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MARC HERZLICH

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MINIMAL SURFACES, THE DIRAC OPERATOR AND THE PENROSE INEQUALITY

Marc HERZLICH

1. Introduction

Asymptotically flat Riemannian manifolds are endowed with a rather intriguing numerical invariant, called the *mass*. It was defined by physicists in General Relativity in the early 60s [1] and is now known as the most important global invariant of these manifolds, as shown by the positive mass theorem proved in most cases by R. Schoen and S.-T. Yau with the help of minimal surfaces [15] and by E. Witten with spinors and the Dirac operator [16] (see also [3, 13]). This fundamental result states that, if scalar curvature is non-negative, then the mass must also be non-negative and cannot vanish unless the manifold is isometric to euclidean flat space. Obtaining better inequalities generally stands as hard work but R. Penrose [14] conjectured in 1973 that, in dimension 3, the mass m of any asymptotically flat manifold of non-negative scalar curvature should satisfy: $m \geqslant \frac{1}{4}\sqrt{A/\pi}$, where A is the area of the *outermost* minimal sphere in M. Moreover, equality case should be achieved if and only if the metric is a standard slice in the exterior Schwarzschild spacetime. Belief in this inequality was at first supported by physical considerations: in the relativistic setting, existence of a minimal surface is seen as strong evidence that the spacetime contains a black hole and the inequality yielded a link between its energy (mass) and its entropy (area).

Penrose's inequality has been recently rigorously proved by G. Huisken and T. Ilmanen [11] and independently by H. Bray [4]. However it still raises some interesting questions: first of all, is there any Penrose-type inequality for higher-dimensional asymptotically flat manifolds? In view of the existing proofs of the positive mass theorem, can we find a spinorial proof of the inequality, using Witten's techniques? And a last one: the *outermost* surface is defined as the only (collection of) minimal sphere(s) S such that the unbounded part of $M \setminus S$ does'nt contain any other minimal surface. Folklore examples show that the exact Penrose bound cannot be obtained with A replaced by the area of an

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arbitrary minimal sphere: imagine a manifold with one large minimal sphere and another smaller sphere in the exterior region of the first one; such an example can easily be made explicit by using metrics globally conformal to the euclidean space (the relevant positive mass theorem for $C^{1,1}$ metrics has been proved by H. Bray and F. Finster [5]). Nevertheless, the existence of a minimal sphere is strong evidence that our asymptotically flat manifold is in some sense far away from the euclidean space, since the latter doesn't contain any compact minimal surface. Its mass being positive due to the positive mass theorem, it is reasonable to ask whether we can also find a lower bound on the mass in this setting.

In this note we intend to bring a partial answer, extending a previous work of the author [8]: our result is valid in any dimension $n \geqslant 3$, uses the Dirac operator and spinors and provides a lower estimate by some function-theoretic quantity, area and a conformal invariant known as the *Yamabe number*. Although the minimal hypersurface must satisfy some extra condition (its Yamabe number must be positive), this still covers a very large class of examples, *e.g.* the case of strictly stable minimal hypersurfaces.

Hereafter, we consider a Riemannian asymptotically flat manifold (M,g), *i.e.* such that the complement of some compact set is diffeomorphic to the complement of a ball in $\mathbb{R}^n$ and the difference between the metric and the euclidean metric in this chart behaves like $r^{-\tau}$, its first derivatives like $r^{-\tau-1}$ and its second derivatives like $r^{-\tau-2}$, where $\tau > (n-2)/2$. If moreover the scalar curvature lives in the Lebesgue space L^1, then its mass is defined [3] as

$$m(g) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j \, d\,\mathrm{vol}_{S_r}$$

S_r being a coordinate sphere and ν its outer unit normal. If M has an (inner) compact boundary ∂M, we define

$$\mathcal{I}(\partial M) = (\mathrm{vol}(\partial M))^{\frac{1}{n-1}} \inf_{f\in C_c^\infty} \frac{\int_M |df|^2}{\int_{\partial M} f^2}.$$

The constant $\mathcal{I}(\partial M)$ is the normalized inverse of the norm of the Sobolev trace injection of $H^1_{-1}(M)$ (the closure of the space of compactly supported smooth functions on M with respect to the norm $\|df\|_{L^2}$) into $L^2(\partial M)$. It is scale-invariant from normalization and positive from abstract functional analysis. Last, the Yamabe invariant $Y(\partial M)$ of the boundary is the infimum of the quantity

$$\frac{\int_{\partial M} \mathrm{Scal}^\gamma \, d\,\mathrm{vol}_\gamma}{\mathrm{vol}_\gamma(\partial M)^{(n-3)/(n-1)}}$$

where γ varies among all metrics conformal to the metric induced by g on the boundary. It is a conformal invariant of the structure induced on the boundary. We can now state:

THEOREM 1.1. — *Let (M,g) be a Riemannian n-dim. asymptotically flat spin manifold with non-negative scalar curvature. Suppose M has a compact (inner) boundary ∂M*

that is minimal and has positive Yamabe invariant. Then, if mass m is defined,

$$m \geqslant \frac{(n-1)}{4\pi(n-2)}\mathscr{I}(\partial M)\left(1+2\left(\sqrt{\frac{n-2}{n-1}}Y(\partial M)\right)^{-1}\mathscr{I}(\partial M)\right)^{-1}\mathrm{vol}(\partial M)^{\frac{n-2}{n-1}}.$$

Since the function-theoretic quantity $\mathscr{I}$ is scale-invariant and the Yamabe quotient is a conformal invariant, the lower bound we obtain depends on the volume of the boundary only in the $\mathrm{vol}(\partial M)^{\frac{n-2}{n-1}}$-term, as expected from scaling considerations. In dimension 3, this statement reduces to the one proved in [8] because the Yamabe number is a topological invariant due to the Gauss-Bonnet theorem. This latter result was meant as a step towards a full proof of the Penrose inequality, since no such proof was available at that time (be careful our present definition of $\mathscr{I}$ differs from the definition of the quantity called σ in that paper by a $\sqrt{\pi}$-term). As in dimension 3, equality case is known in the general setting:

THEOREM 1.2. — *In case of equality above, the Riemannian manifold (M,g) is isometric to the n-dimensional exterior Schwarzschild-slice, of mass m, $M = \mathbb{R}^n \setminus B_0(R)$ with*

$$R = \left(\frac{4\pi m}{(n-1)\omega_{n-1}}\right)^{\frac{1}{n-2}} \quad \textit{and}\; g = \left(1+\left(\frac{R}{r}\right)^{n-2}\right)^{\frac{4}{n-2}} \textit{eucl.}$$

2. Proof of the theorem

The proof of our result mainly elaborates on the proof given in [8]. It relies on three steps: proving a positive mass theorem for a metric with boundary; finding a distinguished scalar flat metric in the conformal class of g that plays for it the same role the euclidean metric plays for the Schwarzschild metric and applying to it the positive mass theorem and estimating the difference between the masses. The novelty lies here in the introduction of the Yamabe invariant and in the treatment of the equality case in section 3 below.

PROPOSITION 2.1. — *Let (M,g) be an asymptotically flat spin manifold with nonnegative scalar curvature and such that the mean curvature H of the boundary satisfies*

$$H \leqslant (\mathrm{vol}_g(\partial M))^{-\frac{1}{n-1}}\sqrt{\frac{n-1}{n-2}}Y(\partial M)$$

then its mass is nonnegative. If mass is zero, then it is isometric to the complement of a round ball in the flat euclidean space.

We shall give only short indications on the proof of this theorem, which uses the spinorial technique introduced by Witten and follows quite closely [8]. If ψ is any asymptotically constant spinor on M and harmonic for the Dirac operator (asymptotic to some constant spinor ψ_0, say),

$$4\pi|\psi_0|^2 m = \int_M |\nabla\psi|^2 + \frac{1}{4}\,\mathrm{Scal}^g\,|\psi|^2 - \sum_n \int_{\partial M} (\lambda_n + \frac{1}{2}\sup H)|\psi_n|^2,$$

where λ_n are the eigenvalues of the Dirac operator on the boundary and ψ_n's are the components of ψ relative to the decomposition of the spinor bundle in eigensubbundles on ∂M. Proving the positive mass theorem then amounts to find a spinor field ψ, asymptotically constant and harmonic, such that the boundary terms above are non-negative. This can be done by looking for ψ such that all ψ_n's corresponding to positive eigenvalues vanish (this is the Atiyah-Patodi-Singer boundary condition; it is elliptic for the Dirac operator of M [2]). The remaining terms are nonnegative because the first eigenvalue λ_1 of the Dirac operator on the boundary satisfies the *Hijazi inequality* [9, 10]:

$$\mathrm{vol}(\partial M)^{\frac{1}{n-1}}|\lambda_1| \geqslant \frac{1}{2}\sqrt{\frac{n-1}{n-2}Y(\partial M)}.$$

The nonnegativity of the boundary term provides both the existence of the required spinor field (by a Lax-Milgram argument) and the positive mass theorem. The equality case follows from section 3 below and we refer to [8] for more details. ■

The construction of the distinguished conformal metric (we shall thereafter denote it by $\bar g = \Phi^{4/(n-2)}$ where $\Phi = 1 + u$) follows from a calculus of variations procedure: we require $\bar g$ to be asymptotically flat and scalar flat, *i.e.*

$$\Delta_g u + \frac{n-2}{4(n-1)}\,\mathrm{Scal}^g(1+u) = 0, \tag{1}$$

and the mean curvature $\bar H$ of its boundary to be constant, equal to $\left(\mathrm{vol}_{\bar g}(\partial M)\right)^{-\frac{1}{n-1}}$ (a relation that any round sphere in the flat space satisfies). This translates as

$$d\Phi(\nu) = \frac{1}{2}\sqrt{\frac{n-2}{n-1}Y(\partial M)}\left(\int_{\partial M}\Phi^{\frac{2(n-1)}{n-2}}\right)^{-\frac{1}{n-1}}\Phi^{n/(n-2)}\,. \tag{2}$$

They are the Euler-Lagrange equations associated to the functional $\mathcal{Q}(f)$ defined as

$$\frac{1}{2}\int_M |df|^2 + \frac{n-2}{8(n-1)}\int_M \mathrm{Scal}^g(1+f)^2 + \frac{1}{4}\sqrt{\frac{n-2}{n-1}Y(\partial M)}\left(\int_{\partial M}(1+f)^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}}$$

on the weighted Sobolev space $H^1_{-1}(M) = \{f \in H^1_{\mathrm{loc}}\,,\, \|df\|_{L^2} + \|\frac{f}{r^*}\|_{L^2} < \infty\}$, where $r^*(x) = (1 + d(o,x)^2)^{\frac{1}{2}}$, o being any fixed base point in M. A minimizer u can be found by the usual convergence procedure. It is a solution of the boundary value problem if and only if we can prove that $1 + u$ is not identically zero on the boundary (as in the classical Yamabe problem, the minimizing sequence can here converge to -1 since the injection of $H^1_{-1}(M)$ into $L^q(\partial M)$, $q = \frac{2(n-1)}{n-2}$, is continuous but not compact). An argument similar to the one given in [8] handles that problem: if u is identically -1 on the boundary, then for any solution h in the weighted space of

$$\Delta h + \frac{n-2}{4(n-1)}\,\mathrm{Scal}^g\, h = 0 \text{ on } M,\, dh(\nu) = -1 \text{ on } \partial M,$$

we may find an $\varepsilon \neq 0$ such that $\mathscr{Q}(u+\varepsilon h) < \mathscr{Q}(u)$, thus contradicting the minimal property of u. Once this is done, the maximum principle shows $1+u$ never vanishes, neither on the boundary nor in M, hence providing us with the desired solution. ■

End of the proof of Theorem 1.1. From a straightforward computation which we leave to the reader, one gets that $m(g) - m(\bar{g}) = \frac{n-1}{2\pi(n-2)}\mathscr{Q}(u)$. We shall now prove:

$$\min \mathscr{Q} \geqslant \frac{\mathscr{I}(\partial M)}{2+4\left(\sqrt{\frac{n-2}{n-1}Y(\partial M)}\right)^{-1}\mathscr{I}(\partial M)} \operatorname{vol}_g(\partial M)^{\frac{n-2}{n-1}}.$$

Indeed, suppose $\inf \mathscr{Q} \leqslant \eta \operatorname{vol}_g(\partial M)^{\frac{n-2}{n-1}}$ where η is a small positive constant. We shall now prove that η cannot be smaller than an expression involving the Sobolev ratio $\mathscr{I}$. From the assumption and Hölder's inequality, we can find u such that

$$\frac{1}{2}\int_M |du|^2 + \frac{1}{4}\sqrt{\frac{n-2}{n-1}Y(\partial M)} \operatorname{vol}_g(\partial M)^{\frac{-1}{n-1}}\left(\int_{\partial M}(1+u)^2\right) \leqslant \eta \operatorname{vol}_g(\partial M)^{\frac{n-2}{n-1}}.$$

As $(1+u)^2 \geqslant 1 - \frac{1}{\varepsilon} + (1-\varepsilon)u^2$, for any $\varepsilon > 0$,

$$\frac{1}{2}\int_M |du|^2 + \frac{1}{4}\sqrt{\frac{n-2}{n-1}Y(\partial M)}\left((1-\varepsilon^{-1})\operatorname{vol}_g(\partial M)^{\frac{n-2}{n-1}}\right)$$
$$+\frac{1}{4}\sqrt{\frac{n-2}{n-1}Y(\partial M)}\left((1-\varepsilon)\operatorname{vol}_g(\partial M)^{-\frac{1}{n-1}}\int_{\partial M}u^2\right) \leqslant \eta \operatorname{vol}_g(\partial M)^{\frac{n-2}{n-1}}.$$

so that

$$\int_M |du|^2 + \frac{1-\varepsilon}{2}\operatorname{vol}_g(\partial M)^{-\frac{1}{n-1}}\sqrt{\frac{n-2}{n-1}Y(\partial M)}\int_{\partial M}u^2$$
$$\leqslant \left(2\eta - \frac{1}{2}\sqrt{\frac{n-2}{n-1}Y(\partial M)}\left(1-\varepsilon^{-1}\right)\right)\operatorname{vol}_g(\partial M)^{\frac{n-2}{n-1}}.$$

The left-hand side is nonnegative if $\frac{1}{2}\sqrt{\frac{n-2}{n-1}Y(\partial M)}(\varepsilon-1) \leqslant \mathscr{I}(\partial M)$, and taking the maximum of the right hand side gives the required inequality. ■

3. The case of equality

If the mass is equal to the lower bound of the theorem, the proof of the last lemma above shows that the scalar curvature of g must vanish, for, if not the case, the quotient

$$\mathscr{I}(\partial M)^{-1}(\operatorname{vol}(\partial M))^{\frac{1}{n-1}} \inf_{f\in C_c^\infty} \frac{\int_M |df|^2}{\int_{\partial M} f^2}$$

would be strictly smaller than 1, thus contradicting the definition of $\mathscr{I}(\partial M)$. Moreover, the difference between the masses equals the mass of g, yielding that the mass of $\bar{g}$ vanishes. Equality in the positive mass theorem above implies $(M,\bar{g})$ is a flat (curvature zero) manifold, its boundary has constant mean curvature

$$\bar{H} = (\mathrm{vol}_{\bar{g}}(\partial M))^{-\frac{1}{n-1}} \sqrt{\frac{n-1}{n-2} Y(\partial M)}$$

and achieves the equality case in the Hijazi inequality. From [10], the induced metric on the boundary must then be Einstein with positive scalar curvature and by Obata's theorem, it is a Yamabe metric [12]. From the Gauss equation, we get

$$\mathrm{Ric}^{\bar{g},\partial M} = \frac{\mathrm{Scal}^{\bar{g},\partial M}}{n-1}\bar{g} = (\bar{H})\theta^{\bar{g},\partial M} - \theta^{\bar{g},\partial M} \circ \theta^{\bar{g},\partial M}$$

where $\theta^{\bar{g},\partial M}$ is the second fundamental form of ∂M in the metric $\bar{g}$. Let (e_i) be any diagonalizing basis for $\theta^{\bar{g},\partial M}$ and (μ_i) the associated system of eigenvalues. From the Gauss equation above for the Ricci curvature, we get that

$$\bar{H}\mu_i - \mu_i^2 = \frac{\mathrm{Scal}^{\bar{g},\partial M}}{n-1}.$$

Writing

$$\mathrm{Scal}^{\bar{g},\partial M} = \kappa^2(n-1)(n-2),$$

and using the fact that $\bar{g}$ on the boundary is Yamabe and that the mean curvature is known, we see that any eigenvalue μ_i is a root of the polynomial

$$X^2 - (n-1)\kappa X - (n-2)\kappa^2.$$

It has two roots, κ and $(n-2)\kappa$, hence *a priori* two possible values for the μ_i. But the sum of the eigenvalues is the mean curvature $\bar{H}$ which is equal to $\kappa(n-1)$. This yields that the boundary is totally umbilic in $\bar{g}$ (all eigenvalues of the second fundamental form are equal), and the Gauss equation again shows that it has constant curvature. The boundary must then be isometric to a quotient of the round sphere.

If $v(r)$ is the volume of distance spheres from the boundary, the Gromov-Heintze-Karcher inequality then shows that $v(r)/r^{n-1}$ is a monotone non-increasing function which starts off at value $\mathrm{vol}(S^{n-1}/\Gamma,\mathrm{can})$ and ends at $\mathrm{vol}(S^{n-1},\mathrm{can})$ (from asymptotic flatness). Hence Γ is trivial. Gluing in a flat ball produces a complete flat manifold with zero mass. The classical positive mass theorem then implies that is is isometric to euclidean space. This enables us to conclude that M has the required topology and $\bar{g}$ is the euclidean metric.

Moreover, the metric g is globally conformal to $\bar{g}$, it has vanishing scalar curvature and the boundary is minimal, whence we get that the conformal factor $1+v$ (v belonging to the weighted Sobolev space) relating the metric g and the Schwarzschild-like metric (as defined in the statement of the equality case) is a solution of the boundary value problem: $\Delta v = 0$ on M, $dv(\nu) = 0$ on ∂M. This easily implies that $v = 0$. ∎

4. Final comments

Our results still leave our questions partially unanswered. First, we are still lacking a complete spinorial proof of the Penrose inequality. The method used here and in [8] doesn't seem to bring anymore information when the minimal submanifold is the outermost one: P. Tod (private communication) constructed examples where $\mathscr{I}$ can be made very small while keeping mass constant. Our methods also seem to be useless when the boundary is a torus, as the Dirac operator has a non-trivial kernel there. However, work of M. Cai and G. Galloway shows that mass should also be very large in this case [6, 7].

The appearance of the Yamabe invariant of the boundary in our result may be seen as some evidence in favor of the fact that generalization of the Penrose inequality in higher dimensions might be more complicated than expected. Dimension 3 has indeed the feature that the boundary has a much simpler geometry than in the general case. This appeared crucially in [8] as well as in Bray/Huisken-Ilmanen proofs through an essential use of the Gauss-Bonnet formula. In dimensions $n \geqslant 4$, some extra work is probably needed to gain more intuition and before stating any conjecture.

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Marc HERZLICH
Département des Sciences Mathématiques
Géométrie - Topologie et Algèbre
UMR 5030 CNRS & Université Montpellier II
MONTPELLIER (France)

herzlich@math.univ-montp2.fr