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A WEIERSTRASS-KENMOTSU FORMULA FOR PRESCRIBED MEAN CURVATURE SURFACES IN HYPERBOLIC SPACE

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Abstract

In this article we prove a Weierstrass-Kenmotsu type formula for prescribed mean curvature in hyperbolic space. We also summarize a meromorphic data and representation theorems for mean curvature one surfaces in hyperbolic space.

Introduction

We begin this paper outlying part of its history: In 1979, Kenmotsu, see [5], proved that any C^2 solution E on a simply-connected domain U of the following equation

$$E_{z\bar{z}} = 2\frac{\overline{E}}{1+E\overline{E}}E_z E_{\bar{z}}$$

produces a conformal imersion $X : U^* \to \mathbb{R}^3$ of constant (non-zero) mean curvature. Here, $U^* := U - \{z, E_{\bar{z}} = 0\}$. He proved a similar result for prescribed mean curvature. In this case, the statement is the same but the equation is rather more complicated. As far as we know, it is not known explicit solutions of the above equation, and the equation is not yet solved. The authors in a very recent work, see [10], have derived a similar equation, namely

$$E_{z\bar{z}} = \frac{\overline{E}}{1+E\overline{E}}E_z E_{\bar{z}}. \qquad (*)$$

In fact, every solution of $(*)$ gives rise to a mean curvature one conformal immersion $X : U^* \to \mathbb{H}^3$ into hyperbolic space. We gave the complete struture of a C^2 solution of $(*)$. Indeed, one may express any solution of $(*)$ in terms of meromorphic data (h,T).

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Conversely, given any non constant meromorphic data (h,T) with $h \neq \frac{1}{\alpha T + \beta}, \alpha,\beta \in \mathbb{C}$, there is a natural way to explicit a conformal parametrization of a piece of a surface with mean curvature one into hyperbolic space, involving just one integration $\int h^2 T_z dz$.

In this paper we prove a theorem similar to the result proved by Kenmotsu referred above, for *prescibed mean curvature surface in hyperbolic space*, see theorem 1.2. We note that R. Aiyama and K. Akutagawa have proved an alternative Kenmotsu type representation in the hyperbolic 3-space, see [1]. They also proved a related result in the 3-sphere, see [2]. We note also that M. Kokubu has given a Weierstrass type representation for minimal surfaces in hyperbolic space, see [6].

1. *Notations and statement of the main Theorem*

We begin as in [10] to state the notations we shall use in this paper. We shall focus the half-space model of the hyperbolic 3-space and we shall denote it by $\mathbb{H}^3$, namely

$$\mathbb{H}^3 = \{(u,v,w) \in \mathbb{R}^3,\ w > 0\},$$

equipped with the hyperbolic metric

$$\frac{du^2 + dv^2 + dw^2}{w^2}.$$

Throughout this paper, $U \subset \mathbb{C}$ will be a simply connected domain of the complex plane with coordinate $z = x + iy$, and $X : U \to \mathbb{H}^3$ will be a C^2 conformal immersion of U into $\mathbb{H}^3$. We shall call $M = X(U)$ the surface in $\mathbb{H}^3$. We set

$$X_x = \frac{\partial X}{\partial x},\ X_y = \frac{\partial X}{\partial y}.$$

For any vectors $\vec{u}$ and $\vec{v}$, the notation $\vec{u} \cdot \vec{v}$ (resp. $\langle \vec{u}; \vec{v} \rangle$) stands for the standard euclidean (resp. hyperbolic) inner product of $\vec{u}$ and $\vec{v}$.

For every C^1-function $f : U \to \mathbb{C} \cup \{\infty\}$ the notation f_z (resp. $f_{\bar{z}}$) stands for the derivative of f with respect to z (resp. $\bar{z}$), that is

$$\begin{cases} f_z = \frac{1}{2}(f_x - i f_y) \\ f_{\bar{z}} = \frac{1}{2}(f_x + i f_y) \end{cases} \quad \text{or equivalently} \quad \begin{cases} f_x = f_z + f_{\bar{z}} \\ f_y = i f_z - i f_{\bar{z}}. \end{cases}$$

Let N be the euclidean Gauss map of X such that $(X_x, X_y, N)(z)$ is a positive basis of $\mathbb{R}^3$ for each $z \in U$. That is,

$$N = \frac{X_x \wedge X_y}{|X_x \wedge X_y|}$$

where $|.|$ stands for the euclidean norm. We call N the *oriented euclidean Gauss map* of X, or more briefly the *euclidean Gauss map* of X, and we denote by N_i, $i = 1,2,3$ the coordinate functions of N, that is $N = (N_1,N_2,N_3)$.

DEFINITION 1.1.

1. *Let $\Pi : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ be the standard stereographic projection. We set*

$$E(z) = (\Pi \circ N)(z) = \frac{N_1 + iN_2}{1 - N_3}(z), \ \forall z \in U.$$

Then

$$N = \frac{(2\operatorname{Re} E, 2\operatorname{Im} E, E\overline{E} - 1)}{E\overline{E} + 1}.$$

We also shall call E the oriented euclidean Gauss map of X.

2. *Let $p = X(z) \in M$ be a point on M. Let γ^+ be the half-geodesic issue from p, orthogonal to M and oriented by the normal vector $N(z)$. Let $\omega \in \partial_\infty \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ be the asymptotic boundary of γ^+. We then define a map $G : U \to \mathbb{C} \cup \{\infty\}$ setting $G(z) = \omega$. The map G is the well-known hyperbolic Gauss map of X (or M), see [4].*

We can now state the main result.

THEOREM 1.2. — *Let $X : U \to \mathbb{H}^3$ be a C^3 conformal immersion and let E be the euclidean Gauss map of X. Let us call $\widetilde{H}$ the hyperbolic mean curvature of X with respect to E and let us assume that $E \neq \infty$ anywhere on U. Then E and $\widetilde{H}$ satisfy*

$$(2 + (\widetilde{H} - 1)(1 + E\overline{E}))(1 + E\overline{E})E_{z\bar{z}} - 2(1 + (\widetilde{H} - 1)(1 + E\overline{E}))\overline{E}E_zE_{\bar{z}} - (1 + E\overline{E})^2E_{\bar{z}}\widetilde{H}_z = 0. \tag{1}$$

The induced metric on U is

$$ds^2 = \frac{4|E_{\bar{z}}|^2}{|2 + (\widetilde{H} - 1)(1 + E\overline{E})|^2} \cdot |dz|^2. \tag{2}$$

The Hopf function with respect to E is

$$\widetilde{\Phi} = \frac{4E_zE_z}{(1 + E\overline{E})(2 + (\widetilde{H} - 1)(1 + E\overline{E}))} \tag{3}$$

and the second fundamental form with respect to E is

$$\widetilde{\Pi} = (\widetilde{H} + \operatorname{Re}\widetilde{\Phi})dx^2 - 2\operatorname{Im}\widetilde{\Phi}dxdy + (\widetilde{H} - \operatorname{Re}\widetilde{\Phi})dy^2. \tag{4}$$

Conversely, let $E : U \to \mathbb{C}$ be a C^2-map and let $\widetilde{H} : U \to \mathbb{R}$ be a C^1- map satisfying (1). Let us set $U^* = \{z \in U, E_{\bar{z}} \neq 0\}$.

Then there exists a map $X : U \to \mathbb{H}^3$ such that the restricted map $X_{|U^*}$ is a conformal immersion with Gauss map E and mean curvature $\widetilde{H}$ and (2), (3) and (4) are valid. Furthermore, such a map X is uniquely determined up to an orientation preserving isometry of $\mathbb{H}^3$. More precisely if $\hat{X} : U \to \mathbb{H}^3$ is another such map, then there exist a real number $\lambda > 0$ and a complex number $\alpha \in \mathbb{C}$ such that $\hat{X} = \lambda \cdot X + \alpha$.

In fact, it can be proved that $S - S^*$ is a discret set in case where $\widetilde{H} \equiv 1$, see Lemma 2-3 of [10].

Proof of Theorem 1.2. — We first state some general result about immersed surfaces in $\mathbb{H}^3$. They can be found in [10]. The notations are the same as above.

PROPOSITION 1.3. — *Let $X : U \to \mathbb{H}^3$ be a C^3 conformal immersion. Setting $X := (u,v,w)$, we have:*

$$E = \frac{G - (u + iv)}{w}, \tag{5}$$

and then $u + iv = G - w \cdot E$.

$$G_z = w \cdot E_z \cdot \tag{6}$$

$$w_{\bar{z}} = \frac{\overline{E}}{1+E\overline{E}} \cdot (G_{\bar{z}} - wE_{\bar{z}}) \cdot \tag{7}$$

$$ds^2 = \frac{|G_z - wE_{\bar{z}}|^2}{w^2} \cdot |dz|^2 \cdot \tag{8}$$

Proof. — Let $z \in U$ be any point. Observe that $G(z) = (u + iv)(z)$ implies $N(z) = (0,0,-1)$ and, therefore $E(z) = 0 = (G - (u + iv))(z)$. Thus, we can assume $G(z) \neq (u + iv)(z)$. We set $P = X(z) = (u,v,w)$, $P_0 = (u,v,0)$, $G(z) = G_1 + iG_2 = (G_1,G_2,0)$ and $N = N(z) = (N_1,N_2,N_3)$. Let γ^+ be the half-geodesic issue from P tangent to and oriented by the vector N. Hence, γ^+ is part of a half-circle lying in $\mathbb{H}^3$ with center $C = (c_1,c_2,0)$. Moreover, γ^+ joins P to G.

By construction, there exists a real number $\lambda \in \mathbb{R}$ such that $C = \lambda P_0 + (1 - \lambda)G$. A computation shows that

$$\lambda = \frac{(G_1 - u)^2 + (G_2 - v)^2 + w^2}{2((G_1 - u)^2 + (G_2 - v)^2)}.$$

Now, observe that:

(*i*) There exists a *strictly positive* real number $\alpha > 0$ such that $(N_1,N_2,0) = \alpha \cdot \overrightarrow{P_0G} = \alpha \cdot (G_1 - u, G_2 - v, 0)$ (because G is the asymptotic boundary of γ^+).

(*ii*) N and $\overrightarrow{CP}$ are orthogonal vectors.

Thus, we get the following system

$$\begin{cases} N_1 = 2\dfrac{w(G_1 - u)}{(G_1 - u)^2 + (G_2 - v)^2 - w^2} \cdot N_3 \\ N_2 = 2\dfrac{w(G_2 - v)}{(G_1 - u)^2 + (G_2 - v)^2 - w^2} \cdot N_3 \end{cases}$$

Finally, as $N_1^2 + N_2^2 + N_3^2 = 1$, a simple computation leads to

$$N_3^2 = \left(\frac{\left(\frac{G_1 - u}{w}\right)^2 + \left(\frac{G_2 - v}{w}\right)^2 - 1}{\left(\frac{G_1 - u}{w}\right)^2 + \left(\frac{G_2 - v}{w}\right)^2 + 1} \right)^2.$$

Hence,

$$N = \left(\frac{2(\frac{G_1-u}{w}), 2(\frac{G_2-v}{w}), \left(\frac{G_1-u}{w}\right)^2 + \left(\frac{G_2-v}{w}\right)^2 - 1}{\left(\frac{G_1-u}{w}\right)^2 + \left(\frac{G_2-v}{w}\right)^2 + 1} \right)$$
$$= \Pi^{-1}\left(\frac{G-(u+iv)}{w}\right).$$

We conclude therefore

$$E = \frac{G-(u+iv)}{w}$$

as desired, this achieves the proof of (5).

As N is the euclidean Gauss map of X we have

$$X_x \cdot \Pi^{-1}(E) = X_x \cdot N = 0$$

that is

$$((u+iv)_x, w_x) \cdot (2E, E\overline{E} - 1) = 0.$$

Substituting $u + iv$ by G we easily found

$$w_x = 2\frac{\mathrm{Re}(\overline{E}(G_x - wE_x))}{E\overline{E}+1}.$$

In the same way, one may show that

$$w_y = 2\frac{\mathrm{Re}(\overline{E}(G_y - wE_y))}{E\overline{E}+1}.$$

Now, as X is a conformal immersion we have

$$X_x \cdot X_x = X_y \cdot X_y \quad \text{and} \quad X_x \cdot X_y = 0.$$

On the other hand, we have

$$X_x \cdot X_x = ((u+iv)_x, w_x) \cdot ((u+iv)_x, w_x)$$
$$= ((G-wE)_x, w_x) \cdot ((G-wE)_x, w_x)$$

thus, using the expression of w_x we get

$$X_x \cdot X_x = |G_x - wE_x|^2.$$

Working in the same fashion, one can show that

$$X_y \cdot X_y = |G_y - wE_y|^2$$

Moreover using the expression of w_x and w_y we get

$$X_x \cdot X_y = 0 \Leftrightarrow ((G-wE)_x, w_x) \cdot ((G-wE)_y, w_y) = 0$$
$$\Leftrightarrow \mathrm{Re}((G_x - wE_x)(\overline{G}_y - w\overline{E}_y)) = 0.$$

We deduce that the complex numbers $(G_x - wE_x)$ and $(G_y - wE_y)$ have same modules and their arguments differ from $\pm\frac{\pi}{2}$. Now, using

$$N = \frac{X_x \wedge X_y}{|X_x \wedge X_y|} = \frac{(2E, E\overline{E} - 1)}{E\overline{E} + 1}.$$

and

$$X_x \wedge X_y = \begin{pmatrix} w_y \operatorname{Im}(G - wE)_x - w_x \operatorname{Im}(G - wE)_y \\ w_x \operatorname{Re}(G - wE)_y - w_y \operatorname{Re}(G - wE)_x \\ \operatorname{Im}(\overline{(G - wE)_x}(G - wE)_y) \end{pmatrix},$$

we infer that $(G_y - wE_y) = -i(G_x - wE_x)$. We deduce that $(G_x - wE_x) - i(G_y - wE_y) = 0$, so then

$$G_z = wE_z,$$

as desired.

Now, using the expressions for w_x and w_y and using the relation $G_z = wE_z$ just proved, we easily get

$$w_{\bar{z}} = \frac{\overline{E}}{1 + E\overline{E}}(G_{\bar{z}} - wE_{\bar{z}}).$$

At last, as X is a conformal immersion, the metric ds^2 induced on U by X is given by

$$ds^2 = \frac{X_x \cdot X_x}{w^2} = \frac{|G_x - wE_x|^2}{w^2}.$$

On account of $G_z - wE_{\bar{z}} = G_{\bar{z}} - wE_{\bar{z}} + G_z - wE_z = G_x - wE_x$, we infer that

$$ds^2 = \frac{|G_{\bar{z}} - wE_{\bar{z}}|^2}{w^2}.$$

which concludes the proof. □

PROPOSITION 1.4. — *Let $X : U \to \mathbb{H}^3$ be a C^3 conformal immersion. Let us call $\widetilde{H}$ the mean curvature of X with respect to E. Let $\widetilde{\Pi} = \widetilde{b}_{xx}dx^2 + 2\widetilde{b}_{xy}dxdy + \widetilde{b}_{yy}dy^2$ be the second fundamental form of X with respect to E and let $\widetilde{\Phi} := \frac{\widetilde{b}_{xx} - \widetilde{b}_{yy}}{2} - i\widetilde{b}_{xy}$ be the Hopf function of X. We have*

$$2\frac{G_{\bar{z}}}{1 + E\overline{E}} = (1 - \widetilde{H})(G_{\bar{z}} - wE_{\bar{z}}) \tag{9}$$

$$\widetilde{\Phi} = -2E_z\frac{\overline{G_{\bar{z}} - wE_z}}{w(1 + E\overline{E})} \tag{10}$$

$$\widetilde{\Pi} = (\widetilde{H} + \operatorname{Re}\widetilde{\Phi})dx^2 - 2\operatorname{Im}\widetilde{\Phi}dxdy + (\widetilde{H} - \operatorname{Re}\widetilde{\Phi})dy^2. \tag{11}$$

Proof. — We must calculate the coefficients of the second fundamental form. We have

$$\tilde{b}_{xx} = -\langle X_x, \nabla_{X_x}\tilde{N}\rangle_{\mathbb{H}^3}, \quad \tilde{b}_{xy} = -\langle X_x, \nabla_{X_y}\tilde{N}\rangle_{\mathbb{H}^3},$$
$$\tilde{b}_{yy} = -\langle X_y, \nabla_{X_y}\tilde{N}\rangle_{\mathbb{H}^3}.$$

where ∇ is the covariant derivative in $\mathbb{H}^3$ and $\tilde{N}$ is the hyperbolic unit normal field, that is $\tilde{N} = w \cdot N$. For this, recall that for any vector fields A, B and C we have

$$\langle C; \nabla_B A\rangle = \frac{1}{w^2} C \cdot D_B A + \frac{1}{w^3}\left(-A[w]B \cdot C - B[w]C \cdot A + C[w]A \cdot B\right) \qquad (\top)$$

where $A[w]$ stands for the (euclidean) derivative of w with respect to A, see [7] or [3]. Therefore

$$\begin{aligned} -\tilde{b}_{xx} &= \langle X_x; \nabla_{X_x}\tilde{N}\rangle \\ &= \frac{1}{w^2} X_x \cdot D_{X_x}\tilde{N} + \frac{1}{w^3}\left[-\tilde{N}[w]X_x \cdot X_x - X_x[w]X_x \cdot \tilde{N} + X_x[w]X_x \cdot \tilde{N}\right] \\ &= \frac{-1}{w} b_{xx} - \frac{N_3}{w^2} X_x \cdot X_x. \end{aligned}$$

where b_{xx} is the related coefficient of the *euclidean* second fundamental form. We have

$$\begin{aligned} -b_{xx} &= X_x \cdot N_x \\ &= ((G - wE)_x, w_x) \cdot \left(\left(\frac{2E}{1+E\overline{E}}\right)_x, \left(\frac{E\overline{E}-1}{E\overline{E}+1}\right)_x\right) \\ &= \frac{2}{(1+E\overline{E})^2}\left[\mathrm{Re}((\overline{G}_x - w\overline{E}_x)(E_x - E^2\overline{E}_x))\right. \\ &\quad \left. +2\,\mathrm{Re}(\overline{E}E_x)\,\mathrm{Re}(\overline{E}(G_x - wE_x))\right] \\ &= \frac{2}{1+E\overline{E}}\,\mathrm{Re}(\overline{E}_x(G_x - wE_x)) \end{aligned}$$

In the same way, one can show

$$\tilde{b}_{yy} = \frac{1}{w^2}\left[w b_{yy} + N_3 X_y \cdot X_y\right]$$
$$\tilde{b}_{xy} = \frac{1}{w^2}\left[w b_{xy} + N_3 X_x \cdot X_y\right] = \frac{1}{w} b_{xy}$$

with

$$b_{xy} = \frac{-2}{1+E\overline{E}}\,\mathrm{Re}(\overline{E}_y(G_x - wE_x)) = \frac{-2}{1+E\overline{E}}\,\mathrm{Im}(\overline{E}_x(G_x - wE_x))$$
$$b_{yy} = \frac{-2}{1+E\overline{E}}\,\mathrm{Im}(\overline{E}_y(G_x - wE_x))$$

As

$$\widetilde{H} = \frac{\tilde{b}_{xx} + \tilde{b}_{yy}}{2\langle X_x, X_x\rangle} \quad \text{and} \quad H = \frac{b_{xx} + b_{yy}}{2X_x \cdot X_x}$$

where H is the euclidean mean curvature of X with respect to E, we deduce

$$H = -2\frac{\mathrm{Re}(\overline{E_{\bar{z}}}(G_{\bar{z}} - wE_{\bar{z}}))}{(1+E\overline{E})|G_z - wE_{\bar{z}}|^2} = \frac{-2E_{\bar{z}}}{(1+E\overline{E})(G_{\bar{z}} - wE_{\bar{z}})}$$

$$\widetilde{H} = wH + N_3$$

$$= -2\frac{G_{\bar{z}}}{(1+E\overline{E})(G_{\bar{z}} - wE_{\bar{z}})} + 1$$

from which we infer relation (9).

The relations (10) and (11) can be established from the expressions of $\tilde{b}_{xx}$, $\tilde{b}_{xy}$ and $\tilde{b}_{yy}$. This concludes the proof of the proposition. □

Now we are going to prove Theorem 1.2. Let $X : U \to \mathbb{H}^3$ be a C^3 conformal immersion. From the relation (9) of Proposition 1.4 we deduce

$$G_z = \frac{(\widetilde{H}-1)(1+E\overline{E})}{2+(\widetilde{H}-1)(1+E\overline{E})} \cdot wE_{\bar{z}} \qquad (*)$$

Furthermore, from Proposition 1.3 we get $G_z = wE_z$. Observe that, as X is a C^3 map, G is a C^2 map. Consequently we get $(G_z)_z = (G_{\bar{z}})_z$. We have

$$(G_z)_{\bar{z}} = (G_{\bar{z}})_z \Leftrightarrow w_z E_z + wE_{zz} = \left(\frac{(\widetilde{H}-1)(1+E\overline{E})}{2+(\widetilde{H}-1)(1+E\overline{E})}\right)_z \cdot wE_{\bar{z}}$$
$$+ \frac{(\widetilde{H}-1)(1+E\overline{E})}{2+(\widetilde{H}-1)(1+E\overline{E})} \cdot w_z E_{\bar{z}} + \frac{(\widetilde{H}-1)(1+E\overline{E})}{2+(\widetilde{H}-1)(1+E\overline{E})} \cdot wE_{z\bar{z}}$$

From $(*)$ we have

$$G_{\bar{z}} - wE_{\bar{z}} = \frac{-2}{2+(\widetilde{H}-1)(1+E\overline{E})} \cdot wE_{\bar{z}} \qquad (**)$$

From Proposition 1.3 relation (7) we deduce

$$w_{\bar{z}} = \frac{-2\overline{E}}{1+E\overline{E}} \cdot \frac{1}{2+(\widetilde{H}-1)(1+E\overline{E})} \cdot wE_{\bar{z}}$$

As w is a real function, we have

$$w_z = \overline{(w_{\bar{z}})} = \frac{-2E}{1+E\overline{E}} \cdot \frac{1}{2+(\widetilde{H}-1)(1+E\overline{E})} \cdot w\overline{E_{\bar{z}}}$$

Substituting w_z and $w_{\bar z}$ by their previous expression and simplifying by w we deduce

$$
\begin{aligned}
&(G_{\bar z})_z = (G_z)_{\bar z} \Leftrightarrow \\
&\quad -2\frac{\overline{E}E_zE_{\bar z}}{(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))} + E_{z\bar z} = \\
&\quad \Big(\frac{-2}{2+(\widetilde{H}-1)(1+E\overline{E})}\Big)_z E_{\bar z} + \frac{(\widetilde{H}-1)(1+E\overline{E})}{2+(\widetilde{H}-1)(1+E\overline{E})}E_{z\bar z} \\
&\quad + \frac{(\widetilde{H}-1)(1+E\overline{E})}{2+(\widetilde{H}-1)(1+E\overline{E})}\cdot\frac{-2E}{1+E\overline{E}}\cdot\frac{E_{\bar z}\overline{E_{\bar z}}}{2+(\widetilde{H}-1)(1+E\overline{E})} \\
\Leftrightarrow\ & -2\frac{\overline{E}E_zE_{\bar z}}{(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))} + \frac{2}{2+(\widetilde{H}-1)(1+E\overline{E})}E_{z\bar z} \\
&\quad = \Big(\frac{-2}{2+(\widetilde{H}-1)(1+E\overline{E})}\Big)_z E_{\bar z} - 2\frac{(\widetilde{H}-1)(1+E\overline{E})}{1+E\overline{E}}\cdot\frac{EE_{\bar z}\overline{E_{\bar z}}}{(2+(\widetilde{H}-1)(1+E\overline{E}))^2} \\
\Leftrightarrow\ & -2\overline{E}E_zE_{\bar z}(2+(\widetilde{H}-1)(1+E\overline{E})) + 2(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))E_{z\bar z} = \\
&\quad 2(\widetilde{H}_z(1+E\overline{E}) + (\widetilde{H}-1)(E_z\overline{E}+E\overline{E_{\bar z}})E_{\bar z}(1+E\overline{E}) - 2(\widetilde{H}-1)(1+E\overline{E})EE_{\bar z}\overline{E_{\bar z}} \\
\Leftrightarrow\ & -\overline{E}E_zE_{\bar z}(2+(\widetilde{H}-1)(1+E\overline{E})) + (1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))E_{z\bar z} = \\
&\quad \widetilde{H}_z(1+E\overline{E})^2E_{\bar z} + (\widetilde{H}-1)(1+E\overline{E})\overline{E}E_zE_{\bar z} \\
\Leftrightarrow\ & (1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))E_{z\bar z} - 2(1+(\widetilde{H}-1)(1+E\overline{E}))\overline{E}E_zE_{\bar z} \\
&\qquad -(1+E\overline{E})^2E_{\bar z}\widetilde{H}_z = 0
\end{aligned}
$$

which is the relation (1) stated in Theorem 1.2. Now, the relations (2), (3) and (4) of Theorem 1.2 are easily induced from $(**)$ and the relations of Proposition 1.3. This accomplished the first part of the proof of Theorem 1.2.

Conversely, let $E : U \to \mathbb{C}$ be a C^2 function and let $\widetilde{H} : U \to \mathbb{R}$ be a C^1 function satisfying relation (1). We are looking for a C^3 conformal map $X = (u,v,w) : U \to \mathbb{H}^3$ such that the restricted map $X_{|U^*}$ is an immersion whose euclidean Gauss map is E and whose mean curvature is $\widetilde{H}$. From relations $(**)$ and (7) of Proposition 1.3 we get that w must satisfy the differential equation

$$w_{\bar z} = \frac{-2\overline{E}}{(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))}\cdot wE_{\bar z} \tag{12}$$

Setting $f := \mathrm{Log}w$. the equation (12) has the form $f_{\bar z} = A(z)$ where A is a complex function. As f is a real function the integrability condition is $\mathrm{Re}(A)_y = \mathrm{Im}(A)_x$. But the last condition is equivalent to $\mathrm{Im}(A_z) = 0$. As

$$A = \frac{-2\overline{E}}{(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))}\cdot E_{\bar z},$$

we have

$$\begin{aligned}
A_z\in\mathbb{R} \Leftrightarrow & \overline{(E_{\bar z})}\,E_{\bar z}(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))+\overline{E}(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))E_{z\bar z}\\
& -\overline{E}E_{\bar z}(E_z\overline{E}+E\overline{(E_{\bar z})})(2+(\widetilde{H}-1)(1+E\overline{E}))-\overline{E}E_{\bar z}(1+E\overline{E})(\widetilde{H}_z(1+E\overline{E})\\
& +(\widetilde{H}-1)(E_z\overline{E}+E\overline{(E_{\bar z})}))\in\mathbb{R}\\
\Leftrightarrow & \overline{E}[(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))E_{z\bar z}-\overline{E}E_{\bar z}E_z(2+(\widetilde{H}-1)(1+E\overline{E}))\\
& -\overline{E}E_{\bar z}E_z(\widetilde{H}-1)(1+E\overline{E})-(1+E\overline{E})^2E_{\bar z}\widetilde{H}_z]\in\mathbb{R}\\
\Leftrightarrow & \overline{E}[(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))E_{z\bar z}-2\overline{E}E_{\bar z}E_z(1+(\widetilde{H}-1)(1+E\overline{E}))\\
& -(1+E\overline{E})^2E_{\bar z}\widetilde{H}_z]\in\mathbb{R}
\end{aligned}$$

As E and $\widetilde{H}$ satisfy (1), the integrability condition holds. Henceforth there exists a real function w satisfying (12). In fact, up to a multiplicative complex constant we have

$$w(z)=e^{-4\,\mathrm{Re}\int\frac{\overline{E}E_{\bar z}}{(1+E\overline{E})(2+(\widetilde{H}-1)(1+E\overline{E}))}\cdot d\bar z}$$

In the same way, in view of Propositions 1.3 and 1.4 we are looking for a map $G: U\to\mathbb{C}$ satisfying

$$(S)\qquad\begin{cases} G_z &= wE_z\\ G_{\bar z} &= \frac{(\widetilde{H}-1)(1+E\overline{E})}{2+(\widetilde{H}-1)(1+E\overline{E})}\cdot wE_z\end{cases}$$

The integrability condition of (S) is $(G_z)_{\bar z}=(G_z)_z$. But the last is equivalent to the equation (1) as we have seen in the first part of the proof. Thus, there does exist a complex map G on U satisfying (S). Observe that G is defined up to an additive complex number. Then, define the real functions u and v on U as $u+iv:=G-wE$. Now, we define the map X setting $X:=(u,v,w)$: $U\to\mathbb{H}^3$. It is a straightforward verification to prove that the restricted map $X_{|U^*}$ is a conformal immersion whose euclidean Gauss map is E and with mean curvature $\widetilde{H}$ with respect to E. Furthermore, the relations (2), (3) and (4) are valid on U^*. Further details can be seen in [10].

At last, suppose that $\hat X=(\hat u,\hat v,\hat w): U\to\mathbb{H}^3$ is another such map. Let us call $\hat G$ the related hyperbolic Gauss map. Observe that $\hat w$ must satisfy the equation (12). Therefore there exists a complex number λ such that $\hat w=\lambda w$. Also, $\hat G$ satisfies the system (S) with λw (that is $\hat w$) instead of w. We deduce that there exixts a complex number α such that $\hat G=\lambda G+\alpha$. Therefore we have

$$\begin{aligned}
\hat X &= (\hat u+i\hat v,\hat w)=(\hat G-\hat wE,\hat w)=(\lambda G+\alpha-\lambda wE,\lambda w)\\
&= \lambda(G-wE,w)+(\alpha,0)=\lambda\cdot X+(\alpha,0).
\end{aligned}$$

□

2. *Mean curvature one surfaces in hyperbolic space*

We now specialize for the case where the mean curvature is one. Let $X : U \to \mathbb{H}^3$ be a C^3 conformal immersion such that $E \neq \infty$ along U. Observe that when $\widetilde{H} \equiv 1$ the equation (1) reduces to

$$E_{z\bar{z}} = \frac{\overline{E}}{1 + E\overline{E}} E_z E_{\bar{z}}. \qquad (*)$$

In [10] the authors have considered this equation and they have proved the following results.

THEOREM 2.1 (**Existence of Meromorphic Data**). — *Let $U \subset \mathbb{C}$ be a simply connected domain and let $X : U \to \mathbb{H}^3$ be a non-totally umbilic conformal immersion. Let E be the oriented euclidean Gauss map E of X. Assume that X has mean curvature one with respect to E (therefore, E satisfies equation $(*)$).*

Then, there exist two meromorphic functions h, T on U such that

$$E = h \cdot \overline{\left(\frac{Th_z + hT_z}{h^2 T_z}\right)} \left(T + \overline{\left(\frac{h_z}{Th_z + hT_z}\right)}\right).$$

Furthermore, up to a multiplicative positive constant we have

$$w = \frac{|h^2 T_z|^2}{|Th_z + hT_z|^2 + |h_z|^2}$$
$$G_z = h^2 T_z$$

then, up to the same multiplicative positive constant and up to an additive complex constant we have

$$u + iv = G - wE = G - \frac{h^3 T_z}{|Th_z + hT_z|^2 + |h_z|^2} \cdot \overline{(Th_z + hT_z)}.\left(T + \overline{\left(\frac{h_z}{Th_z + hT_z}\right)}\right).$$

Moreover,

$$ds = |E_{\bar{z}} dz| = \frac{|hT_z h_{zz} - 2h_z^2 T_z - hh_z T_{zz}|}{|hT_z|^2}(1 + |T|^2)|dz|$$
$$\tilde{\Phi} = 2\frac{hT_z h_{zz} - 2h_z^2 T_z - hh_z T_{zz}}{h^2 T_z}$$
$$\tilde{\mathrm{II}} = \mathrm{Re}(\tilde{\Phi}(dz)^2) + ds^2$$
$$K = -4\frac{|h|^4 |T_z|^6}{|hT_z h_{zz} - 2h_z^2 T_z - hh_z T_{zz}|^2 (1 + |T|^2)^4}$$

and, up to a rigid motion in $\mathbb{R}^3$, *we can choose*

$$g = T$$
$$f = -\frac{1}{T_z} \cdot \frac{hT_z h_{zz} - 2h_z^2 T_z - hh_z T_{zz}}{h^2 T_z}$$

where $(g, f\,dz)$ *is the Weierstrass representation of the minimal immersion in* $\mathbb{R}^3$ *associated to* X.

This means that the euclidean map E and the immersion X can be expressed in terms of two analytic functions h and T on U, see Theorem 3-9 of [10]. The following result is a converse one: given two meromorphic functions, assuming some conditions, we can construct a mean curvature one conformal immersion $X : U \to \mathbb{H}^3$, see Theorem 3-11 of [10].

THEOREM 2.2 (**Representation Theorem**). — *Let* S *be a Riemann surface and let* h *and* T *be two meromorphic functions on* S *such that* $h \neq \frac{1}{\alpha T+\beta}$, *for any complex numbers* α, β. *We set:*

$$w(z) = \frac{|h^2 T_z|^2}{|Th_z + hT_z|^2 + |h_z|^2}, \quad E = h \cdot \overline{\left(\frac{Th_z + hT_z}{h^2 T_z}\right)} \left(T + \overline{\left(\frac{h_z}{Th_z + hT_z}\right)}\right)$$

and $S^* = \{z \in S, |E_{\bar{z}}| \neq 0, \infty\}$. *Let us assume that*

$$\int_\gamma h^2 T_z dz = 0 \qquad (\top)$$

for every closed path $\gamma \subset S$ *on which neither* h *nor* T *have poles, that is the 1-form* $h^2 T_z dz$ *has a global primitive* G *on* S. *Set:*

$$(u + iv)(z) := (G - wE)(z), \quad z \in S^*.$$

Then, the function $X := (u, v, w) : S^* \to \mathbb{H}^3$ *defines a mean curvature one conformal immersion whose euclidean Gauss map is* E *and the hyperbolic Gauss map is* G. *Furthermore, the geometric quantities of* X *are given by the last relations of Theorem 2.1.*

We recall that $S - S^*$ is a discret set, see Lemma 2-3 of [10]. As a matter of fact, equation $(*)$ can be solved in terms of meromorphic functions, see Theorem 3-12 of [10]. We found the following representation:

THEOREM 2.3. — *Let* $U \subset \mathbb{C}$ *be a simply connected domain and let* $E : U \to \mathbb{C}$ *be a* C^2 *map, neither holomorphic nor anti-holomorphic, satisfying equation*

$$E_{z\bar{z}} = \frac{\overline{E}}{1 + E\overline{E}} E_z E_{\bar{z}}. \qquad (*)$$

Then, there exists a holomorphic function h and there exists a meromorphic function T on U such that

$$E = h \cdot \overline{\left(\frac{Th_z + hT_z}{h^2 T_z}\right)} \left(T + \overline{\left(\frac{h_z}{Th_z + hT_z}\right)}\right) \cdot$$

Moreover, for any point $z_0 \in U$, then T has a pole there if and only if: h vanishes at z_0, and T and h have same order at this point.

Now we are going to give some examples of complete mean curvature one immersions in $\mathbb{H}^3$. For this, we choose $U = \mathbb{C}$, $h(z) = e^{\gamma z}$ and $T(z) = b + e^z$, where $z \in \mathbb{C}$ and b is a constant complex number. Using Theorem 2.2 it is easily seen that (h,T) give rise to a complete mean curvature one immersion of $\mathbb{C}$ into $\mathbb{H}^3$.

Let us first assume $\gamma = -\frac{1}{2}$. We deduce from Theorem 2.2 that

$$\begin{cases} (u + iv)(z) = z + \dfrac{2}{1 + |b - e^z|^2}((b + e^z)(\overline{b - e^z}) + 1) \\ w(z) = 4\dfrac{|e^z|}{1 + |b - e^z|^2}. \end{cases}$$

Therefore we have $X(z + 2\pi i) = X(z) + (0,2\pi,0)$ for any $z \in \mathbb{C}$. That is, the surface $X(\mathbb{C})$ is invariant under the horizontal translations $(u + iv,w) \to (u + iv,w) + (2\pi i,0)$ of $\mathbb{H}^3$.

In Figure 1-a we draw a piece of a fundamental domain of the surface corresponding to $b = 1/2$. We draw also three fundamental domains of the same surface in Figure 1-b.

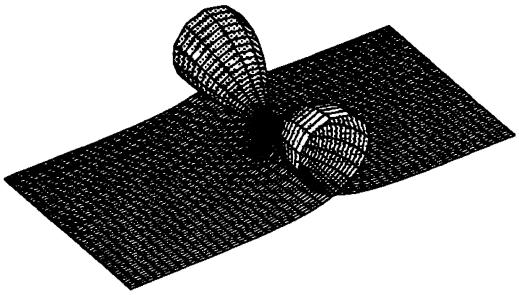

Figure 1-a

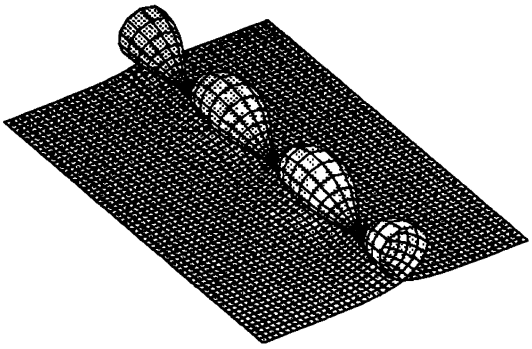

Figure 1-b

In the particular case where $b = 0$ we have

$$\begin{cases} (u + iv)(z) = z - 2\coth(\dfrac{z + \bar{z}}{2}) \\ w(z) = \dfrac{2}{\cosh(\frac{z+\bar{z}}{2})}. \end{cases}$$

Now we have $X(z + iy_0) = X(z) + (0,y_0,0)$ for any $y_0 \in \mathbb{R}$. That is, the surface $X(\mathbb{C})$ is invariant under the continuous group of horizontal translation

$$\{(u,v,w) \to (u,v,w) + (0,y_0,0), y_0 \in \mathbb{R}\},$$

see Figure 2-a. The profile curve, called "courbe des forçats" (see figure 2-b), was studied by Poleni in 1729, see [8].

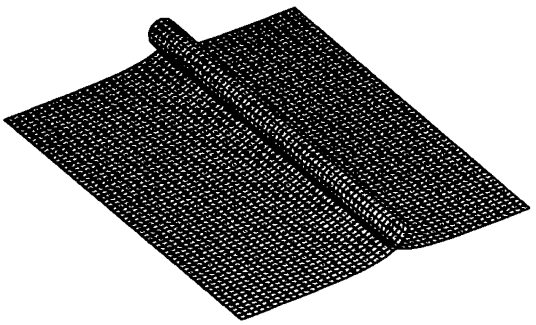

Figure 2-a

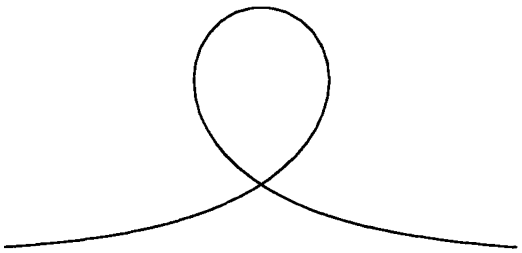

Figure 2-b

It can be shown that this surface is an Enneper cousin dual, see [9], Remark 1-11-(2).

At last, when $\gamma \neq -\frac{1}{2}$ we have

$$\begin{cases} (u+iv)(z) = e^{(2\gamma+1)z}\left(\dfrac{1}{2\gamma+1} - \dfrac{(b+e^z)(\overline{b\gamma+(1+\gamma)e^z}) + \overline{\gamma}}{|b\gamma+(1+\gamma)e^z|^2+|\gamma|^2}\right) \\ w(z) = e^{\gamma z+\overline{\gamma z}} \cdot \dfrac{|e^z|^2}{|b\gamma+(1+\gamma)e^z|^2+|\gamma|^2}. \end{cases}$$

It is easily conferred that $X(z+2\pi i) = (H_{-4\pi\,\mathrm{Im}(\gamma)} \circ R_{4\pi\,\mathrm{Re}(\gamma)})X(z)$, where for any real number λ, we call R_λ the rotation around the w-axis whose argument is λ and we call H_λ the homothety with respect to 0 and ratio e^λ. That is, X is invariant under a discret subgroup of screw motions of $\mathbb{H}^3$ isometric to $\mathbb{Z}$. See Figure 3, where $\gamma = -1+i$ and $b = 1$.

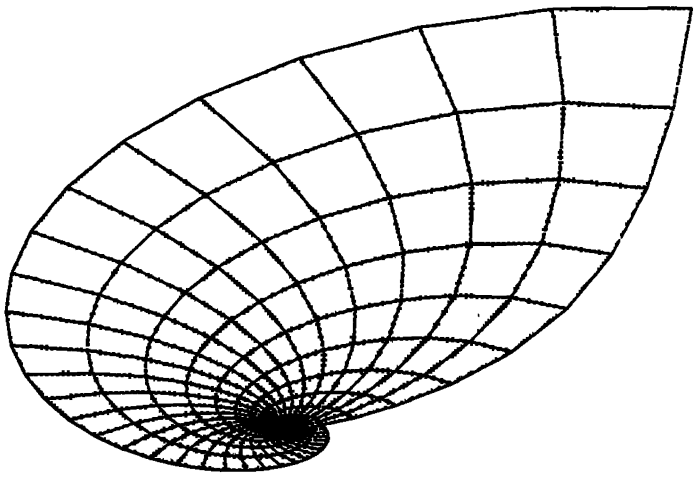

Figure 3

We refer to [10] for a full geometric description of the above surfaces. We observe that M. Umehara and K. Yamada have given some of the previous examples using other techniques.

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