Séminaire de Théorie spectrale et géométrie

XIAONAN MA

Flat vector bundles and analytic torsion forms

Séminaire de Théorie spectrale et géométrie, tome 19 (2000-2001), p. 25-40

<http://www.numdam.org/item?id=TSG_2000-2001__19__25_0>

Séminaire de théorie spectrale et géométrie
GRENOBLE
Volume **19** (2001) 25–40

FLAT VECTOR BUNDLES AND ANALYTIC TORSION FORMS

Xiaonan MA

1. Introduction

Let Z be a compact manifold. Let F be a flat vector bundle over Z. Let $H^\bullet(Z,F) = \oplus_{i=0}^{\dim Z} H^i(Z,F)$ be the cohomologie of the sheaf of locally flat sections of F.

If E is a finite dimensional vector space, set $\det E = \Lambda^{\max}(E)$. Following an established tradition in algebraic geometry, we define the determinant of the cohomology of F to be the line $\lambda(F)$ given by

$$\lambda(F) = \det H^\bullet(Z,F) = \otimes_{i=0}^{\dim Z} (\det H^i(Z,F))^{(-1)^i}.$$

Assume temporarily that h^F is a flat metric on F. Let K be a smooth triangulation of Z. We can define the Reidemeister metric $\| \ \|_{\lambda(F)}^{R,K}$ on $\lambda(F)$. It is a basic result of Franz [13], Reidemeister [29], and de-Rham [30] (see also [25, §8]), that the metric $\| \ \|_{\lambda(F)}^{R,K}$ does not depend on K. The metric $\| \ \|_{\lambda(F)}^{R,K}$ on $\lambda(F)$ is then a topological invariant of F. If $H^\bullet(Z,F) = 0$, it is a positive number, now called the Reidemeister torsion (or R-torsion).

Remark that the Reidemeister torsion is the first topological invariant which is homeomorphic invariant but is not a homotopy invariant. Reidmeister, Franz classified the lens spaces S^n/G up to isometry by their fundamental group, along with R-torsion. In [19], Köhler generalized it to quotients of Grassmannians.

In 1971, Ray and Singer asked whether as for many other real topological invariants, there is an analytic version of the R-torsion.

Let $(\Omega(Z,F),d^F)$ be the de Rham complex of smooth sections of $\Lambda(T^*Z) \otimes F$ over Z. Let g^{TZ} and h^F be smooth metrics on TZ and F. In [28], Ray and Singer constructed the logarithm of the analytic torsion of $(\Omega(Z,F), d^F)$, as a combination of derivatives at 0 of the zeta functions of the Laplacian acting on forms in $\Omega(Z,F)$ of various degrees. We

Classification math.: 58J52, 58J20, 19K56.

can associate a metric on the line $\lambda(F)$ which is the product of the standard L^2 metric on $\lambda(F)$ (obtained by identifying $H^\bullet(Z,F)$ with the harmonic elements of $(\Omega(Z,F),d^F)$), by the Ray-Singer analytic torsion [28]. This metric is called the Ray-Singer metric on $\lambda(F)$, and is denoted $\| \quad \|^{RS}_{\lambda(F)}$. Ray and Singer showed that if $\dim Z$ is odd, then $\| \quad \|^{RS}_{\lambda(F)}$ does not depend on g^{TZ} and h^F, i.e. it is a topological invariant of F. When Z is even dimension and oriented, and if h^F is a flat metric, then the Ray-Singer torsion is equal to 1.

In 1978, Cheeger [9] and Müller [26] proved the famous Ray-Singer conjecture. Namely, if h^F is flat metric on F, then Ray-Singer metric is equal to R-metric. Assume now that Z is odd dimensional, and that only the metric $\| \quad \|_{\det F}$ induced by h^F on $\det F$ is flat. Then the metrics $\| \quad \|^{R,K}_{\lambda(F)}$ and $\| \quad \|^{RS}_{\lambda(F)}$ are still topological invariants. Müller [27] has shown that equality still holds.

In [7], Bismut and Zhang have extended the equlity between Reidemeister and Ray-Singer metrics to any flat vector bundle F. More recently, Bismut and Goette [4] have generalized Bismut-Zhang theorem to the family case, provided there exists a fiberwise Morse function.

This paper is organized as follows: In Section 2, we recall the definition of Ray-Singer analytic torsion. In Section 3, we explain characteristic classes of a flat vector bundle. In Section 4, we construct the analytic torsion form of Bismut and Lott [6]. In Section 5, we explain briefly the Leray spectral sequence associated to a fibration, and a flat vector bundle which will appear naturally in Theorem 6.1. In Section 6, we review the main result of [24], the fonctoriality of analytic torsion forms with respect to the composition of two submersions. In Section 7, we discuss briefly Lott's secondary index.

In the whole note, if the space $V = V^+ \oplus V^-$ is $\mathbb{Z}_2$ graded, and $A \in \mathrm{End}(V)$, then we denote

$$\mathrm{Tr}_s[A] = \mathrm{Tr}A|_{V^+} - \mathrm{Tr}A|_{V^-}. \tag{1.1}$$

2. Ray-Singer analytic torsion

Let Z ba a compact $\mathscr{C}^\infty$ maniflods. Let (F,∇^F) be a flat complex vector bundle on Z, i.e. ∇^F is a connection on F such that its curvature is zero. Let g^{TZ},h^F be smooth metrics on TZ,F.

REMARK 2.1. — *i) (F,∇^F) is flat iff there is a representation of the fundamental group of Z to $GL(m,\mathbb{C})$ ($m = \dim_{\mathbb{C}} F$), $\rho : \pi_1(Z) \to GL(m,\mathbb{C})$, such that F is the corresponding associated bundle $F = \widetilde{Z} \times_{\pi_1(Z)} \mathbb{C}^m$, here $\widetilde{Z}$ is the universal covering of Z.*

ii) h^F is flat iff F can be obtained through a representation of $\pi_1(Z)$ into $U(m)$, and h^F is the metric on F induced by this representation.

Let $\Omega(Z,F) = \oplus_{i=0}^{\dim Z}\Omega^i(Z,F)$ be the vector space of smooth sections over Z of $\Lambda(T^*Z) \otimes F = \oplus_{i=0}^{\dim Z}\Lambda^i(T^*Z) \otimes F$. Let d^F denote the obvious action of ∇^F on $\Omega(Z,F)$.

Then

$$(d^F)^2 = 0. \tag{2.1}$$

By the de Rham theorem, the cohomology groups of the complex $(\Omega(Z,F),d^F)$ are canonically isomorphic to $H^\bullet(Z,F)$ the cohomology of the sheaf of locally flat sections of F.

Let dv_Z be the Riemannian volume form on Z associated to the metric g^{TZ}. Let $\langle\ \rangle_F$ and $\langle\ \rangle_{\Lambda(T^*Z)\otimes F}$ be the corresponding scalar products on F and $\Lambda(T^*M)\otimes F$. Let $*$ be the Hodge operator associated to g^{TZ} acting on $\Lambda(T^*Z)$. The operator $*$ also acts on $\Lambda(T^*Z)\otimes F$. If $\alpha,\alpha' \in \Omega(Z,F)$, set

$$\left\langle \alpha,\alpha' \right\rangle = \int_Z \left\langle \alpha \wedge *\alpha' \right\rangle_F = \int_Z \left\langle \alpha,\alpha' \right\rangle_{\Lambda(T^*Z)\otimes F} dv_Z. \tag{2.2}$$

Let d^{F*} be the formal adjoint of d^F with respect to the scalar product $\langle,\rangle$. Set

$$\begin{aligned} D^Z &= d^F + d^{F*}, \\ K^\bullet(Z,F) &= \mathrm{Ker} D^Z. \end{aligned} \tag{2.3}$$

Then $D^{Z,2} = d^F d^{F*} + d^{F*} d^F : \Omega^q(Z,F) \to \Omega^q(Z,F)$ preserves the $\mathbb{Z}$-graded of $\Omega(Z,F)$. By Hodge theory,

$$K^\bullet(Z,F) \simeq H^\bullet(Z,F). \tag{2.4}$$

Clearly $K^\bullet(Z,F)$ inherits a metric from the scalar product $\langle\ \rangle$. Let $h^{H(Z,F)}$ be the corresponding metric on $H^\bullet(Z,F)$.

Let P be the orthogonal projection operator from $\Omega(Z,F)$ on $K^\bullet(Z,F)$ with respect to the Hermitian product (2.2). Set $P^\perp = 1 - P$. Let N_Z be the number operator of $\Omega(Z,F)$, i.e. N_Z acts by multiplication by q on $\Omega^q(Z,F)$.

Definition 2.1. — *For $s \in \mathbb{C}$, $\mathrm{Re}(s) > \frac{1}{2}\dim Z$, set*

$$\theta^F(s) = -\mathrm{Tr}_s[N_Z (D^{Z,2})^{-s} P^\perp] = \sum_q (-1)^q q \,\mathrm{Tr}[(D^{Z,2}_{|\Omega^q(Z,F)})^{-s} P^\perp]. \tag{2.5}$$

By a result of Seeley, $\theta^F(s)$ extends to a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$.

Definition 2.2. — *The Ray-Singer torsion $T(Z,h^F)$ of the complex $(\Omega(Z,F),d^F)$ is defined by*

$$T(Z,h^F) = \exp\Big(\frac{1}{2}\frac{\partial \theta^F}{\partial s}(0)\Big). \tag{2.6}$$

Let $\lambda(F)$ be the determinant of the cohomology of F to be the following complex line

$$\lambda(F) = \det H^{\bullet}(Z,F) = \otimes_{i=0}^{\dim Z} (\det H^{i}(Z,F))^{(-1)^{i}}.$$

Let $|\ |^{RS}_{\lambda(F)}$ be the L^2 metric on $\lambda(F)$ induced by $h^{H(Z,F)}$.

DEFINITION 2.3. — *Let* $\|\ \|^{RS}_{\lambda(F)}$ *be the Ray-Singer metric on the complex line* $\lambda(F) = \det H^{\bullet}(Z,F)$

$$\|\ \|^{RS}_{\lambda(F)} = |\ |^{RS}_{\lambda(F)} \exp\left\{\frac{1}{2}\frac{\partial\theta^F}{\partial s}(0)\right\}. \tag{2.7}$$

THEOREM 2.1. — *(Cheeger-Müller [9], [26]) Assume that* h^F *is a flat metric on* F. *Then*

$$\|\ \|^{R,K}_{\lambda(F)} = \|\ \|^{RS}_{\lambda(F)}. \tag{2.8}$$

3. Characteristic class of a flat vector bundle

We use the same assumptions and notation of Section 2.

By the definition of flat vector bundles, the usual Chern class of a flat vector bunlde is zero as the curvature of ∇^F is zero. But we still can define odd characteristic classes for a flat vector bundle.

Let $\Omega(Z)$ denote the space of smooth sections of $\Lambda(T^*Z)$. Let $\varphi : \Omega(Z) \to \Omega(Z)$ be the linear map such that for all homogeneous $\omega \in \Omega(Z)$

$$\varphi\omega = (2\pi i)^{-(\deg\omega)/2}\omega. \tag{3.1}$$

DEFINITION 3.1. — *Let* $\omega(F,h^F)$ *be the 1-form on* Z *taking values in self-adjoint endomorphisms of* F,

$$\omega(F,h^F) = (h^F)^{-1}\nabla^F h^F. \tag{3.2}$$

REMARK 3.1. — h^F *is flat iff* $\omega(F,h^F) = 0$.

For $a \in \mathbb{C}$, put

$$f(a) = a\exp(a^2), \tag{3.3}$$

We have

$$f'(a) = (1+2a^2)\exp(a^2). \tag{3.4}$$

Put

$$f(\nabla^F,h^F) = (2i\pi)^{1/2}\varphi\mathrm{Tr}[f(\omega(F,h^F))] \in \Omega(Z). \tag{3.5}$$

THEOREM 3.1 ([6], Theorem 1.11). — $f(\nabla^F,h^F)$ *is a real, closed odd form and its de Rham cohomology class is independent of* h^F. *We will denote it as* $f(F) \in H^{\mathrm{odd}}(Z,\mathbb{R})$.

For a flat vector bundle F, the characteristic class $f(F)$ will play the same role as the Chern character for complex vector bundles on Z.

Let $\mathrm{Pf} : so(m) \to \mathbb{R}$ denote the Pfaffian. Set

$$e(TZ,\nabla^{TZ}) = \mathrm{Pf}\Big[\frac{R^{TZ}}{2\pi}\Big]. \tag{3.6}$$

Let $o(TZ)$ be the orientation bundle of TZ, a flat real line bundle on Z. Then $e(TZ,\nabla^{TZ})$ is an $o(TZ)$ value closed n-form on Z which represents the Euler class $e(TZ)$ of TZ, lying in $H^{\dim Z}(Z,o(TZ))$ [7, (3.17)]. Of course, $e(TZ,\nabla^{TZ}) = 0$, if $\dim Z$ is odd.

Let $p : TZ \to Z$ be the natural projection. Let δ_Z be the current of integration on Z. In [7, Theorem 3.7], Bismut and Zhang constructed a current $\psi(TZ,\nabla^{TZ})$ on TZ with valus in $o(TZ)$ such that

$$\psi(TZ,\nabla^{TZ}) = p^* e(TZ,\nabla^{TZ}) - \delta_Z. \tag{3.7}$$

The restriction of $\psi(TZ,\nabla^{TZ})$ to the sphere bundle of TZ is the Mathai-Quillen form.

Let $h : Z \to \mathbb{R}$ be a Morse function on Z. Let X be the gradient vector field of h with respect to g^{TZ} on Z. We assume that X verifies the Smale tranversality conditions. Then we can define the Milnor metric $\|\ \|^{M,X}_{\lambda(F)}$ on $\lambda(F)$, which is equal to the Reidmeister metric on $\lambda(F)$ when h^F is flat. Remark that if h^F isn't flat, it isn't a topological invariant.

The following theorem was established by Bismut and Zhang [7, Theorem 0.2],

THEOREM 3.2. — *The following identity holds*

$$\log\left(\Big[\frac{\|\ \|^{RS}_{\lambda(F)}}{\|\ \|^{M,X}_{\lambda(F)}}\Big]^2\right) = -\int_Z \mathrm{Tr}[\omega(F,h^F)]\ X^*\psi(TZ,\nabla^{TZ}). \tag{3.8}$$

4. Analytic torsion forms

In this Section, we explain the construction of the analytic torsion form of Bismut-Lott. We use the notation of Section 3.

4.1. Riemann-Roch-Grothendieck type theorem for flat vector bundles

From now on, let $\pi : W \to S$ be a fibration of $\mathscr{C}^\infty$ manifolds with compact fibre Z. Let TZ be the vertical tangent bundle of the fiber bundle, and let T^*Z be its dual bundle. Let F be a flat complex vector bundle on W and let ∇^F denote its flat connection.

Let $H^\bullet(Z,F|_Z) = \oplus_{i=0}^{\dim Z} H^i(Z,F|_Z)$ be the $\mathbb{Z}$-graded vector bundle over S whose fiber over $s \in S$ is the cohomology $H(Z_s,F|_{Z_s})$ of the sheaf of locally flat sections of F on Z_s. By [6, §3 (f)], ∇^F induces a canonical flat connection $\nabla^{H(Z,F|_Z)}$ on $H^\bullet(Z,F|_Z)$ which preserves the $\mathbb{Z}$-grading.

The following Theorem is an analog of the Riemann-Roch-Grothendieck theorem for holomorhpic submersions, in which a holomorphic submersion becomes a smooth fiber bundle, $\bar{\partial}$-flat (i.e. holomorphic) bundles become d-flat bundles, the direct image of F becomes $\sum_{q=0}^{\dim Z}(-1)^q H^q(Z,F|_Z)$, the Chern character becomes the f class and the Todd class becomes the Euler class.

THEOREM 4.1. — *The following identity holds in $H^*(S,\mathbb{R})$.*

$$\sum_{p=0}^{\dim Z}(-1)^p f(H^p(Z,F|_Z)) = \int_Z e(TZ)\, f(F) \tag{4.1}$$

Actually, Bismut and Lott proved it in analytic way. More precisely, equipped the fiber bundle with a horizontal distribution $T^H W$ and a vertical Riemannian metric g^{TZ}, and the flat vector bundle F with a Hermitain metric h^F. In [6, Theorem 3.23], they constructed an even and real form $\mathscr{T}(T^H W, g^{TZ}, h^F)$ such that

$$d\,\mathscr{T}(T^H W, g^{TZ}, h^F) = \int_Z e(TZ,\nabla^{TZ})\, f(\nabla^F, h^F) - f(\nabla^{H(Z,F|_Z)}, h^{H(Z,F|_Z)}). \tag{4.2}$$

On the right hand side of (4.2), the first term is local, and the second term is global along the fibres Z. So $\mathscr{T}(T^H W, g^{TZ}, h^F)$ must be global along the fibres Z. Actually, let $\mathscr{T}^{(0)}(T^H W, g^{TZ}, h^F)$ be the zero degree part of $\mathscr{T}(T^H W, g^{TZ}, h^F)$ in $\Lambda(T^*S)$, let $\theta^F(s)$ be the function on S defined by (2.5) for each fibre Z. Then

$$\mathscr{T}^{(0)}(T^H W, g^{TZ}, h^F) = \log T(Z,h^F) = \frac{1}{2}\frac{\partial \theta^F}{\partial s}(0). \tag{4.3}$$

Remark that if we take the cohomology class of each side for (4.2), we get (4.1). Thus (4.2) refines (4.1) on the differential form level. In the next subsection, we will explain the construction of the analytic torion form $\mathscr{T}(T^H W, g^{TZ}, h^F)$ in details.

4.2. Construction of the analytic torsion form

We use the notation in Section 4.1.

Let $T^H W$ be a sub-bundle of TW such that

$$TW = T^H W \oplus TZ. \tag{4.4}$$

Let P^{TZ} denote the projection from TW to TZ. If $U \in TS$, let U^H be the lift of U in $T^H W$, so that $\pi_* U^H = U$.

Let $E = \oplus_{i=0}^{\dim Z} E^i$ be the smooth infinite-dimensional $\mathbb{Z}$-graded vector bundle over S whose fiber over $s \in S$ is $\mathscr{C}^\infty(Z_s, (\Lambda(T^*Z)\otimes F)_{Z_s})$. That is

$$\mathscr{C}^\infty(S; E^i) = \mathscr{C}^\infty(W, \Lambda^i(T^*Z)\otimes F). \tag{4.5}$$

For $s \in \mathscr{C}^\infty(S;E)$ and U a vector field on S, then the Lie differential L_{U^H} acts on $\mathscr{C}^\infty(S,E)$. Set

$$\nabla^E_U s = L_{U^H} s. \tag{4.6}$$

Then ∇^E is a connection on E which preserves the $\mathbb{Z}$-grading.

If U_1, U_2 are vector fields on S, put

$$T(U_1,U_2) = -P^{TZ}[U_1^H, U_2^H] \in \mathscr{C}^\infty(W,TZ). \tag{4.7}$$

We denote $i_T \in \Omega^2(S,\mathrm{Hom}(E^\bullet, E^{\bullet-1}))$ be the 2-form on S which, to vector fields U_1, U_2 on S, assigns the operation of interior multiplication by $T(U_1,U_2)$ on E. Let d^Z be exterior differentiation along fibers. We consider d^Z to be an element of $\mathscr{C}^\infty(S;\mathrm{Hom}(E^\bullet, E^{\bullet+1}))$. By [6, Proposition 3.4], we have

$$d^W = d^Z + \nabla^E + i_T. \tag{4.8}$$

So d^W is a flat superconnection of total degree 1 on E. We have

$$(d^Z)^2 = 0, \; [\nabla^E, d^Z] = 0. \tag{4.9}$$

Let g^{TZ} be a metric on TZ. Let h^F be a Hermitian metric on F. Let h^E be the metric on E defined by (2.2).

Let $(\nabla^E)^*$, d^{Z*}, $(d^W)^*$ be the formal adjoint of ∇^E, d^Z, d^W with respect to the scalar product $\langle,\rangle_{h^E}$. Set

$$D^Z = d^Z + d^{Z*}, \qquad \nabla^{E,u} = \tfrac{1}{2}(\nabla^E + (\nabla^E)^*). \tag{4.10}$$

Let N_Z be the number operator of E, i.e. acts by multiplication by k on the space $\mathscr{C}^\infty(W,\Lambda^k(T^*Z)\otimes F)$. For $u > 0$, set

$$\begin{aligned} C'_u &= u^{N_Z/2} d^W u^{-N_Z/2}, \quad C''_u = u^{-N_Z/2} (d^W)^* u^{N_Z/2}, \\ C_u &= \frac{1}{2}(C'_u + C''_u), \quad D_u = \frac{1}{2}(C''_u - C'_u). \end{aligned} \tag{4.11}$$

then C''_u is the adjoint of C'_u with respect to h^E. C_u is a superconnection and D_u is an odd element of $\Omega(S,\mathrm{End}(E))$, and

$$C_u^2 = -D_u^2. \tag{4.12}$$

For $X \in TZ$, let $X^* \in T^*Z$ correspond to X by the metric g^{TZ}. Set $c(X) = X^* \wedge -i_X$. By [6, Proposition 3.9], we get

$$C_u = \frac{\sqrt{u}}{2} D^Z + \nabla^{E,u} - \frac{1}{2\sqrt{u}} c(T). \tag{4.13}$$

In fact, C_u is essentially the same as the Bismut superconnection $A_{u/4}$ associated to the vertical signature operator (cf. [6, (3.46)]).

Let g^{TS} be a Riemannian metric on S then $g^{TW} = g^{TZ} \oplus \pi^* g^{TS}$ is a Riemannian metric on W. Let ∇^{TW} denote the corresponding Levi-Civita connection on W. Put $\nabla^{TZ} = P^{TZ}\nabla^{TW}$, a connection on TZ. As shown in [2, Theorem 1.9], ∇^{TZ} is independent of the choice of g^{TS}.

By [6, §3(f)], the flat superconnection d^W induces a canonical flat connection $\nabla^{H(Z,F|_Z)}$ on $H(Z,F|_Z)$. Let $h^{H(Z,F|_Z)}$ be the Hermitian metric on $H(Z,F|_Z)$ as in Section 2. Let P be the orthonormal projection from E on $\mathrm{Ker}(D^Z)$ with respect to the Hermitian product (2.2). Then by [6, Proposition 3.14], we have

$$\nabla^{H(Z,F|_Z)} = P\nabla^E. \tag{4.14}$$

Put

$$f(\nabla^{H(Z,F|_Z)},h^{H(Z,F|_Z)}) = \sum_{q=0}^{\dim Z} (-1)^q f(\nabla^{H^q(Z,F|_Z)},h^{H(Z,F|_Z)}). \tag{4.15}$$

For any $u > 0$, the operator D_u is a fiberwise-elliptic differential operator. Then $f(D_u)$ is a fiberwise trace class operator. For $u > 0$, put

$$\begin{aligned} f(C'_u,h^E) &= (2i\pi)^{1/2}\varphi \mathrm{Tr}_s[f(D_u)], \\ f^\wedge(C'_u,h^E) &= \varphi \mathrm{Tr}_s\Big[\frac{N_Z}{2} f'(D_u)\Big]. \end{aligned} \tag{4.16}$$

The following results are proved in [6, Theorem 3.16],

THEOREM 4.2. — *For any $u > 0$, the form $f(C'_u,h^E)$ is real, odd, and closed. Its de Rham cohomology class is independent of u, T^HW, g^{TZ} and h^F. As $u \to 0$,*

$$f(C'_u,h^E) = \begin{cases} \displaystyle\int_Z e(TZ,\nabla^{TZ}) f(\nabla^F,h^F) + O(u) & \text{if } \dim Z \text{ is even,} \\ O(\sqrt{u}) & \text{if } \dim Z \text{ is odd.} \end{cases} \tag{4.17}$$

As $u \to +\infty$

$$f(C'_u,h^E) = f(\nabla^{H(Z,F|_Z)},h^{H(Z,F|_Z)}) + O(\frac{1}{\sqrt{u}}). \tag{4.18}$$

Put

$$\begin{aligned} \chi(Z) &= \sum_{i=0}^{\dim Z} (-1)^i \mathrm{rk}\, H^i(Z,\mathbb{R}), \\ \chi'(Z,F) &= \sum_{i=0}^{\dim Z} (-1)^i i\, \mathrm{rk}\, H^i(Z,F|_Z). \end{aligned} \tag{4.19}$$

Then $\chi(Z)$ is the Euler characteristic number of TZ. And $\chi(Z)$, $\chi'(Z,F)$ are locally constant functions on S.

The following results are proved in [6, Theorems 3.20 and 3.21],

THEOREM 4.3. — *For any $u > 0$, the form $f^{\wedge}(C'_u,h^E)$ is real and even. Moreover,*

$$\frac{\partial}{\partial u} f(C'_u,h^E) = \frac{1}{u} d f^{\wedge}(C'_u,h^E). \tag{4.20}$$

As $u \to 0$,

$$f^{\wedge}(C'_u,h^E) = \begin{cases} \frac{1}{4} \dim Z \operatorname{rk}(F)\chi(Z) + O(u) & \text{if } \dim Z \text{ is even,} \\ O(\sqrt{u}) & \text{if } \dim Z \text{ is odd.} \end{cases} \tag{4.21}$$

As $u \to +\infty$

$$f^{\wedge}(C'_u,h^E) = \frac{1}{2}\chi'(Z,F) + O\left(\frac{1}{\sqrt{u}}\right). \tag{4.22}$$

DEFINITION 4.1. — *The analytic torsion form $\mathcal{T}(T^H W,g^{TZ},h^F)$ is a form on S which is given by*

$$\begin{aligned} \mathcal{T}(T^H W,g^{TZ},h^F) = -\int_0^{+\infty} \Big[& f^{\wedge}(C'_u,h^E) - \frac{1}{2}\chi'(Z,F) f'(0) \\ & - \left(\frac{1}{4} \dim Z \operatorname{rk}(F)\chi(Z) - \frac{1}{2}\chi'(Z,F)\right) f'\left(\frac{i\sqrt{u}}{2}\right)\Big]\frac{du}{u}. \end{aligned} \tag{4.23}$$

The following results are proved in [6, Theorem 3.23],

THEOREM 4.4. — *The form $\mathcal{T}(T^H W,g^{TZ},h^F)$ is even and real. Moreover,*

$$d\,\mathcal{T}(T^H W,g^{TZ},h^F) = \int_Z e(TZ,\nabla^{TZ}) f(\nabla^F,h^F) - f(\nabla^{H(Z,F|_Z)},h^{H(Z,F|_Z)}). \tag{4.24}$$

From [6, Theorem 3.24], we know how $\mathcal{T}(T^H W,g^{TZ},h^F)$ depends on its arguments. Remark that if $\dim Z$ is odd or if h^F is flat, then, $\mathcal{T}(T^H W,g^{TZ},h^F)$ in $Q^S/Q^{S,0}$ is independent of $T^H W$. If $\dim Z$ is odd and $H(Z,F|_Z) = 0$ then $\mathcal{T}(T^H W,g^{TZ},h^F)$ is a closed form whose de Rham cohomology class is independent of $T^H W, g^{TZ}$, and h^F.

Now, one of the important problems is to understand the analytic torison form $\mathcal{T}(T^H W, g^{TZ}, h^F)$. More precisely, how to generalize Cheeger-Müller Theorem, or more generally, Bismut-Zhang Theorem to the family case. If the fibration has a fiberwise Morse function, Bismut and Goette [4] confirmed it. For the topological side of this problem, we refer [12], [16], [17].

4.3. Torsion form of a flat complex

Let W be a $\mathscr{C}^\infty$ manifold. Let

$$(E,v) : 0 \to E^0 \xrightarrow{v} E^1 \xrightarrow{v} \cdots \xrightarrow{v} E^n \to 0. \tag{4.25}$$

be a flat complex of complex vector bundles on W. That is $\nabla^E = \oplus_{i=0}^n \nabla^{E^i}$ is a flat connection on $E = \oplus_{i=0}^n E^i$ and v is a flat chain map, meaning by

$$(\nabla^E)^2 = 0, v^2 = 0, \nabla^E v = 0. \tag{4.26}$$

Then $v + \nabla^E$ is a flat superconnection of total degree 1. By [6, §2(a)], the cohomology $H(E,v)$ of the complex is a vector bundle on W, and let $\nabla^{H(E,v)}$ be the flat connection on $H(E,v)$ induced by ∇^E. Let

$$d(E) = \sum_{i=0}^n (-1)^i i \operatorname{rk} E^i, \tag{4.27}$$

$$d(H(E,v)) = \sum_{i=0}^n (-1)^i i \operatorname{rk} H^i(E,v).$$

Let $h^E = \oplus h^{E_i}$ be a metric on $E = \oplus E_i$. Let v^* be the formal adjoint of v with respect to h^E. Let N be the number operator on E, i.e. N acts by multiplication by i on E^i. As in (4.15), set

$$f(\nabla^E, h^E) = \sum_{i=0}^n (-1)^i f(\nabla^{E^i}, h^{E^i}), \tag{4.28}$$

$$f(\nabla^{H(E,v)}, h^{H(E,v)}) = \sum_{i=0}^n (-1)^i f(\nabla^{H^i(E,v)}, h^{H(E,v)})$$

For $u > 0$, let

$$D_u = \frac{1}{2}\sqrt{u}(v^* - v) + \frac{1}{2}\omega(E,h^E). \tag{4.29}$$

By [6, Theorems 2.9, 2.13],

$$\frac{\partial}{\partial u}\operatorname{Tr}_s[f(D_u)] = \frac{1}{u} d\operatorname{Tr}_s[\frac{1}{2} N f'(D_u)]. \tag{4.30}$$

As $u \to +\infty$,

$$(2\pi i)^{1/2}\varphi\operatorname{Tr}_s[f(D_u)] = f(\nabla^{H(E,v)}, h^{H(E,v)}) + O(\frac{1}{\sqrt{u}}), \tag{4.31}$$

$$\varphi\operatorname{Tr}_s[\frac{1}{2} N f'(D_u)] = \frac{1}{2} d(H(E,v)) + O(\frac{1}{\sqrt{u}}).$$

In the same principle of Section 4.2, the following Theorem [6, Theorem 2.22] provides a finite dimensional version of Theorem 4.4,

THEOREM 4.5. — *The following integral is well defined*

$$T_f(v + \nabla^E, h^E) = -\int_0^{+\infty} \Big[\varphi\operatorname{Tr}_s[\frac{1}{2} N f'(D_u)] - \frac{1}{2} d(H(E,v)) - [d(E) - d(H(E,v))] f'\big(\frac{i\sqrt{u}}{2}\big)\Big]\frac{du}{u} \tag{4.32}$$

Moreover $T_f(v+\nabla^E,h^E)$ *is an even and real form, and*

$$dT_f(v+\nabla^E,h^E) = f(\nabla^E,h^E) - f(\nabla^{H(E,v)},h^{H(E,v)}). \tag{4.33}$$

Let (F,∇^F) be a flat complex vector bundle on W. Let $0 \subset F^0 \subset \cdots \subset F^n = F$ be a filtration of F such that $\nabla^F(F^i) \subset F^i$. Let $\mathrm{Gr}^i F = F^i/F^{i-1}$, then we have a flat complex of complex vector bundles:

$$G_i : 0 \to F^i \xrightarrow{v} F^{i+1} \xrightarrow{v} \mathrm{Gr}^{i+1}F \to 0. \tag{4.34}$$

Let h^E, $h^{\mathrm{Gr}F} = \oplus_i h^{\mathrm{Gr}^i F}$ be Hermitian metrics on F,$\mathrm{Gr}F = \oplus_i \mathrm{Gr}^i F$. Let h^{F^i} be the metric on F^i induced by h^F. Let $h^{G_i} = h^{F^{i-1}} \oplus h^{F^i} \oplus h^{\mathrm{Gr}^i F}$ be the metric on $G_i = F^{i-1} \oplus F^i \oplus \mathrm{Gr}^i F$. Let $T(v+\nabla^{G_i},h^{G_i})$ be the form on W defined by (4.32) associated to (4.34).

DEFINITION 4.2. — *The torsion form of the filtered flat complex vector bundle F is defined by*

$$T(F,\mathrm{Gr}F,h^F,h^{\mathrm{Gr}F}) = \sum_{i=0}^{n-1} T(v+\nabla^{G_i},h^{G_i}). \tag{4.35}$$

5. Leray spectral sequence

Let $\pi_1 : Z \to Y$ be a fibration of compact manifolds with compact fibre X. Let F be a flat complex vector bundle on Z. Let

$$\Lambda(T^*Z) = F^0(\Lambda(T^*Z)) \supset F^1(\Lambda(T^*Z)) \supset \cdots \supset F^{\dim Y+1}(\Lambda(T^*Z)) = \{0\}. \tag{5.1}$$

be the standard filtration of $\Lambda(T^*Z)$. In fact $F^p\Lambda^q(T^*Z)$ are the forms which can be written as a finite sum of forms of the shape $\omega\wedge\pi^*\eta$ for $\omega \in \Lambda^{q-k}(T^*Z)$, $\eta \in \Lambda^k(T^*Y)$ for some $k \geqslant p$. The filtration (5.1) induces a corresponding filtration of the complex $(\Omega(Z,F),d^F)$ such that $F^p\Omega(Z,F) = \mathscr{C}^\infty(Z,F^p\Lambda(T^*Z)\otimes F)$. We also get a corresponding filtration on $H^\bullet(Z,F)$. Set

$$\begin{aligned} \mathrm{Gr}^p H^\bullet(Z,F) &= \frac{F^p H^\bullet(Z,F)}{F^{p+1}H^\bullet(Z,F)}, \\ \mathrm{Gr}^\bullet H^\bullet(Z,F) &= \oplus_{p=0}^{\dim Y} \mathrm{Gr}^p H^\bullet(Z,F). \end{aligned} \tag{5.2}$$

Let (E_r,d_r) be the spectral sequence associated to the filtration (5.1) on the filtered complex $(\Omega(Z,F),d^F)$ [14, §3.5]. Then, we get

$$\begin{aligned} (E_0^{\bullet,\bullet},d_0) &= (\Omega^\bullet(Y,\Omega^\bullet(X,F_{|X})),d^F_{|X}), \\ (E_1^{\bullet,\bullet},d_1) &= (\Omega^\bullet(Y,H(X,F_{|X})),d_{|Y}^{H(X,F_{|X})}), \\ E_2^{p,q} &= H^p(Y,H^q(X,F_{|X})). \end{aligned} \tag{5.3}$$

And E_2 is a finite dimensional $\mathbb{Z}$-graded vector space. More generally, for any $r \geqslant 0$, E_{r+1} is the cohomology of the complex (E_r, d_r). And for $r > \dim Z$,

$$(E_r^{\bullet,\bullet}, d_r) = (\mathrm{Gr}^\bullet H^\bullet(Z,F), 0). \tag{5.4}$$

By [15, Theorem 3.7.3], there is a functor of Leray spectral sequence associated to the fibration $\pi_1 : Z \to Y$. By [24, Theorem 2.1], (E_r, d_r) $(r \geqslant 2)$ calculates the Leray spectral sequence.

6. Functoriality of analytic torsion form

Let W, V, S be smooth manifolds. Let $\pi_1 : W \to V$, $\pi_2 : V \to S$ be smooth fibrations of manifolds with compact fibre X, Y. Then $\pi_3 = \pi_2 \circ \pi_1 : W \to S$ is a smooth fibration with compact fibre Z with $\dim Z = n$. Let (F, ∇^F) be a flat complex vector bundle over W. Then we have the diagram of smooth fibrations:

$$\begin{tikzcd}
X \arrow[r] & Z \arrow[r] \arrow[d, "\pi_1"'] & W \arrow[d, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
 & Y \arrow[r] & V \arrow[r, "\pi_2"] & S
\end{tikzcd}$$

Let $H^\bullet(X, F_{|X}) = \oplus_{i=0}^{\dim X} H^i(X, F_{|X})$, $H^\bullet(Z, F_{|Z})$, $H^\bullet(Y, H^\bullet(X, F_{|X}))$ be the $\mathbf{Z}$-graded vector bundles over V, S, S whose fiber over $a \in V$, $s \in S$ are the cohomologies $H^\bullet(X_a, F_{|X_a})$, $H^\bullet(Z_s, F_{|Z_s})$, $H^\bullet(Y_s, H^\bullet(X, F_{|X}))$ of the sheaf of locally flat sections of F, F, $H(X, F_{|X})$ on X_a, Z_s, Y_s.

Let Q^S be the vector space of real even forms on S. Let $Q^{S,0}$ be the vector space of real exact even forms on S.

Let $T_1^H W, T_2^H V, T_3^H W$ be sub-bundles of TW, TV, TW with respect to π_1, π_2, π_3 as in (4.4). Let E be the smooth infinite-dimensional $\mathbb{Z}$-graded vector bundle over S whose fiber over $s \in S$ is $\mathscr{C}^\infty(Z_s, (\Lambda(T^*Z) \otimes F)_{Z_s})$. For $s \in S$, let $(E_{r,s}, d_{r,s})$ be the Leray spectral sequence with respect to $\pi_1 : Z_s \to Y_s, F$.

PROPOSITION 6.1. — *[24, Proposition 3.2] There are flat complex vector bundles $E_r^{p,q}$ $(r \geqslant 2, p,q \in \mathbb{N})$, and $d_r : E_r^{p,q} \to E_r^{p-r,q+1-r}$ such that the fiber of complex $(E_r = \oplus_{p,q} E_r^{p,q}, d_r)$ over $s \in S$ is the Leray spectral sequence $(E_{r,s} = \oplus_{p,q} E_{r,s}^{p,q}, d_r)$.*

By [6, §2(a)], there is also a canonical connection $\nabla^{E_r} = \oplus_{p,q} \nabla^{E_r^{p,q}}$ on $E_r = \oplus_{p,q} E_r^{p,q}$ induced by d^W.

Let g^{TZ}, g^{TX}, g^{TY} be metrics on TZ, TX, TY. Let h^F be a Hermitian metric on F.

Let $h^{H(X,F|_X)}$, $h^{H(Z,F|_Z)}$, $h^{H(Y,H(X,F|_X))}$ be the L^2-metrics on $H^\bullet(X,F|_X)$, $H^\bullet(Z,F|_Z)$, $H^\bullet(Y,H^\bullet(X,F|_X))$ with respect to g^{TX},h^F; g^{TZ},h^F and $g^{TY},h^{H(X,F|_X)}$ defined in Section 1.2.

Let $\nabla^{TX},\nabla^{TY},\nabla^{TZ}$ be the connections on (TX,g^{TX}), (TY,g^{TY}), (TZ,g^{TZ}) defined in Section 4.2. Let $T^HZ = T_1^HW \cap TZ$. Let $\pi_1^*\nabla^{TY}$ be the connection on T^HZ induced by ∇^{TY}. Then ${}^0\nabla^{TZ} = \pi_1^*\nabla^{TY} \oplus \nabla^{TX}$ is a connection on $TZ = T^HZ \oplus TX$. Let $\widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})$ be the Chern-Simons $n-1$ forms on Z with values in $o(TZ)$ such that

$$d\,\widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ}) = e(TZ,{}^0\nabla^{TZ}) - e(TZ,\nabla^{TZ}). \tag{6.1}$$

Let $\mathscr{T}(T_1^HW,g^{TX},h^F)$, $\mathscr{T}(T_2^HV,g^{TY},h^{H(X,F|_X)})$, $\mathscr{T}(T_3^HW,g^{TZ},h^F)$ be the analytic torsion forms corresponding to π_1,π_2,π_3. Let h^{E_2} be the metric on E_2 induced by $h^{H(Y,H(X,F|_X))}$. Let h^{E_r} $(r \geqslant 3)$ be the L^2 metric on E_r as in Section 4.3. Set

$$\begin{aligned} T(H(Z,F|_Z),E_\infty,h^{H(Z,F|_Z)},h^{E_\infty}) \\ = \sum_{k=0}^{\dim Z}(-1)^kT(H^k(Z,F|_Z),\oplus_{p+q=k}E_\infty^{p,q},h^{H(Z,F|_Z)},h^{E_\infty}). \end{aligned} \tag{6.2}$$

By (4.9), $d_r + \nabla^{E_r}$ is a flat superconnection of total degree 1 on E_r.

Definition 6.1. — *Set*

$$\begin{aligned} T(E_2,H(Z,F|_Z),h^{E_2},h^{H(Z,F|_Z)}) = \sum_{r=2}^{\infty} T(d_r + \nabla^{E_r},h^{E_r},h^{E_{r+1}}) \\ - T(H(Z,F|_Z),E_\infty,h^{H(Z,F|_Z)},h^{E_\infty}). \end{aligned} \tag{6.3}$$

In fact, by [6, Theorem 2.24], $T(.,.) \in Q^S/Q^{S,0}$ doesn't depend on the choice of h^{E_r} $(r \geqslant 2)$ on E_r.

Theorem 6.1. — *[24, Theorem 0.1] The following identity holds in $Q^S/Q^{S,0}$,*

$$\begin{aligned} \mathscr{T}(T_3^HW,g^{TZ},h^F) = \int_Y e(TY,\nabla^{TY})\mathscr{T}(T_1^HW,g^{TX},h^F) \\ + \mathscr{T}(T_2^HV,g^{TY},h^{H(X,F|_X)}) + T(E_2,H(Z,F|_Z),h^{E_2},h^{H(Z,F|_Z)}) \\ - \int_Z \widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})\,f(\nabla^F,h^F). \end{aligned} \tag{6.4}$$

Assume now that S is a point. Then we have a submersion $\pi_1 : Z \to Y$ with fibre X. Let

$$\begin{aligned} \lambda(F) = \otimes_{i=0}^{\dim Z}(\det H^i(Z,F))^{(-1)^i}, \\ \lambda(H^\bullet(X,F|_X)) = \otimes_{i,j=0}^{\dim Z}(\det H^i(Y,H^j(X,F|_X)))^{(-1)^{i+j}}. \end{aligned} \tag{6.5}$$

be the determinant of the cohomologies of F, $H^\bullet(X,F_{|X})$. By [18], we have a canonical nonzero section $\sigma \in \lambda^{-1}(H(X,F_{|X})) \otimes \lambda(F)$.

Let $\| \ \|_{\lambda(H(X,F_{|X}))}$, $\| \ \|_{\lambda(F)}$ be the Ray-Singer metrics on $\lambda(H(X,F_{|X}))$, $\lambda(F)$ associated to the metrics $g^{TY}, h^{H(X,F_{|X})}$, and g^{TZ}, h^F. Let $\| \ \|_{\lambda^{-1}(H(X,F_{|X}))\otimes\lambda(F)}$ be the corresponding Ray-Singer metric on $\lambda^{-1}(H(X,F_{|X})) \otimes \lambda(F)$. Let $T(X,h^F)$ be the Ray-Singer analytic torsion on the fibre X associated to the metrics g^{TX}, h^F.

By [6, Theorems 2.25 and 3.29], and (3.5), we can reformulate Theorem 6.1

$$\log(\|\sigma\|_{\lambda^{-1}(H(X,F_{|X}))\otimes\lambda(F)}) = \int_Y e(TY,\nabla^{TY}) \log T(X,h^F) - \frac{1}{2}\int_Z \widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ}) \mathrm{Tr}[(h^F)^{-1}\nabla^F h^F]. \tag{6.6}$$

If Z is oriented, odd dimensional, and h^F is a flat metric, let $g_\varepsilon^{TZ} = \varepsilon^2 g^{TZ} + \pi^* g^{TY}$. Let $T_\varepsilon(Z,h^F)$ be the Ray-Singer analytic torsion associated to g_ε^{TZ}. In [10], [11], Dai and Melrose have calculated the asymptotics of $T_\varepsilon(Z,h^F)$ as $\varepsilon \to 0$. In [21], Lück, Schick and Thielmann have generalized it to the case that F is unimodular, and that Z is odd or even. In fact, by using [7, Theorems 0.1, 0.2], [27], they show their main result [21, Theorem 0.2] follows from the corresponding result on Reidemeister torsion which is essentially a problem of finite dimensional linear algebra.

So the equation (6.6) extends the results of [11], [21], to the general case, where F is not necessary unimodular.

7. Lott's secondary index

Trying to understand the analytic torsion form in algebraical way, in [20], Lott defined a secondary K-group for flat complex Hermitian vector bundles on a $\mathscr{C}^\infty$ manifold. Lott defined also the direct image (secondary index) in his secondary K-group for a $\mathscr{C}^\infty$ fibration with compact fibre, and the real analytic torsion form is one part of his secondary index. We can consider it as a $\mathscr{C}^\infty$ analogue of Gillet-Soulé's arithmetic K-Theory in Arakelov goemetry.

Let W be a $\mathscr{C}^\infty$ manifold. The abelian group $\hat{K}^0(W)$ is generated by triples (F,h^F,η), where (F,∇^F) is a flat complex vector bundle on W, h^F is a Hermitian metric on F, and $\eta \in \Omega^{ev}(W)/\mathrm{image}(d)$, subject to the following relations: If

$$\mathscr{E} : 0 \to F_1 \to F_2 \to F_3 \to 0 \tag{7.1}$$

is an exact sequence of flat complex vector bundles on W, h^{F_i} are hermitian metrics, $\eta_i \in \Omega^{ev}(W)/\mathrm{Im}(d)$, and we form $E_i := (F_i,h^{F_i},\eta_i)$, then $E_2 \sim E_1 + E_3$ if

$$\eta_2 = \eta_1 + \eta_3 + T(\mathscr{E},h^{\mathscr{E}}), \tag{7.2}$$

where $\mathscr{T}(\mathscr{E},h^{\mathscr{E}})$ is the torsion form of Theorem 4.4 associated to (7.1) equipped with the metric $h^{\mathscr{E}}$ induced by h^{F_i}.

Lott shows that

$$(F,h^F,\eta) \mapsto f(\nabla^F,h^F) - d\eta$$

extends to a map $c' : \hat{K}^0(W) \to \Omega^{\mathrm{odd}}(W)$, and he defines

$$\bar{K}^0(W) := \ker(c') \,. \tag{7.3}$$

The assignment $W \mapsto \bar{K}^0(W)$ yields a homotopy invariant contravariant functor from the category of manifolds to abelian groups.

We now consider a smooth fibre bundle $\pi_3 : W \to S$ with compact fibre Z. We further choose a horizontal distribution T^HW. Lott defines the push-forward $(\pi_3)_! : \bar{K}^0(W) \to \bar{K}^0(S)$ by the assignment:

$$(F,h^F,\eta) \mapsto \sum_q (-1)^q (H^q(Z,F|_Z),h^{H^q(Z,F|_Z)},0) + (0,0,\int_Z e(TZ,\nabla^{TZ}) \wedge \eta - \mathcal{T}(T^HW,g^{TZ},h^F)) \,. \tag{7.4}$$

Lott proves well-definedness and independence of T^HW and g^{TZ}.

In [8], Bunke shows that Theorem 6.1 actually implies the functoriality of Lott's secondary indices [20]. More precisely,

THEOREM 7.1. — *Let $\pi_1 : W \to V$, $\pi_2 : V \to S$ be smooth fibrations of manifolds with compact fibre X,Y. We have $(\pi_3)_! = (\pi_2)_! \circ (\pi_1)_!$ as maps from $\bar{K}^0(W)$ to $\bar{K}^0(S)$.*

REMARK 7.1. — *Let R be a commutative ring with unitary element. Lott also defined $\bar{K}^0_R(W)$ for the tripes $(\mathcal{F},h^{F_\mathbb{C}},\eta)$, where*

1. *$\mathcal{F}$ is a local system of finitely generated right-R-modules,*
2. *$h^{F_\mathbb{C}}$ is a hermitean metric of the corresponding flat complex vector bundle $(F_\mathbb{C},\nabla^{F_\mathbb{C}})$, and*
3. *$\eta \in \Omega^{ev}(M)/\mathrm{image}(d)$,*

By the same proof, Theorem 7.1 still holds.

References

[1] BERLINE N., GETZLER E. and VERGNE M., *Heat kernels and the Dirac operator*, Grundl. Math. Wiss. 298, Springer, Berlin-Heidelberg-New York 1992.

[2] BISMUT J.-M., The index Theorem for families of Dirac operators: two heat equation proofs, *Invent.Math.*,83 (1986), 91-151.

[3] BISMUT J.-M., Families of immersions, and higher analytic torsion, *Astérisque* 244, 1997.

[4] BISMUT J.-M. and GOETTE S., Families torsion and Morse functions, *Astérisque* 275, 2001.

[5] BISMUT J.-M. and LEBEAU G., Complex immersions and Quillen metrics, *Publ. Math. IHES.*, Vol. 74, 1991, 1-297.

[6] BISMUT J.-M. and LOTT J., Flat vector bundles, direct images and higher real analytic torsion, *J.A.M.S.* 8(1995), 291-363.

[7] BISMUT J.-M. and ZHANG W., An extension of a Theorem by Cheeger and Müller, *Astérisque* 205, 1992.

[8] BUNKE U., On the functoriality of Lott's secondary analytic index, math.DG/0003171.

[9] CHEEGER J., Analytic torsion and the Heat Equation, *Ann of Math*, 109 (1979), 259-322.

[10] DAI X., Geometric Invariants and Their Adiabatic Limits, *Proc. Symposia Pure Math.* 54 (1993), part II, 145-156.

[11] DAI X., Melrose R.B., Adiabatic limit of the analytic torsion, Preprint.

[12] DWYER W., WEISS M., WILLIAMS B., A Parametrized Index Theorem for the Algebraic K-Theory Euler Class, http://www.math.uiuc.edu/K-theory/0086/index.html.

[13] FRANZ W., Uber die Torsion einer überdeckrung, *J. Reine Angew. Math.* 173 (1935), 245-254.

[14] GRIFFITHS P., HARRIS J., *Principles of Algebraic Geometry*, New-York,Wiley 1978.

[15] GROTHENDIECK A., Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* 9, 1957, 119-221.

[16] IGUSA K., Parametrized Morse theory and its applications. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 643–651, Math. Soc. Japan, Tokyo, 1991

[17] KLEIN J., Higher Franz-Reidemeister torsion: low-dimensional applications. Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), 195–204, Contemp. Math., 150, Amer. Math. Soc., Providence, RI, 1993.

[18] KNUDSEN P.F., MUMFORD D., The projectivity of the moduli space of stable curves. I, Preliminaries on "det" and "div", *Math. Scand.* 39 (1976), 19-55.

[19] KÖHLER K., Equivariant Reidemeister torsion on symmetric spaces, *Math. Ann.* 307 (1997), 57-69.

[20] LOTT J., Secondary analytic indices, *Regulars in analysis, Geometry and Number Theory.* N.Schappacher, A Reznikov(ed), Progress in Math. 171. Birkhäuse 2000.

[21] LÜCK W., SCHICK T., and THIELMANN T., Torsion and fibrations, *J. Reine Angew. Math*, 498 (1998), 1-33.

[22] MA X., Formes de torsion analytique et familles de submersions I, *Bull.Soc.Math. France*, 127 (1999), 541-621.

[23] MA X., Formes de torsion analytique et familles de submersions II, *Asian J of Math*, 4 (2000), 633-668.

[24] MA X., Functoriality of real analytic torsion forms. *Israel J of Math*, to appear.

[25] MILNOR J., Whitehead torsion. Bull. *Amer. Math. Soc.* 72 1966 358–426.

[26] MÜLLER W., Analytic torsion and R-torsion of Riemannian manifolds. Adv. in Math. 28 (1978), 233–305.

[27] MÜLLER W., Analytic torsion and R-torsion for unimodular representations, *J.A.M.S*, 6 (1993), 721-753.

[28] RAY D.B., Singer I.M., R-torsion and the Laplacian on Riemannian Manifolds, *Adv. in Math*, 7 (1971), 145-210.

[29] REIDEMEISTER K., Homotopieringe und Linsenraüm, *Hamburger Abhandl*, 11 (1935), 102-109.

[30] DE RHAM G., Complexes à automorphismes et homéomorphie différentiable. *Ann. Inst. Fourier Grenoble* 2 (1950), 51–67 (1951).

Xiaonan MA
Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin (Germany)
xiaonan@mathematik.hu-berlin.de

Address after Dec 15, 2001
Centre de Mathématiques
École polytechnique
91128 Palaiseau Cedex (France)