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# A GENERALIZATION OF FRENET'S FRAME FOR NON-DEGENERATE QUADRATIC FORMS WITH ANY INDEX

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## 1. Introduction

### 1.1. Statement of the problem

Let us consider a curve  $c: I \rightarrow \mathbb{R}^n$ ,  $I$  being an open interval of  $\mathbb{R}$ , and  $\mathbb{R}^n$  furnished with its Euclidian structure.

We suppose the curve  $c$  to be regular, in the sense that the iterative derivatives  $c^{(1)}(t), \dots, c^{(n)}(t)$  are independent vectors for all  $t \in I$ . Under this assumption, it is well known that one may build, through a Gram-Schmidt orthonormalization process, a particular frame associated to the curve  $c$ , called the Frenet's frame (for details, cf. [Sp] for example), and deduce from that curvature and torsion.

In fact, we may restrict our hypothesis of regularity into a weaker one, which we shall call  $r$ -regularity (for  $1 \leq r \leq n$ ):  $c^{(1)}(t), \dots, c^{(r)}(t)$  are independent, and  $c^{(r+1)}(t) \in F_r(t) := \langle c^{(1)}(t), \dots, c^{(r)}(t) \rangle$ ,  $\forall t \in I$ .

Indeed, it is clear that, in this case, the subspace  $F_r(t)$  is independent of  $t$  (we can see this using Taylor formulae), and thus, we may identify  $F_r$  with  $\mathbb{R}^r$ , so that we are led to the former case.

Our aim in this article is to generalize this construction of a canonical frame field associated to each sufficiently regular curve in  $\mathbb{R}^n$  furnished with an arbitrary non degenerate quadratic form.

REMARK 1.1 Some constructions were already studied in particular cases (see for example [Y-C W] and [D], chapter 13 problem 8 p. 329). Also, some authors introduce auxiliary datas along the curve in order to manage with isotropic vectors (see for example [D-B] chapter 3 and [D-J]). Here, we want to focus on a "canonical" construction, without any auxiliary choices, which applies to any Minkowski space and more generally to any pseudo-Riemannian manifold (see chapter 9).

### 1.2. Some definitions and notations

Let us consider a curve  $c: I \rightarrow \mathbb{R}^n$ , where  $\mathbb{R}^n$  is furnished with a fixed non-degenerate quadratic form  $\langle \cdot, \cdot \rangle$ . In all this paper, for any  $t \in I$ , we will denote by  $F_k(t)$  the space generated by the iterative derivatives  $c^{(1)}(t), \dots, c^{(k)}(t)$ , and by  $g_k(t)$  the Gram's determinant of these vectors, i.e. the determinant of the  $(k, k)$  matrix  $(\langle c^{(i)}(t), c^{(j)}(t) \rangle)$ ,  $i, j \in \{1, \dots, k\}$ .

DEFINITION 1.2 A curve will be said " $r$ -pseudo-regular" if

1. is  $r$ -regular, i.e.  $F_{r-1} \subsetneq F_r \equiv F_{r+1}$
2. for all  $k \leq r$ , the function  $g_k$  is either positive, identically zero, or negative.

From now on, the curve  $c$  will always be assumed pseudo-regular.

We may then define a strictly increasing finite sequence  $(a_k)$  (with  $a_0 = 0$ ), corresponding to the successive integers such that  $g_{a_k} \neq 0$ .

We denote by  $b_k := a_{k+1} - a_k$ . Then, for any integer  $i$  from 1 to  $b_k - 1$ , we must have  $g_{a_k+i} \equiv 0$ , so that the spaces  $F_{a_k+i}$  are degenerate for the restricted form.

We denote by  $K_{a_k+i}$  the kernel of the restricted form on  $F_{a_k+i}$ .

We will prove in the next chapter that we have only 3 possibilities

$$K_{a_k+i} \begin{cases} \subseteq K_{a_k+i+1} \\ \equiv K_{a_k+i+1} \\ \supseteq K_{a_k+i+1} \end{cases} \quad \text{with respectively} \quad \dim K_{a_k+i+1} = \begin{cases} \dim K_{a_k+i} + 1 \\ \dim K_{a_k+i} \\ \dim K_{a_k+i} - 1 \end{cases}$$

It allows us to define a sequence  $(d_k)$  by supposing that

- we have a strictly increasing sequence  $K_{a_k+1} \subseteq K_{a_k+2} \subseteq \dots \subseteq K_{a_k+d_k}$
- this sequence is maximal, i.e.  $K_{a_k+d_k} \not\subseteq K_{a_k+d_k+1}$ , so that

$$K_{a_k+d_k} \equiv K_{a_k+d_k+1} \quad \text{or} \quad K_{a_k+d_k} \subseteq K_{a_k+d_k+1}.$$

Now, it is clear that we have a direct sum decomposition

$$F_{a_k+i} = F_{a_k} \oplus \underbrace{K_{a_k+i}}_{\text{kernel}} \quad \forall 1 \leq i \leq d_k.$$

Finally, the last notations we need in this article are the following:

◊ we denote by  $k_{\max}$  the unique integer such that the last term of the sequence  $(a_k)$  is  $a_{k_{\max}}$ , i.e.  $k_{\max}$  satisfies  $g_i \equiv 0$  for any integer  $i$  from  $a_{k_{\max}} + 1$  to  $r$ , and  $g_{a_{k_{\max}}} \neq 0$ .

◊ by convention, we will denote by  $b_{a_{k_{\max}}}$  the integer  $r - a_{k_{\max}}$ .

Let us remark that:

1.  $k_{\max}$  may be equal to 0 (with our convention  $g_0 = 1$ ). In this case, where all subspaces  $F_i$  are degenerate, the curve  $c$  is said totally isotropic.
2.  $g_r$  may be null or not. In fact, it is clear that  $g_r \neq 0 \iff r = a_{k_{\max}}$ . Then we will have to distinguish two cases, according to  $r = a_{k_{\max}}$  or not.

### 1.3. Statement of the main result

With the notations above, we will prove that  $b_k = 2d_k + 1$ , and that we have

$$K_{a_k+d_k+1} \equiv K_{a_k+d_k}.$$

More precisely, we will obtain the following sequence:

$$K_{a_k+1} \subseteq \dots \subseteq K_{a_k+d_k} \equiv K_{a_k+d_k+1} \supsetneq K_{a_k+d_k+2} \supseteq \dots \supseteq K_{a_k+b_k-1}.$$

Finally, we are able to construct a basis adapted to the moving flag  $F_1 \subset F_2 \subset \dots \subset F_r$ , canonical in a sense that we will precise in chapter 3, given by the following theorem:

**THEOREM 1.3** *Given a  $r$ -regular, pseudo-regular, curve  $c: I \rightarrow \mathbb{R}^n$ , and a non-degenerate quadratic form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^n$ , there exists a unique moving basis  $\{v_1(t), \dots, v_r(t)\}$  on  $F_r(t)$  with the following properties:*

1.  $\{v_1(t), \dots, v_r(t)\}$  is adapted to the flag  $F_1 \subset \dots \subset F_r$ , i.e.  $F_i(t)$  is generated by  $\{v_1(t), \dots, v_i(t)\}$  for any integer  $i$  from 1 to  $r$ , and any  $t \in I$
2. the  $(r, r)$  matrix  $U$  of the restriction of  $\langle \cdot, \cdot \rangle$  to  $F_r$  with respect to the basis  $\{v_1, \dots, v_r\}$  is

$$U = \begin{pmatrix} \mathcal{U}_0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \dots & \dots & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots & \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots & \\ \vdots & \dots & \dots & \ddots & \mathcal{U}_{k_{\max}-1} & 0 & \\ 0 & \dots & \dots & \dots & 0 & 0_{r-a_{k_{\max}}} & \end{pmatrix}$$

where  $\mathcal{U}_k = (-1)^{d_k} \epsilon_k U_k$ ,  $\epsilon_k = \pm 1$  is the sign of  $g_{a_{k+1}} (g_{a_k})^{-1}$  (remark that  $g_{a_k} \neq 0$  for  $0 \leq k \leq k_{\max} - 1$ ),  $U_k$  the  $(b_k, b_k)$  matrix defined by

$$U_k = \begin{pmatrix} 0 & \dots & \dots & \dots & \dots & 0 & (-1)^{d_k} \\ \vdots & & & & 0 & / & 0 \\ \vdots & & & 0 & -1 & 0 & \vdots \\ \vdots & & 0 & 1 & 0 & & \vdots \\ \vdots & 0 & -1 & 0 & & & \vdots \\ 0 & / & & & & \vdots & \\ (-1)^{d_k} & 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

and  $0_{r-a_{k_{\max}}}$  is the null matrix of type  $(r - a_{k_{\max}}, r - a_{k_{\max}})$

3. the moving basis  $\{v_1(t), \dots, v_r(t)\}$  satisfies  $V' = \Delta V$ , i.e.

$$\begin{pmatrix} v'_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v'_r \end{pmatrix} = \begin{pmatrix} \Delta_0 & Lc_0 & 0 & \dots & \dots & \dots & 0 \\ \widetilde{Lc}_0 & \Delta_1 & Lc_1 & \ddots & \dots & \dots & \vdots \\ 0 & \widetilde{Lc}_1 & \ddots & \ddots & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \widetilde{Lc}_{h-1} & \Delta_h & Lc_h \\ 0 & \dots & \dots & \dots & \dots & 0 & \Gamma \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v_r \end{pmatrix}$$

with  $h = k_{\max} - 1$ , and the  $(r,r)$  matrix  $\Delta$  admits a block-decomposition which main diagonal is made with:

(i)  $\Delta_i$  is a  $(b_i, b_i)$  matrix,  $b_i = 2d_i + 1$ , with a decomposition

$$\begin{pmatrix} T_i & O_{d_i} \\ D_i & \widetilde{T}_i \end{pmatrix}$$

where

•  $T_i$  is the  $(d_i, d_i + 1)$  matrix with

$$T_i = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

•  $\widetilde{T}_i$  is the  $(d_i + 1, d_i)$  matrix with

$$\widetilde{T}_i = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

•  $D_i$  is a  $(d_i + 1, d_i + 1)$  matrix depending on  $d_i$  functions  $Y_{i,j}$  ( $j=1, \dots, d_i$ ) on  $I$ , in the form

$$\begin{pmatrix} 0 & \dots & \dots & \dots & 0 & Y_{i,d_i} & 0 \\ 0 & \dots & \dots & 0 & Y_{i,d_i-1} & 0 & Y_{i,d_i} \\ 0 & \dots & 0 & Y_{i,d_i-2} & 0 & Y_{i,d_i-1} & 0 \\ 0 & \dots & / & 0 & / & 0 & \dots \\ 0 & / & 0 & / & 0 & \dots & \dots \\ Y_{i,1} & 0 & Y_{i,2} & 0 & \dots & \dots & 0 \\ 0 & Y_{i,1} & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

(ii)  $\Gamma$  is a  $(r - a_{k_{\max}}, r - a_{k_{\max}})$  matrix depending on  $r - a_{k_{\max}} - 1$  functions  $\zeta_i$  ( $i = 1, \dots, r - a_{k_{\max}} - 1$ ) on  $I$ , in the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & \dots & \dots & & 1 \\ \zeta_1 & \zeta_2 & \dots & \zeta_{r-a_{k_{\max}}-1} & 0 \end{pmatrix}$$

Notice that there is no matrix  $\Gamma$  if (and only if)  $a_{k_{\max}} = r$ .

(iii) •  $Lc_i = \varkappa_i L_i$ , where  $L_i$  is the unique  $(b_i, b_{i+1})$  matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

and  $\varkappa_i$  is a positive function on  $I$ .

•  $\tilde{L}c_i = -\varepsilon_i \varepsilon_{i+1} \varkappa_i \tilde{L}_i$ , where  $\tilde{L}_i$  is the unique  $(b_{i+1}, b_i)$  matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

and the function  $\varkappa_i$  is the same as in the matrix  $Lc_i$ .

## 2. A preliminary study of kernels

Let us recall that for  $1 \leq i \leq b_k - 1$ , the space  $F_{a_k+i}$  is degenerate for the form  $\langle \cdot, \cdot \rangle$ , with kernel  $K_{a_k+i}$ . Since  $F_{a_k+i+1} = F_{a_k+i} \oplus \langle c^{(a_k+i+1)} \rangle$ , it is clear that we have either

$$\dim K_{a_k+i+1} = \dim K_{a_k+i} + 1, \quad \dim K_{a_k+i+1} = \dim K_{a_k+i}, \quad \text{or} \quad \dim K_{a_k+i+1} = \dim K_{a_k+i} - 1.$$

More precisely, we have the following result:

LEMMA 2.1 *Either  $K_{a_k+i} \subset K_{a_k+i+1}$  or  $K_{a_k+i} \equiv K_{a_k+i+1}$  or  $K_{a_k+i} \supset K_{a_k+i+1}$ .*

*Proof.* If  $c^{(a_k+i+1)}$  is orthogonal to  $F_{a_k+i}$ , we then have either  $\langle c^{(a_k+i+1)}, c^{(a_k+i+1)} \rangle = 0$ , in which case  $K_{a_k+i+1} = K_{a_k+i} \oplus \langle c^{(a_k+i+1)} \rangle$ , or  $|c^{(a_k+i+1)}|^2 \neq 0$ , and then  $K_{a_k+i} \equiv K_{a_k+i+1}$ . Now, if  $c^{(a_k+i+1)}$  is not orthogonal to  $F_{a_k+i}$ , let  $z \in K_{a_k+i+1}$ . We have  $z = y + \lambda c^{(a_k+i+1)}$ , with  $y \in F_{a_k+i}$ , and  $\lambda \in \mathbb{R}$ . We have  $\langle y + \lambda c^{(a_k+i+1)}, x \rangle = 0 \forall x \in F_{a_k+i+1}$ ; particulary, for  $x \in K_{a_k+i}$ , we obtain  $\langle \lambda c^{(a_k+i+1)}, x \rangle = 0$ . Thus, if  $c^{(a_k+i+1)}$  is orthogonal to  $K_{a_k+i}$ , we have  $K_{a_k+i} \subseteq K_{a_k+i+1}$ . If  $c^{(a_k+i+1)}$  is not orthogonal to  $K_{a_k+i}$ , the equality

$\langle \lambda c^{(a_k+i+1)}, x \rangle = 0, \forall x \in K_{a_k+i}$  implies  $\lambda = 0$ . In this case, we deduce  $z = y + \lambda c^{(a_k+i+1)} = y \in F_{a_k+i}$  and  $z$  satisfies  $\langle z, x \rangle = 0 \forall x \in F_{a_k+i+1} \supset F_{a_k+i}$ . Thus  $z = y \in K_{a_k+i}$ , and then  $K_{a_k+i+1} \subseteq K_{a_k+i}$ .  $\square$

Let us suppose that, for  $i = 1, \dots, d_k - 1$ , we have  $\dim K_{a_k+i+1} = \dim K_{a_k+i} + 1$ , in other words, that  $K_{a_k+1} \subsetneq K_{a_k+2} \subsetneq \dots \subsetneq K_{a_k+d_k}$ , and suppose besides that  $d_k$  is the first index  $i$  for which the kernel  $K_{a_k+i}$  is not increasing.

Then we must have either  $K_{a_k+d_k+1} \equiv K_{a_k+d_k}$  or  $K_{a_k+d_k+1} \subset K_{a_k+d_k}$ . We know that we may write  $F_{a_k+i} = F_{a_k} \oplus K_{a_k+i}$  for  $i = 1$  until  $d_k$ , and we will denote by  $e_{k;i} := \pi^{K_{a_k-i}}(c^{(a_k+i)})$ , where  $\pi^{K_{a_k+i}}: F_{a_k+i} \rightarrow K_{a_k+i}$  is the natural projection for that direct sum.

Let us notice that  $e_{k;i} \notin K_{a_k+i-1}$ , since otherwise, we would have  $c^{(a_k+i)} \in F_{a_k} \oplus K_{a_k+i-1} = F_{a_k+i-1}$ , which contradicts the hypothesis on the flag  $(F_j)$ .

**LEMMA 2.2** *The family  $\{e_{k;1}, \dots, e_{k;d_k}\}$  is a basis of  $K_{a_k+d_k}$ .*

*Proof.* This family is free. Indeed, if  $\sum_{i=1}^{d_k} \lambda_i e_{k;i} = 0$  with coefficients  $\lambda_i$  not all nil, let  $m$  the greatest index  $i$  such that  $\lambda_m \neq 0$ . We then have  $\lambda_m e_{k;m} = -\sum_{i=1}^{m-1} \lambda_i e_{k;i} \in K_{a_k+m-1}$ , so  $e_{k;m} \in K_{a_k+m-1}$ , which is impossible. Thus, the family  $\{e_{k;1}, \dots, e_{k;d_k}\}$  forms a basis of  $K_{a_k+d_k}$  (since it is clear that  $\dim K_{a_k+d_k} = d_k$ ).  $\square$

**LEMMA 2.3** *We have  $\langle c^{(a_k+d_k+1)}, e_{k;i} \rangle = 0, \forall i = 1, \dots, d_k$ .*

*Proof.* Let us write  $c^{(a_k+d_k)}$  under the form  $c^{(a_k+d_k)} = e_{k;d_k} + \sum_{i=1}^{a_k} \alpha_i c^{(i)}$ . Then we have  $c^{(a_k+d_k+1)} = e'_{k;d_k} + \sum_{i=1}^{a_k} \alpha'_i c^{(i)} + \sum_{i=1}^{a_k} \alpha_i c^{(i+1)}$ . Thus,  $\langle c^{(a_k+d_k+1)}, e_{k;i} \rangle = \langle e'_{k;d_k}, e_{k;i} \rangle$ . Now,  $\langle e'_{k;d_k}, e_{k;i} \rangle + \langle e_{k;d_k}, e'_{k;i} \rangle = \underbrace{\langle e_{k;d_k}, e_{k;i} \rangle}'_{=0} = 0$ . So  $\langle c^{(a_k+d_k+1)}, e_{k;i} \rangle = -\langle e_{k;d_k}, e'_{k;i} \rangle$ . But  $e'_{k;i} \in F_{a_k+i+1}$ : indeed, we may write  $c^{(a_k+i)} = e_{k;i} + \sum_{l=1}^{a_k} \beta_l c^{(l)}$ , and then  $e'_{k;i} = c^{(a_k+i+1)} - \sum_{l=1}^{a_k} \beta'_l c^{(l)} - \sum_{l=1}^{a_k} \beta_l c^{(l+1)} \in F_{a_k+i+1}$ .

We deduce that  $\langle c^{(a_k+d_k+1)}, e_{k;i} \rangle = 0 \forall i = 1, \dots, d_k - 1$ . On the other hand, we have  $\langle e_{k;d_k}, e_{k;d_k} \rangle = 0$ , so  $\langle e'_{k;d_k}, e_{k;d_k} \rangle = 0$ . It results that  $\langle c^{(a_k+d_k+1)}, e_{k;d_k} \rangle = 0$ .  $\square$

Using this lemma together with Lemma 2.1, we get

**COROLLARY 2.4** *We have  $K_{a_k+d_k+1} \equiv K_{a_k+d_k}$ .*

Then we may complete our family  $\{e_{k;1}, \dots, e_{k;d_k}\}$  into a basis of  $F_{a_k+d_k+1}$ , adding a vector  $e_{k;d_k+1}$  defined in the following way: the quotient space  $F_{a_k+d_k+1}/K_{a_k+d_k}$  is not de-



generate (we quotient the space  $F_{a_k+d_k+1}$  by its kernel), and contains  $F_{a_k+d_k}/K_{a_k+d_k}$ , which is also non-degenerate.

Thus, the orthogonal space of  $F_{a_k+d_k}/K_{a_k+d_k}$  inside  $F_{a_k+d_k+1}/K_{a_k+d_k}$  is a supplementary subspace of dimension 1, and we get the orthogonal decomposition

$$F_{a_k+d_k+1}/K_{a_k+d_k} = F_{a_k+d_k}/K_{a_k+d_k} \oplus (F_{a_k+d_k}/K_{a_k+d_k})^\perp.$$

This supplementary subspace gives us a vector  $e_{k;d_k+1}$  (unique modulo the kernel  $K_{a_k+d_k}$ ), that we may choose unitary (i.e.  $\langle e_{k;d_k+1}, e_{k;d_k+1} \rangle = \pm 1$ ), and we denote it by  $\epsilon^k$ , and such that the family  $\{e_{k;1}, \dots, e_{k;d_k+1}\}$  is a basis of  $F_{a_k+d_k+1}$ .

*Remarks.*

1. The vector  $e_{k;d_k+1}$  is not isotropic, otherwise it should belong to the kernel  $K_{a_k+d_k+1}$ , and we would then have  $\dim K_{a_k+d_k+1} = \dim K_{a_k+d_k} + 1$ , which is not the case.
2. The quotient space  $F_{a_{k+1}}/F_{a_k}$  is of type  $(d_k, d_k + 1)$  or  $(d_k + 1, d_k)$ . Thus,  $\epsilon^k$  is the signature of this quotient space.
3. In order to fix the ideas, we may consider the vector  $e_{k;d_k+1}$  as the unitary projection of the vector  $c^{(a_k+d_k+1)}$  onto the quotient space  $(F_{a_k+d_k+1}/K_{a_k+d_k})^\perp$ .

Lemma 2.3 may be generalized in the following:

LEMMA 2.5 *We have*

$$\langle c^{(a_k+d_k+m)}, e_{k;i} \rangle = 0, \forall 1 \leq m \leq d_k, \forall i = 1, \dots, d_k + 1 - m.$$

*Proof.* Lemma 2.3 tells us that the result is true for  $m = 1$ . Suppose the lemma true up to  $m - 1$ ,  $2 \leq m \leq d_k$ . Remark that, since  $e'_{k;i} \in F_{a_k+i+1}$ , we may decompose

$$e'_{k;i} = \sum_{l=1}^{a_k} \lambda_l c^{(l)} + \sum_{l=1}^{i+1} \nu_l e_{k;l}, \text{ with } a_k + 2 \leq a_k + i + 1 \leq a_k + d_k + 2 - m \leq a_k + d_k.$$

But for any  $l \leq a_k$ ,  $\langle e'_{k;i}, c^{(l)} \rangle + \langle e_{k;i}, c^{(l+1)} \rangle = \underbrace{\langle e_{k;i}, c^{(l)} \rangle'}_{=0} = 0$ , and so,

$$\langle e'_{k;i}, c^{(l)} \rangle = -\langle e_{k;i}, \underbrace{c^{(l+1)}}_{\in F_{i+1} \subseteq F_{a_k+1}} \rangle = 0.$$

Thus, the vector  $e_{k;i}$  is orthogonal to the space  $F_{a_k}$ , and therefore, since the space  $F_{a_k}$  is non degenerate, we have  $\lambda_l = 0, \forall 1 \leq l \leq a_k$ . We obtain  $e'_{k;i} = \sum_{l=1}^{i+1} \nu_l e_{k;l}$ , and we immediately deduce, according to our induction hypothesis, that  $\langle c^{(a_k+d_k+m-1)}, e'_{k;i} \rangle = 0$ .

It follows, according to the equality

$$\langle c^{(a_k+d_k+m)}, e_{k;i} \rangle + \underbrace{\langle c^{(a_k+d_k+m-1)}, e'_{k;i} \rangle}_{=0} = \underbrace{\langle c^{(a_k+d_k+m-1)}, e_{k;i} \rangle'}_{=0 \text{ by hypothesis}},$$

that  $\langle c^{(a_k+d_k+m)}, e_{k;i} \rangle = 0$ .  $\square$

LEMMA 2.6  $\forall 1 \leq i \leq d_k - 1$ , there is a function  $C_i$  such that  $e'_{k;i} = e_{k;i+1} + C_i e_{k;1}$ .  
Moreover, there are some functions  $\mathcal{C}_1, \dots, \mathcal{C}_{d_k}, \delta_k$  such that  $e'_{k;d_k} = \delta_k e_{k;d_k+1} + \sum_{l=1}^{d_k} \mathcal{C}_l e_{k;l}$ .

*Proof.* We have seen, in the proof of Lemma 2.5, that  $\pi^{F_{a_k}}(e'_{k;i}) = 0, 1 \leq i \leq d_k$ , where  $\pi^{F_{a_k}}$  is the projection  $F_{a_k+i+1} \rightarrow F_{a_k}$ .

Let us then write  $c^{(a_k+i)} = e_{k;i} + \sum_{l=1}^{a_k} \phi_l^{(i)} c^{(l)}$ . We obtain, for  $1 \leq i \leq d_k - 1$ ,  $e'_{k;i} = c^{(a_k+i+1)} - \sum_{l=1}^{a_k} \phi_l^{(i)'} c^{(l)} - \sum_{l=1}^{a_k} \phi_l^{(i)} c^{(l+1)} = e_{k;i+1} - \phi_{a_k}^{(i)} e_{k;1} + \zeta$ , where  $\zeta \in F_{a_k}$ .

Thus, since  $\pi^{F_{a_k}}(e'_{k;i}) = 0$ ,  $e'_{k;i} = e_{k;i+1} - \phi_{a_k}^{(i)} e_{k;1}$ . Thus we obtain the result with  $C_i = -\phi_{a_k}^{(i)}$ .

For  $i = d_k$ , the result is clear since  $\pi^{F_{a_k}}(e'_{k;d_k}) = 0$ . More precisely, we may write  $c^{(a_k+d_k+1)} = \delta_k e_{k;d_k+1} + \sum_{l=1}^{d_k} \theta_l e_{k;l} + \sum_{l=1}^{a_k} \varphi_l c^{(l)}$ . Then the equality

$$e'_{k;d_k} = c^{(a_k+d_k+1)} - \sum_{l=1}^{a_k} \phi_l^{(d_k)'} c^{(l)} - \sum_{l=1}^{a_k} \phi_l^{(d_k)} c^{(l+1)}$$
 implies, as above, that

$$e'_{k;d_k} = \delta_k e_{k;d_k+1} + \sum_{l=1}^{d_k} \theta_l e_{k;l} - \phi_{a_k}^{(d_k)} e_{k;1}. \quad \square$$

LEMMA 2.7  $\forall 1 \leq m \leq d_k + 1$ , we have  $\langle c^{(a_k+d_k+m)}, e_{k;d_k+2-m} \rangle \neq 0$ .

*Proof.* The result is true for  $m=1$ : indeed, we may write  $c^{(a_k+d_k+1)} = \sum_{l=1}^{a_k} \varphi_l c^{(l)} + \sum_{l=1}^{d_k} \theta_l e_{k;l} + \delta_k e_{k;d_k+1}$ , and then, if  $\langle c^{(a_k+d_k+1)}, e_{k;d_k+1} \rangle = 0$ , we must have  $\delta_k = 0$ , so that  $c^{(a_k+d_k+1)} \in F_{a_k+d_k}$ , a contradiction.

Suppose the result true for an integer  $m \leq d_k$ , i.e.  $\langle c^{(a_k+d_k+m)}, e_{k;d_k+2-m} \rangle \neq 0$ .

We have

$$\langle c^{(a_k+d_k+m+1)}, e_{k;d_k+1-m} \rangle + \langle c^{(a_k+d_k+m)}, e'_{k;d_k+1-m} \rangle = \underbrace{\langle c^{(a_k+d_k+m)}, e_{k;d_k+1-m} \rangle'}_{=0 \text{ according to lemma 2.5}}.$$

In other words, if  $m \geq 2$  we have

$$\begin{aligned} \langle c^{(a_k+d_k+m+1)}, e_{k;d_k+1-m} \rangle &= -\langle c^{(a_k+d_k+m)}, e'_{k;d_k+1-m} \rangle \\ &= -\langle c^{(a_k+d_k+m)}, e_{k;d_k+2-m} + C_{d_k+1-m} e_{k;1} \rangle \end{aligned}$$

Now, according to Lemma 2.6 and the induction hypothesis, this is equal to  $-\langle c^{(a_k+d_k+m)}, e_{k;d_k+2-m} \rangle$ , which is not null.

If  $m = 1$ , we obtain

$$\begin{aligned} \langle c^{(a_k+d_k+2)}, e_{k;d_k} \rangle &= -\langle c^{(a_k+d_k+1)}, e'_{k;d_k} \rangle \\ &= -\langle c^{(a_k+d_k+1)}, \delta_k e_{k;d_k+1} + \sum_{l=1}^{d_k} \mathcal{C}_l e_{k;l} \rangle \\ &= -\langle c^{(a_k+d_k+1)}, \delta_k e_{k;d_k+1} \rangle \\ &\neq 0. \end{aligned}$$

□

Remark that Lemmas 2.5 and 2.7 imply that

$$\boxed{K_{a_k+d_k+m} \equiv K_{a_k+d_k+1-m}, \forall 1 \leq m \leq d_k}.$$

To end this section, let us prove the following lemma relating the signs  $\epsilon^k$  and  $\epsilon_k$ .

**LEMMA 2.8** *The sign  $\epsilon^k$  of  $\langle e_{k;d_k+1}, e_{k;d_k+1} \rangle$  is related to the sign  $\epsilon_k$  of the quotient  $g_{a_{k-1}}(g_{a_k})^{-1}$  by the formula  $\epsilon^k = (-1)^{d_k} \epsilon_k$ .*

*Proof.*

Let us denote by  $B_k^{[\text{int}]}$  the basis of  $F_{a_k}$  defined by the recurrence

$$\begin{cases} \bullet B_{k+1}^{[\text{int}]} = B_k^{[\text{int}]} \cup \{e_{k;1}, e_{k;1}^{(1)}, \dots, e_{k;1}^{(b_k-1)}\} \text{ if } a_{k+1} \neq a_k + 1, \text{ i.e. if } b_k \neq 1 \\ \bullet B_{k+1}^{[\text{int}]} = B_k^{[\text{int}]} \cup \{\pi^{F_{a_k} \cdot a_k + 1} (c^{(a_k+1)})\} \text{ if } a_{k+1} = a_k + 1, \text{ i.e. if } b_k = 1 \end{cases}$$

We denote by  $g_k^{[\text{int}]}$  the Gram's determinant of the basis  $B_k^{[\text{int}]}$ . Then, an easy computation shows that

$$\langle e_{k;1}^{(d_k)}, e_{k;1}^{(d_k)} \rangle^{2d_k+1} = (-1)^{d_k} g_{a_{k+1}}^{[\text{int}]} (g_{a_k}^{[\text{int}]})^{-1}. \quad (\star)$$

From Lemma 2.6, we may deduce that  $e_{k;1}^{(d_k)} = \delta_k e_{k;d_k+1} + \kappa$ , for some function  $\kappa$  on  $K_{a_k+d_k}$ , so that  $\langle e_{k;1}^{(d_k)}, e_{k;1}^{(d_k)} \rangle = \delta_k^2 \langle e_{k;d_k+1}, e_{k;d_k+1} \rangle$ . Consequently,

$$\begin{aligned} \epsilon^k &:= \text{sgn} \langle e_{k;d_k+1}, e_{k;d_k+1} \rangle \\ &= \text{sgn} \langle e_{k;1}^{(d_k)}, e_{k;1}^{(d_k)} \rangle \\ &= (-1)^{d_k} \text{sgn} (g_{a_{k-1}}^{[\text{int}]} (g_{a_k}^{[\text{int}]})^{-1}) \end{aligned} \quad (\star\star)$$

where the last equality results from equation ( $\star$ ).

Now, since the matrix  $P^{(k)}$  of change of basis from the basis  $\{c^{(1)}, \dots, c^{(a_k)}\}$  of  $F_{a_k}$  to the basis  $B_k^{[\text{int}]}$  is an upper triangular matrix with coefficient 1 everywhere on the main diagonal, and since we clearly have  $(P^{(k)})^t \text{Gram}\{c^{(1)}, \dots, c^{(k)}\} P^{(k)} = \text{Gram}\{B_k^{[\text{int}]}\}$ , we deduce that  $[\det(P^{(k)})]^2 g_{a_k} = g_k^{[\text{int}]}$ , ie  $g_{a_k} \equiv g_k^{[\text{int}]}$ .

Thus, the equation ( $\star\star$ ) gives us the result.  $\square$

### 3. The geometrical foundations of the construction

#### 3.1. First step: when the kernel grows

Since the space  $F_{a_k+2d_k+1}$  is not degenerate, we have  $\dim F_{a_k+2d_k} + \dim F_{a_k+2d_k}^{\perp a_k+2d_k+1} = \dim F_{a_k+2d_k+1} = a_k + 2d_k + 1$ ; in other words,  $\dim F_{a_k+2d_k}^{\perp a_k+2d_k+1} = 1$ .

Thus, the space  $F_{a_k+2d_k}^{\perp a_k+2d_k+1}$  is generated by a vector  $n_{k;1}$ , necessarily null since  $F_{a_k+2d_k}$  is degenerate. Moreover, let us remark that  $n_{k;1} \in F_{a_k+1}$ . Indeed, recall that if we write  $e_{k;1} = \pi^{K_{a_k+1}}(c^{(a_k+1)})$ , we have  $e_{k;1} \in K_{a_k+1} \subset K_{a_k+2} \subset \dots \subset K_{a_k+d_k} \equiv K_{a_k+d_k+1}$ , so  $e_{k;1} \in F_{a_k+d_k+1}^{\perp a_k+2d_k+1}$ , and on the other hand, the Lemma 2.5 ensures us that  $\langle c^{(a_k+d_k+m)}, e_{k;1} \rangle = 0 \forall 1 \leq m \leq d_k$ , so that  $e_{k;1} \in F_{a_k+2d_k}^{\perp a_k+2d_k+1}$ .

Since the space  $F_{a_k+2d_k}^{\perp a_k+2d_k+1}$  has dimension 1, there exists a smooth function  $\lambda_{(k)}(t)$  satisfying  $n_{k;1} = \lambda_{(k)} e_{k;1}$ , which naturally implies  $n_{k;1} \in F_{a_k+1}$ .

Remark now that the equalities

$$\diamond \underbrace{\dim F_{a_k+2d_k-1}}_{a_k+2d_k-1} + \dim F_{a_k+2d_k-1}^{\perp a_k+2d_k+1} = \underbrace{\dim F_{a_k+2d_k+1}}_{a_k+2d_k+1}$$

$$\diamond \underbrace{\dim F_{a_k+2d_k-1}}_{a_k+2d_k-1} + \dim F_{a_k+2d_k-1}^{\perp a_k+2d_k} = \underbrace{\dim F_{a_k+2d_k}}_{a_k+2d_k} + \underbrace{\dim F_{a_k+2d_k-1} \cap K_{a_k+2d_k}}_{\equiv K_{a_k+2d_k} \subset K_{a_k+2d_k-1}} \text{ imply}$$

that  $\dim F_{a_k+2d_k-1}^{\perp a_k+2d_k+1} = 2 = \dim F_{a_k+2d_k-1}^{\perp a_k+2d_k}$ . Thus we deduce that  $F_{a_k+2d_k-1}^{\perp a_k+2d_k} \equiv F_{a_k+2d_k-1}^{\perp a_k+2d_k+1} \equiv K_{a_k+2d_k-1}$ .

Therefore, we may choose a vector  $n_{k;2}$  such that  $K_{a_k+2d_k-1} = \langle n_{k;1} \rangle \oplus \langle n_{k;2} \rangle$ .

Let us remark that  $n'_{k;1}$  is a priori the most natural candidate, since  $\forall 1 \leq m \leq a_k + 2d_k - 1$ ,  $\langle n'_{k;1}, c^{(m)} \rangle + \underbrace{\langle n_{k;1}, c^{(m+1)} \rangle}_{=0} = \underbrace{(\langle n_{k;1}, c^{(m)} \rangle)'}_{=0}$ .

The above construction may be pursued in the same way for the spaces  $F_{a_k+2d_k-m}$  (for any integer  $m$  from 1 to  $d_k$ ), and leads us to introduce some vectors  $n_{k;2}, \dots, n_{k;d_k+1}$  which are nothing but the successive derivatives of  $n_{k;1}$ .

Remark nevertheless that for  $m \leq d_k - 1$ , the previous computations show that  $\dim F_{a_k+2d_k-m}^{\perp a_k+2d_k+1} = \dim F_{a_k+2d_k-m}^{\perp a_k+2d_k-m+1} = 1 + \dim K_{a_k+2d_k-m+1} = \dim K_{a_k+2d_k-m}$ , so that

$F_{a_k+2d_k-m}^{\perp a_k+2d_k+1} \equiv K_{a_k+2d_k-m}$ , which forces the introduced vectors  $n_{k;1}, \dots, n_{k;d_k}$ , belonging to a kernel, to be isotropic.

On the contrary, for  $m = d_k$ , we have  $K_{a_k+d_k+1} \equiv K_{a_k+d_k}$ , so  $\dim F_{a_k+d_k}^{\perp a_k+2d_k+1} = \dim F_{a_k+d_k}^{\perp a_k+d_k+1} = 1 + \dim K_{a_k+d_k+1} > \dim K_{a_k+d_k}$ .

Thus the space  $F_{a_k+d_k}^{\perp a_k+2d_k+1}$  contains strictly the kernel  $K_{a_k+d_k}$ , and the introduced vector  $n_{k;d_k+1}$  (which we will denote by  $\pi_k$  in the next sections), generating a supplementary space of  $K_{a_k+d_k}$  in  $F_{a_k+d_k}^{\perp a_k+2d_k+1}$ , is not isotropic.

Moreover, it is important to note that, since  $n_{k;1} \in F_{a_k+1}$ , we have  $n_{k;i} = n_{k;1}^{(i-1)} \in F_{a_k+i}$  for  $1 \leq i \leq d_k + 1$ , so that  $n_{k;i} \in K_{a_k+i}$  for  $i \leq d_k$ . In particular, for  $i \leq d_k$ , it is clear that  $\{n_{k;1}, \dots, n_{k;i}\}$  is a basis of  $K_{a_k+i}$ , and thus

$$K_{a_k+i} \equiv \langle n_{k;1}, \dots, n_{k;i} \rangle \equiv K_{a_k+2d_k-i+1}$$

Finally, let us remark that, by construction, it is clear that we have

$$\diamond K_{a_k+2d_k-m} \equiv F_{a_k+2d_k-m}^{\perp a_k+2d_k-m+1} \equiv F_{a_k+2d_k-m}^{\perp a_k+2d_k+1} \equiv \underbrace{F_{a_k+2d_k-m+1}^{\perp a_k+2d_k+1}}_{\equiv K_{a_k+2d_k-m+1}} \oplus \langle n_{k;m+1} \rangle \text{ for } m < d_k,$$

and

$$\diamond F_{a_k+d_k}^{\perp a_k+d_k+1} \equiv F_{a_k+d_k}^{\perp a_k+2d_k+1} \equiv \underbrace{F_{a_k+d_k+1}^{\perp a_k+2d_k+1}}_{\equiv K_{a_k+d_k+1} \equiv K_{a_k+d_k}} \oplus \langle n_{k;d_k+1} \rangle.$$

Before beginning the second step, we may wonder what choice of vector  $n_{k;1}$  seems the most judicious, in other words, knowing that one may write  $n_{k;1} = \lambda_{(k)} e_{k;1}$  for some function  $\lambda_{(k)}(t)$ , the problem is to find some function  $\lambda_{(k)}$  which makes the choice of the vector  $n_{k;1}$  the most natural one.

In view of what we have just seen above, the only function  $\lambda_{(k)}$  which seems to impose itself is the function which would allow to normalize the non-isotropic vector  $n_{k;d_k+1}$ , i.e. the unique positive function such that we have  $\langle n_{k;1}^{(d_k)}, n_{k;1}^{(d_k)} \rangle = \epsilon^k$ , where  $\epsilon^k = \pm 1$  is the sign of  $\langle e_{k;d_k+1}, e_{k;d_k+1} \rangle$ . Since  $n_{k;1}^{(d_k)} = \lambda_{(k)} e_{k;1}^{(d_k)} + \varkappa$  for some vector  $\varkappa$  in the kernel  $K_{a_k+d_k}$ , it is clear that  $\langle n_{k;1}^{(d_k)}, n_{k;1}^{(d_k)} \rangle = \lambda_{(k)}^2 \langle e_{k;1}^{(d_k)}, e_{k;1}^{(d_k)} \rangle$ . Now, recall that we have seen in the previous section that  $\langle e_{k;1}^{(d_k)}, e_{k;1}^{(d_k)} \rangle^{2d_k+1} = (-1)^{d_k} g_{a_{k+1}}(g_{a_k})^{-1}$ . Thus, we deduce that the function  $\lambda_{(k)}$  which we are looking for must satisfy  $\lambda_{(k)}^{2(2d_k+1)} = (-1)^{d_k} \epsilon^k g_{a_k}(g_{a_{k+1}})^{-1} = \epsilon^k g_{a_k}(g_{a_{k+1}})^{-1}$ .

In other words, we are led to the following

$$\text{Formula 3.1: } g_{a_k}(g_{a_{k+1}})^{-1} = (\lambda_{(k)}^2 \epsilon_k)^{2d_k+1}$$

If besides, we impose  $\lambda_{(k)} > 0$ , by analogy with the Frenet's frame in Riemannian Geometry (where this last condition determines in some sorts the orientation of

the frame), the function  $\lambda_{(k)}$  is then defined in a unique way by the previous formula.

### 3.2. Second step: when the kernel decreases

It is important to note that  $F_{a_k+d_k-1}^{\perp a_k+d_k} \equiv K_{a_k+d_k} \subsetneq F_{a_k+d_k-1}^{\perp a_k+2d_k+1}$ . Indeed, we have  $\dim F_{a_k+d_k-1}^{\perp a_k+2d_k+1} = d_k + 2$ . Our first aim is to find a priori a vector  $u_{k;d_k}$  such that

$$F_{a_k+d_k-1}^{\perp a_k+2d_k+1} \equiv \underbrace{F_{a_k+d_k}^{\perp a_k+2d_k+1}}_{\equiv K_{a_k+d_k}} \oplus \langle u_{k;d_k} \rangle.$$

Then we must choose some vector  $u_{k;d_k}$ , orthogonal to the space  $F_{a_k+d_k-1}$ , but not belonging to the space  $F_{a_k+d_k}^{\perp a_k+2d_k+1}$  (i.e. such that  $\langle u_{k;d_k}, n_{k;d_k} \rangle \neq 0$ ); moreover, it seems natural, if possible, to want to choose  $u_{k;d_k}$  isotropic.

For those reasons, the previous strategy, which would have consisted to take for vector  $u_{k;d_k}$  the vector  $n_{k;1}^{(d_k+1)}$ , does not seem to be the most judicious anymore, since now, the kernel  $K_{a_k+d_k-1}$  does not coincide with the space  $F_{a_k+d_k-1}^{\perp a_k+2d_k+1}$  anymore. In particular, we lose the argument which, in the first step, ensured us that the introduced vectors  $n_{k;i}$  were indeed isotropic.

This may be expressed by the fact that the hyperbolic plane generated by the vectors  $n_{k;d_k}$  and  $n_{k;1}^{(d_k+1)}$  is furnished with a metric of the form

$$\begin{pmatrix} 0 & -\epsilon^k \\ -\epsilon^k & \kappa \end{pmatrix}$$

for some function  $\kappa \in \mathcal{C}^\infty(I, \mathbb{R})$  (taking into account the fact that

$$\langle n_{k;d_k}, n_{k;1}^{(d_k+1)} \rangle = \langle n_{k;1}^{(d_k-1)}, n_{k;1}^{(d_k+1)} \rangle = \underbrace{\langle n_{k;1}^{(d_k-1)}, n_{k;1}^{(d_k)} \rangle'}_{=0} - \langle n_{k;1}^{(d_k)}, n_{k;1}^{(d_k)} \rangle = -\epsilon^k).$$

Then, it seems more judicious to account for the "obstruction of the vector  $n_{k;1}^{(d_k+1)}$  to be isotropic" by defining the vector  $u_{k;d_k}$  so that we have  $n_{k;1}^{(d_k+1)} = u_{k;d_k} + Y_{k,d_k} n_{k;d_k}$ , where  $Y_{k,d_k}$  is the function  $-\frac{\epsilon^k}{2} \kappa = (-1)^{d_k-1} \frac{\epsilon^k}{2} \kappa$ .

The hyperbolic plane  $\mathcal{H}_{k,d_k}$  generated by the vectors  $n_{k;d_k}$  and  $u_{k;d_k}$  is then furnished with the metric

$$\begin{pmatrix} 0 & -\epsilon^k \\ -\epsilon^k & 0 \end{pmatrix}$$

For the same reasons, we may hope to end naturally the whole process by introducing for  $2 \leq i \leq d_k$  some vectors  $u_{k;d_k+1-i}$ , such that

$$\diamond F_{a_k+d_k-i}^{\perp a_k+2d_k+1} \equiv \underbrace{F_{a_k+d_k-i+1}^{\perp a_k+2d_k+1}}_{\equiv K_{a_k+d_k-i+1}} \oplus \langle u_{k;d_k+1-i} \rangle$$

$$\diamond \langle u_{k;d_k+1-i}, n_{k;d_k+1-i} \rangle = (-1)^i \epsilon^k$$

- ◇  $\langle u_{k;d_k+1-i}, n_{k;d_k+m-i} \rangle = 0, \forall 2 \leq m \leq i$
- ◇  $\langle u_{k;d_k+1-i}, u_{k;d_k+1-j} \rangle = 0, \forall 1 \leq j \leq i$
- ◇  $u'_{k;d_k+1-i} = u_{k;d_k+2-i} + Y_{k;d_k+2-i} n_{k;d_k+3-i} + Y_{k;d_k+1-i} n_{k;d_k+1-i}$ .

*Remarks.*

• We have  $\dim K_{a_k+d_k-i+2} = d_k - i + 2 < d_k + i - 1 = \dim F_{a_k+d_k-i+2}^{\perp a_k+2d_k+1}$ , and thus  $K_{a_k+d_k-i+2} \subseteq F_{a_k+d_k-i+2}^{\perp a_k+2d_k+1}$ .

• We have  $u_{k;d_k+1-i} \in F_{a_k+d_k+i+1} \forall 1 \leq i \leq d_k$ .

We thus have obtained a decomposition of the space  $F_{a_{k+1}} \equiv F_{a_k+2d_k+1}$  in the sum

$$F_{a_{k+1}} = F_{a_k} \oplus \langle n_{k;d_k+1} \rangle \oplus \mathcal{H}_{k,1} \oplus \cdots \oplus \mathcal{H}_{k,d_k},$$

each of the hyperbolic planes  $\mathcal{H}_{k,i} = \langle n_{k,i}, u_{k,i} \rangle$  being furnished with a metric in the form

$$\begin{pmatrix} 0 & (-1)^{i-1} \epsilon_k \\ (-1)^{i-1} \epsilon_k & 0 \end{pmatrix}$$

#### 4. The construction of our basis

Let us suppose that we have already built a basis  $\mathcal{B}_k$  of  $F_{a_k}$ , satisfying the conditions given by the Theorem 1.3 (with  $k$  possibly 0 and the conventions  $a_0 = 0, F_0 = \{0\}, g_0 = 1$ , and  $\mathcal{B}_0 = \emptyset$ ).

We want to build  $\mathcal{B}_{k+1}$  on  $F_{a_{k+1}}$ .

##### 4.1. The case where $k < k_{\max}$

In this case, there exists an integer  $a_{k+1}$ ; Then we will complete the basis of  $F_{a_k}$  into a basis of  $F_{a_{k+1}}$ .

First, if  $a_{k+1} = a_k + 1$ , i.e. if  $F_{a_{k+1}}$  is not degenerate, then the orthogonal space  $F_{a_k}^{\perp a_k+1}$  has dimension 1, and  $F_{a_{k+1}} = F_{a_k} \oplus F_{a_k}^{\perp a_k+1}$ .

Then we may complete the family  $\mathcal{B}_k$  with some unitary vector generating the space  $F_{a_k}^{\perp a_k+1}$ , and we define  $\lambda_{(k)}(t) > 0$  to be the function such that the vector

$$\pi_k := \lambda_{(k)} \pi_{F_{a_k}^{\perp a_k+1}}(c^{(a_k+1)})$$

has norm  $\epsilon_k$ . Note that we clearly have  $\lambda_{(k)}^2 \epsilon_k = g_{a_k}(g_{a_{k+1}})^{-1}$ .

Now, if  $a_{k+1} \neq a_k + 1$ , we must make a more subtle analysis.

Let  $n_{k,1} := \lambda_{(k)} e_{k,1}$  a vector generating the kernel  $K_{a_{k+1}}$ ,  $\lambda_{(k)}$  being the smooth function defined by

$$\begin{cases} (\lambda_{(k)}^2 \epsilon_k)^{2d_k+1} = g_{a_k}(g_{a_{k+1}})^{-1} \\ \lambda_{(k)} > 0. \end{cases}$$

Let  $n_{k;i} := n_{k;1}^{(i-1)}$  the  $(i-1)$ -th derivative of  $n_{k;1}$  for  $2 \leq i \leq d_k$ , and put  $\pi_k := n_{k;1}^{(d_k)}$ .

Recall that the function  $\lambda_{(k)}$  has been chosen in order to have

$$\langle \pi_k, \pi_k \rangle = (-1)^{d_k} \epsilon_k = \pm 1.$$

PROPOSITION 4.1 *The family  $\mathcal{B}_k \cup \{n_{k;1}, \dots, n_{k;d_k}, \pi_k\}$  is a basis of the space  $F_{a_k+d_k+1}$*

*Proof.* Since  $\mathcal{B}_k$  is a basis of  $F_{a_k}$  which is non-degenerate, and since the vectors  $\{n_{k;1}, \dots, n_{k;d_k}, \pi_k\}$  are all orthogonal to  $F_{a_k}$ , it is clear that it is sufficient to prove that the family  $\{n_{k;1}, \dots, n_{k;d_k}, \pi_k\}$  is free.

Suppose that  $\sum_{i=1}^{d_k} \alpha_i n_{k;i} + \beta \pi_k = 0$ . Then the equality  $\left| \sum_{i=1}^{d_k} \alpha_i n_{k;i} + \beta \pi_k \right|^2 = 0$  implies that  $\beta = 0$ .

Now, suppose that the coefficients  $\alpha_1, \dots, \alpha_{d_k}$  are not all null. Denoting by  $p$  the greatest index  $i$  such that  $\alpha_i \neq 0$ , we have  $\sum_{i=1}^p \alpha_i n_{k;i} = 0$ , and thus  $n_{k;p} = -\frac{1}{\alpha_p} \sum_{i=1}^{p-1} \alpha_i n_{k;i} \in K_{a_k+p-1}$ , which is absurd, since  $n_{k;p}$  may be written  $n_{k;p} = \zeta_{(k)} e_{k;p} + \xi_p$ , with  $\xi_p \in K_{a_k+p-1}$ , and  $e_{k;p} \notin K_{a_k+p-1}$ .

Therefore, we have  $\alpha_1 = \dots = \alpha_{d_k} = 0$ . □

It remains to complete this family into a basis of  $F_{a_k+1}$ .

PROPOSITION 4.2 *For each integer  $i$  from 1 to  $d_k$ , there exists a vector  $u_{k;i}$ , uniquely defined modulo the kernel  $K_{a_k+i-1}$ , satisfying the following conditions:*

- (1)  $u_{k;i}$  is orthogonal to the space  $F_{a_k}$
- (2)  $\langle u_{k;i}, n_{k;j} \rangle = 0 \forall 1 \leq j \leq d_k, j \neq i$
- (3)  $\langle u_{k;i}, n_{k;i} \rangle = (-1)^{1-i} \epsilon_k$
- (4)  $\langle u_{k;i}, \pi_k \rangle = 0$
- (5)  $\langle u_{k;i}, u_{k;j} \rangle = 0 \forall i \leq j \leq d_k$

*Proof.* Let us write

$$u_{k;i} = \nu_i c^{(a_k+2d_k+2-i)} + \sum_{j=1}^{d_k} \eta_{k,j} n_{k;j} + \xi_k \pi_k + \sum_{j=i+1}^{d_k} \mu_{k,j} u_{k;j} + \sum_{l=0}^{k-1} \left( \sum_{j=1}^{d_l} \eta_{l,j} n_{l;j} + \xi_l \pi_l + \sum_{j=1}^{d_l} \mu_{l,j} u_{l;j} \right).$$

We choose below the coefficients  $\eta_{l,j}, \xi_l, \mu_{l,j}$  so that  $u_{k;i}$  satisfies the conditions (1), ..., (5) in the proposition.



$$(1) \quad \forall 0 \leq l \leq k-1, \forall 1 \leq j \leq d_l$$

$$\bullet \langle u_{k,i}, n_{l,j} \rangle = v_i \langle c^{(a_k+2d_k+2-i)}, n_{l,j} \rangle + (-1)^{j-1} \epsilon_l \mu_{l,j} = 0.$$

$$\text{So we choose } \mu_{l,j} = (-1)^j \epsilon_l v_i \langle c^{(a_k+2d_k+2-i)}, n_{l,j} \rangle.$$

$$\bullet \langle u_{k,i}, \pi_l \rangle = v_i \langle c^{(a_k+2d_k+2-i)}, \pi_l \rangle + (-1)^{d_l} \epsilon_l \xi_l = 0.$$

$$\text{So we choose } \xi_l = (-1)^{d_l-1} \epsilon_l v_i \langle c^{(a_k+2d_k+2-i)}, \pi_l \rangle.$$

$$\bullet \langle u_{k,i}, u_{l,j} \rangle = v_i \langle c^{(a_k+2d_k+2-i)}, u_{l,j} \rangle + (-1)^{j-1} \epsilon_l \eta_{l,j} = 0.$$

$$\text{So we choose } \eta_{l,j} = (-1)^j \epsilon_l v_i \langle c^{(a_k+2d_k+2-i)}, u_{l,j} \rangle.$$

$$(2) \quad \bullet \forall 1 \leq j \leq i-1, \langle u_{k,i}, n_{k,j} \rangle = v_i \langle c^{(a_k+2d_k+2-i)}, n_{k,j} \rangle = 0 \text{ according to Lemma 2.5}$$

$$\bullet \forall i+1 \leq j \leq d_k, \langle u_{k,i}, n_{k,j} \rangle = v_i \langle c^{(a_k+2d_k+2-i)}, n_{k,j} \rangle + (-1)^{j-1} \epsilon_k \mu_{k,j} = 0.$$

$$\text{So we choose } \mu_{k,j} = (-1)^j \epsilon_k v_i \langle c^{(a_k+2d_k+2-i)}, n_{k,j} \rangle.$$

$$(3) \quad \langle u_{k,i}, n_{k,i} \rangle = v_i \langle c^{(a_k+2d_k+2-i)}, n_{k,i} \rangle = (-1)^{i-1} \epsilon_k.$$

$$\text{So we choose } v_i = \frac{(-1)^{i-1} \epsilon_k}{\langle c^{(a_k+2d_k+2-i)}, n_{k,i} \rangle}.$$

Recall that  $\langle c^{(a_k+2d_k+2-i)}, n_{k,i} \rangle \neq 0$  according to Lemma 2.7.

$$(4) \quad \langle u_{k,i}, \pi_k \rangle = v_i \langle c^{(a_k+2d_k+2-i)}, \pi_k \rangle + (-1)^{d_k} \epsilon_k \xi_k = 0.$$

$$\text{So we choose } \xi_k = (-1)^{d_k-1} \epsilon_k v_i \langle c^{(a_k+2d_k+2-i)}, \pi_k \rangle.$$

$$(5) \quad \bullet \forall i+1 \leq j \leq d_k, \langle u_{k,i}, u_{k,j} \rangle = v_i \langle c^{(a_k+2d_k+2-i)}, u_{k,j} \rangle + (-1)^{j-1} \epsilon_k \eta_{k,j} = 0.$$

$$\text{So we choose } \eta_{k,j} = (-1)^j \epsilon_k v_i \langle c^{(a_k+2d_k+2-i)}, u_{k,j} \rangle.$$

$$\begin{aligned} \bullet \langle u_{k,i}, u_{k,i} \rangle &= \langle u_{k,i}, v_i c^{(a_k+2d_k+2-i)} + \sum_{j=1}^{d_k} \eta_{k,j} n_{k,j} + \xi_k \pi_k + \sum_{j=i+1}^{d_k} \mu_{k,j} u_{k,j} \\ &\quad + \sum_{l=0}^{k-1} \left( \sum_{j=1}^{d_l} \eta_{l,j} n_{l,j} + \xi_l \pi_l + \sum_{j=1}^{d_l} \mu_{l,j} u_{l,j} \right) \rangle \\ &= v_i \langle u_{k,i}, c^{(a_k+2d_k+2-i)} \rangle + (-1)^{i-1} \epsilon_k \eta_{k,i} \\ &= v_i \langle v_i c^{(a_k+2d_k+2-i)} + \sum_{j=1}^{d_k} \eta_{k,j} n_{k,j} + \xi_k \pi_k + \sum_{j=i+1}^{d_k} \mu_{k,j} u_{k,j} \\ &\quad + \sum_{l=0}^{k-1} \left( \sum_{j=1}^{d_l} \eta_{l,j} n_{l,j} + \xi_l \pi_l + \sum_{j=1}^{d_l} \mu_{l,j} u_{l,j} \right), c^{(a_k+2d_k+2-i)} \rangle \\ &\quad + (-1)^{i-1} \epsilon_k \eta_{k,i} \end{aligned}$$

$$\begin{aligned}
 &= \nu_i^2 |c^{(a_k+2d_k+2-i)}|^2 + \sum_{j=i+1}^{d_k} (-1)^j \epsilon_k \eta_{k,j} \mu_{k,j} + (-1)^{d_k-1} \epsilon_k \xi_k^2 \\
 &+ \sum_{j=i+1}^{d_k} (-1)^j \epsilon_k \mu_{k,j} \eta_{k,j} \\
 &+ \sum_{l=0}^{k-1} \left( \sum_{j=1}^{d_l} (-1)^j \epsilon_l \eta_{l,j} \mu_{l,j} + (-1)^{d_l-1} \epsilon_l \xi_l^2 + \sum_{j=1}^{d_l} (-1)^j \epsilon_l \eta_{l,j} \mu_{l,j} \right) \\
 &+ 2(-1)^{i-1} \epsilon_k \eta_{k,i} \\
 &= 0
 \end{aligned}$$

Notice that the last equation defines the coefficient  $\eta_{k,i}$ . □

Thus we have succeeded in completing our initial basis  $\mathcal{B}_k$  of  $F_{a_k}$  into a basis  $\mathcal{B}_k \cup \{n_{k;1}, \dots, n_{k;d_k}, \pi_k, u_{k;d_k}, \dots, u_{k;1}\}$  of  $F_{a_{k+1}}$ , where the vectors  $u_{k;i}$  are defined modulo the kernel  $K_{a_k+i-1}$ , and the metric has the following matrix in that basis:

$$\begin{pmatrix}
 (-1)^{d_0} \epsilon_0 U_0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & \dots & 0 \\
 0 & (-1)^{d_1} \epsilon_1 U_1 & \dots & 0 & \dots & \dots & \vdots & \dots & \dots & \vdots \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \dots & \dots & \vdots \\
 0 & \dots & 0 & (-1)^{d_{k-1}} \epsilon_{k-1} U_{k-1} & 0 & \dots & 0 & \dots & \dots & 0 \\
 0 & \dots & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \epsilon_k \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -\epsilon_k & 0 \\
 0 & \vdots & \vdots & 0 & 0 & \vdots & 0 & \vdots & 0 & \vdots \\
 0 & \dots & \dots & \dots & \dots & 0 & (-1)^{d_k} \epsilon_k & 0 & \dots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & \dots & \dots & \dots & 0 & -\epsilon_k & 0 & \vdots & \dots & \vdots \\
 0 & \dots & \dots & 0 & \epsilon_k & 0 & \dots & 0 & \dots & 0
 \end{pmatrix}$$

In other words, we have obtained a matrix

$$\begin{pmatrix}
 (-1)^{d_0} \epsilon_0 U_0 & & & \\
 & \ddots & & \\
 & & & (-1)^{d_k} \epsilon_k U_k
 \end{pmatrix}$$

#### 4.2. The case where $k = k_{\max}$

Recall that we have denoted by  $r$  the unique integer such that  $F_{r-1} \subsetneq F_r \equiv F_{r+1}$ .

If  $r = a_{k_{\max}}$ , then the construction is finished at level  $k_{\max}$ : we have the basis  $\mathcal{B}_r$  of  $F_r$  satisfying the conditions of Theorem 1.3.

If  $r > a_{k_{\max}}$ , then for any integer  $i$  between  $a_{k_{\max}} + 1$  and  $r$ , the space  $F_i$  must be degenerate.

Furthermore, we have the following result:

LEMMA 4.3 *If  $a_k = a_{k_{\max}} < r$ , then  $K_{a_k+1} \subseteq K_{a_k+2} \subseteq \dots \subseteq K_r$*

*Proof.* Suppose the result false. Then, with our previous notation for  $d_k$ , this means that  $r \geq a_k + d_k + 1$ .

Then we may complete the basis  $\mathcal{B}_k$  of  $F_{a_k}$  into a basis of  $F_r$ , adding vectors  $\{n_{k;1}, \dots, n_{k;d_k}, \pi_k, u_{k;d_k}, \dots, u_{k;a_k+2d_k-r+2}\}$ , in the same way that we have done in the previous section. It is clear that we must have  $\langle c^{(a_k+d_k+1)}, \pi_k \rangle \neq 0$ , otherwise, we should have  $c^{(a_k+d_k+1)} \in F_{a_k+d_k}$ , and consequently,  $F_{a_k+d_k} \equiv F_{a_k+d_k+1}$ , which would imply  $r \leq a_k + d_k$ .

Remark that the proof of Lemma 2.7 is still true as long as one may start the induction. We thus conclude that  $\langle c^{(a_k+d_k+m)}, n_{k;d_k+2-m} \rangle \neq 0$  for any integer  $m$  with  $2 \leq m \leq d_k + 1$ . In particular, we have  $\langle c^{(r+1)}, n_{k;a_k+2d_k-r+1} \rangle \neq 0$ . But, since  $c^{(r+1)} \in F_{r+1} \equiv F_r$ , we have  $\langle c^{(r+1)}, n_{k;a_k+2d_k-r+1} \rangle = 0$ . (Remark that  $a_k + 2d_k + 1 > r$ , otherwise there would exist an integer  $a_{k+1}$  defined by  $a_{k+1} := a_k + 2d_k + 1$ , which would give a contradiction with the fact that  $k = k_{\max}$ ).  $\square$

Thus we obtain a basis of  $F_r$  adding to the family  $\mathcal{B}_k$  some vectors  $n_{k;1}, \dots, n_{k;r-a_{k_{\max}}}$ , where  $n_{k;i} \in K_{a_k+i} - K_{a_k+i-1}$ .

In this basis, the metric is

$$\begin{pmatrix} (-1)^{d_0} \epsilon_0 U_0 & & & \\ & \ddots & & \\ & & (-1)^{d_{k_{\max}-1}} \epsilon_{k_{\max}-1} U_{k_{\max}-1} & \\ & & & O_{r-a_{k_{\max}}} \end{pmatrix}$$

In conclusion, we may say that the process described in this paper must then come to an end either with a non-degenerate final space  $F_r$ , or with a degenerate space  $F_r$ , in which case, the residual kernel is constituted with vectors of type " $n_{k_{\max};i}$ ".

## 5. The matrix of derivatives

### 5.1. The case where $k < k_{\max}$

In this section, we are going to choose explicitly the vectors  $u_{k;i}$ , which were previously defined only modulo the kernel  $K_{a_k+i-1}$ . We choose them so that the basis  $\mathcal{B}_k \cup \{n_{k;1}, \dots, n_{k;d_k}, \pi_k, u_{k;d_k}, \dots, u_{k;1}\}$  satisfies the conditions (3) given in Theorem 1.3.

- By construction, it is clear that we have  $n'_{k;i} = n_{k;i+1}$  for any integer  $i$  from 1 to  $d_k$ .
- Since  $\pi'_k \in F_{a_k+d_k+2}$ , we may write

$$\pi'_k = \alpha_{k,d_k}^{(0)} u_{k;d_k} + \beta_k^{(0)} \pi_k + \sum_{j=1}^{d_k} \gamma_{k,j}^{(0)} n_{k;j} + \sum_{l=0}^{k-1} \left( \sum_{j=1}^{d_l} \alpha_{l,j}^{(0)} u_{l;j} + \beta_l^{(0)} \pi_l + \sum_{j=1}^{d_l} \gamma_{l,j}^{(0)} n_{l;j} \right).$$

$$\diamond \forall 0 \leq l \leq k-1, \forall 1 \leq j \leq d_l,$$

$$\langle \pi'_k, n_{l;j} \rangle = -\langle \pi_k, n'_{l;j} \rangle = \begin{cases} -\langle \pi_k, n_{l;j+1} \rangle & \text{if } j < d_l \\ -\langle \pi_k, \pi_l \rangle & \text{if } j = d_l \end{cases} = 0.$$

Then we deduce that  $\alpha_{l,j}^{(0)} = 0$  for any integer  $j$  from 1 to  $d_l$ .

$$\diamond \forall 0 \leq l \leq k-1,$$

$$\langle \pi'_k, \pi_l \rangle = -\langle \pi_k, \pi'_l \rangle = -\langle \pi_k, u_{l;d_l} + \gamma_{l,d_l} n_{l;d_l} \rangle = 0.$$

Thus, we have  $\beta_l^{(0)} = 0$ .

$$\diamond \forall 0 \leq l \leq k-1, \forall 1 \leq j \leq d_l,$$

$$\langle \pi'_k, u_{l;j} \rangle = -\langle \pi_k, u'_{l;j} \rangle = 0$$

since  $u'_{l;j} \in F_{a_l+2d_l+3-j} \subseteq F_{a_l+2d_l+2} \subseteq F_{a_{k-1}+2d_{k-1}+2} \equiv F_{a_k+1}$ . So, we have  $\gamma_{l,j}^{(0)} = 0 \forall 1 \leq j \leq d_l$ .

$$\diamond \langle \pi'_k, n_{k;d_k} \rangle = -\langle \pi_k, n'_{k;d_k} \rangle = -\langle \pi_k, \pi_k \rangle = (-1)^{d_k-1} \epsilon_k.$$

Thus,  $\alpha_{k,d_k}^{(0)} = 1$ .

$$\diamond \langle \pi'_k, \pi_k \rangle = \frac{1}{2} \underbrace{\langle \pi_k, \pi_k \rangle}'_{=(-1)^{d_k} \epsilon_k} = 0.$$

Thus,  $\beta_k^{(0)} = 0$ .

$$\text{Therefore, } \pi'_k = u_{k;d_k} + \sum_{j=1}^{d_k} \gamma_{k,j}^{(0)} n_{k;j}.$$

Recall that the vector  $u_{k;d_k}$  is defined modulo the kernel  $K_{a_k+d_k-1}$ ; this leads us to substitute to the vector  $u_{k;d_k}$  the vector  $v_{k;d_k}$  defined by  $v_{k;d_k} = u_{k;d_k} +$

$\sum_{j=1}^{d_k-1} \gamma_{k,j}^{(0)} n_{k;j}$ , and then, denoting by  $Y_{k,d_k} := \gamma_{k,d_k}^{(0)}$ , we obtain:

$$\pi'_k = v_{k;d_k} + Y_{k,d_k} n_{k;d_k}.$$

- $v_{k;d_k}$  being defined in this way, let us compute  $v'_{k;d_k}$ .

For this, let us write

$$u'_{k;d_k} = \alpha_{k;d_{k-1}} u_{k;d_{k-1}} + \alpha_{k;d_k} u_{k;d_k} + \beta_k \pi_k + \sum_{j=1}^{d_k} \gamma_{k,j} n_{k;j} + \sum_{l=0}^{k-1} \left( \sum_{j=1}^{d_l} \alpha_{l,j} u_{l;j} + \beta_l \pi_l + \sum_{j=1}^{d_l} \gamma_{l,j} n_{l;j} \right).$$

$$\diamond \forall 0 \leq l \leq k-1, \forall 1 \leq j \leq d_l,$$

$$\langle u'_{k;d_k}, n_{l;j} \rangle = - \langle u_{k;d_k}, \underbrace{n'_{l;j}}_{\in F_{a_1+j+1} \subseteq F_{a_{k-1}+d_{k-1}+1}} \rangle = 0.$$

Thus,  $\alpha_{l,j} = 0 \forall 0 \leq l \leq k-1, \forall 1 \leq j \leq d_l$ .

$$\diamond \langle u'_{k;d_k}, \pi_l \rangle = - \langle u_{k;d_k}, \underbrace{\pi'_l}_{\in F_{a_1+d_1+2} \subseteq F_{a_{k-1}+d_{k-1}+2}} \rangle = 0.$$

Thus,  $\beta_l = 0$ .

$$\diamond \forall 0 \leq l \leq k-1, \forall 1 \leq j \leq d_l,$$

$$\langle u'_{k;d_k}, u_{l;j} \rangle = - \langle u_{k;d_k}, \underbrace{u'_{l;j}}_{\in F_{a_1+2d_1+3-j} \subseteq F_{a_{k-1}+2d_{k-1}+2}} \rangle = 0.$$

Thus,  $\gamma_{l,j} = 0 \forall 0 \leq l \leq k-1, \forall 1 \leq j \leq d_l$ .

$$\diamond \langle u'_{k;d_k}, n_{k;d_k} \rangle = - \langle u_{k;d_k}, n'_{k;d_k} \rangle = - \langle u_{k;d_k}, \pi_k \rangle = 0.$$

Thus,  $\alpha_{k;d_k} = 0$ .

$$\diamond \langle u'_{k;d_k}, n_{k;d_{k-1}} \rangle = - \langle u_{k;d_k}, n'_{k;d_{k-1}} \rangle = - \langle u_{k;d_k}, n_{k;d_k} \rangle = (-1)^{d_k} \epsilon_k.$$

Thus,  $\alpha_{k;d_{k-1}} = 1$ .

$$\diamond \langle u'_{k;d_k}, \pi_k \rangle = - \langle u_{k;d_k}, \pi'_k \rangle = - \langle u_{k;d_k}, u_{k;d_k} + Y_{k;d_k} n_{k;d_k} \rangle = (-1)^{d_k} \epsilon_k Y_{k;d_k}.$$

Thus,  $\beta_k = Y_{k;d_k}$ .

$$\diamond \langle u'_{k;d_k}, u_{k;d_k} \rangle = \frac{1}{2} \underbrace{\langle u_{k;d_k}, u_{k;d_k} \rangle}'_{=0} = 0.$$

Thus,  $\gamma_{k;d_k} = 0$ .

From all these computations, it results that  $u'_{k;d_k} = u_{k;d_{k-1}} + Y_{k;d_k} \pi_k + \sum_{j=1}^{d_{k-1}} \gamma_{k,j} n_{k;j}$ .

Reminding that the vector  $u_{k;d_{k-1}}$  is defined modulo the kernel  $K_{a_k+d_{k-2}}$ , we are led to substitute to the vector  $u_{k;d_{k-1}}$  the vector  $u_{k;d_{k-1}} := u_{k;d_{k-1}} + \sum_{j=1}^{d_{k-2}} \gamma_{k,j} n_{k;j}$ , and then, putting  $Y_{k;d_{k-1}} := \gamma_{k,d_{k-1}}$ , we obtain:

$$u'_{k;d_k} = u_{k;d_{k-1}} + Y_{k;d_k} \pi_k + Y_{k;d_{k-1}} n_{k;d_{k-1}}.$$

- Now, suppose that, for  $i = d_k - 2, \dots, 1$ , we have made a particular choice, for any  $i + 1 \leq h \leq d_k$ , of vector  $u_{k;h} + \sum_{j=1}^{h-1} \gamma_{k,j} n_{k;j}$ , that we will denote by  $u_{k;h}$ , satisfying the following equation

$$u'_{k;h} = u_{k;h-1} + Y_{k,h} n_{k;h+1} + Y_{k,h-1} n_{k;h-1}, \quad \forall 1 \leq h \leq i.$$

Let us then write

$$u'_{k;i+1} = \alpha_{k,i} u_{k;i} + \sum_{j=i+1}^{d_k} \alpha_{k,j} u_{k;j} + \beta_k \pi_k + \sum_{j=1}^{d_k} \gamma_{k,j} n_{k;j} + \sum_{l=0}^{k-1} \left( \sum_{j=1}^{d_l} \alpha_{l,j} u_{l;j} + \beta_l \pi_l + \sum_{j=1}^{d_l} \gamma_{l,j} n_{l;j} \right).$$

- ◊ exactly the same computations as above show easily that all coefficients  $\alpha_{l,j}, \beta_l, \gamma_{l,j}$  equal to 0, for any integer  $l$  from 0 to  $k-1$ , and any integer  $j$  from 1 to  $d_l$ .
- ◊  $\forall i + 1 \leq j \leq d_k$ ,

$$\langle u'_{k;i+1}, n_{k;j} \rangle = -\langle u_{k;i+1}, n'_{k;j} \rangle = 0.$$

Thus,  $\alpha_{k,j} = 0$ .

- ◊  $\langle u'_{k;i+1}, n_{k;i} \rangle = -\langle u_{k;i+1}, n'_{k;i} \rangle = -\langle u_{k;i+1}, n_{k;i+1} \rangle = (-1)^{i-1} \epsilon_k$ .

Thus, we deduce that  $\alpha_{k,i} = 1$ .

- ◊  $\langle u'_{k;i+1}, \pi_k \rangle = -\langle u_{k;i+1}, \pi'_k \rangle = 0$ .

Thus,  $\beta_k = 0$ .

- ◊  $\forall i + 2 \leq j \leq d_k$ ,

$$\begin{aligned} \langle u'_{k;i+1}, u_{k;j} \rangle &= -\langle u_{k;i+1}, u'_{k;j} \rangle \\ &= -\langle u_{k;i+1}, u_{k;j-1} + Y_{k,j} n_{k;j+1} + Y_{k,j-1} n_{k;j-1} \rangle \\ &= 0 \text{ except for } i + 1 = \begin{cases} j + 1 \\ j - 1 \end{cases} \end{aligned}$$

So, we have  $\gamma_{k,j} = 0 \forall i + 3 \leq j \leq d_k$ .

- ◊  $\langle u'_{k;i+1}, u_{k;i+2} \rangle = -\langle u_{k;i+1}, u_{k;i+1} + Y_{k,i+2} n_{k;i+3} + Y_{k,i+1} n_{k;i+1} \rangle$   
 $= (-1)^{i+1} \epsilon_k Y_{k,i+1}$

Thus,  $\gamma_{k,i+2} = Y_{k,i+1}$ .

- ◊  $\langle u'_{k;i+1}, u_{k;i+1} \rangle = \frac{1}{2} \underbrace{\langle u_{k;i+1}, u_{k;i+1} \rangle}'_{=0} = 0$ .

Thus,  $\gamma_{k,i+1} = 0$ .

Therefore, we may write  $u'_{k;i+1} = u_{k;i} + Y_{k,i+1} n_{k;i+2} + \sum_{j=1}^i \gamma_{k,j} n_{k;j}$ .

But, the vector  $u_{k;i}$  being defined modulo the kernel  $K_{a_{k+i-1}}$ , we may substitute to this vector the vector  $u_{k;i} := u_{k;i} + \sum_{j=1}^{i-1} \gamma_{k,j} n_{k;j}$ , so that, denoting by  $Y_{k,i} := \gamma_{k,i}$ , we obtain:

$$u'_{k;i+1} = u_{k;i} + Y_{k,i+1} n_{k;i+2} + Y_{k,i} n_{k;i}.$$

- Finally, it remains us to compute the derivative  $u'_{k;1}$ . According to whether the space  $F_{a_{k+1}}$  is degenerate or not, we are going to introduce a new vector  $\chi$ , which will generate the kernel (in the degenerate case), or the orthogonal space  $F_{a_{k+1}}^{\perp a_{k+1}+1}$  (in the non-degenerate case).

More precisely, we choose the vector  $\chi$  to be

- the null vector if  $r = a_{k+1}$
- the unitary projection of the vector  $c^{(a_{k-1}+1)}$  on the space  $F_{a_{k+1}}^{\perp a_{k+1}+1}$  if  $F_{a_{k+1}}$  is not degenerate, i.e. if  $a_{k+2} = a_{k+1} + 1$
- $\lambda_{(k+1)} e_{k+1;1}$ , if  $F_{a_{k-1}+1}$  is degenerate, where  $e_{k+1;1}$  is the projection of the vector  $c^{(a_{k+1}+1)}$  onto the kernel  $K_{a_{k+1}+1}$ , and  $\lambda_{(k+1)}$  is the function defined by

$$\begin{cases} (\lambda_{(k+1)}^2 \epsilon_{k+1})^{2a_{k+1}+1} = g_{a_{k+1}} (g_{a_{k+2}})^{-1} \\ \lambda_{(k+1)} > 0 \end{cases}$$

if  $k+1 \neq k_{\max}$ , or some function which we will define in the next section if  $k+1 = k_{\max}$ .

Let us then write  $u'_{k;1} = \varkappa_k \chi + \sum_{l=0}^k \left( \sum_{j=1}^{d_l} \alpha_{l,j} u_{l;j} + \beta_l \pi_l + \sum_{j=1}^{d_l} \gamma_{l,j} n_{l;j} \right)$ .

- ◊ The same computations as we have made above show that all coefficients are equal to 0, except  $\varkappa_k$ ,  $\gamma_{k,2}$ , and  $\gamma_{k-1,1}$ .
- ◊  $\langle u'_{k;1}, u_{k;2} \rangle = -\langle u_{k;1}, u_{k;1} + Y_{k,2} n_{k;3} + Y_{k,1} n_{k;1} \rangle = -\epsilon_k Y_{k,1}$ .  
Thus,  $\gamma_{k,2} = Y_{k,1}$ .
- ◊  $\langle u'_{k;1}, u_{k-1;1} \rangle = -\langle u_{k;1}, \varkappa_{k-1} n_{k;1} + Y_{k-1,1} n_{k-1;2} + \kappa n_{k-2;1} \rangle = -\epsilon_k \varkappa_{k-1}$   
for some function  $\kappa$  on  $\mathbb{R}$ .  
Thus,  $\gamma_{k-1,1} = -\epsilon_{k-1} \epsilon_k \varkappa_{k-1}$ .

Therefore, we obtain  $u'_{k;1} = \varkappa_k \chi + Y_{k,1} n_{k;2} - \epsilon_{k-1} \epsilon_k \varkappa_{k-1} n_{k-1;1}$ .

Relatively to the basis  $\mathcal{B}_{k+1} \equiv \mathcal{B}_k \cup \{n_{k;1}, \dots, n_{k;d_k}, \pi_k, u_{k;d_k}, \dots, u_{k;1}\}$ , the matrix of derivatives looks like

$$\begin{pmatrix}
 \Delta_0 & Lc_0 & \dots & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 \widetilde{Lc}_0 & \Delta_1 & Lc_1 & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 0 & \widetilde{Lc}_1 & \ddots & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 0 & \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 0 & \dots & \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 0 & \dots & \dots & \widetilde{Lc}_{k-2} & \Delta_{k-1} & \widetilde{Lc}_{k-1} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 0 & \dots & \dots & \dots & \dots & \widetilde{Lc}_{k-1} & 0 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & \dots & \dots & \dots & \dots & 0 \\
 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & Y_{k,d_k} & 0 & 1 & \dots & \dots & 0 \\
 0 & \dots & \dots & \dots & 0 & Y_{k,d_{k-1}} & 0 & Y_{k,d_k} & 0 & 1 & 0 & \dots & 0 & \dots & 0 \\
 0 & \dots & \dots & 0 & Y_{k,d_{k-2}} & 0 & Y_{k,d_{k-1}} & 0 & 0 & 0 & 1 & \dots & 0 & \dots & 0 \\
 0 & \dots & \dots & \dots & 0 & / & / & 0 & / & 0 & \dots & \dots & \dots & \dots & 1 & 0 \\
 0 & \dots & \dots & \dots & \dots & Y_{k,1} & 0 & Y_{k,2} & 0 & \dots & \dots & \dots & \dots & \dots & 1 & 0 \\
 0 & \dots & \dots & \dots & \dots & \dots & 0 & Y_{k,1} & 0 & \dots & \dots & \dots & \dots & \dots & 0 & \underline{\underline{x_k}}
 \end{pmatrix}$$

where the last column represents the coefficients of the vector  $\chi \in F_{a_{k+1}+1} - F_{a_{k+1}}$ .

In other words, we have obtained a matrix in the form

$$\begin{pmatrix}
 \Delta_0 & Lc_0 & 0 & \dots & \dots & \dots & 0 \\
 \widetilde{Lc}_0 & \Delta_1 & Lc_1 & \ddots & \dots & \dots & \vdots \\
 0 & \widetilde{Lc}_1 & \ddots & \ddots & \ddots & \dots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 \vdots & & & & & & 0 \\
 0 & \dots & \dots & 0 & \widetilde{Lc}_{k-1} & \Delta_k & Lc_k
 \end{pmatrix}$$

### 5.2. The case where $k = k_{\max}$

If  $r = a_{k_{\max}}$ , there is nothing more to do. If  $a_{k_{\max}} < r$ , exactly in the same way as we have done above, we put  $n_{k_{\max};1} = \lambda_{(k_{\max})} e_{k_{\max};1}$ , and for any integer  $i$  from 2 to  $r - a_{k_{\max}}$ ,  $n_{k_{\max};i} := n_{k_{\max};1}^{(i-1)}$ .

Here, the only problem lies in the choice of the function  $\lambda_{(k_{\max})}$ , noting that the previous formula does not apply anymore, since now, there does not exist a non-zero " $g_{a_{k_{\max}+1}}$ ".

In fact, the most natural choice of  $\lambda_{(k_{\max})}$  is given by the following proposition.



PROPOSITION 5.1 *There exists a unique choice of function  $\lambda_{(k_{\max})}$  such that*

$$\begin{cases} n'_{k_{\max}; r-a_{k_{\max}}} \in K_{r-1} \\ \lambda_{(k_{\max})}(0) = 1 \end{cases}$$

*Proof.* An easy computation shows that

$$\begin{aligned} n'_{k_{\max}; r-a_{k_{\max}}} &= (\lambda_{(k_{\max})} e_{k_{\max}; 1})^{(r-a_{k_{\max}})} \\ &= \lambda_{(k_{\max})} e_{k_{\max}; 1}^{(r-a_{k_{\max}})} + (r-a_{k_{\max}}) \lambda'_{(k_{\max})} e_{k_{\max}; 1}^{(r-a_{k_{\max}}-1)} \\ &\quad + \underbrace{\sum_{i=2}^{r-a_{k_{\max}}} \mathcal{E}_{r-a_{k_{\max}}}^i \lambda_{k_{\max}}^{(i)} e_{k_{\max}; 1}^{(r-a_{k_{\max}}-i)}}_{\in K_{r-1}}. \end{aligned}$$

Moreover, since  $e_{k_{\max}; 1}^{(r-a_{k_{\max}})} \in K_{r+1} \equiv K_r$ , we may write it as  $\sum_{i=1}^{r-a_{k_{\max}}} \nu_i e_{k_{\max}; 1}^{(i-1)}$ , for some functions  $\nu_i$ , so that  $n'_{k_{\max}; r-a_{k_{\max}}} \in K_{r-1} \iff \lambda_{(k_{\max})} \nu_{r-a_{k_{\max}}} + (r-a_{k_{\max}}) \lambda'_{(k_{\max})} = 0$ .

Therefore,  $\lambda_{(k_{\max})}$  is the unique solution of the above differential equation with initial condition  $\lambda_{(k_{\max})}(0) = 1$ .  $\square$

From this, it results that for the curves for which the final space  $F_r$  is degenerate, we may introduce a third kind of functions  $\zeta_i$ , such that the matrix of derivatives admits a decomposition

$$\begin{pmatrix} \Delta_0 & Lc_0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \widetilde{Lc}_0 & \Delta_1 & Lc_1 & \ddots & \dots & \dots & \dots & \dots & \dots & \vdots \\ 0 & \widetilde{Lc}_1 & \ddots & & \ddots & \dots & \dots & \dots & \dots & \vdots \\ \vdots & \ddots & & \ddots & & \ddots & \dots & \dots & \dots & \vdots \\ 0 & \dots & 0 & \widetilde{Lc}_{k_{\max}-2} & \Delta_{k_{\max}-1} & Lc_{k_{\max}-1} & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots & \vdots & \dots & \dots & \ddots & 0 \\ \vdots & \dots & \dots & \dots & \vdots & 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 & \zeta_1 & \dots & \dots & \zeta_{r-1-a_{k_{\max}}} & 0 \end{pmatrix}$$

i.e. the matrix

$$\Delta = \begin{pmatrix} \Delta_0 & Lc_0 & 0 & \dots & \dots & \dots & 0 \\ \widetilde{Lc}_0 & \Delta_1 & Lc_1 & \ddots & \dots & \dots & \vdots \\ 0 & \widetilde{Lc}_1 & \ddots & \ddots & \ddots & \cdot & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \widetilde{Lc}_{h-1} & \Delta_h & Lc_h \\ 0 & \dots & \dots & \dots & \dots & 0 & \Gamma \end{pmatrix},$$

with  $h = k_{\max} - 1$ .

### 6. The change of basis

Notice that one may get our new basis  $\{v_1, \dots, v_r\}$  of  $F_r$  from the canonical basis  $\{c^{(1)}, \dots, c^{(r)}\}$  by an upper triangular matrix, the main diagonal of which is

$$\underbrace{(\lambda_{(0)}, \dots, \lambda_{(0)})}_{b_0} \underbrace{(\lambda_{(1)}, \dots, \lambda_{(1)})}_{b_1} \dots \underbrace{(\lambda_{(k_{\max})}, \dots, \lambda_{(k_{\max})})}_{r - a_{k_{\max}}}$$

Besides, we have the following result relating the functions  $\lambda_{(i)}$  and  $\varkappa_i$ :

PROPOSITION 6.1  $\varkappa_i = \frac{\lambda_{(i)}}{\lambda_{(i+1)}} \begin{cases} \forall 0 \leq i \leq k_{\max} - 1 \text{ if } r \neq a_{k_{\max}} \\ \forall 0 \leq i \leq k_{\max} - 2 \text{ if } r = a_{k_{\max}} \end{cases}$

*Proof.*

★ First case: if  $a_{i+1} \neq a_i + 1$ .

In this case,  $\chi = n_{i+1;1} = \lambda_{(i+1)} e_{i+1;1}$ . We have  $u'_{i;1} = \varkappa_i \lambda_{(i+1)} e_{i+1;1} + Y_{i,d} n_{i;2} - \epsilon_{i-1} \epsilon_i \varkappa_{i-1} n_{i-1;1}$ . But  $c^{(a_{i+1}+1)} = e_{i+1;1} + y$ , where  $y \in F_{a_{i+1}}$ . So,  $u'_{i;1} = \varkappa_i \lambda_{(i+1)} c^{(a_{i+1}+1)} + z$  (1), for some vector  $z \in F_{a_{i+1}}$ . Furthermore, we know, looking at the change of basis above, that we may write  $u_{i;1} = \lambda_{(i)} c^{(a_{i+1})} + v$ , with  $v \in F_{a_{i+1}-1}$ . Thus,  $u'_{i;1} = \lambda_{(i)} c^{(a_{i+1}+1)} + w$  (2), with  $w \in F_{a_{i+1}}$ .

Equations (1) and (2) together with the fact that  $c^{(1)}, \dots, c^{(a_{i+1}+1)}$  are independent give us the result.

★ Second case: if  $a_{i+1} = a_i + 1$ .

The above arguments are still valid; we just have to write now

$$\chi = \pi_{i+1} := \lambda_{(i+1)} \pi_{F_{a_{i+1}}}^{\perp a_{i+1}+1} (c^{(a_{i+1}+1)}).$$

□

Using **Formula 3.1**, we may deduce

$$\text{Formula 6.1: } \kappa_i^2 = |g_{a_i}|^{\frac{1}{2d_i-1}} |g_{a_{i+1}}|^{(-\frac{1}{2d_i-1} - \frac{1}{2d_{i+1}+1})} |g_{a_{i+2}}|^{\frac{1}{2d_{i+1}+1}}$$

## 7. Parametrization of curves

The study that we have led in this paper allows us to give a natural parametrization to each pseudo-regular curve  $c$  in the following way:

**PROPOSITION 7.1** *Given any  $t_0$  in  $I$ ,*

1. *If  $\|c^{(1)}\|^2 \neq 0$ , i.e. if  $a_1 = 1$ , there exists a unique parametrization  $\gamma = c \circ \varphi$  such that*

$$\begin{cases} \langle \gamma^{(1)}, \gamma^{(1)} \rangle = \pm 1 \\ \varphi(0) = t_0 \\ \varphi' > 0 \end{cases}$$

2. *If  $k_{\max} \neq 0$  and  $a_1 \neq 1$  (i.e.  $\|c^{(1)}\|^2 = 0$ ), there exists a unique parametrization  $\gamma = c \circ \varphi$  such that*

$$\begin{cases} |g_{a_1}| = 1 \\ \varphi(0) = t_0 \\ \varphi' > 0 \end{cases}$$

3. *If  $k_{\max} = 0$ , i.e. if  $c$  is totally isotropic, there exists a unique parametrization  $\gamma = c \circ \varphi$  such that*

$$\begin{cases} \gamma^{(r+1)} \in F_{r-1} \\ \varphi(0) = t_0 \\ \varphi' > 0 \\ \varphi'(0) = 1 \end{cases}$$

*Proof.*

1. If  $\|c^{(1)}\|^2 \neq 0$ , i.e. if  $a_1 = 1$ , then, putting  $\gamma = c \circ \varphi$ , the vector  $\gamma^{(1)} = \varphi' \cdot c^{(1)}$  is unitary if and only if  $\varphi' = \pm \lambda_{(0)}$  by definition of  $\lambda_{(0)}$ .

The only solution with  $\varphi' > 0$  is then  $\varphi' = \lambda_{(0)}$  with initial condition  $\varphi(0) = t_0$ .

2. If  $\|c^{(1)}\|^2 = 0$ , and there exists an integer  $a_1$  such that  $g_{a_1} \neq 0$ , then it is easy to show that we have the following result (where we omit the proof):

**LEMMA 7.2** *Let  $\gamma = c \circ \varphi$  a parametrization of the curve  $c$ , and let us denote by  $g_{a_1}^{[c]}$  (respectively  $g_{a_1}^{[\gamma]}$ ) the Gram's determinant of  $F_{a_1}$  relatively to the curve  $c$  (respectively to the curve  $\gamma$ ).*

Then we have  $g_{a_1}^{[y]} = \varphi'^{2b_0} g_{a_1}^{[c]}$ .

Consequently, we may deduce

$$\begin{aligned} |g_{a_1}^{[y]}| = 1 &\iff \varphi'^{2b_0} |g_{a_1}^{[c]}| = 1 \\ &\iff \varphi'^{2b_0} = \lambda_{(0)}^{2b_0} \\ &\iff \varphi' = \lambda_{(0)} \text{ if we choose } \varphi' > 0 \end{aligned}$$

3. If  $c$  is totally isotropic, i.e. if  $k_{\max} = 0$ , then the proposition (5.1) gives us a unique function  $\lambda_{(0)}$  such that the vector  $n_{0;1} := \lambda_{(0)} e_{0;1} = \lambda_{(0)} c^{(1)}$  satisfies  $\begin{cases} n_{0;1}^{(r)} \in F_{r-1} \\ \lambda_{(0)}(0) = 1 \end{cases}$ . Therefore, if we put  $y = c \circ \varphi$  with  $\varphi' = \lambda_{(0)}$ , we obtain

$$\begin{aligned} y^{(1)} &= \varphi' c^{(1)} \\ &= \lambda_{(0)} c^{(1)} \\ &= n_{0;1} \end{aligned}$$

and thus,  $y^{(r+1)} \in F_{r-1}$ , and  $\varphi'(0) = 1$ . □

*Remarks.*

(1) Those parametrizations may be seen in a natural way as parametrizations "by arc-length", in the sense that they are nothing else but the parametrizations  $y = c \circ \varphi$  which give us  $y^{(1)} = \nu_1$ .

In other words, for the curve  $y$ , the new function  $\lambda_{(0)}$  is identically 1, so that the triangular matrix of change of basis from the basis  $\{y^{(1)}, \dots, y^{(r)}\}$  to the basis  $\{\nu_1, \dots, \nu_r\}$  begins with a block of coefficients 1 on its main diagonal.

(2) In the two first cases, the parametrization may be called unitary, since it allows us to norm the first non-isotropic vector that we meet.

In the third case, the parametrization is given by a function  $\lambda_{(0)}$  satisfying a first order differential equation. That is why this parametrization may be called an affine parametrization.

Note that the result we have obtained gives a generalization of the well-known result that any null geodesic has an affine parametrization, since a null geodesic is nothing but a pseudo-regular curve, 1-regular, with  $k_{\max} = 0$ .

(3) The functions which appear in the generalized Frenet's frame depend on the parametrization of the curve. We call generalized curvatures the functions which appear in the generalized Frenet's frame for the good parametrization of the curve that we have defined in this chapter.

### 8. Remarks on the invariants $\varkappa_k$ , $Y_{k,i}$ and $\zeta_k$ .

As for Frenet's frames, the curve is characterized by its (generalized) curvatures, i.e. the invariants  $Y_{l,i}$ ,  $\varkappa_l$ ,  $\zeta_l$ . More precisely:

**THEOREM 8.1** *Let  $\varkappa_k > 0$ ,  $Y_{k,i}$  and  $\zeta_k$  be smooth real functions, and  $(a_k)_{k \leq r}$  a sequence of integer. There exists a curve for which the subspaces  $F_{a_k}$  are not degenerate, and having the functions  $Y_{k,i}$ ,  $\varkappa_k$ ,  $\zeta_k$  for (generalized) curvatures. Furthermore, any two such curves differ by an isometry (i.e. a translation followed by an element of the orthogonal group).*

The proof of this proposition works exactly in the same way as in the Riemannian case, so we omit it (for details, cf. [Sp] second volume p. 1.43).

Notice also that the invariants  $\varkappa_k$  and  $Y_{k,i}$  are not exactly of the same nature. To see that, it is sufficient to remark that the functions  $\varkappa_k$  are positive by definition (see proposition 6.1), whereas the functions  $Y_{k,i}$  have, a priori, no sign.

As an example, let us consider the curve  $c$  of  $\mathbb{R}^{1,3}$  defined by

$$c : \mathbb{R} \rightarrow \mathbb{R}^{1,3} \\ t \mapsto \frac{1}{4} \left( \frac{4}{3} t^3 + t, 2t^2, \frac{4}{3} t^3 - t \right)$$

We have:

$$c^{(1)}(t) = \frac{1}{4} (4t^2 + 1, 4t, 4t^2 - 1)$$

$$c^{(2)}(t) = (2t, 1, 2t)$$

$$c^{(3)}(t) = (2, 0, 2)$$

Hence, the vectors  $c^{(1)}$ ,  $c^{(2)}$ ,  $c^{(3)}$  are clearly independent.

Respectively to this basis, the Gram's matrix of  $c$  is

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

In this example, we have (with our usual notations):

$$a_0 = 0, \quad a_1 = 3, \quad d_0 = 1, \quad n_{0,1} = c^{(1)}, \quad \pi_0 = c^{(2)}.$$

Since  $(c^{(2)})' = c^{(3)} + 0 \cdot c^{(1)}$ , we get  $Y_{0,1} = 0$ .

### 9. Generalizing the construction to arbitrary pseudo-Riemannian manifolds

Let  $(M, g)$  be any  $n$ -dimensional pseudo-Riemannian manifold. Let  $c : I \rightarrow M$  be a smooth curve in  $M$ , where  $I$  is some open interval in  $\mathbb{R}$ , containing 0.

The tangent space  $T_x M$  at  $c(0) = x \in M$  is given with a non-degenerate quadratic form  $g_x$ . For any  $t \in I$ , we may consider the parallel transport  $\tau(t): T_x M \rightarrow T_{c(t)} M$  along  $c$  with respect to the Levi-Civita connection  $D$  of  $g$ . At any point  $c(t)$ , the curve  $c$  admits a velocity vector  $\dot{c}(t) \in T_{c(t)} M$ , and iterated derivatives:

$$D^{(k)} \dot{c} = \underbrace{D_{dt}(D_{dt}(\dots(D_{dt} \dot{c})))}_k.$$

Then  $c^{(k)}(t) := \tau(t)^{-1} D^{(k-1)} \dot{c}$  are vectors in  $T_x M$ , and we may build a unique curve  $\gamma: I \rightarrow T_x M$  such that  $\gamma(0) = 0$  and  $\gamma^{(k)}(t) = c^{(k)}(t)$ .

Now, we may translate our construction to  $c$  by applying it to the curve  $\gamma$ . For that, we denote by  $F_k^c(t)$  (respectively  $F_k^\gamma(t)$ ) the space generated by  $\{\dot{c}(t), \dots, D^{(k-1)} \dot{c}(t)\}$  (respectively  $\{c^{(1)}(t), \dots, c^{(k)}(t)\}$ ).

**DEFINITION 9.1** *The curve  $c$  is said "pseudo-regular" if*

1.  $c$  is  $r$ -regular, i.e.  $F_{r-1}^c \cong F_r^c \cong F_{r+1}^c$
2. for any integer  $k \leq r$ , the Gram's determinant of  $F_k^c(t)$ , i.e. the determinant of the  $(k, k)$  matrix  $(\langle D^{(i)} \dot{c}(t), D^{(j)} \dot{c}(t) \rangle, i, j \in \{0, \dots, k-1\})$ , is either positive, identically zero, or negative.

Since the parallel transport  $\tau(t)$  is a linear isometry, we may deduce that

**PROPOSITION 9.2** *The curve  $c$  is pseudo-regular if and only if the curve  $\gamma$  is.*

Therefore, if the curve  $c$  is pseudo-regular, our previous work allows us to build a canonical frame  $\{v_1, \dots, v_r\}$  inside  $T_x M$ , associated to the curve  $\gamma$ . Consequently, if we put  $w_i(t) := \tau(t)(v_i(t))$ , we obtain a moving frame  $\{w_1, \dots, w_r\}$  for the curve  $c$ , with the same invariants  $Y_{k,j}, \kappa_k, \varsigma_k$  (which is a straightforward consequence of the fact that  $D_{dt} w_i(t) = \tau(t)(v_i'(t))$  since  $\tau(t)$  is linear).

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