\otimes_{i=1}^{m+2k}(\sigma_i'\circ\rho)^*\det\xi\big)^6.\end{aligned} \tag{4.12}$$

By using Poincaré residue morphism (cf. [21, p. 147]) and the fact that the determinant of Grothendick–Knudsen–Mumford [22] is an exact functor, we have the canonical isomorphism (cf. [15, (1.29)]), which is the protagonist of this section

$$\mathscr{L}_n|_{S'} \to \mathscr{L}_n'\otimes\big(\det\pi_*\mathscr{O}_{\Sigma_{Z/S'}}\big)^{12\cdot\mathrm{rk}(\xi)}. \tag{4.13}$$

For $k\in\mathbb{N}^*$, we define

$$\begin{aligned}C_0 &= -6\log(\pi),\\ C_k &= -6(1+k)\log(2)-6(1+2k)\log(\pi)-6\log((2k)!).\end{aligned} \tag{4.14}$$

Theorem 4.1 ([17, Theorem C3] and [15, Theorem 1.2]). — *Under the assumption* (4.8)*, the norm* $\|\cdot\|_{\mathscr{L}_n}$ *extends continuously over* S*. If, moreover, we require* (4.10)*, then under* (4.13)*, we have*

$$\|\cdot\|_{\mathscr{L}_n}|_{S'} = \exp(k\cdot\mathrm{rk}(\xi)\cdot C_{-n})\cdot\|\cdot\|_{\mathscr{L}_n'}. \tag{4.15}$$

The proof of this theorem, given in [15, § 3.1], relies on Theorems 2.1, 2.4, a result of Bismut [4, Theorems 0.2, 0.3] (describing the behaviour of Quillen metric for families endowed with smooth metric coming from a Kähler metric on the total space of the family) and the results of Freixas from [19], [20] (establishing Theorem 4.1 for a specific family of surfaces, endowed with complete constant scalar curvature metrics, and using a different definition for the analytic torsion).

Now, let's describe how this result implies the compatibility of our definition of the analytic torsion and the definition of Takhtajan–Zograf, [29]. We describe their definition first.

Fix a compact Riemann surface $\overline{M}$ and $D_M \subset \overline{M}$, $\#D_M = m$, $m < +\infty$. We denote $M := \overline{M} \setminus D_M$. Suppose that $(\overline{M}, D_M)$ is stable, i.e. the genus $g(\overline{M})$ of $\overline{M}$ satisfies

$$2g(\overline{M}) - 2 + m > 0, \tag{4.16}$$

then, by the uniformization theorem (cf. [14, Chapter IV], [2, Lemma 6.2]), there is exactly one complete metric $g^{TM}_{\rm hyp}$ of constant scalar curvature -1 on M with cusps at D_M. We call this metric the *canonical hyperbolic metric*. We denote by $\|\cdot\|^{\rm hyp}_M$ the norm induced by $g^{TM}_{\rm hyp}$ on $\omega_M(D)$ over M. Then, as we explain in [16, § 2.1], the triple $(\overline{M}, D_M, g^{TM}_{\rm hyp})$ is a surface with cusps (see Section 2), in particular, the analytic torsion $T(g^{TM}_{\rm hyp}, (\|\cdot\|^{\rm hyp}_M)^{2n})$ is well-defined.

We denote by $Z_{(\overline{M},D_M)}(s), s \in \mathbb{C}$ the Selberg zeta-function, given for $\mathrm{Re}(s) > 1$ by

$$Z_{(\overline{M},D_M)}(s) = \prod_{\gamma} \prod_{k=0}^{\infty} \Big(1 - e^{-(s+k)l(\gamma)}\Big)^2, \tag{4.17}$$

where γ runs over the set of all primitive non-oriented closed geodesics on $(M, g^{TM}_{\rm hyp})$, and $l(\gamma)$ is the length of γ. The function $Z_{(\overline{M},D_M)}(s)$ admits a meromorphic extension to the whole complex s-plane with a simple zero at $s = 1$ (see for example [12, (5.3)]).

Let $\zeta(s) := \sum_{k=1}^{\infty} k^{-s}$ be the Riemann zeta function. For $k \in \mathbb{N}^*$, we put

$$\begin{aligned} c_0 &= 4\zeta'(-1) - \tfrac{1}{2} + \log(2\pi), \\ c_k &= \textstyle\sum_{l=0}^{k-1} (2k-2l-1)\big(\log\big(2k+2kl-l^2-l\big) - \log(2)\big) + \\ &+ (2k+1)\log(2\pi) + 4\zeta'(-1) - 2(k+\tfrac{1}{2})^2 - 4\textstyle\sum_{l=1}^{k-1}\log(l!) - 2\log(k!). \end{aligned} \tag{4.18}$$

For $k \in \mathbb{N}$, we denote by $B_k : \mathbb{N}^2 \to \mathbb{R}$, $E : \mathbb{N}^2 \to \mathbb{R}$ the following functions

$$\begin{aligned} B_k(g,m) &= \exp\left(\big(2 - 2g(\overline{M}) - m\big)\,\frac{c_k}{2}\right), \\ E(g,m) &= \exp\left(\big(g(\overline{M}) + 2 - m\big)\,\frac{\log(2)}{3}\right). \end{aligned} \tag{4.19}$$

For surfaces of constant scalar curvature -1 and (ξ, h^ξ) trivial, for $l \in \mathbb{Z}$, $l < 0$, Takhtajan–Zograf in [29, (6)] proposed the analogue[(1)] of the analytic torsion defined via (4.17) by

$$\begin{aligned} T_{TZ}\left(g^{TM}_{\mathrm{hyp}}, 1\right) &= E\left(g(\overline{M}), m\right) \cdot B_0\left(g(\overline{M}), m\right) \cdot Z'_{(\overline{M}, D_M)}(1), \\ T_{TZ}\left(g^{TM}_{\mathrm{hyp}}, \left(\|\cdot\|^{\mathrm{hyp}}_M\right)^{2l}\right) &= B_{-l}\left(g(\overline{M}), m\right) \cdot Z_{(\overline{M}, D_M)}(-l+1). \end{aligned} \tag{4.20}$$

Theorem 4.2 (Compatibility theorem). — *For any surface with cusps $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$, for which g^{TM}_{hyp} has constant scalar curvature -1, the following identity holds*

$$T\left(g^{TM}_{\mathrm{hyp}}, \left(\|\cdot\|^{\mathrm{hyp}}_M\right)^{2n}\right) = T_{TZ}\left(g^{TM}_{\mathrm{hyp}}, \left(\|\cdot\|^{\mathrm{hyp}}_M\right)^{2n}\right). \tag{4.21}$$

For $m = 0$, i.e. when surfaces have no cusps, Theorem 4.2 was proved by D'Hoker–Phong [12, (7.30)], [13, (3.6)] (see also [28], [10, (50)] and [25, (9)]). Our proof is based on their result. In fact, we obtain Theorem 4.2 by studying the "limit" of the identity (4.21) on a special degenerating family of surfaces without cusps, obtained from desingularization of the clutching of M with a number of fixed 1-punctured tori, attached to the points D_M. By the results of D'Hoker–Phong, our definitions of Quillen metric coincide on the nonsingular fibers of this family. The continuous extension at the singular fiber of our definition coincides with our version of the Quillen metric of the normalization by Theorem 4.1. This norm of the normalization contains, in particular, the left-hand side of (4.21). The same continuity theorem holds for the Quillen metric defined using Takhtajan–Zograf's version of the analytic torsion by the results of Freixas, [19], [20] (which are based on the precise study of the degeneration of the Selberg zeta function, done by Wolpert [31]). By using those results, in [15, § 3.3], after passing to the limit in (4.21), as one approaches the singular fiber of the family, we get Theorem 4.2.

Now we can finally describe precisely the relation between Theorem 3.2 and the curvature theorem of Takhtajan–Zograf [29]. Consider a universal family $\pi : \overline{\mathscr{C}}_{g,m} \to \overline{\mathscr{M}}_{g,m}$ of m-pointed stable curves of genus g. Denote by ω_π the relative canonical line bundle of π, and by $\omega_\pi(D)$ the associated twisted relative canonical line bundle.

Endow the non-singular fibers of this family with a metric of constant scalar curvature -1, and denote by $\|\cdot\|^{\omega}_{\pi}$ (resp. $\|\cdot\|_{\pi}$) the induced norm

(1) The constant in front of Selberg zeta function didn't appear in [29], as their result is independent of it. This normalization was introduced by Freixas in [19, Definition 2.2] and [20, Definition 4.2].

on ω_π (resp. $\omega_\pi(D)$). This data induces the Quillen metric on the line bundles $\det(R^\bullet \pi_* \omega_\pi(D)^n)$ for $n \leqslant 0$ by Section 2. We denote by $\mathscr{L}_n^{g,m}$ the adaptation of the line bundle (3.3) to current situation, and by $\|\cdot\|_{\mathscr{L}_n^{g,m}}$ the associated norm, (3.2).

Theorem 3.2 in this situation, expresses the first Chern form of $(\mathscr{L}_n^{g,m}, \|\cdot\|_{\mathscr{L}_n^{g,m}})$ in terms of the pushforward of $c_1(\omega_\pi(D), (\|\cdot\|_\pi^\omega)^2)^2$. This pushforward was identified by Wolpert in [30, Corollary 5.11] as a multiple of the Weil–Petersson form. By those results, the compatibility of our definitions, given by Theorem 4.2, and another result of Wolpert, [33, Theorem 5], which expresses the curvature of the Wolpert norm through Takhtajan–Zograf's forms, we see that that Theorem 3.2 in this particular case, restricted to non-singular locus, gives exactly the curvature theorem of Takhtajan–Zograf [29, Theorem 1], relating the curvature of the Quillen norm, defined through (4.20), the Weil–Petersson form and the Takhtajan–Zograf's forms. So, we may say that our curvature theorem extends the result of Takhtajan–Zograf to the "boundary" $\partial \mathscr{M}_{g,m} := \overline{\mathscr{M}}_{g,m} \setminus \mathscr{M}_{g,m}$ of the moduli space $\mathscr{M}_{g,m}$.

In context of K3 surfaces with involution, similar results are obtained by Yoshikawa in [34, 35].

We believe that an advantage of our approach (besides the fact that it gives a result in the case of non-constant scalar curvature) is that by introducing a more general definition of the analytic torsion for surfaces, and studying precisely the relation between this definition and the classical definition of Ray–Singer, we were able to put the curvature theorem of Takhtajan–Zograf in the framework of the curvature theorem of Bismut–Gillet–Soulé and related results from Quillen metric theory. This gives us a hope that our results can be extended to higher dimensions.

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