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Around heat decay on forms and relations of nilpotent Lie groups

AROUND HEAT DECAY ON FORMS AND RELATIONS OF NILPOTENT LIE GROUPS

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Abstract

One knows that the large time heat decay exponent on a nilpotent group is given by half the growing rate of the volume of its large balls. This work deals with the similar problem of trying to interpret geometrically the heat decay on (1-)forms. We will show how it is (partially) related to the depth of the relations required to define the group.

Contents

1. Introduction

Let (M, g) be a compact riemannian manifold with fundamental group $\Gamma = \pi_1(M)$. We denote by $\tilde{M}$ its universal cover and $\tilde{g}$ the pull back metric. The de Rham differential $d : \Omega^p(\tilde{M}) \to \Omega^{p+1}(\tilde{M})$ acts between p and $p + 1$-forms of $\tilde{M}$.

We are interested in the spectrum of the p-Laplacian $\Delta_p = d\delta + \delta d$ acting on L^2 p-forms of $\tilde{M}$. More precisely, one would like to know what kind of geometric information about M or Γ is encoded in the near zero spectrum of Δ_p.

The first result of this kind deals with the invertibility of Δ_0, the Laplacian on functions.

THEOREM 1.1 (Brooks [2]). — *0 belongs to the spectrum of Δ_0 iff Γ is amenable. (That means that Γ may be exhausted by parts Γ_i such that $\dfrac{\#(\partial\Gamma_i)}{\#(\Gamma_i)} \to 0$.)*

Roughly, the argument that Δ_0 is not invertible if Γ is amenable is based on the idea that one can approach the constant function on $\tilde{M}$ by compactly supported test functions with small differential with respect to their integral norm. These approximative units can also be used to cut of the pull back of any harmonic *form* of M. This shows therefore that if Γ is amenable and $H^p(M, \mathbb{R}) \neq 0$ then 0 belongs to the spectrum of Δ_p on $\tilde{M}$. This is satisfied for any $p \leq \dim M$ in the case we will study of $M = G/\Gamma$, with Γ a cocompact group of a nilpotent Lie group G (see [10]).

Note also that in these examples, 0 is never *embedded* in $\mathrm{Sp}(\Delta_p)$. There is no harmonic L^2-form on $\tilde{M} = G$, otherwise they would have to be invariant through the Killing direction associated to the non-vanishing center of these groups. More precise information about the density of $\mathrm{Sp}(\Delta_p)$ near 0 is obtained with the help of the notions of Γ-dimension, and Γ-trace.

Recall (see [1]) that in the case of a Γ-invariant smoothing operator S, acting on $\Omega^*(\tilde{M})$, one can consider the average diagonal trace density of its Schwarz kernel K_S, namely

$$\mathrm{Tr}_\Gamma(S) = \int_{\mathscr{F}} \mathrm{Tr}(K_S(x, x))\,\mathrm{dvol}\,,$$

where $\mathscr{F}$ stands for any fundamental domain of the Γ-action. For example S can be the heat operator $e^{-t\Delta_p}$ or the spectral projection $\Pi(\Delta_p \leq \lambda)$. Actually we can first take profit of the orthogonal splitting of Δ_p in δd and $d\delta$ to slightly precise the analysis. So the basic operators in concern will be δd restricted to $H = (\ker d)^\perp$ and the associated

heat $e^{-t\delta d}$ on H. Since these the second operator is the Laplace transform of the former, the two asymptotics of $\text{Tr}_\Gamma(e^{-t\delta d})$ when $t \to +\infty$ and of $F_{\delta d}(\lambda) = \text{Tr}_\Gamma(\Pi(\delta d \leqslant \lambda))$ when $\lambda \to 0$ are related in the following way (see appendix of [14]). One has

$$\text{Tr}_\Gamma(e^{-t\Delta_p}\Pi_H) \asymp t^{-\alpha_p/2} \text{ when } t \to +\infty$$

iff

$$F_{\delta d}(\lambda) \asymp \lambda^{\alpha_p/2} \text{ when } \lambda \to 0.$$

A more general definition of α_p, called the pth Novikov-Shubin number of M, is

$$\alpha_p = 2 \liminf_{\lambda \to 0}(\ln F_{\delta d}(\lambda)/\ln \lambda) \in [0, +\infty] \tag{1}$$

(also it seems there is no known geometric example where this liminf is not an actual limit.)

The first one, α_0, describing heat decay on functions is known.

THEOREM 1.2 (Varopoulos [27], Gromov). — $\alpha_0 < +\infty$ *iff Γ has polynomial growing that is iff Γ is (virtually) nilpotent, and then*

$$\alpha_0 = \text{growing rate of } \Gamma = \lim_{R \to +\infty} \frac{\ln \text{Vol} \cdot (B(0, R))}{\ln R} \in \mathbb{N}.$$

It appears then that α_0 is a rough invariant of M, independent of the metric g, depending only here on the large scale structure of $\Gamma = \pi_1(M)$. For the other α_p, one has

THEOREM 1.3 (Gromov-Shubin, Efremov [14]). — α_p *are homotopy invariants of M.*

In fact it turns out that α_p depends on the (rational) homotopy type of M up to degree p. In particular the next one α_1, describing heat on 1-forms, depends only on $\pi_1(M)$, and is thus of particular interest. We will mainly focus on its study here, although some sections will deal with related fields.

The (very partial) results are actually disseminated along the paper. We will illustrate them with (carefully chosen) examples. This work rely on the use of constructions that have already been exposed in [25]. Anyway we recall them first.

2. Some differential geometry on C-C manifolds

2.1. Dilations and differential on graded groups

As time increases, heat spreads in larger and larger domains. It is therefore convenient to re-scale the phenomena. This requires some dilation acting on the space.

Typical groups admitting such a structure are graded nilpotent (Lie) groups. These are groups G such that their Lie algebra $\mathfrak{g}$ splits in

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i \text{ with } [\mathfrak{g}_1, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}.$$

We call r the *rank* of G. The multiplication by i on $\mathfrak{g}_i$ induces a family of dilations h_ε on G by $h_\varepsilon(\exp X_i) = \varepsilon^i(\exp X_i)$ for $X_i \in \mathfrak{g}_i$. The function $w = i$ on $\mathfrak{g}_i$ is called the *weight* of vectors, and similarly on covectors. It can be extended to the whole differential algebra of G. Forms of weight p are spanned by

$$\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_k} \text{ with } \sum_{j=1}^{k} w(\theta^{i_j}) = p.$$

De Rham's differential d is not homogeneous with respect to the weight. In fact it splits like

$$d = d_0 + d_1 + \cdots + d_r \tag{2}$$

with components d_k increasing the weight by k. Indeed on functions $d_0 f = 0$ since there is no 1-forms of weight 0, whereas $d_k f$ for $1 \leqslant k \leqslant r$ is just $d f$ restricted to vectors of weight k. Now, if α is a (left) invariant form of weight p one has $d(f\alpha) = df \wedge \alpha + f d\alpha$. Then for $k \geqslant 1$, $d_k(f\alpha) = d_k f \wedge \alpha$, while

$$d_0(f\alpha) = f d_0 \alpha = f d\alpha$$

is an *algebraic* (order 0) operator. The assertion that for invariant forms $d\alpha = d_0\alpha$ is of the same weight as α is exactly dual to the property that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, as is seen starting from invariant 1-forms for which $d\theta(X, Y) = -\theta([X, Y])$ on invariant vectors.

Now let G be endowed with an invariant metric such that the $\mathfrak{g}_i$ are mutually orthogonal. If α is a form of weight p, one has point-wise $||h_\varepsilon^* \alpha|| = \varepsilon^p ||\alpha||$, so that

$$||d(h_\varepsilon^* \alpha)|| \geqslant \varepsilon^p ||d_0 \alpha|| = ||d_0(h_\varepsilon^* \alpha)||.$$

Therefore d_0, the zero order part of d, invisible on functions and $\mathbb{R}^n$, is an obstruction of decreasing the differential of forms through dilations. In other words, the spectral spaces $E(\delta d \leqslant \lambda)$ have to contract on ker d_0 in some sense when $\lambda \to 0$.

2.2. Cutting d_0 out of d

One would like to find a way of working directly on ker d_0 when dealing with small spectrum problems on our groups. The spectral sequence technique achieves this from the algebraic viewpoint. In fact the previous discussion pointed out a natural decreasing filtration of $\Omega^* G$ by the spaces F^p of forms of weight $\geqslant p$. The decomposition (2) says that d respects it. Thus we have a filtered complex giving rise to a sequence of spaces converging to the (graded part of) de Rham cohomology of G.

We briefly recall this technique, d_0 is interpreted as d acting on F^p/F^{p+1}. Let $E_0 =$ ker $d_0/$ Im d_0 denotes its cohomology. This is the bundle which fiber is the Lie algebra cohomology $H^*(\mathfrak{g}, \mathbb{R})$. We get then d acting on the quotient space E_0, whose cohomology is called E_1, etc. Note that this process produces successive quotients by image of *differential* operators, not very easy to handle analytically. The fact that d_0 is algebraic will allow to perform part of the construction *inside* the de Rham complex. This will be useful to keep track of the original problem of analyzing the small spectrum of d.

In fact the constructions apply to more general spaces than graded groups called Carnot-Carathéodory manifolds which we introduce now.

2.2.1. C-C manifolds and graded groups.

DEFINITION 2.1. — *An equiregular Carnot-Carathéodory structure on a manifold M consists in an (increasing) filtration of TM by bundles H_i such that*

$$[H_1, H_i] \subset H_{i+1} \text{ with } H_r = TM \text{ for some } r. \tag{3}$$

Remarks 2.2

– Strictly speaking this definition is more general than the usual one given in [13] for instance. Here we don't require TM to be generated by H_1 through brackets, that is H_1 to be an Hörmander distribution and $H_{i+1} = [H_1, H_i]$. This will allow us to change the grading if necessary.

– The equiregularity assumption means that we suppose the distributions H_i are bundles. Their dimensions are not allowed to jump.

The link with our previous discussion is the well known fact that an equiregular C-C manifold admits a tangent graded nilpotent Lie algebra $\mathfrak{g}_{x_0}$ at each point. Indeed due to (3) the usual bracket on TM admits a quotient map

$$[\ ,\ ]_0 : H_p/H_{p-1} \times H_q/H_{q-1} \to H_{p+q}/H_{p+q-1}.$$

which is an algebraic operator (order 0), since $[X, fY]_0 = \Pi_{H_{p+q}/H_{p+q-1}}(f[X, Y] + (X.f)) = f[X, Y]_0$. Therefore $[\ ,\ ]_0$ defines a graded Lie algebra structure on

$$\mathfrak{g}_{x_0} = \bigoplus_{i=1}^{r} \mathfrak{g}_i \text{ with } \mathfrak{g}_i = H_i/H_{i-1} \text{ at } x_0. \tag{4}$$

We recover also a decreasing filtration of $\Lambda^* T^* M$. Firstly vectors in H_p are called of weight $\leqslant p$, and dually a differential form of degree k is of weight $\geqslant p$ iff it vanishes on all sets of k vectors of total weight $< p$. Let F_p be the bundle of forms of weight $\geqslant p$. Again F_p is d-stable as come from (3), and we get a filtered complex on $\Omega^* M$. As before $d_0 = d$, acting on F_p/F_{p-1}, is an algebraic operator. This zero order part of d is identified with the previous d_0 we introduced on $\mathfrak{g}_{x_0}$.

Examples 2.3

– Let D^k be a distribution of k-planes in $\mathbb{R}^n$. Generically in the jets of the vector fields generating it, D^k will not be integrable and even will be an Hörmander distribution. This gives a lot of (true) C-C structures on $\mathbb{R}^n$.

– We just describe some classical cases. An hyperplane distribution D^{2n} in $\mathbb{R}^{2n+1}$ is called a contact structure when the Lie bracket $[\ ,\ ]_0$ is non-degenerate on D. Reducing this bilinear form gives that $G_{x_0} = \exp \mathfrak{g}_{x_0}$ is isomorphic to the Heisenberg group. Even more, thanks to Darboux' theorem the contact structure itself is locally isomorphic to its tangent group G_{x_0}.

– A generic D^2 in $\mathbb{R}^4$, called an Engel's structure, also gives rise to a unique tangent $\mathfrak{g}_{x_0}$. This is the 4-dimensional Lie algebra of rank 3 with the following relations $[X,Y]_0 = Z$, $[X,Z]_0 = T$ and 0 elsewhere. Here $D = H_1 = \text{span}(X,Y)$, $H_2 = [D,D] = \text{span}(X,Y,Z)$, and $H_3 = \mathbb{R}^4 = [D,[D,D]] = \text{span}(X,Y,Z,T)$. Again D is an Hörmander distribution, but we can look this structure differently. Consider $H_1' = \text{span}(X)$, $H_2' = \text{span}(X,Y) = D$, $H_3' = H_2$, $H_4' = H_3 = \mathbb{R}^4$. This C-C structure has the same model group G_{x_0} as before, but with a different grading. This will give another way of rescaling the asymptotic problems we consider.

2.2.2. One homotopy and three complexes.

We now show how to retract the de Rham complex of a C-C manifold M on its E_0 part. Recall that $E_0 = \ker d_0 / \text{Im } d_0 = H^*(\mathfrak{g}_{x_0})$, where $\mathfrak{g}_{x_0}$ is tangent Lie algebra at x_0. In order to work on bundles we will assume now that $\dim(E_0)$ is constant (always satisfied on an open dense set of M).

DEFINITION 2.4. — *A C-C structure is E_0-regular if E_0 is a bundle.*

Choose a metric on M. Let V_i be the orthogonal supplement of H_i in H_{i-1}. By (4), this fixes an isometry between $T_{x_0}M = \bigoplus_{i=1}^r V_i$ and $\mathfrak{g}_{x_0} = \bigoplus_{i=1}^r \mathfrak{g}_i$ and fixes the weight on TM. This allows to see d_0 as acting between spaces of (true, not quotiented) forms on M of given pure weight. We identify E_0 with $\ker d_0 \cap \ker \delta_0 = \ker d_0 \cap (\text{Im } d_0)^\perp$, where δ_0 is the metric adjoint of d_0. We also get a partial inverse of d_0 by $d_0^{-1} = (\delta_0 d_0)^{-1}\delta_0$. We can use this d_0^{-1} as a partial inverse of d.

Define the following retraction on $\Omega^* M$

$$r = \text{Id} - d_0^{-1}d - dd_0^{-1}. \tag{5}$$

This is an homotopical equivalence (preserving d and the cohomology) whose zero order term is $r_0 = \text{Id} - d_0^{-1}d_0 - d_0 d_0^{-1} = \Pi_{E_0}$ a projection, whereas $r - r_0 = -d_0^{-1}(d - d_0) - (d - d_0)d_0^{-1}$ strictly increases the weight and is therefore nilpotent. This incites to iterate the homotopy r in order to retract the de Rham complex on the smallest possible space. The maps r^k actually converge toward a map that certainly have to be both an homotopical equivalence and a projection to a sub-complex along an other. The following lemma is useful to identify quickly the limit spaces and operators.

LEMMA 2.5. — *The map $d_0^{-1}d$ induces an isomorphism on Im d_0^{-1}, whose inverse is a differential operators P.*

Proof. — ([25]) On Im d_0^{-1}, one can write $d_0^{-1}d = \mathrm{Id} + D$ where $D = d_0^{-1}(d - d_0)$ is nilpotent since it strictly increases the weight of forms. One has then

$$P = (d_0^{-1}d)^{-1} = \sum_{k=0}^{d(M)} (-1)^k D^k$$

where $d(M) = \mathrm{weight}(\Lambda^{\max}T^*M)$. □

Define $Q = Pd_0^{-1}$ with P as in the lemma. One has then.

THEOREM 2.6. — *Let M be a E_0-regular C-C space.*

1. *Ω^*M splits in the direct sum of two sub-complexes*

$$E = \ker d_0^{-1} \cap \ker(d_0^{-1}d) \text{ and } F = \mathrm{Im}\, d_0^{-1} + \mathrm{Im}(dd_0^{-1}).$$

The projection Π_F on F along E is given by the differential operator $Qd + dQ$, with Q as above.

2. *The homotopies r^k converge to Π_E the projection on E along F. It is the homotopical equivalence given by the differential operator $\mathrm{Id} - \Pi_F = \mathrm{Id} - Qd - dQ$.*

3. *One has*

$$\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0} \text{ and } \Pi_E\Pi_{E_0}\Pi_E = \Pi_E, \tag{6}$$

saying that E and E_0 are in bijection, and that Π_E restricted to E_0 and Π_{E_0} restricted to E are inverse maps of each other. In particular the complex (E, d) is conjugated to another one (E_0, d_c) with $d_c = \Pi_{E_0}d\Pi_E\Pi_{E_0}$.

The constructions are summarized in the diagram

A short proof is given in [25](thm 1). We don't repeat it here, but insist on particular points.

Observe that E is a space of forms satisfying some differential equations $(E = \ker d_0^{-1} \cap \ker(d_0^{-1}d))$ that projects bijectively onto the algebraic $E_0 = \ker d_0 \cap \ker d_0^{-1}$. Even more, the equation $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ says that Π_E restricted to E_0 is $\mathrm{Id} + a$ $(E_0)^{\perp}$ part. In other words E is a particular space of liftings (extensions) of E_0. For

computations, we notice from (5) that the retraction r preserves the space $(\ker d_0)^{-1} = \ker \delta_0 \supset E_0$ on which

$$r = \mathrm{Id} - d_0^{-1}(d - d_0)$$

with $d_0^{-1}(d - d_0)$ strictly increasing the weight. Starting from some form α of weight p in E_0 this gives by iteration the successive extensions in greater weight by

$$\begin{cases} (\Pi_E \alpha)_p = \alpha \\ (\Pi_E \alpha)_{p+k+1} = -d_0^{-1}\left(\sum_{l=1}^{r} d_l (\Pi_E \alpha)_{p+k+1-l}\right) \end{cases} \tag{7}$$

where d_l is the part of d that increases the weight by l.

Remarks 2.7

– At this point it is clear that this construction is everything but a new idea, merely an hold one in a fancy dress. Namely this Π_E is a realization of the first homotopical equivalence that is predicted by the general fact that the spectral sequence starting on E_0 will finally compute (the graded part of) the cohomology. The map d_c describes the remaining obstructions of extending a form in E_0 to a true closed one, put all together.

– Formally at least it is tentative to carry on with the retraction process by replacing d_0 by $d_{c,1}$, the part of d_c that increases the weight by one. Choosing a supplement space to $\ker d_{c,1}$ as the L^2 closure of $\mathrm{Im}\, d_{c,1}$, leads to a (now not bounded) partial inverse $d_{c,1}^{-1}$ of $d_{c,1}$ and an another retraction $r_1 = \mathrm{Id} - d_{c,1}^{-1} d_c - d_c d_{c,1}^{-1}$. It should be iterated and so on. Of course this does not seem realistic analytically. Anyway some analysis (yet a little bit mysterious) related to the spectral sequence structure is working. We will describe it in section 5.

– We didn't take care of the invariance of d_c on the choice of metric, mainly because it is not invariant in general ! This doesn't matter here. Again this construction is taken as a convenient approximation of the underlying invariant spectral sequence, which is hopefully related to the (even homotopically) invariant problem of studying the asymptotic heat decays.

We conclude this general section with a remark on duality. As observed in [25], the complex (E_0, d_c) is Hodge $*$-dual. This should look rather surprising since the retraction map $r = \mathrm{Id} - d_0^{-1} d - d d_0^{-1}$ breaks the symmetry between d_0 and δ_0. Anyway, we have

PROPOSITION 2.8.

1. $*\delta_0 = (-1)^{k+1} d_0 *$ on E_0^k and $*$ preserves E_0.

2. $*E$ is orthogonal to F, equivalently the pairing $(\alpha, \beta) \to \int_M \alpha \wedge \beta$ vanishes on $E \times F$. One has $*\Pi'_E = \Pi_E *$, where Π'_E is the formal adjoint of Π_E.

3. $*\delta_E = (-1)^{k+1} d_E *$ on k-forms. Similarly $*\delta_c = (-1)^{k+1} d_c *$ on E_0^k.

Proof.

– The first observation comes from the same (point-wise) duality standing at the Lie algebra level on $\mathfrak{g}_{x_0}$.

– By definition $F = \operatorname{Im} \delta_0 + \operatorname{Im} d\delta_0$ and therefore

$$*(F^\perp) = *(\ker d_0 \cap \ker d_0\delta) = \ker \delta_0 \cap \ker d\delta_0 = E.$$

Then,

$$(\alpha, \Pi'_E\beta) = (\Pi_E\alpha, \beta) = \int_M \Pi_E\alpha \wedge *\beta$$

$$= \int_M \Pi_E\alpha \wedge \Pi_E * \beta = \int_M \alpha \wedge \Pi_E * \beta$$

$$= (\alpha, *^{-1}\Pi_E * \beta).$$

– From $d_E = d\Pi_E = \Pi_E d$, we find that

$$*\delta_E = *\Pi'_E\delta = \Pi_E * \delta = (-1)^{k+1}\Pi_E d* = (-1)^{k+1}d_E*,$$

and similarly for δ_c starting from $d_c = \Pi_{E_0}d_E\Pi_{E_0}$ and using that Π_{E_0} commutes with $*$ and is self-adjoint, being an orthogonal projection. □

2.3. A few examples

We briefly describe some specific cases of the previous constructions.

• On $\mathbb{R}^n$, or M^n with the trivial C-C structure ($H_1 = TM$), one has $d_0 = 0$, and $d_c = d$.

• Let (M^{2n+1}, H) be a contact structure. Let θ be such that $H = \ker \theta$. Ω^*M splits in $\Omega^* H + \theta \wedge \Omega^* H$ and the 0 order part of $d(\alpha + \theta \wedge \beta)$ is readily seen to be $d\theta \wedge \beta$. From this we get that E_0^k are primitive horizontal k-forms if $k \leqslant n$, and the vertical co-primitive ones ($\in \ker(d\theta \wedge \operatorname{Id})$) if $k \geqslant n + 1$. These bundles E_0^k are of pure C-C weight, namely k if $k \leqslant n$ and $k + 1$ if $k \geqslant n + 1$.

This implies that the d_c complex consists in first order operators, except an order 2 one in degree n, due to the "jump" of weights between E_0^n (weight n) and E_0^{n+1} (weight $n + 2$). This is the contact complex.

• Let $M = G$ be the Engel group (see examples 2.3). Recall this is the 4 dimensional Lie group, with relations $[X, Y] = Z$, $[X, Z] = T$, and other brackets vanishing. In the dual base of 1-forms, this translates in

$$d_0\theta_X = d_0\theta_Y = 0, \quad d_0\theta_Z = -\theta_X \wedge \theta_Y \quad \text{and} \quad d_0\theta_T = -\theta_X \wedge \theta_Z.$$

We find then

$$
\begin{cases}
E_0^0 = C(G) \ : \ \text{functions on } G, \\[2pt]
E_0^1 = \text{span}\,(\theta_X, \theta_Y) \ : \ \text{horizontal one forms,} \\[2pt]
E_0^2 = \text{span}\,(\theta_Y \wedge \theta_Z, \theta_X \wedge \theta_T) = *E_0^2, \\[2pt]
E_0^3 = *E_0^1 = \text{span}\,(\theta_Y \wedge \theta_Z \wedge \theta_T, \theta_X \wedge \theta_Z \wedge \theta_T) \\[2pt]
E_0^4 = \Omega^4 G.
\end{cases}
$$

We compute d_c. As there is nothing to lift on functions $\Pi_E f = f$, and

$$
d_c f = \Pi_{E_0^1} d f = d_{H_1} f = (X.f)\theta_X + (Y.f)\theta_Y.
$$

Using (7), we can compute $\Pi_E \alpha = \tilde{\alpha}$ for $\alpha = \alpha(X)\theta_X + \alpha(Y)\theta_Y \in E_0^1$. One gets

$$
\begin{aligned}
-d_0(\tilde{\alpha}(Z)\theta_Z) &= \tilde{\alpha}(Z)\theta_X \wedge \theta_Y \\
&= d_1\alpha = (X.\alpha(Y) - Y.\alpha(X))\theta_X \wedge \theta_Y,
\end{aligned}
$$

thus $\tilde{\alpha}(Z) = X.\alpha(Y) - Y.\alpha(X)$. Again with (7),

$$
\begin{aligned}
-d_0(\tilde{\alpha}(T)\theta_T) &= \tilde{\alpha}(T)\theta_X \wedge \theta_Z \\
&= (X.\tilde{\alpha}(Z) - Z.\alpha(X))\theta_X \wedge \theta_Z,
\end{aligned}
$$

and $\tilde{\alpha}(T) = X.\tilde{\alpha}(Z) - Z.\alpha(X) = X^2.\alpha(Y) - (XY + Z).\alpha(X)$. Now $d_c\alpha$ is the restriction of $d\tilde{\alpha}$ to E_0^2, that is

$$
d_c\alpha = (Y.\tilde{\alpha}(Z) - Z.\alpha(Y))\theta_Y \wedge \theta_Z + (X.\tilde{\alpha}(T) - T.\theta(X))\theta_X \wedge \theta_T.
$$

The full complex d_c may be completed either by $*$-duality or computing Π_E in degree 2. The result can be read out from the following diagram, adding all possible travels between points gives the various components of the liftings and d_c

(8)
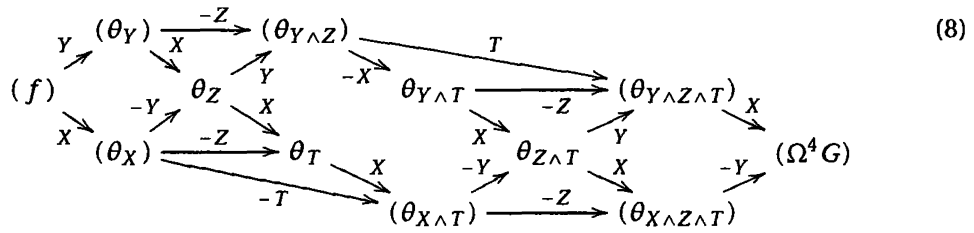
It should be noted that d_c doesn't depend on the various possible choices of grading on G (see 2.3), if keeping the same basis for $E_0 = \ker d_0 / \operatorname{Im} d_0$.

Now, what about d_c on a general Engel structure $D^2 \subset \mathbb{R}^4$ (or TM^4)? Unlike the contact case these are not locally diffeomorphic to their tangent group G. That means that we can't find a local system of vector fields X, Y generating D^2 and satisfying *exactly* the previous bracket relations. Even so, thanks to $[\ , \ ]_0$, they can be satisfied up to vectors of lower C-C weight. The conclusion is that the general d_c will be a perturbation of the

above d_c on G by differential operators of lower C-C weight. This is of course a general feature of the construction.

• We close this series of examples with a glimpse toward the nice case of C-C structures given by a generic 4 dimensional distribution D^4 in $\mathbb{R}^7$ (or TM^7). Here again we are in an exceptional situation where there are finitely many isomorphic type of possible tangent graded group G_{x_0}.

We see this. Its Lie algebra $\mathfrak{g}_{x_0}$ is generated by $\mathfrak{g}_1 = D$ and $\mathfrak{g}_2 = TM/D$ and determined by its curvature $d_0 : \Lambda^1(TM/D)^* \to \Lambda^2 D^{4*}$ given by $d_0\theta(X,Y) = -\theta([X,Y]_0)$. This map is injective iff $[\ ,\ ]_0 : \Lambda^2 D \to TM/D$ is surjective, that is the distribution D is bracket generating. In that (generic) case $L = \operatorname{Im} d_0$ is a 3 dimensional subspace in $\Lambda^2 D^*$. This is a famous case where L is determined up to isomorphism by the signature of the quadratic form $q(\omega)d\mathrm{vol}_D = \omega \wedge \omega$ restricted to it. The distribution D is called elliptic if q is positive definite on L (changing the orientation of D if necessary). This is an open condition, but not dense due to the other open possibility of an hyperbolic $(2,1)$ signature!

If D is elliptic, our assumption by now, one knows there exists a unique conformal class of metric g on D such that L becomes $\Lambda^{2,+}D^*$ the space of $*$-self-dual 2 forms of D^4. It is convenient to describe L by seeing D as the quaternions $\mathbb{H}$ in which case $L = \operatorname{span} \{d\theta_l\}_{1 \leqslant l \leqslant 3}$ is such that

$$d_0\theta_l = g(J_l \cdot, \cdot),$$

for $J_l = i, j, k \in \mathbb{H}$. This is an example of an Heisenberg type group or H-group (see [19], [7]), associated here to the quaternion-hyperbolic group $Sp(2,1)$.

The cohomology E_0 of this structure is easily computed.

1. Again $E_0^1 = \Lambda^1 D^*$ are the horizontal 1-forms.

2. E_0^2 splits in two weights $E_0^{2,(2)} = \Lambda^2 D^*/L \simeq \Lambda^{2,-}D^*$, the anti self-dual part of $\Lambda^2 D^*$. The weight 3 part $E_0^{2,(3)}$ is generated by $\theta_1 \wedge J_1\alpha - \theta_2 \wedge J_2\alpha$ and $\theta_2 \wedge J_2\alpha - \theta_3 \wedge J_3\alpha$ for $\alpha \in \Lambda^1 D^*$.

3. E_0^3 is seen to be of pure weight 4. It is the 14-dimensional space of 3-forms $y = \sum_{i=1}^3 \theta_i \wedge \beta_i$ with $\beta_i \in \Lambda^2 D^*$ satisfying $d_0 y = \sum_{1 \leqslant i \leqslant 3}(d\theta_i, \beta_i) d\mathrm{vol}_D = 0$ modulo $\operatorname{Im} d_0$ generated by $d_0(\theta_1 \wedge \theta_2) = -\theta_1 \wedge d\theta_2 + \theta_2 \wedge d\theta_1$ and the two others coming from permutation.

4. The missing degrees are obtained by duality.

We summarize the information in the following diagram, with the degree of forms

along x and (weight – degree) along y

$$
\begin{array}{c}
E_0^{5,(8)} \xrightarrow{d_c} E_0^{6,(9)} \xrightarrow{d_c} \Omega^7 M \qquad (9)\\[4pt]
{}^{d_c}\nearrow \qquad \nearrow^{d_c}\\
E_0^{4,(6)} \xrightarrow{d_c} E_0^{5,(7)} \qquad {}^{d_c}\\
{}^{d_c}\nearrow \qquad \nearrow\\
E_0^{2,(3)} \xrightarrow{d_c} E_0^{3,(4)}\\
{}^{d_c}\nearrow \qquad \nearrow\\
f \xrightarrow{d_D} \Lambda^1 D^* \xrightarrow{d^-} \Lambda^{2,-} D^* \qquad {}^{d_c}
\end{array}
$$

The interesting feature here is that the bottom line looks like an elliptic complex on the 4 dimensional D, except that D is not integrable here. Anyway the ellipticity of this first order part of d_c will be helpful in in determining the heat decays of forms on the tangent group.

2.4. Geometric comments in degree $\leqslant 2$

We give some precisions on the structure of d_c in degree $\leqslant 2$.

• We start with $E_0^1 = H^1(\mathfrak{g}_{x_0})$ on C-C manifold M. If D is bracket generating, that is an Hörmander distribution in TM, then $\mathfrak{g}_{x_0}$ will be generated by its vectors of weight 1. (These particular graded groups are often called filtered groups.) This is equivalent to the fact that d_0 is injective on 1 forms of weight $\geqslant 1$. Indeed d_0 restricted to one forms is dual to the bracket map $[\ ,\ ]_0 : \mathfrak{g}_{x_0} \wedge \mathfrak{g}_{x_0} \to \mathfrak{g}_{x_0}$ and this one is surjective onto vectors of weight $\geqslant 1$ precisely when D is bracket generating.

As a consequence, $E_0^1 = \ker d_0 / \operatorname{Im} d_0$ consists only of forms of weight 1, which can be identified with $\Lambda^1 D^*$, the partial 1-forms on D. Then $d_c f$ is simply the restriction of df to vectors in D, a first order operator.

• Assume again that $\mathfrak{g}_{x_0}$ is filtered, that is $D = \mathfrak{g}_1$ is bracket generating. We now see how $E_0^2 = H^2(\mathfrak{g}_{x_0})$ is linked to the relations defining G_{x_0}. Let $\widetilde{G}$ be the free Lie group generated over D. The map $\Pi : \widetilde{G} \to G_{x_0}$ is surjective since D is bracket generating. Therefore one has $G_{x_0} = \widetilde{G}/N$ where $N = \Pi^{-1}(0)$ interprets as the normal subgroup of relations of G (with respect to $\widetilde{G}$). At the Lie algebra level $N = \exp \mathfrak{n}$ with $\mathfrak{n} = \ker \pi$ is the ideal of relations of $\mathfrak{g}_{x_0}$.

Since $\mathfrak{n}$ is an ideal, $r = \mathfrak{n}/[\mathfrak{n}, \widetilde{\mathfrak{g}}]$ may be viewed as its space of generators, isomorphic to $R = N/(N, \widetilde{G})$ generating the relations N of G_{x_0}. Now a (classical) fact is that

PROPOSITION 2.9. — $\Lambda^1 r^*$ is naturally isomorphic to $H^2(\mathfrak{g}_{x_0})$.

Proof. — This is an homological consequence of the short exact sequence $\mathfrak{n} \to \widetilde{\mathfrak{g}} \to \mathfrak{g}_{x_0}$. We give the principle in our case. Consider $d_0 : \Lambda^1 \widetilde{\mathfrak{g}}^* \to \Lambda^2 \widetilde{\mathfrak{g}}^*$ given by $d_0 \alpha(X, Y) = -\alpha([X, Y])$.

The space $\Lambda^1 r^*$ identifies with 1-forms on $\mathfrak{n}$ vanishing on $[\mathfrak{n}, \tilde{\mathfrak{g}}]$. Let $\alpha \in \Lambda^1 r^*$, and $\tilde{\alpha}$ be any extension of α to $\Lambda^1 \tilde{\mathfrak{g}}^*$. Then $d_0 \tilde{\alpha}$ vanishes on $\mathfrak{n} \times \tilde{\mathfrak{g}}$ and is therefore the pull-back of a 2-form β on $\mathfrak{g}_{x_0}$. One checks easily that $\beta \in \ker d_{0,\mathfrak{g}_{x_0}}$, and that $\beta \to \beta + d_0 \gamma$ when the extension $\tilde{\alpha}$ is changed (by a one form vanishing on $\mathfrak{n}$). We get therefore a map

$$[d_0] : \Lambda^1 r^* \to H^2(\mathfrak{g}_{x_0}) = \ker d_0 / \operatorname{Im} d_0$$

$$\alpha \to [\beta = d_{0,\tilde{\mathfrak{g}}} \tilde{\alpha}].$$

- Injectivity of $[d_0]$. If $[d_0]\alpha = 0$ then $\exists \gamma \in \Lambda^1 \mathfrak{g}_{x_0}$ such that $d_0(\tilde{\alpha} - \Pi^* \gamma) = 0$. But since D generates $\tilde{\mathfrak{g}}$, we knows that $\tilde{\alpha} - \Pi^* \gamma$ is a form of weight 1. Therefore by restriction on $\mathfrak{n}$, of weight $\geqslant 2$, we get $\alpha = 0$.

- The surjectivity of $[d_0]$ relies on the fact that $H^2(\tilde{\mathfrak{g}}) = 0$.

This can been taken as a definition of $\tilde{\mathfrak{g}}$, meaning that there is one construction of $\tilde{\mathfrak{g}}$ which is precisely done to achieve this ! (Starting with D, vectors of weight 1, the space of covectors of weight $k + 1$ in $\tilde{\mathfrak{g}}$ is taken to be $\ker d_0$ among the *two* forms of weight $\leqslant k + 1$ already constructed.) More geometrically, to $\beta \in \Lambda^2 \tilde{\mathfrak{g}}^* \cap \ker d_0$, a two cocycle, corresponds an extension of $\tilde{\mathfrak{g}}$ by $\mathbb{R}$, $\mathfrak{g}' = \tilde{\mathfrak{g}} \oplus \mathbb{R}\beta$ with $[,]'$ defined by $[(x_1, t_1), (x_2, t_2)]' = ([x_1, x_2], t_1 + t_2 + \beta(x_1, x_2))$. By universality of $\tilde{\mathfrak{g}}$, this extension has to be trivial, a product, which corresponds to a d_0-exact β.

We finish the proof. If β is a d_0-closed two form of $\mathfrak{g}_{x_0}$, its pull-back $\Pi^* \beta$ on $\tilde{\mathfrak{g}}$ is still d_0-closed and therefore d_0-exact. Let $\alpha \in \Lambda^1 \tilde{\mathfrak{g}}^*$ be such that $\Pi^* \beta = d_0 \alpha$. One checks easily that the restriction of α on $\mathfrak{n}$ depends only on the cohomology class of β, and vanishes on $[\mathfrak{n}, \tilde{\mathfrak{g}}]$. It is therefore a one form on $r = \mathfrak{n}/[\mathfrak{n}, \tilde{\mathfrak{g}}]$. $\quad\square$

The previous proposition will give a geometric interpretation, in terms of relations of a group G, of the pinching of heat decay on 1-forms that will be first related to the algebraic $E_0^2 = H^2(\mathfrak{g})$. The use of the free Lie algebra $\tilde{\mathfrak{g}}$ is also helpful in understanding d_c itself on 1-forms on a fixed filtered group G.

We see this. Let $\Pi : \tilde{G} \to G$ as before and $\alpha \in E_0^1(G)$. One has $\Pi^* \alpha \in E_0^1(\tilde{G})$ and since $E_0^2(\tilde{G})$ vanishes there will be no obstruction in extending α on vectors of increasing weights to a closed one β. By injectivity of d_0 on $\tilde{\mathfrak{g}}_{\geqslant 2}$ this extension is unique. So one must have $\beta = \Pi_E \Pi^* \alpha$ as given by the iteration of the retraction map r on $\tilde{G}$ by (5). More concretely β is determined by

$$\beta([X_1, X_2]) = X_1.\beta(X_2) - X_2.\beta(X_1) \quad \text{and} \quad \beta = \Pi^* \alpha \text{ on } \mathfrak{g}_1. \tag{10}$$

We note that in this extension process the components of β stay invariant functions along $\mathfrak{n}$

$$Y.\beta(X) = 0 \tag{11}$$

for $Y \in \mathfrak{n}$, as proved from (10) by recurrence on weight of X and using that $\mathfrak{n}$ is an ideal. Observe that the value of β on $\mathfrak{n}$ is determined by its value on the generators of $\mathfrak{n}$ because for any $Y \in \mathfrak{n}$

$$\beta([X, Y]) = X.\beta(Y). \tag{12}$$

This β is a canonical form, but in order to push its restriction to n on $E_0^2(G)$ with proposition 2.9, we need to choose a supplementary subspace of $[n, \tilde{g}]$ in n, or equivalently a set of generators of relations of G. Depending on this choice, we finally get $d_c \alpha$.

3. Near-cohomology

So far we have only some formal hint in section 2.1 that the asymptotic spectral problem we consider has something to do with the previous constructions. The notion of near-cohomology, introduced in [14], is the main tool that allows to relate explicitly the problems.

3.1. Definition and first applications

Here is a brief presentation of this notion (see [14, 15] for details). Let M be a complete riemannian manifold. An Hilbert complex over M, consists in a sequence of Hilbert spaces E over M together with $d : D(d) \subset E \to E$ such that d are closed densely defined operators and $d^2 = 0$. On the quotiented Hilbert space $F = E / \ker d$, one considers the family of closed cones

$$C_\varepsilon = \{\alpha \in F, \|d\alpha\|_E \leqslant \varepsilon \|\alpha\|_F\}$$

depending of $\varepsilon > 0$. These are shrinking (towards $\{0\}$) as $\varepsilon \to 0$. By definition, the near-cohomology of (E, d) consists in this family of cones up to equivalence induced by dilatational changes of $\varepsilon \to K\varepsilon$. That means that two Hilbert complexes (E, d) and (E', d') have the same near cohomology if for ε small enough there exists a constant $K > 0$ and bounded maps $f : C_\varepsilon \to C'_{K\varepsilon}$ and $g : C'_\varepsilon \to C_{K\varepsilon}$ invertible on their images.

One observation of Gromov and Shubin is that bounded homotopical equivalence between two Hilbert complexes induces equivalence of their near-cohomologies.

DEFINITION 3.1. — Two Hilbert complexes (E, d), (E', d') are homotopy equivalent, if there exists bounded maps $f : E \to E'$ and $g : E' \to E$ such that $fd = d' f$ on $D(d)$, $gd' = dg$ on $D(d')$, $g \circ f = \mathrm{Id}_E + dA + Bd$ on $D(d)$, resp. $f \circ g = \mathrm{Id}_F + d'A' + B'd'$ on $D(d')$ for bounded operators A, A', B, B'.

THEOREM 3.2 (prop 4.1 [14]). — Let (E, d), (E', d') be two homotopy equivalent Hilbert complexes. Then, for ε small enough, f and g induce maps $[f] : C_\varepsilon \to C'_{K\varepsilon}$ and $[g] : C'_\varepsilon \to C_{K\varepsilon}$ invertible on their images.

Proof. — We recall the proof as it is short. Let $\alpha \in C_\varepsilon$. Then certainly

$$\|d'(f\alpha)\| = \|f d\alpha\| \leqslant \|f\| \|d\alpha\| \leqslant \varepsilon \|f\| \|\alpha\|. \tag{13}$$

We need to control α by $\overline{f\alpha}$, the projection of $f\alpha$ in $F = E / \ker d' \simeq (\ker d')^\perp$. One has $\overline{f\alpha} = f\alpha + \beta$ with $\beta \in \ker d'$, so that

$$g(\overline{f\alpha}) = g(f\alpha) + g\beta = \alpha + dA\alpha + Bd\alpha + g\beta,$$

with $dA\alpha + g\beta \in \ker d$. Therefore, as $\alpha \in (\ker d)^{\perp}$,

$$\|\alpha\| \leqslant \|\alpha + dA\alpha + g\beta\| \leqslant \|g\overline{f\alpha}\| + \|B\|\varepsilon\|\alpha\|.$$

Finally

$$\|\alpha\| \leqslant \frac{\|g\|}{1 - \varepsilon\|B\|}\|\overline{f\alpha}\|,$$

giving the injectivity of $\overline{f}$ acting on C_ε, and the result together with (13).   □

One can't apply directly this theorem to the complexes (E, d) and (E_0, d_c) described in theorem 2.6. This is because one of the relevant map here, Π_E, being a differential operator, is not bounded in L^2. Yet we observe that its un-boundedness occurs on high energy forms, and we are precisely interested in the bottom of the spectrum. So we are leaded to consider an intermediate cut-off de Rham complex $(E(\Delta \leqslant 1), d)$, where $E(\Delta \leqslant 1)$ is the spectral space associated to $[0, 1]$ by the Laplacian Δ.

To use this remark, we first notice that the previous theorem 3.2 applies to the full de Rham complex and the cut-off one's, because $\Pi_{E(\Delta \leqslant 1)} = \mathrm{Id} - dA - Ad$ where $A = \delta\Delta^{-1}\Pi_{E(\Delta > 1)}$ is bounded. Thus, they have equivalent near-cohomology (observe that although they have exactly the same small spectrum their C_ε are distinct). But now by ellipticity of Δ, the spectral projection $\Pi_{E(\Delta \leqslant 1)}$ is a smoothing operator and therefore the differential operator $\Pi_E = \mathrm{Id} - Qd - dQ$ and $\Pi_{E_0}\Pi_E$ becomes bounded on $E(\Delta \leqslant 1)$, making theorem 3.2 usable here.

THEOREM 3.3 ([25]). — *Let M be a complete E_0-regular C-C manifold. Then the de Rham complex, (E, d) and (E_0, d_c) have equivalent near-cohomologies.*

Proof. — To complete the previous discussion, we make short comments on the closures of (E_0, d_c) and (E, d).

- Firstly, d_c, being a differential operator, its formal adjoint δ_c has dense (initial) domain C_0^∞ in L^2. Therefore d_c is closable, for instance by $\overline{d_c} = (d_c)^{**}$ (see eg [23]).

- About (E, d). Given a closed extension $\overline{d}$ of de Rham's differential, its restriction to $\overline{E} = \ker d_0^{-1} \cap \ker d_0^{-1}\overline{d}$ may be seen as a closed extension of (E, d).   □

The boundedness of differential operators on $E(\Delta \leqslant 1)$ has also a geometric consequence on the shape of the cones $C_\varepsilon(E(\Delta \leqslant 1), d)$.

We denote by $[\Pi_E]$ and $[\Pi_F]$ the actions of the retractions Π_E and Π_F of theorem 2.6 on the quotient space $\Omega^*M/\ker d$. One has still $[\Pi_F] = \mathrm{Id} - [\Pi_E]$ with $[\Pi_E]$ a projection, so that they induce a splitting of $\Omega^*M/\ker d$ in $[E] \oplus [F]$ (possibly degenerated) with $[E] = \mathrm{Im}[\Pi_E] = (E + \ker d)/\ker d$ and $[F] = \mathrm{Im}[\Pi_F] = (F + \ker d)/\ker d$.

PROPOSITION 3.4. — *The cut-off cones $C_\varepsilon(E(\Delta \leqslant 1), d) \subset \Omega^*M/\ker d \simeq (\ker d)^{\perp}$ are uniformly shrinking around $[E]$ relatively to $[F]$. Precisely, $\exists C$ such that*

$$\|\Pi_F\alpha\|_{\Omega^*M/\ker d} \leqslant C\varepsilon\|\alpha\|$$

for all $\alpha \in C_\varepsilon(E(\Delta \leqslant 1), d)$.

Proof. — Recall that $\Pi_F = Qd + dQ$, so that for $\alpha \in C_\varepsilon(E(\Delta \leqslant 1), d)$ one has

$$\|\Pi_F \alpha\|_{\Omega^* M / \ker d} \leqslant \|Qd\alpha\|$$
$$\leqslant C\|d\alpha\| \leqslant C\varepsilon\|\alpha\|,$$

where C is a bound for Q on $E(\Delta \leqslant 1) \supset d(E(\Delta \leqslant 1))$. $\square$

Remarks 3.5

– Notice that $[\Pi_E]$ and $[\Pi_F]$ are not orthogonal projections.

– As $F = \operatorname{Im} d_0^{-1} + \operatorname{Im} dd_0^{-1}$ is pinched between $\operatorname{Im} d_0^{-1}$ and $\operatorname{Im} d_0^{-1} + \operatorname{Im} d$ one has $[F] = [\operatorname{Im} d_0^{-1}]$. Also $E = \ker d_0^{-1} \cap \ker d_0^{-1}d \subset \ker d_0^{-1}$ so that $[E] \subset [\ker d_0^{-1}]$.

We close this section with an application of theorem 3.3 to d_c-harmonic decomposition on *compact* C-C manifolds.

PROPOSITION 3.6. — *Let M be a compact E_0-regular C-C manifold. Then* $\operatorname{Im} d_c$ *and* $\operatorname{Im} \delta_c$ *are closed in L^2 and we have an orthogonal Hodge-de Rham splitting*

$$E_0 = \mathcal{H}_c \oplus \operatorname{Im} d_c \oplus \operatorname{Im} \delta_c,$$

where $\mathcal{H}_c = \ker d_c \cap \ker \delta_c$, *is the space of d_c-harmonic forms, isomorphic to de Rham's cohomology of M.*

Remark 3.7 This d_c-harmonic representation of the cohomology is probably more interesting when E_0 is of pure C-C weight (like on one forms), since in that case $\mathcal{H}_c$ will be invariant under the natural C-C dilations of the riemannian metric $g \to \lambda g_{V_1} + \lambda^2 g_{V_2} + \cdots + \lambda^r g_{V_r}$, with $V_i = H_i \ominus H_{i-1}$. In that pure weight case, one can also obtain analytic information on the regularizing properties of d_c, showing in particular the smoothness of $\mathcal{H}_c$ (see section 5.1).

Proof. — On a compact manifold, 0 is isolated in the (discrete) spectrum of the de Rham's Laplacian Δ. Therefore, on $(\ker d)^\perp$

$$\|d\alpha\|^2 = (\Delta\alpha, \alpha) \geqslant \lambda_1 \|\alpha\|^2$$

for some $\lambda_1 > 0$. Then $C_\varepsilon(d) = \{0\}$ for $\varepsilon < \lambda_1$, and the near-cohomology of M vanishes. By theorem 3.3 the same is true at the level of (E_0, d_c). This implies the same kind of bound for $q_c = \|d_c\|^2$, showing that $\operatorname{Im} d_c$ is closed, and the same for $\operatorname{Im} \delta_c$ by $*$-duality. $\square$

3.2. From cones to spectral densities

An interesting feature of near-cohomology lies in its stability under general bounded maps, whereas the spectral decomposition of an operator (like the Laplacian) is only preserved by isometries. This is the advantage of working with the quadratic form $q(\alpha) =$

$\|d\alpha\|^2$ on $\Omega^* M /$ ker d, up to multiplicative constants, instead of doing the (much more) fine study of the exact spectral decomposition of Δ on forms. Yet, the cones $C_\varepsilon = \{\alpha \mid q(\alpha) \leqslant \varepsilon^2 \|\alpha\|^2\}$ encoded the "rough" information we are looking for of the asymptotic spectral density of Δ at 0.

We first observe that $E(0 < \delta d \leqslant \varepsilon^2)$ is a closed linear space in $C_\varepsilon \subset (\ker d)^\perp$. Moreover, one has $q(\alpha) > \varepsilon^2 \|\alpha\|^2$ on its orthogonal $E(\delta d > \varepsilon^2) = E(0 < \delta d \leqslant \varepsilon^2)^\perp \cap (\ker d)^\perp$. Therefore, $C_\varepsilon \cap E(\delta d > \varepsilon^2) = \{0\}$. A consequence is that *any* linear space L lying in C_ε projects *injectively* into $E(0 < \delta d \leqslant \varepsilon^2)$, showing in some sense that this space is the "largest" possible one inside C_ε.

3.2.1. Γ-dimension and the rational case.

In order to translate numerically this observation, one needs a notion of dimension working on some infinite dimensional linear spaces. There is an classical one, called Γ-dimension, in the case a discrete group of isometries Γ is acting co-compactly on the manifold (eg a Galois covering of a compact manifold). A complete exposition of this may be found in Atiyah's original work [1] (or [21] for a survey of its properties and applications).

Briefly, in the case we are concerned in, let M be a compact manifold and $\tilde{M}$ its universal cover. Let P be a positive Γ-invariant Hermitian operator acting on some Γ-invariant Hilbert space H over $\tilde{M}$. We assume H to be a subspace of $L^2(\tilde{M}, V)$, where V is the pull-back of some finite dimensional vector bundle on M. Then H admits a Γ-invariant Hilbert base $(e_i)_{i \in I}$ on which Γ-acts freely (H is called a free Γ-module). Choose one vector by Γ-orbit. (For instance one can take an Hilbert base of H restricted to a fundamental domain $\mathscr{F}$ of $\tilde{M}$.) The Γ-trace of P is defined by

$$\mathrm{Tr}_\Gamma(P) = \sum_{i \in \Gamma \backslash I} (Pe_i, e_i),$$

The definition is shown to be independent of the choices of e_i (lemma 2.2 in [21]). If L is a closed Γ-invariant subspace of H, its Γ-dimension is defined by $\dim_\Gamma L = \mathrm{Tr}_\Gamma \Pi_L$, where Π_L denotes the orthogonal projection onto L.

The basic properties of $\dim_\Gamma$ we need here are

- first its invariance under Γ-invariant bounded injection (proved using polar decomposition see eg lemma 2.3 in [21]),

- its link with kernel density in the case of a smoothing projection Π_L. For example, if $L = E(\Delta \leqslant \lambda)$ then

$$\dim_\Gamma L = \mathrm{Tr}_\Gamma \Pi_L = \int_{\mathscr{F}} \mathrm{Tr}(K(x, x)) \mathrm{dvol}, \tag{14}$$

where $K(x, y)$ is the smooth Schwartz kernel of Π_L, and Tr is the standard finite dimensional trace on $\mathrm{End}(\Lambda^* T^* M)$.

Applying this to a previous remark that any linear subspace in C_λ projects injectively into $E(0 < \delta d \leqslant \lambda^2)$, one obtain the following variational principle ([14])

$$F_{\delta d}(\lambda^2) = \dim_\Gamma E(0 < \delta d \leqslant \lambda^2) = \sup_{L \in \mathscr{A}} \dim_\Gamma L, \tag{15}$$

where $\mathscr{A}$ is the set of all Γ-invariant closed linear subspaces $L \subset C_\lambda(d)$. Hence the spectral distribution function of δd is encoded in the Γ-linear "thickness" of the near-cohomology cones. Moreover, theorem 3.2 and the invariance of $\dim_\Gamma$ trough injective maps, implies Gromov-Shubin's result that Γ-homotopical equivalent complexes (E, d) and (E', d') on $\tilde{M}$ must have equivalent density functions

$$F_d(C^{-1}\lambda) \leqslant F_{d'}(\lambda) \leqslant F_d(C\lambda), \tag{16}$$

for small λ and some $C > 0$, where $F_d(\lambda)$ is defined in general by

$$F_d(\lambda) = \sup_{L \in \mathscr{A}} \dim_\Gamma L, \tag{17}$$

with the same $\mathscr{A}$ as above. In particular they have the same Novikov-Shubin exponents

$$\alpha_d = \liminf_{\lambda \to 0} \frac{\ln F_d(\lambda)}{\ln \lambda} \tag{18}$$

(but also same limsup, or any other dilatationally invariant limit). This applies to theorem 3.3, *if* the manifold admits a cocompact discrete action.

DEFINITION 3.8. — *A graded nilpotent Lie group G is called rational if it admits a cocompact discrete group Γ.*

For such a group rationality is equivalent to being able to find a basis of $\mathfrak{g}$ with brackets given by rational coefficients (see [8]). Then a cocompact group Γ is given by the exponential of a corresponding integral lattice in $\mathfrak{g}$.

THEOREM 3.9 ([25]). — *Let G be a rational graded nilpotent Lie group. Then de Rham's complex and (E_0, d_c) have the same Novikov-Shubin exponents.*

These exponents do not depend on the choice of Γ here. One can endow G with a (left) invariant metric, in which case all operators and spaces become G-invariant and formulae (15) and (14) reduce to

$$F_{\delta d}(\lambda^2) = \text{vol}(\Gamma \backslash G) \, \text{Tr}(K_{\Pi_{E(0 < \delta d \leqslant \lambda^2)}}(e, e)).$$

Hence $\displaystyle \liminf_{\lambda \to 0} \frac{\ln F_{\delta d}(\lambda^2)}{\ln \lambda}$ is independent of Γ.

We remark also that theorem 3.9 is true on any fundamental cover of a compact E_0 regular C-C manifold. We don't state it in this generality since for applications we will need some dilation respecting the C-C structure, that is more or less to work on a graded group.

3.2.2. Extension to non-rational groups.

Although "geometric" nilpotent groups (associated to semi-simple geometries) tend to be rational, they are certainly negligible in the variety of nilpotent Lie groups. Moreover the question of the asymptotic heat decay makes sense on any group, and it seems unlikely this rough invariant should depend so sharply on the existence of a rational structure. (Observe for instance that the value on functions is the growth rate of G, which turns out to be an integer for all nilpotent Lie groups.) Therefore it is natural to extend the previous result to non-rational groups. The point is to use a counterpart to Γ-dimension for G-invariant Hilbert spaces. We describe two different approaches.

• The first one relies on the use of unitary representation theory of nilpotent Lie groups. On such a group G, the Fourier-Plancherel transform gives an isometry between $L^2(G)$ with $L^2(\hat{G})$ the unitary dual of G (see eg |8| for a complete exposition). We briefly explain how the Plancherel measure on $\hat{G}$ gives rise to a relevant measure on G-invariant Hilbert spaces in $L^2(G)$.

A bounded (left) G-invariant map P acting on $L^2(G)$ has a Schwarz kernel $K(x, y)$ (a distribution on $G \times G$) which is invariant $K(gx, gy) = K(x, y)$. Therefore P acts as a convolution product $P f = k * f$ which "diagonalizes" trough Fourier-Plancherel transform in $\pi(P f) = \pi(k) \circ \pi(f)$, where π is an irreducible unitary representation of G (in bijection with covectors ξ in $\mathfrak{g}^*$ modulo the coadjoint action of $\mathfrak{g}$ by Kirillov's orbit method). Recall that for $f \in L^2(G)$, $\hat{f} : \pi \to \pi(f)$ consists in a measurable field of Hilbert-Schmidt operators acting for each π on the associated Hilbert space H_π. For $f \in L^1(G)$, one has

$$\pi(f) = \int_G f(x) \pi(x^{-1}) dx.$$

Plancherel formula is

$$\| f \|_2^2 = \int_G |f(x)|^2 dx = \| \hat{f} \|_2^2 = \int_{\hat{G}} \| \pi(f) \|_{HS}^2 d\mu(\pi) \tag{19}$$

with $\| \pi(f) \|_{HS}^2 = \mathrm{Tr}(\pi(f)^* \pi(f))$, while Fourier inversion formula reads

$$f(e) = \int_{\hat{G}} \mathrm{Tr}(\pi(f)) d\mu(\pi) \tag{20}$$

for $f \in \mathscr{S}(G)$. Then to a bounded P is associated a field of bounded $\pi(k) \in \mathrm{End}(H_\pi)$. If moreover P is an orthogonal projection onto a G-invariant closed space L, these $\pi(k)$ have to be orthogonal projections onto spaces $L_\pi \subset H_\pi$. We can now define the G-dimension of L as

$$\dim_G L = \int_{\hat{G}} \dim(L_\pi) d\mu(\pi).$$

For a smoothing Π_L Fourier inversion formula gives

$$\dim_G L = k(e) = K_{\Pi_L}(e, e),$$

to be compared with its Γ-dimension if G is rational. Using the previous discussion one find also that $\dim_G$ is preserved trough injective bounded G-invariant maps has needed. All this leads to an extension of theorem 3.9 to any graded nilpotent Lie group.

THEOREM 3.10. — *Theorem 3.9 holds without rationality assumption, replacing the Γ by the G-dimension.*

• The second approach of the previous result uses more elementary tools, and follows Atiyah's presentation of the Γ-trace in [1] section 4. Let G be a graded nilpotent group. The main fact we need is that although G has no compact quotient in general, it admits a nice covering. This has been observed in a close form by Helffer and Nourrigat in [16], lemma 4.5.2.

LEMMA 3.11. — *Let G be a graded nilpotent Lie group. There are a relatively compact domain D and a discrete set $Z \subset G$, such that $Z^{-1} = Z$ and*

$$G = \bigsqcup_{z \in Z} zD.$$

Proof. — The Lie algebra $\mathfrak{g}$ splits in $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_r$ with $[\mathfrak{g}_1, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$. Choose an integral *additive* lattice $Z_i \subset \mathfrak{g}_i$, together with a fundamental domain D_i of Z_i in $\mathfrak{g}_i$. Consider $D = \exp(D_1 \times D_2 \cdots \times D_r)$ and $Z = \exp(Z_1 \times Z_2 \cdots \times Z_r)$. Now at the Lie algebra level, given $y = (y_1, \cdots, y_r) \in \mathfrak{g}$, the equation

$$(z_1, \cdots, z_r).(x_1, \cdots, x_r) = (y_1, \cdots, y_r)$$

leads to a triangular system whose diagonal terms are

$$z_i + x_i = y_i + \text{(non linear) function of } z_j, x_k \text{ with } j, k < i.$$

Hence it admits a unique solution with $z \in Z$ and $x \in D$. (This proof adapts on any nilpotent Lie group.) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Box$

This gives a discretization of $L^2(G)$ in $\ell^2(Z) \otimes L^2(D)$, which allows to follow the classical construction of Γ-trace as given in [1] section 4 (or [21]). For instance a bounded G-invariant operator P may be called Z-Hilbert Schmidt if its kernel belongs to $L^2(Z \backslash G \times G)$, which identifies either to $L^2(D \times G)$ or $L^2(G \times D)$ by the symmetry $Z^{-1} = Z$. Its Z-Hilbert Schmidt norm may be defined by

$$\|P\|_{ZHS}^2 = \int_{G \times D} |K(x, y)|^2 dxdy = \|P\chi_D\|_{HS}^2 = \sum_{e_i} \|(P\chi_D)e_i\|^2$$

$$= \int_{D \times G} |K(x, y)|^2 dxdy = \|\chi_D P\|_{HS}^2 = \sum_{e_i} \|(\chi_D P)e_i\|^2,$$

for e_i an Hilbert base of $L^2(G)$ and χ_D the characteristic function of D. It is easily seen to be independent on the choice of D and $*$-invariant $\|P\|_{ZHS} = \|P^*\|_{ZHS}$. We check the

first point, if D' is another "fundamental domain", one has

$$\|P\|^2_{DHS} = \int_G \left(\int_D |K(x,y)|^2 \left(\sum_{z \in Z} \chi_{zD'} \right)(y) dy \right) dx$$

$$= \int_G \sum_{z \in Z^{-1}=Z} \int_{zD} |K(x,y)|^2 \chi_{D'}(y) dy dx$$

$$= \int_G \left(\int_G |K(x,y)|^2 \chi_{D'}(y) dy \right) dx = \|P\|^2_{D'HS}$$

A positive hermitian P is of Z-trace class if $\mathrm{Tr}_Z(P) = \sum_{i \in I}(Pe_i, e_i)$ converges where (e_i) an Hilbert base of $L^2(D)$, equivalently $\chi_D P \chi_D$ is of trace class on $L^2(G)$. The relation with the Z-HS class being

$$\mathrm{Tr}_Z(P^*P) = \mathrm{Tr}(\chi_D P^* P \chi_D) = \|P\chi_D\|^2_{HS} = \|P\|^2_{ZHS}.$$

Again if Π_L is a smoothing G-invariant projection on L, $\dim_Z L = \mathrm{Tr}_Z \Pi_L$ reduces to $\mathrm{vol}(D) K_{\Pi_L}(e, e)$ as in the case of Γ-dimension in (14). So we get a version of the previous theorem 3.10 using this "Z-dimension" instead.

Observe that in all considered Γ, G, or Z approaches of dim, one can uses only G-invariant closed spaces to determine the cones thickness function $F(\lambda)$ in formula (17). Indeed the maximal (through injective projection) Hilbert space in $C_\lambda(d)$ is $E(0 < \delta d \leqslant \lambda^2)$ a G-invariant space, not only a Γ or Z-invariant one.

3.3. The algebraic pinching of heat decay

We can now give a first application of these constructions to our initial problem of estimating the heat decay on (1-)forms of graded groups. We define the C-C weight of G, with respect to its grading, as $N(G) = \sum_{k=1}^{r} k \dim(\mathfrak{g}_k)$. This $N(G)$ is the weight of $\Lambda^{max} T^* G$, occurring in the Jacobian of the dilations of G. When the group G is filtered, $N(G)$ is also its Hausdorff dimension, that is the growing rate of the volume of large balls in G. We also consider the splitting of $E_0^k = H^k(\mathfrak{g})$, the k-forms of E_0, in its various C-C weights p

$$E_0^k = \bigoplus_{p=N_k^{min}}^{N_k^{max}} E_0^{k,p}.$$

E_0^k is said of pure weight p if $p = N_k^{min} = N_k^{max}$.

DEFINITION 3.12. — *We will use* $\beta_k = N(G)/\alpha_k$ *as a convenient renormalization of* α_k, *the k-th Novikov-Shubin exponent of* G *(twice the heat decay on coclosed k-forms of* G *by* (1)*).*

THEOREM 3.13 ([25]). — *(Algebraic pinching of β_k.) Let G be a graded nilpotent Lie group G. If $E_0^k = H^k(\mathfrak{g})$ has pure weight N_k then,*

$$\delta N_k^{\min} = \max(N_{k+1}^{\min} - N_k, 1) \leqslant \beta_k \leqslant N_{k+1}^{\max} - N_k = \delta N_k^{\max},$$

giving an estimation of $\beta_k = N(G)/\alpha_k$ from the weight gaps δN_k.

Remark 3.14 This is of course consistent with the case of functions $k = 0$ mentioned in theorem 1.2. Namely for a filtered Lie group (bracket generated by $D = \mathfrak{g}_1$), one has $E_0^1 \simeq \Lambda^1 D^*$ of pure weight 1 (section 2.4). Therefore $\beta_0 = 1$ and we recover that $\alpha_0 = N(G)$ is the growth of G.

Proof. — By theorem 3.10, we can use (E_0, d_c) to estimate β_k. Then the result relies on an homogeneity argument of d_c trough the dilations h_r of G. Observe that on E_0^k,

$$d_c = d_c^{\delta N_k^{\min}} + \cdots + d_c^{\delta N_k^{\max}}, \tag{21}$$

where d_c^i increases the weight by i.

We recall that for a form α of weight $N(\alpha)$, the integral L^2-norms transforms as $\|h_r^* \alpha\|_G = r^{N(\alpha) - N(G)/2} \|\alpha\|_G$. We now see how the near-cohomology cones $C_\lambda(d_c)$ rescale through h_r^*. Firstly, by the homogeneity assumption of E_0^k, the space $E_0^k / \ker d_c \simeq (\ker d_c)^\perp$ is preserved. By the previous remarks, one has then for $\alpha \in E_0^k \cap C_\lambda(d_c)$ and $r \geqslant 1$,

$$\|d_c(h_r^* \alpha)\| = \|h_r^* d_c \alpha\| \leqslant r^{N_{k+1}^{\max} - N(G)/2} \|d_c \alpha\|$$
$$\leqslant \lambda r^{N_{k+1}^{\max} - N(G)/2} \|\alpha\|$$
$$= \lambda r^{\delta N_k^{\max}} \|h_r^* \alpha\|.$$

Therefore

$$h_r^*(C_\lambda(d_c)) \subset C_{\lambda r^{\delta N_k^{\max}}}(d_c).$$

In the opposite direction, for $\varepsilon \leqslant 1$ and $\alpha \in E_0^k \cap C_\lambda(d_c)$

$$\|d_c(h_\varepsilon^* \alpha)\| = \|h_\varepsilon^* d_c \alpha\| \leqslant \varepsilon^{\max(N_k^{\min}, N_k + 1) - N(G)/2} \|d_c \alpha\|$$
$$\leqslant \lambda \varepsilon^{\max(N_k^{\min}, N_k + 1) - N(G)/2} \|\alpha\|$$
$$\leqslant \lambda \varepsilon^{\delta N_k^{\min}} \|h_\varepsilon^* \alpha\|,$$

giving

$$h_\varepsilon^*(C_\lambda(d_c)) \subset C_{\lambda \varepsilon^{\delta N_k^{\min}}}(d_c).$$

Putting this together we have found that, for $r \geqslant 1$

$$h_r^*(C_{r^{-\delta N_k^{\max}}}(d_c)) \subset C_1(d_c) \subset h_r^*(C_{r^{-\delta N_k^{\min}}}(d_c)). \tag{22}$$

The next point to check is the effect of dilations on dimension of G-invariant Hilbert spaces $L \subset E_0$. We have for $n \in \mathbb{N}$,

$$\dim_Z(h_n^* L) = \dim_{h_n Z}(L) = n^{N(G)} \dim_Z(L). \tag{23}$$

We use the discrete dilations h_n because we will need that $h_n(Z) \subset Z$. (We could avoid this working with the less elementary G-dimension). For Γ-dimension this formula is a direct consequence of its behavior through finite covering. The proof is the same here. Since $n^{N(G)/2 - N_k} h_n^*$ is an isometry on $L^2(E_0^k)$, we have

$$\Pi_{h_n^* L} = h_n^* \Pi_L h_n^{*-1}.$$

Therefore,

$$
\begin{aligned}
\dim_Z(h_n^* L) &= \mathrm{Tr}_Z(\Pi_{h_n^* L}) = \mathrm{Tr}(\chi_D h_n^* \Pi_L h_n^{*-1} \chi_D) \\
&= \mathrm{Tr}(h_n^* \chi_{h_n D} \Pi_L \chi_{h_n D} h_n^{*-1}) = \mathrm{Tr}(\chi_{h_n D} \Pi_L \chi_{h_n D}) \\
&= \dim_{h_n Z}(L),
\end{aligned}
$$

Now by lemma 3.11, $h_n D$ is a "fundamental domain" for the new "integral lattice" $h_n Z$ which splits, for $n \in \mathbb{N}$, in $n^{N(G)}$ domains D' for Z, giving (23). This is applied to G-invariant Hilbert linear sub-spaces $L \subset C_\lambda(d_c)$ and leads with (22) and (17) to the following pinching for the cone thickness function F_{d_c}

$$F_{d_c}(n^{-\delta N_k^{\max}}) \leqslant n^{-N(G)} F_{d_c}(1) \leqslant F_{d_c}(n^{-\delta N_k^{\min}}),$$

for $n \in \mathbb{N}$ and finally

$$C F_{d_c}(1) \lambda^{N(G)/\delta N_k^{\min}} \leqslant F_{d_c}(\lambda) \leqslant C' F_{d_c}(1) \lambda^{N(G)/\delta N_k^{\max}}, \tag{24}$$

for $\lambda > 0$ (even real) and some positive constants C, C'.

The last observation is that all these numbers are finite and strictly positive. Finiteness comes from the fact that this is true "upstairs" for the homotopically equivalent de Rham complex. Indeed by ellipticity of the de Rham Laplacian, spectral projectors have smooth kernels, and therefore of finite diagonal trace. Non-vanishing amounts to the fact that 0 belongs to the spectrum of Δ on k-forms for any k on nilpotent groups. This can be viewed at the (E_0, d_c) level. The cones $C_\lambda(d_c)$ don't vanish, because any sufficiently dilated non-closed smooth form of E_0 lies in it, as d_c (unlike d) strictly increases the weight. We note that d_c can't vanish identically in degree $k < \dim G$, because this complex is a resolution of functions of G and has to be of maximal length. $\square$

We can give a more geometric meaning of the previous theorem 3.13 in the case of one forms. Namely we have seen in section 2.4 that when the group is filtered,

- firstly E_0^1 identifies with $\Lambda^1 D^*$ forms of weight 1,

- $E_0^2 = H^2(\mathfrak{g})$ may be interpreted as $\Lambda^1 r^*$, the dual to the space $R = N/(N, \tilde{G})$ of generators of the relations N of G, with respect to the free Lie group $\tilde{G}$.

Therefore theorem 3.13 translates into the following result.

COROLLARY 3.15. — *Let G be a filtered nilpotent Lie group. Then $\beta_1 + 1 = N(G)/\alpha_1 + 1$ is pinched between the minimal and maximal weight of the relations giving a presentation of G from the free Lie group. In particular $1 \leqslant \beta_1 \leqslant r$ if G has rank r.*

4. Examples

The previous result gives an estimation of heat decay from the knowledge of the weights of the Lie algebra cohomology $H^*(\mathfrak{g})$. This pinching relies therefore on finite dimensional linear systems computations (that may of course become incomprehensible as the dimension increases).

4.1. Quadratically presented groups

The first class of nilpotent groups on which the previous results applies sharply are quadratically presented groups. At the Lie algebra level they are groups admitting a presentation using only linear combinations of bracket of first order in the generators $D = \mathfrak{g}_1$. By the previous discussion this may also be checked dually by showing that $H^2(\mathfrak{g})$ contains only 2-forms of weight 2 (horizontal 2-forms in $\Lambda^2 D^*$).

For all these groups one has $\beta_1 = 1$, meaning that $\alpha_1 = N(G)$ (in the strong sense that on $\Omega^1 G$, the heat kernel satisfies $K_t(x, x) \asymp t^{-N(G)/2}$ when $t \to +\infty$, as follows from (24)). Thus heat decay on one forms is the quickest possible and the same as on functions. Fortunately these groups exist.

• The simplest examples are given by the Heisenberg groups $\mathbf{H}^{2n+1}$ of dimension $2n + 1 \geqslant 5$. They may be presented by the relations $[X_i, X_j] = [Y_i, Y_j] = 0$ with $1 \leqslant i, j \leqslant n$, $[X_i, Y_j] = 0$ for $i \neq j$ and (*for $n > 1$*), $[X_i, Y_i] = [X_j, Y_j]$ (= T the generator of the center). In fact using the Lie algebra computation in section 2.3 and theorem 3.13, we have $\alpha_p = N(G)$ for all $0 \leqslant p \leqslant 2n$ except $\alpha_n = N(G)/2$. In particular the value $\alpha_1(\mathbf{H}^3) = 2$ (first computed by Lott [20]) is half the Hausdorff dimension. This corresponds geometrically to the fact that $\mathbf{H}^3$ has a *cubic* presentation : $[X, [X, Y]] = [Y, [X, Y]] = 0$.

• There are in fact many other examples. In [13] section 4.2, Gromov introduced the class of two step graded groups that possess a Ω-regular legendrian two plane P. That means there exists X_1, X_2 in D such that $[X_1, X_2] = 0$ and the map

$$\Omega : D \to \mathrm{End}(P, [D, D])$$

$$X \to [X, \cdot] \text{ acting on } P = \mathrm{span}(X_1, X_2),$$

is surjective. (Giving $T_1, T_2 \in \mathfrak{g}_2 = [D, D]$, there exists X in D such that $[X, X_1] = T_1$ and $[X, X_2] = T_2$.) One can show easily (see [13] 4.2 A") that this condition is generically satisfied (is a Zariski open dense set) for graded groups associated to distributions $D^k \subset$

$\mathbb{R}^n$ as long as $n \leqslant 3k/2 - 1$. (The necessity of this bound is a dimension counting in the previous formula noting that $P \subset \ker \Omega$.)

The geometric interest of this class of groups is that Gromov has shown they are quadratically fillable in the sense that closed horizontal curves y can be filled by horizontal disks of area controlled by $\mathrm{length}(y)^2$. (This is a global non-linear hard property relying infinitesimally on the presence of a lot of flexible legendrian planes.) Our observation is simply that

PROPOSITION 4.1. — Ω-regular groups are quadratically presented, and thus have $\beta_1 = 1$.

Proof. — We have to show that $H^2(\mathfrak{g})$ doesn't contain 2-forms of weight 3, that is the map

$$d_0 : \Lambda^{2,(3)}\mathfrak{g}^* = D^* \wedge \mathfrak{g}_2^* \longrightarrow \Lambda^{3,(3)}\mathfrak{g}^* = \Lambda^3 D^*$$

$$\sum \alpha_i \wedge \theta_i \longrightarrow -\sum \alpha_i \wedge d_0\theta_i = \sum \alpha_i \wedge \theta_i([\cdot,\cdot]) \tag{25}$$

is injective. Let $\beta = \sum \alpha_i \wedge \theta_i$ be in $\ker d_0$. We first show that α_i vanishes on $P = \mathrm{span}(X_1, X_2)$. We fix a base θ_i of $\mathfrak{g}_2^*$ in the previous formula. Given T_{i_0} dual to θ_{i_0} we can find by Ω-regularity a vector $X \in D$ such that $[X, X_1] = 0$ and $[X, X_2] = T_{i_0}$ (and still $[X_1, X_2] = 0$). Hence

$$0 = d_0\beta(X_1, X_2, X) = \sum \alpha_i(X_1) \wedge \theta_i([X_2, X]) = -\alpha_{i_0}(X_1),$$

and similarly on X_2. Now taking X as before and $Y \in D$ such that $[Y, X_2] = 0$, one gets

$$0 = d_0\beta(X_2, X, Y) = \sum \alpha_i(Y) \wedge \theta_i([X_2, X]) = -\alpha_{i_0}(Y),$$

so that α_{i_0} vanishes on $\ker [\cdot, X_2]$. For the last step, choose $Y \in D$ such that $[Y, X_1] = T_{i_0}$ and $[Y, X_2] = 0$ then, for any $X \in D$

$$0 = d_0\beta(X_1, X, Y) = \sum \alpha_i(X) \wedge \theta_i([Y, X_1]) = \alpha_{i_0}(X).$$

This shows that $\alpha_{i_0} = 0$, and finally $\beta = 0$. $\qquad\square$

The previous fact may be seen more geometrically using Gromov's result that these groups are quadratically fillable. If any horizontal curve can be filled by an horizontal disk then certainly, for any 1-form α on G, the vanishing of $d\alpha$ on $\Lambda^2 D$ is sufficient to conclude that α is exact. Namely $f(m) = \int_y \alpha$ where y is any horizontal path from 0 to m, is well defined in that case and satisfies $df = \alpha$ along D. Hence $\theta = \alpha - df$ is a vertical form such that $d_0\theta = d\alpha$ on $\Lambda^2 D$ vanishes, and finally $\alpha = df$. This argument means that the quadratic filling property implies that they are no closed two forms of weight > 2. In particular $E_0^2 = H^2(\mathfrak{g})$ has to be of pure weight 2, because invariant 2-forms on G coming from $H^2(\mathfrak{g}) = \ker d_0/ \mathrm{Im}\, d_0$ are always closed but not exact in G. We recover therefore that G is quadratically presented.

• Gromov's quadratically fillable groups have "a lot" of integrable legendrian planes $P = (X, Y) \subset D$ which lead to simple quadratic relations $[X, Y] = 0$ and allows a quadratic (in weight) integration of 2-forms. Anyway, the presence of these flat planes is not a necessary condition for a group to have a quadratic presentation. Here is a "sporadic" example with no flat plane at all but which is quadratically presentable anyway. This is a J-group corresponding to a $D^8 \subset \mathbb{R}^{15}$. We have to describe a 7-dimensional subspace L of curvature in $\Lambda^2 \mathbb{R}^8$, such that

$$\theta \in \mathbb{R}^7 \rightarrow d\theta = g_D(J(\theta),) \in L$$

with $-J(\theta)^2 = \|\theta\|^2 \mathrm{Id} = J(\theta)^* J(\theta)$. This map J can be realized either as a particular Clifford action, or the left multiplication of the imaginary Cayley numbers $\mathrm{Im}\, \mathbb{O}$ on $\mathbb{O}$. It can been checked, using direct calculations with the J_i, $1 \leqslant i \leqslant 7$ given in tables of Clifford algebras (see eg [18] section 11), or representation theory (see [26]) that the map d_0 in (25) is injective (even an isomorphism, since both spaces have dimension 56). Therefore this group is quadratically presented, although it contains no integrable horizontal plane since $[X, Y] = 0$ implies that $X \perp J_i Y$ for $1 \leqslant i \leqslant 7$, which gives collinear X and Y.

• The previous example $D^8 \subset \mathbb{R}^{15}$ goes beyond the generic bound $n \leqslant 3k/2 - 1$ under which a generic $D^k \subset \mathbb{R}^n$ is Ω-regular hence quadratically presented. Unfortunately, due to the eight periodicity of their structure, Clifford modules fail to produce a linearly increasing number of orthogonal complex structures on $\mathbb{R}^k$ for $k > 8$ ($\mathbb{R}^{16k}$ has only 8 more such complex structures than $\mathbb{R}^k$). In fact it doesn't seem easy to improve the previous bound $n \leqslant 3k/2 - 1$ in general. Here is another (too rough?) method of construction of quadratic groups, independent of Ω-regularity, but which leads to the same bound.

PROPOSITION 4.2. — *Any increasing family of extensions of $D = \mathbb{R}^k$ by*

$$L_p = \mathrm{span}(\omega_1, \omega_2, \cdots, \omega_p) \subset \Lambda^2 D^*,$$

leads to quadratically presented groups if

1. each ω_p is of rank $\geqslant 2(p + 1)$,

2. $\omega_p \notin L_{p-1} + \{\alpha \wedge \beta \in \Lambda^2 D^*\}$.

Proof. — As in (25), we have to show that $\sum_{i=1}^{p} \alpha_i \wedge \omega_i = 0$ with $\alpha_i \in D^*$ implies $\alpha_i = 0$. This is clear for a single term by assumption ω_1 of rank $\geqslant 4$. Suppose we have a non trivial relation $\alpha_p \wedge \omega_p = \sum_{i=1}^{p-1} \alpha_i \wedge \omega_i$, with $\alpha_p \neq 0$ by recurrence hypothesis.

We will show that all α_i are multiple of α_p. Firstly $\omega_p \wedge \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p = 0$, and since $\mathrm{rk}(\omega_p) \geqslant 2(p+1)$, this implies $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p = 0$. Therefore some α_{i_0} with $1 \leqslant i < p$ is a linear combination of the others. Now $\omega_p \bigwedge_{i \neq i_0} \alpha_i = 0$ implies again $\bigwedge_{i \neq i_0} \alpha_i = 0$ and

gives another dependent α_{i_1}. We repeat this until we obtain $\alpha_p \wedge \omega_p = \displaystyle\sum_{i=1}^{p-1} c_i \alpha_p \wedge \omega_i$

for some constants c_i. This implies that $\omega_p = \displaystyle\sum_{i=1}^{p-1} c_i \omega^i + \alpha_p \wedge \beta_p$, which is contrary to the assumption 2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remarks 4.3

- Notice that the first hypothesis implies, and is generically satisfied if $2p + 2 \leqslant k$, while the second one is generic if $p - 1 + 2(k - 2) + 1 < \frac{k(k-1)}{2} = \dim(\Lambda^2 D^*)$. This is seen to be implied by the first condition, which amounts finally to $n = p + k \leqslant 3k/2 - 1$ as with Ω-regularity.

- The second condition means that L_p doesn't contain any rank 2 form $\alpha \wedge \beta$. This is certainly a necessary condition for quadratic presentation. Otherwise $\alpha \wedge \theta$ and $\beta \wedge \theta$, with $d_0\theta = \alpha \wedge \beta \in L_p$, will be a d_0-closed but not exact 2-forms of weight 3 in the corresponding extension G of D. Another equivalent formulation is that G can't admit a surjective morphism onto the 3-dimensional Heisenberg group $\mathbf{H}^3$.

• So far we have only considered 2-step groups, but quadratically presented groups of arbitrarily high rank do exist. Interest in that field came from its relation with Sullivan's rational homotopy theory, where these groups are 1-formal 1-minimal. In particular since Kähler manifolds M have formal minimal models, the Malcev's completion $\Pi_1(M) \otimes \mathbb{Q}$ of $\Pi_1(M)$ are examples of such groups, at the condition they are finite dimensional, that is nilpotent. We refer to [12] for details on these problems.

The simplest example of a rank 3 quadratically presented group is given by a generic 3-step distribution $D^5 \subset \mathbb{R}^8$ studied in [3]. Its structure is given by $D = \mathrm{span}_{1 \leqslant i \leqslant 5}(X_i)$, $\mathfrak{g}_2 = [D, D] = \mathrm{span}(Y_1, Y_2)$, and $\mathfrak{g}_3 = \mathbb{R}Z$ with the dual relations

$$dy_1 = x_1 \wedge x_3 + x_2 \wedge x_4, \ dy_2 = x_1 \wedge x_4 + x_2 \wedge x_5, \ dz = x_1 \wedge y_1 + x_2 \wedge y_2.$$

Observe that the third cubic form $x_1 \wedge y_1 + x_2 \wedge y_2$ is closed from the two first one, and thus may be written dz in order to kill it in $H^2(\mathfrak{g})$. It turns out (see [3]) that there are no other weight $\geqslant 3$ d_0-closed 2-forms, and that this group is quadratically presented.

• Lastly, a series of arbitrarily high rank quadratically presented group has been described by Chen in [5]. Given $(n, k) \in \mathbb{N} \times \mathbb{N}^*$, these $k + 1$-rank groups are generated by $D = \{X_1, \cdots, X_n, Y_\alpha, |\alpha| = k\}$, with $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The group structure is given by

$$dx_i = 0, \quad dy_\alpha = 0 \text{ if } |\alpha| = k, \quad dy_\alpha = \sum y_{\alpha+e_i} \wedge x_i \text{ if } |\alpha| < k,$$

with $e_i = (0, \cdots, 1, \cdots, 0)$. One has $\mathfrak{g}_j = D^{[j]} = \mathrm{span}(Y_\alpha, |\alpha| = k - j + 1)$. These examples have still $\beta_1 = 1$, showing that this function may be independent of the rank of the group.

4.2. Some higher rank groups

We now give examples of groups with relations of higher weight.

- Let $G^{k,r}$ be the "free" nilpotent group of rank r with k generators. This is the quotient of the infinite dimensional free Lie group $\widetilde{G}^k$ in k generators by all its elements of weight $> r$. Is is the largest r-step filtered group, since any other one may be presented as a quotient of it. By definition the relations of $G^{k,r}$ with respect to $\widetilde{G}^k$ are generated by all words of weight $r + 1$ in $\widetilde{G}^k$. Therefore corollary 3.15 gives

$$\beta_1(G^{k,r}) = r,$$

the upper possible bound for r-step groups, and thus the lowest possible $\alpha_1 = N(G)/r$ and heat decay $N(G)/2r$ (with respect to the Hausdorff dimension $N(G)$).

- These $G^{k,r}$ are helpful in precising the pinching of β_1 for groups associated to generic distributions $D^k \subset \mathbb{R}^n$. Let $n(k,r) = \dim(G^{k,r})$ (a general formula using Möbius function may be found eg in [13], section 4.1.B).

PROPOSITION 4.4. — *For a fixed k, one has $r - 1 \leqslant \beta_1(G) \leqslant r$ for groups G associated to a generic $D^k \subset \mathbb{R}^n$, if r is such that $n(k, r - 1) \leqslant n < n(k, r)$.*

Proof. — In fact these groups have only relations of weight r and $r + 1$ generically.

Let $(X_i)_{1 \leqslant i \leqslant k}$ be a polynomial germ of k independent vectors fields around 0 in $\mathbb{R}^n$. Define $D^k = \mathrm{span}(X_i, 1 \leqslant i \leqslant k)$, and G the graded Lie group associated to this C-C structure at 0 (by the quotiented Lie bracket $[\ ,\ ]_0$ see section 2.2.1). Recall that if $H_i = [H, H_{i-1}]$, and $H_1 = D^k$ one has $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$, with $\mathfrak{g}_i = H_i/H_{i-1}$, at 0. Generically, on a Zariski open dense set in the $(j - 1)$-jet of (X_i) at 0, one has

$$\mathfrak{g}^j = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_j$$

of maximal possible dimension, that is $\min(n(k, j), n)$. Therefore $\mathfrak{g}^j$ is isomorphic for $j \leqslant r - 1$ to the Lie algebra of $G^{k,j}$, and G has no relation of weight $< r$. For $j = r$, one has $\mathfrak{g}^r$ of dim n (meaning D^k is an Hörmander r-step distribution in $\mathbb{R}^n$), and then $\mathfrak{g} = \mathfrak{g}^r$ is of rank r. This gives only relations of weight $\leqslant r + 1$. □

- In the opposite direction to the previous "generic" situation, nilpotent groups associated to complex semi-simple geometries may have increasing rank but still a presentation of bounded (although not quadratic) depth.

For instance, consider $N_n \subset SL(n, \mathbb{C})$ the nilpotent group of Id +strictly upper triangular matrices. It is a rank $(n-1)$ group whose Lie algebra is generated by $D = \mathrm{span}(X_i = E_{i,i+1} \in M_n(\mathbb{C}), 1 \leqslant i \leqslant n - 1)$. $N_3 = \mathbf{H}^3$ is the 3 dimensional Heisenberg group, already encountered (it is cubically presented and has $\beta_1 = 2$. For all $n \geqslant 4$, N_n has a presentation with both quadratic an cubic relations, which are

$$[X_i, X_j] = 0 \text{ for } j - i > 1, \text{ and } [X_i, [X_i, X_{i+1}]] = [X_{i+1}, [X_i, X_{i+1}]] = 0.$$

Dually, $H^2(N_n)$ is generated by $\theta_{X_i} \wedge \theta_{X_j}$ for $j - i > 1$ and $\theta_{X_i} \wedge \theta_{Y_i}$, $\theta_{X_{i+1}} \wedge \theta_{Y_i}$ where θ_{Y_i} is the dual form to $Y_i = E_{i,i+2} = [X_i, X_{i+1}]$. All this comes from the general description by Kostant of the structure of the cohomology of (maximal) nilpotent Lie algebras in semisimple Lie algebra (which splits in multiplicity one factors through the adjoint action of the maximal torus) (see [4],[9]).

Therefore, for these groups of increasing rank, we have $1 \leqslant \beta_1 \leqslant 2$. We see that $\beta_1(N_4) = 2$.

Proof. — For $f \in C_0^\infty(N_4)$, consider $\alpha = f\theta_{X_2}$. We observe that $d_c\alpha$ has weight 3. This is because the weight 2 part of $E_0^2 = H^2(\mathfrak{n}_4)$ is spanned by $\theta_{X_1} \wedge \theta_{X_3}$, and thus

$$(d_c\alpha)_2 = d\alpha(X_1, X_3) = X_1.\alpha(X_3) - X_3.\alpha(X_1) = 0.$$

As a consequence, for a generic f, we have found a non-closed smooth form in $E_0^1 = \Lambda^1 D^*$ such that $d_c\alpha$ has weight 3. We will see in proposition 5.6 this implies $\beta_1 \geqslant 2$. □

• We close this section with a first example showing that $\beta_1 = N(G)/\alpha_1$ is not necessarily an integer when $N(G)$ the Hausdorff dimension of a filtered group G.

We consider the 4-dimensional Engel group G (studied in section 2.3). We observe that with its filtered weight N, one has $N(G) = 2 + 2 + 3 = 7$, while $H^2(\mathfrak{g})$ is generated by $\theta_Y \wedge \theta_Z$, of weight 3 and $\theta_X \wedge \theta_T$, of weight 4. Hence, one has the pinching $2 \leqslant \beta_1^N(G) \leqslant 3$, leading to

$$7/3 \leqslant \alpha_1(G) \leqslant 7/2. \tag{26}$$

Now, we can change the grading, using instead N' such that $N'(X) = 1$, $N'(Y) = 2$, $N'(Z) = 3$ and $N'(T) = 4$. In that case $H^2(\mathfrak{g})$ becomes homogeneous of pure weight 5, while $E_0^3 = H^3(\mathfrak{g}) = \text{span}(\theta_{X \wedge Z \wedge T}, \theta_{Y \wedge Z \wedge T})$ has now mixed weight 8 and 9. Applying theorem 3.13 at the level of 2 forms, we find that $8 - 5 = 3 \leqslant \beta_2^{N'}(G) \leqslant 9 - 5 = 4$, giving

$$10/4 \leqslant \alpha_2(G) \leqslant 10/3 \tag{27}$$

since $N'(G) = 10$. This gives a pinching of the spectral density of δd on $\Omega^2(G)/\ker d$ which is $*$-conjugated to $d\delta$ acting on $\Omega^2(G)/\ker \delta$. This last one is itself conjugated by δ to the spectrum of δd on $\Omega^1(G)/\ker d$. We get finally that $\alpha_2(G) = \alpha_1(G)$ and that the second pinching (27) is strictly sharper than the first one (26). In particular $\beta_1^N(G) = 7/\alpha_1(G)$ can't be an integer using the "Hausdorff" scaling N, the relevant one on functions.

• Still on that group, we can use the computations of section 2.3 to understand more precisely some aspect of the asymptotic heat diffusion on its 1-forms. We restrict our study to forms on G whose components along the left invariant vectors fields X, Y, Z, T are invariant functions along Z, T, that is pull-back functions by $\Pi : G \to \mathbb{R}^2 = G/(Z, T)$. We note $\Omega^*_{\mathbb{R}^2} G = C(\mathbb{R}^2) \otimes \Lambda^* \mathfrak{g}^*$ this class of forms, and look at the asymptotic behavior of the "commutative" heat, or spectrum of Δ_G as acting on $L^2_{\mathbb{R}^2}(\Omega^*_{\mathbb{R}^2} G)$. Like in the true $L^2(G)$ situation, the de Rham complex on G has the same restricted $\mathbb{R}^2$-asymptotics as (E_0, d_c) (equivalent $\mathbb{R}^2$-near cohomology in fact, as given by the same proofs as before).

Now, the action of d_c on $\Omega^1_{\mathbb{R}^2} G$ may be read from (8), by cutting the Z and T arrows. We see that d_c factorizes here in $P d_{\mathbb{R}^2}$ where

$$d_{\mathbb{R}^2}(f\,\theta_X + g\theta_Y) = (X.g - Y.f)\theta_Z,$$

and

$$P(f\theta_Z) = (X^2.f)\theta_{X\wedge T} + (Y.f)\theta_{Y\wedge Z}.$$

Hence, on $E^1_{0,\mathbb{R}^2} \cap (\ker d_c)^{\perp}$, the operator $\delta_c d_c$ is conjugated by $d_{\mathbb{R}^2}$ to $P^*P = X^4 + Y^2$ acting on the one dimensional bundle θ_Z. Its spectrum can be described by Fourier analysis, in $\mathbb{R}^2$ here. The Fourier transform of the spectral space $E(P^*P \leqslant \lambda^2)$ are the L^2 functions whose supports are contained in the set

$$FS(\lambda) = \{(x,y) \in \mathbb{R}^2, x^4 + y^2 \leqslant \lambda^2\}.$$

When $\lambda \to 0$, this strange collection of flying saucers are shrinking at speed λ along y and only $\sqrt{\lambda}$ along x. Therefore the rescaling $(X,Y) \to (\lambda^{-1/2}X, \lambda^{-1}Y)$ associated to the previous weight N' appears naturally here. Observe anyway that the $FS(\lambda)$ have no usable self-similar limit in this rescaling (they converge to the union of two segments along x and y). In fact one finds easily that the asymptotic area of $FS(\lambda)$ is $-\lambda^2 \ln \lambda$ when $\lambda \to 0$. This measure of $FS(\lambda)$ is also the density of $E(P^*P \leqslant \lambda^2)$, $E(\delta_c d_c \leqslant \lambda^2)$, and finally $E(\delta d \leqslant \lambda^2)$ as acting on $L^2(\mathbb{R}^2)$. We get finally by Laplace transform that the asymptotic $\mathbb{R}^2$-heat decay on 1-forms is equivalent to $\dfrac{\ln t}{t^{1/2}}$ as $t \to +\infty$ (to be compared to the standard heat decay in $1/t$ on $\mathbb{R}^2$).

Lastly, one can obtain some information on the anisotropic aspects of large scale heat diffusion here. For $\varepsilon > 0$, consider the orthogonal splitting

$$L^2(\mathbb{R}^2) = L^2_\varepsilon(\mathbb{R}^2) \oplus (L^2_\varepsilon(\mathbb{R}^2))^{\perp},$$

with $L^2_\varepsilon(\mathbb{R}^2)$ the space of L^2 functions whose Fourier transforms are supported in $D_\varepsilon = \{(x,y), |y| \leqslant \varepsilon|x|\}$. We note that for each ε, $\text{area}(FS(\lambda) \cap D_\varepsilon) \asymp -\lambda \ln \lambda$ while $\text{area}(FS(\lambda) \cap D_\varepsilon^{\perp}) \asymp \lambda = o(-\lambda \ln \lambda)$, showing that most the spectral measure of $E(P^*P \leqslant \lambda^2)$ is supported on D_ε when $\lambda \to 0$. Recall that the d_c Laplacian on horizontal 1-forms is conjugated to P^*P acting on θ_Z by the map

$$\delta_{\mathbb{R}^2}(f\theta_Z) = (Y.f)\theta_X - (X.f)\theta_Y.$$

Since for functions $f \in L^2_\varepsilon(\mathbb{R}^2)$, one has

$$\|Y.f\|_2 = \|y\hat{f}\|_2 \leqslant \varepsilon\|x\hat{f}\|_2 = \varepsilon\|X.f\|_2,$$

we see that most the spectral measure of $E(\delta_c d_c \leqslant \lambda^2)$, and finally $E(\delta d \leqslant \lambda^2)$ is supported on forms α such that $\|\alpha(X)\|_2 \leqslant \varepsilon\|\alpha(Y)\|_2$, that is forms close to the Y-direction. Topologically this direction corresponds to the largest Massey product we can find in the 1-minimal model of G. Namely, one has $\theta_X, \theta_Y \in H^1(\mathfrak{g})$ with

$$\begin{cases} -\theta_X \wedge \theta_Y = d_0\theta_Z \\ -\theta_X \wedge \theta_Z = d_0\theta_T \\ -\theta_X \wedge \theta_T \in H^2(\mathfrak{g}), \end{cases}$$

Analytically, this sequence is recognized in (8) as the X^3 component of d_c from θ_Y to $\theta_X \wedge \theta_T$, the link being done by taking the elliptic symbol of d_c in the X direction. This shows that some information on the multiplicative structure of $H^*(\mathfrak{g})$, not only its weight, is analytically encoded in (E_0, d_c) even at the commutative level. The effect here is that we can "hear" the exceptional direction Y.

5. Towards a refined pinching

The pinching of α_1 we gave relies on very few analysis, general facts on the Hilbert-Schmidt and trace class operators, and the ellipticity of the de Rham Laplacian. Anyway it is certainly not sufficient to obtain the exact value of α_1, β_1 on groups of non homogeneous presentation. To improve (even partially) the result we have to be more careful on the analytic properties of d_c as related to the convergence of the underlying spectral sequence and filtered complex.

5.1. C-C ellipticity and applications

To that purpose we introduce some class of analytic regularity that will fit to the vector valued operators on graded groups we are dealing with. It relies on the notion of (maximal) hypoellipticity as defined for scalar operators.

Let G be a graded group, and P a left invariant differential operator on $C^\infty(G)$ of order p with respect to natural grading in TG. P is said maximally hypoelliptic if locally $\|Pf\|_2 + \|f\|_2$ controls all derivatives of f of weight $\leq p$. One can show (see [22, 17]) that this is equivalent to the existence for P of a parametrix Q of order $-p$. That means the kernel of Q is an homogeneous distribution on G of order $p - N(G)$ near the diagonal, and such that $Q \circ P = \text{Id} + \mathcal{S}$ for some smoothing operator $\mathcal{S}$. This amounts also to the partial invertibility of P at the convolution level in $q * p = \delta_0 + s$ with $s \in \mathcal{S}(G)$ (where the convolution product on G is defined by $(f * g)(x) = \int_G f(y^{-1}x)g(y)dy$). This last criteria makes sense on more general operators than differential ones. On a filtered group the basic example of a second order maximally hypoelliptic operator is given by the horizontal sub-Laplacian defined on functions by

$$\Delta_H f = -\sum_{i=1}^k X_i^2 f, \tag{28}$$

where $D = \text{span}(X_i, 1 \leq i \leq k) = \mathfrak{g}_1$ generates $\mathfrak{g}$. Starting from this positive self-adjoint operator, one can define $|\nabla| = \Delta_H^{1/2}$ which can be shown to be an order one hypoelliptic pseudo-differential operator on G (see [11] section 3) invertible in the previous sense. On a general graded group, not necessarily filtered, such an order one scalar hypoelliptic operator can also be constructed. One can either replace Δ_H with the operator P of [16] corollaire 0.2, or use use the one of [6] section 6. We denote by $|\nabla|^{-1}$ its parametrix, in both left and right senses by self-adjointness of $|\nabla|$.

We are now ready to define the class of regularity we need. We extend $|\nabla|$ as acting diagonally on the full exterior algebra $\Omega^* G$. We call N the induced weight function there.

DEFINITION 5.1. — *An operator P acting on (part) of $\Omega^* G$ is called C-C elliptic if $P^\nabla = |\nabla|^{-N} P |\nabla|^N$ is a maximally hypoelliptic operator in the previous sense.*

In the scalar case this is just hypoellipticity. Thus $P = \Delta_H$ is such an operator, while the full de Rham Laplacian on functions is not (except in $\mathbb{R}^n$). Indeed if G is a r-step group, Δ is a $2r$-order operator in the graded sense, as it contains X_r^2 terms with $X_r \in \mathfrak{g}_r$, but Δ do not control $2r$-derivatives along $\mathfrak{g}_1$ for $r > 1$. One interesting feature of this class is that even if de Rham Laplacian is not in it, the de Rham complex itself is, as (E_0, d_c).

THEOREM 5.2 ([25]). — *Let G be a graded nilpotent Lie group. Then the de Rham complex and (E_0, d_c) are C-C elliptic.*

That means the Laplacians associated to d^∇ and d_c^∇

$$\Box_d = d^\nabla (d^\nabla)^* + (d^\nabla)^* d^\nabla \quad \text{and} \quad \Box_{d_c} = d_c^\nabla (d_c^\nabla)^* + (d_c^\nabla)^* d_c^\nabla \tag{29}$$

(*not* $(\Delta)^\nabla$ and $(\Delta_c)^\nabla$) are maximally hypoelliptic. Observe that they are all 0 order operators ! Indeed d and d_c are homogeneous operators on $\Omega^* G$ in the sense the components of d and d_c that increase the weight of forms by k are precisely order k operators (the whole construction of section 2 is homogeneous in that sense).

Proof. — Thanks to analysts' work a short proof of thm 5.2 is possible.

It relies on an hypoellipticity criteria (called Rockland's condition) first proved for differential operators on general graded groups by Helffer and Nourrigat [16], and extended to the pseudo-differential setting by Christ and all in [6]. Given a (pseudo-differential) operator P of order k on G, call P_k its homogeneous part of order k. Then the maximal hypoellipticity of P is implied by the injectivity, for all non trivial irreducible unitary representations π of G, of $\pi(P_k)$ on $\mathscr{S}_\pi \subset H_\pi$ the space of C^∞ vectors of the representation. (If $G = \mathbb{R}^n$ this is the well known ellipticity criteria on the injectivity of the principal symbol of P in all directions. In general we recall (see [8] or [17]) that Kirillov has shown that these representations are parameterized by non-zero covectors $\xi \in \mathfrak{g}^*$ modulo the co-adjoint action of G on $\mathfrak{g}^*$.)

We will need to know that for $0 \neq \pi \in \hat{G}$ one can find some $X \in \mathfrak{g}$ such that $\pi(X)$ is the scalar (anti-self adjoint) operator $i\,\mathrm{Id}$. Indeed, by irreducibility of π the center $Z(G)$ of G acts by scalars on H_π (given by a character $Z(G) \to U(1)$). If it is trivial, consider the quotiented action on $Z(G/Z(G))$ and so on until, by nilpotency of G, some non trivial scalar action is found.

We come to the proof. Let π and X as above. that the 0 order part of $\pi(\Box_d)$ and $\pi(\Box_{d_c})$ are injective on the C^∞ vectors $\mathscr{S}_\pi \subset H_\pi$. These spaces really identify with the standard Schwarz space $\mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ with $\mathbb{R}^n \simeq G/H$ when π is realized as an induced representation from G/H. The differential operators P on G transform as differential operators with polynomial coefficients on $\mathbb{R}^n$, while by definition $|\nabla|$ is such

that $\pi(|\nabla|)$ is invertible and preserves $\mathscr{S}_\pi$ (in the filtered part we have just $\pi(|\nabla|)$ = $\pi(\Delta_H)^{1/2}$). Hence the injectivity of $\pi(\square_d)$ on $\mathscr{S}_\pi$ is equivalent to the injectivity of the system $\pi(d^\nabla) + \pi(d^\nabla)^*$ there. This is certainly implied by

$$\ker \pi(d^\nabla) \cap \mathscr{S}_\pi = \pi(d^\nabla)(\mathscr{S}_\pi).$$

This is equivalent by dropping the conjugation by $|\nabla|^N$, which preserves $\mathscr{S}_\pi$, to the vanishing of the smooth cohomology of the complex $\pi(d)$

$$\ker \pi(d) \cap \mathscr{S}_\pi = \pi(d)(\mathscr{S}_\pi). \tag{30}$$

Consider $\mathscr{L}_X$ the Lie derivative along X, with X as above. As a differential operator on $\Omega^* G$ it can decomposed in $\mathscr{L}_X = X \operatorname{Id} + ad(X)$ where

$$ad(X)\alpha = -\sum \alpha(\cdot, [X, \cdot], \cdot) = (\mathscr{L}_X)_0 = i_X d_0 + d_0 i_X$$

is the algebraic part of $\mathscr{L}_X$ when expressed in a left invariant base. Since $\pi(X) = i\operatorname{Id}$, one has $\pi(\mathscr{L}_X) = i\operatorname{Id} + ad(X)$ on H_π with $ad(X)$ nilpotent. It is therefore invertible of inverse

$$P_X = \pi(\mathscr{L}_X)^{-1} = \sum_{k \geqslant 0} i^{k-1} ad^k(X).$$

From Cartan's formula $\mathscr{L}_X = i_X d + d i_X$ we get $[\mathscr{L}_X, d] = [\mathscr{L}_X, i_X] = 0$, so that $[P_X, \pi(d)] = [P_X, i_X] = 0$, and finally

$$\operatorname{Id} = (P_X i_X)\pi(d) + \pi(d)(P_X i_X),$$

which gives (30). The proof for (E_0, d_c) is a formal consequence of this one. We recall that by theorem 2.6, $d_c = \Pi_{E_0} \Pi_E d \Pi_E \Pi_{E_0}$, with $\Pi_{E_0} \Pi_E \Pi_{E_0} = \operatorname{Id}$ on E_0. Composing (31) with $\pi(\Pi_{E_0} \Pi_E)$ on the left and $\pi(\Pi_E \Pi_{E_0})$ on the right we get on E_0

$$\operatorname{Id} = Q_X \pi(d_c) + \pi(d_c) Q_X,$$

with $Q(X) = \pi(\Pi_{E_0} \Pi_E) P_X i_X \pi(\Pi_E \Pi_{E_0})$. As before this implies the injectivity of $\pi(\square_{d_c})$. $\qquad\qquad\square$

Remark 5.3 The role of the conjugation by $|\nabla|^N$ in this proof appears very formal. Anyway it is crucial to put everything in a Sobolev scale and work with an homogeneous operator.

The twisted d_c^∇ has a rather strange harmonic theory, due to the fact that $(d_c^\nabla)^* = |\nabla|^N \delta_c |\nabla|^{-N}$ is not $\delta_c^\nabla = |\nabla|^{-N} \delta_c |\nabla|^N$. On the other hand we observed in remark 3.7 that the d_c-harmonic theory is not so attractive, because not scale invariant, except in the case we have $E_0^k = H^k(\mathfrak{g})$ of pure weight N_k. This is precisely the assumption we did in the previous pinching results. In that case the C-C regularity d_c is actually useful for the d_c harmonic theory itself. We remark that C-C ellipticity is preserved by any conjugation by an invertible fixed $|\nabla|^{N_0}$. So one can use any shifted weight function $N - N_0$ instead

of N. In the case E_0^k is of pure weight N_k the interesting one is $N' = N - N_k$, since then on E_0^k

$$d_c^\nabla + (d_c^\nabla)^* = |\nabla|^{-N'} d_c + |\nabla|^{N'} \delta_c = P(d_c + \delta_c), \tag{31}$$

with P of *strictly* negative order $\leqslant -1$, because d_c increases the weight by 1 at least.

We now show some applications of these techniques. Let again G be a graded group G and some k with $E_0^k = H^k(\mathfrak{g})$ of pure weight N_k. The C-C ellipticity of (E_0, d_c) allows to define some number r_k describing the analytic order of convergence of the spectral sequence associated to d on k-forms. Starting form the splitting (21) of d_c in $d_c^{\delta N_k^{\min}} + \cdots + d_c^{\delta N_k^{\max}}$, we consider for $r \in [N_k^{\min}, N_k^{\max}]$ its cut-off

$$d_c^{[r]} = d_c^{\delta N_k^{\min}} + \cdots + d_c^r.$$

By theorem 5.2, we know that the complex

$$E_0^{k-1} \xrightarrow{\ d_c\ } E_0^k \xrightarrow{\ d_c^{[r]}\ } E_0^{k+1}$$

is C-C elliptic on E_0^k if $r = \delta N_k^{\max}$.

DEFINITION 5.4. — *Define r_k as the smallest integer r such that the previous complex is C-C elliptic on E_0^k.*

In view of the proof of theorem 5.2 and (30) this r_k is the smallest number r such that $\ker \pi(d_c^{[r]}) \cap \mathscr{S}_\pi = \pi(d_c)(\mathscr{S}_\pi)$ for all non-trivial $\pi \in \hat{G}$. From a purely algebraic viewpoint, this r_k is actually the order of convergence, at the level of representation, of the underlying spectral sequence associated to the natural filtered complex here (see section 2). In other words, r_k is also the smallest integer such that for all π, any smooth k-form $\alpha \in \mathscr{S}_\pi$ of weight p which satisfies $\pi(d)\alpha$ of weight $\geqslant p + r$ is necessarily $\pi(d)$-closed (and finally exact) up to a form of weight $p + 1$.

THEOREM 5.5 ([25]). — *Let G, E_0^k as in theorem 3.13. Then for r_k as above*

$$\delta N_k^{\min} \leqslant \beta_k \leqslant r_k \leqslant \delta N_k^{\max}.$$

We will give examples of groups with $r_k < \delta N_k^{\max}$ in the next section.

Proof. — We have to show that $\beta_k \leqslant r_k$. We use the same near-cohomology approach as in the proof of theorem 3.13. Firstly since for $\alpha \in E_0^k$

$$\|d_c^{[r_k]}\alpha\|^2 = \|d_c^{\delta N_k^{\min}}\alpha\|^2 + \cdots + \|d_c^{r_k}\alpha\|^2 \leqslant \|d_c\alpha\|^2,$$

we certainly have that the near-cohomology cones thickness functions of d_c and $d_c^{[r_k]}$ satisfy

$$F_{d_c}(\lambda) \leqslant F_{d_c^{[r_k]}}(\lambda).$$

Hence the same homogeneity argument as in the proof of thm 3.13 gives the result if we show that $F_{d_c^{[r_k]}}(\lambda)$ is finite. This is a consequence of the C-C ellipticity assumption. Namely the same proof as theorem 5.2 together with (31) shows that $|\nabla|^{-N'} d_c^{[r_k]} + |\nabla|^{N'} \delta_c = P(d_c^{[r_k]} + \delta_c)$ is an order 0 maximally hypoelliptic pseudo-differential operator. That means (see [6]) it is bounded and invertible in L^2 up to a smoothing term

$$QP(d_c^{[r_k]} + \delta_c) = \text{Id} + \mathscr{S},$$

for some bounded (order 0) Q and smoothing $\mathscr{S}$. We have already observed that P is of order $\leqslant -1$, hence $B = |\nabla| QP$ is bounded and satisfies

$$B(d_c^{[r_k]} + \delta_c) = |\nabla| + \mathscr{S}. \tag{32}$$

Therefore there exists some C such that

$$\||\nabla| \alpha\| \leqslant C(\|d_c^{[r_k]} \alpha\| + \|\delta_c \alpha\| + \|\alpha\|),$$

giving on $\ker \delta_c$ the control of $F_{d_c^{[r_k]}}(\lambda)$ by $F_{|\nabla|}(C\lambda + C)$. But this last function is finite because $|\nabla|$ is a first order maximally hypoelliptic operator with smoothing spectral projectors (for any p, $|\nabla|^{-k}$ has a C^p kernel for k large enough by the Sobolev embedding theorems).   □

In the opposite direction to the previous result, one can sometimes improve the bound from below of β_k, if $d_c^{[r]}$ is sufficiently degenerated.

PROPOSITION 5.6. — *Let G, E_0^k as above. Suppose that for some r, there exists non-closed forms in $D(d_c) \cap \ker d_c^{[r]}$, then $\beta_k \geqslant r + 1$.*

Proof. — Consider the closed quadratic form $q(\alpha) = \|d_c \alpha\|^2$ on $(\ker d_c)^{\perp}$. Its restriction to $D(d_c) \cap \ker d_c^{[r]}$ is still closed, positive, and therefore associated to a self-adjoint Δ_r (see [23] thm VIII.15). By hypothesis $\Delta_r \neq 0$. Then for some λ, $L = E(0 < \Delta_r \leqslant \lambda^2)$ is a non zero G-invariant linear space in the near-cohomology cone $C_\lambda(d_c)$. Since d_c is homogeneous of order $\geqslant r + 1$ on $D(d_c) \cap \ker d_c^{[r]}$ we have, as in the proof of theorem 3.13, that $h_\varepsilon^\sim(L) \subset C_{\lambda \varepsilon^{r+1}}(d_c)$. This gives the bound from below for β_k.   □

Remark 5.7 We have already used this for the triangular group N_4 in section 4.2.

We can interpret the assumption in proposition 5.6 in terms of representation theory. Namely by the Fourier-Plancherel isomorphism between G and $\hat{G}$, the existence of some non-zero L^2 form $\alpha \in D(d_c) \cap \ker d_c^{[r]}$ implies that $\pi(\alpha)$ is a field of Hilbert-Schmidt operators in $\ker \pi(d_c^{[r]})$ with $\pi(\alpha) \neq 0$ on a set of representations of strictly positive Plancherel measure. Then the cohomology $\ker \pi(d_c^{[r]}) / \text{Im } \pi(d_c)$ is non-vanishing for "a lot" of representations of maximal dimensions (called generic). This is opposite (but not complementary) to the case of definition 5.4 and theorem 5.5, where we required this cohomology to vanish on all representations.

We close this section with an application the d_c Hodge-de Rham decomposition on compact E_0 regular C-C manifolds M. We have seen in proposition 3.6 that the we have a closed splitting in $L^2(\Omega^* M)$

$$E_0 = \mathcal{H}_c \oplus \mathrm{Im}\, d_c \oplus \mathrm{Im}\, \delta_c,$$

where $\mathcal{H}_c = \ker d_c \cap \ker \delta_c$. Again we restrict to the case of $E_0^k = H^k(\mathfrak{g}_{x_0})$ is of pure weight N_k (see also remark 3.7). We assume also that C-C structure is given by a bracket generating distribution D (the filtered case).

PROPOSITION 5.8. — *Let M, E_0^k as above. Then $\mathcal{H}_c = \ker d_c \cap \ker \delta_c$ consists of smooth forms.*

To see this we define a differential operator of the form $\square_c = P(d_c + \delta_c)$, with P chosen such that $\square_c$ becomes homogeneous. Fix any horizontal connection ∇_D on $\Omega^* M$. let K be the maximal order of the components of $d_c + \delta_c$ on E_0^k. Then we consider

$$\square_c = \sum_p \nabla_D^{K-p}(d_c^p + \delta_c^p),$$

where d_c^p (resp. δ_c^p) is the components of d_c that increases (resp. decreases) the C-C weight by p. This $\square_d$ is a differential operator of order K. We show that

LEMMA 5.9. — *$\square_c$ is maximally hypoelliptic on M (in the bracket generating case).*

We recall this means that $\square_c \alpha$ controls the full the K-horizontal jet of α (in L^2 norms). One consequence of this is that weak solutions to $\square_c \alpha = 0$ are smooth, giving proposition 5.8.

Proof. — Some characteristic features of maximal hypoellipticity of differential operators are its stability under perturbations of lower order and the fact, in the filtered case, it can be checked on the model group G_{x_0} with the freezed operator $\square_{c,x_0}$ (see [17]). We have already observed that d_c at x_0 may be viewed as a perturbation of d_c on G_{x_0} (by Cartan's formula $d\alpha(X_i) = \sum X_i \alpha(\cdot) - \alpha(\cdot, [X_i, \cdot], \cdot)$ and the fact that $[\ ,\ ]_0$ is the lower part of $[\ ,\ ]$). Hence we just have to check the result on the tangent group. There $\square_c$ looks like an algebraic version of $d_c^\nabla + (d_c^\nabla)^*$ in (31) with the shift weight $N' = N - N_k$, except the scalar pseudo-differential $|\nabla|$ is replaced by the full first order horizontal jet ∇_D. The same proof as in theorem 5.2 actually gives the injectivity of $\pi(\square_c)$ since on $\mathcal{S}_\pi$, $\ker \pi(\square_c) = \ker \pi(d_c) \cap \ker \pi(\delta_c)$ and we have already seen that $\ker \pi(d_c) \cap \mathcal{S}_\pi = \pi(d_c)(\mathcal{S}_\pi)$. □

Remark 5.10 The last part of this proof applies on any graded group, not only the filtered ones. It can therefore be taken as an alternative approach to C-C ellipticity using only differential operators, in this case of E_0^k of homogeneous weight.

5.2. Last examples

We start with a series of groups illustrating the "analytic" pinching of β_1 obtained in theorem 5.5. We show that some groups may have (arbitrarily) high order relations and still r_1 and thus $\beta_1 = 1$.

5.2.1. First inaudible relations.

Consider some quadratically presented group G of rank $r > 3$. The examples of Carlson-Toledo and Chen have been described in section 4.1. Let I be any ideal in $\mathfrak{g}$ generated by elements of weight $\geqslant 3$, $N = \exp I$ and consider $H = G/N$. The relations of H as referred to the free Lie group, are generated by the quadratic relations of G and the generators of weight $\geqslant 3$ of N. It may therefore have high order relations, but anyway they are "inaudible" in the heat decay.

PROPOSITION 5.11. — *For such groups H, $r_1 = 1 = \beta_1(H)$, and finally $\alpha_1(H) = N(H)$. In particular these values are the same as on quadratic groups.*

Proof. — We use theorem 5.5. Let π_H be a non-trivial irreducible unitary representation of H. Suppose α is a form in $\mathfrak{h}_1^*\cap\mathscr{S}_\pi$ such that $\pi_H(d_c^{[1]})\alpha = 0$. This means that α has an extension $\widetilde{\alpha}$ (given by $\pi_H(\Pi_E)\alpha$) such that $\pi_H(d)\widetilde{\alpha}$ is of weight $\geqslant 3$. Consider $P : G \to H$. Then $P^*(\pi_H) = \pi_G$ is an irreducible unitary representation of G and $\pi_G(d)(P^*\widetilde{\alpha}) = P^*(\pi_H(d)\widetilde{\alpha})$ is still of weight $\geqslant 3$. But since G is quadratically presented this implies that $\pi_G(d_c^G)(P^*\widetilde{\alpha}) = 0$, and finally $P^*\widetilde{\alpha}$ can be written $\pi_G(d_c^G) f$ for some $f \in \mathscr{S}_\pi$. Restricting this to $\mathfrak{g}_1$, gives $\alpha = \pi_F(d_c) f$ and the result. $\qquad\square$

These examples show that the spectral sequence may converge in the L^2 sense quicker than in the algebraic one. More precisely $H^2(\mathfrak{h})$ always contain forms of weight $\geqslant 3$ here, dual to the relations we added by proposition 2.9. View as left invariant forms in H they are closed, therefore locally exact. Thus we obtain one forms α such that $d\alpha \in H^2(\mathfrak{h})$ are non-zero forms of weight $\geqslant 3$. In other words $\Pi_{E_0}\alpha$, the restriction of α to $D = \mathfrak{h}_1$ satisfies $d_c\alpha$ of weight $\geqslant 3$. This shows, as claimed, that the convergence rank of the spectral sequence as a local tool is actually greater than in L^2.

We can explain (partially) more geometrically why these high order relations do not have an L^2 trace. We have to study

$$E_2 = \{\alpha \in E_0^1(H) = \Omega^1 D \mid d_c\alpha \text{ is of weight } \geqslant 3\}. \tag{33}$$

PROPOSITION 5.12. — *Up to closed forms, E_2 contains only forms whose components are polynomial functions (of bounded degree) in the coordinates of G.*

Thus certainly E_2 do not contain any non-closed L^2 form. We can't make an L^2 "wave packet" of them.

Proof. — We first show than the components of $d_c\alpha$ are polynomial functions. Then lifting α to $\Omega^1 H$ (with π_E) and integrating $d\tilde{\alpha}$ with Poincaré's lemma along polynomial vector fields gives the result.

Let $\Pi : G \to H$ and $\alpha \in E_2$. Again $\Pi^*\alpha$ is in $E_2(G)$ which are d_c^G-closed because G is quadratically presented. That means $\Pi^*\alpha$ has a true closed extension β on G. Recall that by section 2.4, $d_c\alpha$ interprets as the components of β along the generators of I the ideal of relations of H with respect to G. So we have to show these components are polynomial. This is consequence of (12). Namely, suppose G is of rank r. Applying (12) to $Y \in I$ of weight r, gives $X.\beta(Y) = 0$ because $[X, Y] = 0$ for any X. Therefore $\beta(Y)$ is constant. Applying now (12) to $Y \in I$ of weight $r-1$ gives then that $X.\beta(Y)$ are constants functions, since $[X, Y]$ is in Y and of weight r. And so on, all components of β along I are polynomials. $\square$

5.2.2. Around H-groups.

The previous groups give us first example of "inaudible" relations. We give another illustration of the fact that "audible" relations have to be sufficiently flexible. We consider again H-type groups (see 4.1). A complete study may be found in [19] or [7]. Let $D = \mathbb{R}^n$, and $\mathbb{R}^k$ be endowed with scalar products. A Clifford module structure on D is a linear map

$$J : \mathbb{R}^k \to GL(n)$$

such that $J(\theta)^2 = -\|\theta\|^2\,\mathrm{Id}$ and $J(\theta)^* = -J(\theta)$. Consider then

$$L : \mathbb{R}^k \to \Lambda^2 D^*$$

$$\theta \to g_D(J(\theta)\cdot, \cdot).$$

The H-type group associated to J is the two step group G determined by the extension of D by $\mathrm{Im}\,L$. That means $\mathfrak{g} = D \oplus T$ with the Lie bracket structure $D \times D \to T = \mathbb{R}^{k*}$ given by $\theta([X, Y]) = -L(\theta)(X, Y) = -\langle J(\theta)X, Y\rangle$. When $k = 1, 3, 7$ such groups appears as the maximal nilpotent (Iwasawa) subgroups of the rank 1 semi-simple groups $SU(n, 1)$, $Sp(n, 1)$ and F_4^{-20} (acting on the Cayley plane).

PROPOSITION 5.13. — *Let G be a H-type group, then either $\beta_1(G) = 1$ or $\beta_1(G) = 2$ and G is the 3 dimensional Heisenberg group, or the 7 dimensional quaternionic group (associated to an elliptic $D^4 \subset \mathbb{R}^7$ see section 4.1).*

Proof. — Observe that D is necessarily of even dimension $n = 2p$, and that the $p = 1$ case corresponds to to the Heisenberg group $\mathbf{H}^3$ we have already met ($\beta_1 = 2$ by cubic presentation). So we restrict now to $p \geqslant 2$. We first show how the use of theorem 5.5 reduce things to a problem on the Heisenberg group $\mathbf{H}^{2p+1}$. We study r_1. By the orbit method non-trivial irreducible representation of G are of two types, up to the coadjoint action.

– The first ones are trivial on T and come from some character $\pi : D \to U(1)$ with $\pi(V) = e^{i\langle\xi, V\rangle}$ and non zero $\xi \in D$. We show that $\ker(\pi(d_c^{[1]})) \subset \mathrm{Im}\,\pi(d_c)$ on

$\Lambda^1 D^*$. Namely $\pi(d_c^{[1]})\alpha = 0$ means that $i\xi \wedge \alpha \in \operatorname{Im} L$ (because $d_c^{[1]}$ is the quotient of the horizontal differential by $\operatorname{Im} L$). But $\operatorname{Im} L$ contains only non-degenerate forms of top rank $2p \geqslant 4$ here. Therefore $\xi \wedge \alpha = 0$ and $\alpha = iC\xi = \pi(d_c)C$.

— The other (generic) ones are induced by a vertical θ, and factors trough a representation π of the Heisenberg group $\mathbf{H}^{2p+1}$ with the contact form θ and $d\theta = g_D(J(\theta)\cdot, \cdot)$. We orthogonally split $\operatorname{Im} L = \mathbb{R}d\theta \oplus L'$, and observe that by the Clifford structure $L' \subset \Lambda^{2,0}D^* \oplus \Lambda^{0,2}D^*$ with respect to $J(\theta)$ (while $d\theta \in \Lambda^{1,1}D^*$). Again the equation $\pi(d_c^{[1]})\alpha = 0$ means $\pi(d_D)\alpha \in \operatorname{Im} L$, and our system can finally be interpreted on $\mathbf{H}^{2p+1}$, as

$$\pi(d_c^{\mathbf{H}})\alpha \in L' \subset \Lambda^{(2,0)+(0,2)}D^* \quad \text{and} \quad \pi(\delta_c^{\mathbf{H}})\alpha = 0, \tag{34}$$

with $\alpha \in \mathscr{S}_\pi$ a smooth vector of the representation. We arrived at the turning.

LEMMA 5.14. — *Equation* (34) *has a non trivial solution iff there exists a non zero* $\alpha \in D^*$ *such that*

$$\alpha^{1,0} \wedge \Lambda^{1,0}D^* \subset L'^{(2,0)}.$$

In real notations $\alpha \wedge \beta - J\alpha \wedge J\beta \in L'$ *for all* $\beta \in D^*$.

Note that in this case the forms $\alpha \wedge \beta - J\alpha \wedge J\beta \in L' \subset \operatorname{Im} L$ are of rank 4. Since here $\operatorname{Im} L$ contains only top rank forms one has necessarily D of dimension 4. Moreover counting dimensions shows that $\dim D - 2 \leqslant \dim(\operatorname{Im} L) - 1 = k - 1$, ie $k \geqslant 3$, and finally $k = 3$ since this is the maximal number of complex structures on $D = \mathbb{R}^4$. Then certainly G has to be the seven dimensional quaternionic group Q_7. If not (34) has no solution and theorem 5.5 gives that $r_1 = 1$ and thus $\beta_1 = 1$.

Reversely Q_7 satisfies the condition of lemma 5.14, because $L'^{(2,0)} = \Lambda^{(2,0)}D^* = \mathbb{C}dZ_1 \wedge dZ_2$. It remains to see that $\beta_1(Q_7) = 2$. By proposition 5.6 it is sufficient to find some non d_c-closed $\alpha \in \Omega^1 D \cap D(d_c)$ such that $d_c^{[1]}\alpha = 0$. This is achieved by making a wave packet of solutions α_π of (34). Namely fix some compact set K of positive Plancherel measure in the generic representations of Q_7. Let P_{α_π} be the orthogonal projection on $\alpha_\pi \subset \mathscr{S}_\pi$. Fourier inversion formula (20) leads to consider for $g \in Q_7$

$$\alpha(g) = \int_{K \subset \hat{G}} \operatorname{Tr}(\pi(g)P_{\alpha_\pi})d\mu(\pi) = \int_K \langle \pi(g)\alpha_\pi, \alpha_\pi \rangle_{H_\pi} d\mu(\pi).$$

α_π being smooth vectors, $g \to \alpha(g)$ is a smooth form. Moreover

$$\|\alpha_\pi\|^2 = \int_K \|P_{\alpha_\pi}\|^2_{HS}d\mu(\pi) = \mu(K) \in]0, +\infty[,$$

and for any derivative

$$\|X^J\alpha\|^2 = \int_K \|\pi(X^J)\alpha_\pi\|^2 d\mu(\pi) < \infty.$$

Of course $d_c^{[1]}\alpha = \int_K \langle \pi(d_c^{[1]})\alpha_\pi, \alpha_\pi \rangle d\mu(\pi) = 0$ while $d_c\alpha \neq 0$ because $\pi(d_c)\alpha_\pi \neq 0$ by C-C ellipticity of d_c. $\qquad\square$

We are left with the proof of lemma 5.14

Proof. — We drop the π and $\mathbf{H}$ to lighten notations and work on $\mathbf{H}^{2p+1}$ (for $p \geq 2$) with the contact complex d_c. A know fact is that the Laplacian $\Delta_c = (p - k)\delta_c d_c + (p - k + 1)d_c\delta_c$ preserves the bigrading of $\Lambda^k D^*$ (for $k < p$). Applying this to $\Delta_c\alpha$ gives here

$$(\Delta_c\alpha, \alpha) = (p - 1)\|d_c^{2,0}\alpha^{1,0}\|^2 + (p - 1)\|d_c^{0,2}\alpha^{0,1}\|^2 \text{ by (34)}$$

$$= (\Delta_c\alpha^{1,0}, \alpha^{1,0}) + (\Delta_c\alpha^{0,1}, \alpha^{0,1})$$

$$= p\|\delta_c\alpha^{1,0}\|^2 + (p - 1)\|d_c^{2,0}\alpha^{1,0}\|^2 + (p - 1)\|d_c^{1,1}\alpha^{1,0}\|^2$$

$$+ p\|\delta_c\alpha^{0,1}\|^2 + (p - 1)\|d_c^{0,2}\alpha^{0,1}\|^2 + (p - 1)\|d_c^{1,1}\alpha^{0,1}\|^2,$$

leading to the vanishing of $\delta_c\alpha^{1,0}$, $d_c^{1,1}\alpha^{1,0}$, $\delta_c\alpha^{0,1}$ and $d_c^{1,1}\alpha^{0,1}$. By [24], this is equivalent to the holomorphy of each component of $\alpha^{1,0}$, meaning $\overline{Z_i}.\alpha(Z_j) = 0$ for $Z_{i,j} \in D^{1,0}$ (resp. anti-holomorphy of $\alpha^{0,1}$). At the representation level holomorphic functions are generated by $f = e^{-\sum_{i=1}^{p} r_i^2/2} \in \mathcal{S}(\mathbb{R}^p) = \mathcal{S}_\pi$, the vacuum state of the harmonic oscillator. Hence there exists a fixed $\beta^{1,0}$ such that $\alpha^{1,0}(x) = f(x)\beta^{1,0}$. Differentiating gives

$$\pi(d_c)\alpha^{1,0} = \pi(d_c^{2,0})\alpha^{1,0} = \sum(\pi(Z_i)f)\theta_i \wedge \beta^{1,0} \in L^{'(2,0)},$$

but the functions $\pi(Z_i)f = (\frac{\partial}{\partial x_i} - x_i)f = -2x_i f$ are independent giving that each $\theta_i \wedge \beta^{1,0}$ belongs to $L^{'(2,0)}$. $\square$

We observe that this result gives rise to a group in the $\beta_1 = 1$ class not quadratically presented, and yet of a different type that in the previous section. Consider the 6 dimensional G_6 quotient of Q_7 by a direction of its center, say T_3 associated to θ_3, J_3. This G_6 is the tangent group to a generic $D^4 \subset \mathbb{R}^6$. It is a two dimensional extension of $D = \mathbb{R}^4$ with 2 orthogonal complex structures J_1 and J_2.

Consider again E_2 as in (33). This E_2 has a lot of non-closed sections. For instance in view of the proof of lemma 5.14, it contains any form $\alpha \in \Lambda^1 D^*$ whose $(1,0)$ components with respect to J_3 are J_3 holomorphic functions on D invariant along T_1, T_2. Here E_2 contains many other forms than polynomials of bounded degree like in proposition 5.12. Thus the cubic relations of G_6 (dual to the forms $\theta_1 \wedge J_1\alpha - \theta_2 \wedge J_2\alpha$ by section 2.3) won't be solved (without adding others) in any finite dimensional extension of G_6. They are inaudible in the heat decay anyway. The fact that E_2 has no L^2 section while plenty of local ones interprets here as a vanishing theorem similar for instance of the vanishing of L^2 holomorphic functions on $\mathbb{C}$. Observe these J_3 holomorphic forms were not controlled in the representation associated to the T_3 direction on Q_7, but can be here since we have removed it!

Lastly we complete the study of the other Novikov-Shubin exponents α_p of Q_7. In view of (9) the algebraic pinching theorem 3.13 gives $\beta_3(Q_7) = 2$. By duality (see discussion around (27)) we are left with the study of $\beta_2(Q_7) = \beta_4(Q_7)$. We show that theorem 5.5 gives $\beta_4(Q_7) = r_4 = 1$, that is the system $d_c^{[1]} + \delta_c$ is C-C elliptic on $E_0^{4,(6)}$.

Proof. — If $\alpha \in \mathcal{S}_\pi$ belongs to $\ker \pi(d_c^{[1]} + \delta_c)$, then by C-C ellipticity of d_c, $\beta = \pi(d_c)\alpha$ is non trivial if α is. Moreover $\beta \in E_0^{5,(8)} \cap \ker \pi(d_c)$, and this space is Hodge-

$*$ conjugated to $\ker \pi(d^{-*})$ from $\Lambda^{2,-}D^*$ to $\Lambda^1 D^*$. The elliptic symbol (on degenerated representations) of this map is well known to be injective. At the level of generic representations, one computes easily that $\pi(d^- d^{-*})$ is an inversible Folland-Stein (or Tanaka) operator, giving the result. $\qquad \square$

Of course these C-C ellipticity results transplant, like in lemma 5.9, on any elliptic distribution $D^4 \subset TM^7$ with M^7 compact, without integrability condition on the structure.

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