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SOME RECENT RESULTS ABOUT THE
$\mathrm{SL}_n(\mathbb{C})$–**REPRESENTATION SPACES OF KNOT GROUPS**

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ABSTRACT. — This survey reviews some facts about about the representation and character varieties of knot groups into $\mathrm{SL}_n(\mathbb{C})$ with $n \geqslant 3$ are presented. This concerns mostly joint work of the author with L. Ben Abdelghani, O. Medjerab, V. Muños and J. Porti.

1. Introduction

Since the foundational work of Thurston [57, 58] and Culler and Shalen [12], the varieties of representations and characters of three-manifold groups in $\mathrm{SL}_2(\mathbb{C})$ have been intensively studied, as they reflect geometric and topological properties of the three-manifold. In particular they have been used to study knots $k \subset S^3$, by analysing the $\mathrm{SL}_2(\mathbb{C})$-character variety of the fundamental group of the knot complement $S^3 - k$ (these are called *knot groups*).

Much less is known about the character varieties of three-manifold groups in other Lie groups, notably for $\mathrm{SL}_n(\mathbb{C})$ with $n \geqslant 3$. There has been an increasing interest for those in the last years. For instance, inspired by the A-coordinates in higher Teichmüller theory of Fock and Goncharov [20], some authors have used the so called Ptolemy coordinates for studying spaces of representations, based on subdivisions of ideal triangulations of the three-manifold. Among others, we mention the work of Dimofty, Gabella, Garoufalidis, Goerner, Goncharov, Thurston, and Zickert [16, 17, 23, 24, 25]. Geometric aspects of these representations, including volume and rigidity, have been addressed by Bucher, Burger, and Iozzi in [10], and by Bergeron,

Falbel, and Guilloux in [6], who view these representations as holonomies of marked flag structures. We also recall the work Deraux and Deraux–Falbel in [13, 14, 15] to study CR and complex hyperbolic structures.

An extended version of this survey which contains more explanations and examples is published in the RIMS Kôkyûroku lecture series [30].

2. Definitions

DEFINITION 2.1. — *Let $k \subset S^3$ be a smooth knot. The* knot exterior *is the compact manifold $C := C(k) = \overline{S^3 \setminus V(k)}$ where $V(k)$ is a tubular neighborhood of k. The* knot group *is $\Gamma_k := \pi_1(C)$.*

In what follows we will make use of the following properties of knot groups:

- We have $H_1(C(k); \mathbb{Z}) \cong \mathbb{Z}$. A canonical surjection $\varphi \colon \Gamma_k \to \mathbb{Z}$ is given by $\varphi(\gamma) = \mathrm{lk}(\gamma, k)$ where lk denotes the linking number in S^3 (see [11, 3.B]).
- The knot exterior is aspherical: we have $\pi_n(C(k)) = 0$ for $n > 1$ i.e. $C(k)$ is an Eilenberg–MacLane space $K(\Gamma_k, 1)$ (see [11, 3.F]). As a cosequence, the (co-)homology groups of Γ and $C(k)$ are naturally identified, and for a given Γ_k-module M we have $H^*(C(k); M) \cong H^*(\Gamma_k; M)$, and $H_*(C(k); M) \cong H_*(\Gamma_k; M)$.

It follows that every *abelian* representation factors through $\varphi \colon \Gamma_k \to \mathbb{Z}$. Here we call ρ *abelian* if its image is abelian. We obtain for each non-zero complex number $\eta \in \mathbb{C}^*$ an abelian representation $\eta^\varphi \colon \Gamma_k \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$ given by $\gamma \mapsto \eta^{\varphi(\gamma)}$.

2.1. Representation and character varieties

The general reference for representation and character varieties is Lubotzky's and Magid's book [46]. Let $\Gamma = \langle \gamma_1, \ldots, \gamma_m \rangle$ be a finitely generated group.

DEFINITION 2.2. — *A* $\mathrm{SL}_n(\mathbb{C})$-*representation is a homomorphism* $\rho \colon \Gamma \to \mathrm{SL}_n(\mathbb{C})$. *The* $\mathrm{SL}_n(\mathbb{C})$-*representation variety is*

$$R_n(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{C})) \subset \mathrm{SL}_n(\mathbb{C})^m \subset M_n(\mathbb{C})^m \cong \mathbb{C}^{n^2 m} \,.$$

The representation variety $R_n(\Gamma)$ is an affine algebraic set. It is contained in $\mathrm{SL}_n(\mathbb{C})^m$ via the inclusion $\rho \mapsto \big(\rho(\gamma_1), \ldots, \rho(\gamma_m)\big)$, and it is the set of solutions of a system of polynomial equations in the matrix coefficients.

Given two representations $\rho_1 : \Gamma \to \mathrm{GL}_m(\mathbb{C})$ and $\rho_2 : \Gamma \to \mathrm{GL}_n(\mathbb{C})$ we define the *direct sum* $\rho_1 \oplus \rho_2 : \Gamma \to \mathrm{GL}_{m+n}(\mathbb{C})$ and the *tensor product* $\rho_1 \otimes \rho_2 : \Gamma \to \mathrm{GL}_{mn}(\mathbb{C})$. For $\gamma \in \Gamma$ these representations are given by

$$(\rho_1 \oplus \rho_2)(\gamma) = \left(\begin{array}{c|c} \rho_1(\gamma) & 0 \\ \hline 0 & \rho_2(\gamma) \end{array} \right) \quad \text{and} \quad (\rho_1 \otimes \rho_2)(\gamma) = \rho_1(\gamma) \otimes \rho_2(\gamma),$$

respectively. Here, $A \otimes B$ denotes the *Kronecker product* of $A \in \mathrm{GL}_m(\mathbb{C})$ and $B \in \mathrm{GL}_n(\mathbb{C})$. The *dual representation* $\rho^* : \Gamma \to \mathrm{GL}(n)$ of $\rho : \Gamma \to \mathrm{GL}(n)$ is defined by $\rho^*(\gamma) = {}^t\rho(\gamma)^{-1}$ where ${}^tA$ is the transpose of the matrix A. (See also Lemme 5.6.)

DEFINITION 2.3. — *We call a representation $\rho : \Gamma \to \mathrm{GL}_n(\mathbb{C})$ reducible if there exists a nontrivial subspace $V \subset \mathbb{C}^n$, $0 \neq V \neq \mathbb{C}^n$, such that V is $\rho(\Gamma)$-stable. The representation ρ is called* irreducible *if it is not reducible. A* semisimple *representation is a direct sum of irreducible representations.*

The group $\mathrm{SL}_n(\mathbb{C})$ acts by conjugation on $R_n(\Gamma)$. More precisely, for $A \in \mathrm{SL}_n(\mathbb{C})$ and $\rho \in R_n(\Gamma)$ we define $(A.\rho)(\gamma) = A\rho(\gamma)A^{-1}$ for all $\gamma \in \Gamma$. Moreover, we let $\mathrm{O}(\rho) = \{A.\rho \mid A \in \mathrm{SL}_n(\mathbb{C})\}$ denote the *orbit* of ρ. In what follows we will write $\rho \sim \rho'$ if there exists $A \in \mathrm{SL}_n(\mathbb{C})$ such that $\rho' = A.\rho$, and we will call ρ and ρ' *equivalent*. For $\rho \in R_n(\Gamma)$ we define its *character* $\chi_\rho : \Gamma \to \mathbb{C}$ by $\chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma))$. If ρ and ρ' are equivalent then $\chi_\rho = \chi_{\rho'}$. The converse does not always hold and we have the following lemma:

LEMMA 2.4. — *Let $\rho \in R_n(\Gamma)$ be a representation. The orbit $\mathrm{O}(\rho)$ is closed if and only if ρ is semisimple. Moreover, let ρ, ρ' be semisimple. Then $\rho \sim \rho'$ if and only if $\chi_\rho = \chi_{\rho'}$.*

Proof. — See Theorems 1.27 and 1.28 in Lubotzky's and Magid's book [46]. □

The *algebraic quotient* or *GIT quotient* for the action of $\mathrm{SL}_n(\mathbb{C})$ on $R_n(\Gamma)$ is called the *character variety*. This quotient will be denoted by $X_n(\Gamma) = R_n(\Gamma) /\!/ \mathrm{SL}_n(\mathbb{C})$. The character variety is not necessary an irreducible affine algebraic set. For an introduction to algebraic invariant theory see Dolgachev's book [18]. *Geometric invariant theory* is concerned with an algebraic action of a group G on an algebraic variety. For a point $v \in V$ the orbit $G\,v$ will be denoted by $\mathrm{O}(v)$. The action of G on V induces an action of G on the coordinate algebra $\mathcal{O}(V)$ of the variety V given by

$g \cdot f(v) = f(g^{-1}v)$, for all $g \in G$, and $v \in V$. The invariant functions of the G-action on V are

$$\mathcal{O}(V)^G = \{f \in \mathcal{O}(V) \mid g \cdot f = f \text{ for all } g \in G\}.$$

The commutative algebra $\mathcal{O}(V)^G$ is interpreted as the algebra of functions on the *GIT quotient* $V /\!/ G$. The main problem is to prove that the algebra $\mathcal{O}(V)^G$ is finitely generated. This is necessary if one wants the quotient to be an affine algebraic variety. We are only interested in *affine varieties* V and in *reductive groups* G. In this situation Nagata's theorem applies (see [18, Sec. 3.4]), and the algebra $\mathcal{O}(V)^G$ is finitely generated. Reductive groups include all finite groups and all classical groups (see [18, Chap. 3]). If $f_1, \ldots, f_N$ generate the algebra $\mathcal{O}(V)^G$ then a model for the quotient is given by the image of the map $t \colon V \to V /\!/ G \subset \mathbb{C}^N$ given by $t(v) = (f_1(v), \ldots, f_N(v))$. The GIT quotient $V /\!/ G$ parametrizes the set of closed orbits (see [18, Corollary 6.1]).

Work of C. Procesi [52] implies that there exists a finite number of group elements $\{\gamma_i \mid 1 \leqslant i \leqslant M\} \subset \Gamma$ such that the image of $t \colon R_n(\Gamma) \to \mathbb{C}^M$ given by

$$t(\rho) = \big(\chi_\rho(\gamma_1), \ldots, \chi_\rho(\gamma_M)\big)$$

can be identified with the affine algebraic set $X_n(\Gamma) \cong t(R_n(\Gamma))$, see also [46, p. 27]. This justifies the name *character variety*.

Examples 2.5.

(1) Let F_2 be the free group on the two generators x and y. Then it is possible to show that $X_2(F_2) \cong \mathbb{C}^3$ and that $t \colon R_2(F_2) \to \mathbb{C}^3$ is given by $t(\rho) = \big(\chi_\rho(x), \chi_\rho(y), \chi_\rho(xy)\big)$. See Goldman's article [51, Chap. 15] and the article of Gonzàlez-Acuña and Montesinos-Amilibia [27] for more details.

(2) We obtain $X_n(\mathbb{Z}) \cong \mathbb{C}^{n-1}$ More precisely, $R_n(\mathbb{Z}) \cong \mathrm{SL}_n(\mathbb{C})$ and $t \colon R_n(\mathbb{Z}) \to \mathbb{C}^{n-1}$ maps the matrice $A \in \mathrm{SL}_n(\mathbb{C})$ onto the coefficients of the characteristic polynomial of A (see [18, Example 1.2]).

(3) Explicit coordinates for $X_3(F_2)$ are also known: $X_3(F_2)$ is isomorphic to a degree 6 affine hyper-surface in $\mathbb{C}^9$ (see S. Lawton [45] and P. Will [62]).

(4) If Γ is a finite group then $X_n(\Gamma)$ is finite for all n. This follows since Γ has up to equivalence only finitely many irreducible representations, and every representation of a finite group is semisimple (see [53]).

2.2. Tangent spaces and group cohomology.

The general reference for group cohomology is Brown's book [9]. In order to shorten notation we will sometimes write $\mathrm{SL}(n)$ and $\mathfrak{sl}(n)$ instead of $\mathrm{SL}_n(\mathbb{C})$, and $\mathfrak{sl}_n(\mathbb{C})$.

The following construction was presented by A. Weil [61]. For $\rho \in R_n(\Gamma)$ the Lie algebra $\mathfrak{sl}(n)$ turns into a Γ-module via $\mathrm{Ad} \circ \rho$, i.e. for $X \in \mathfrak{sl}(n)$ and $\gamma \in \Gamma$ we have $\gamma \cdot X = \mathrm{Ad}_{\rho(\gamma)}(X) = \rho(\gamma) X \rho(\gamma)^{-1}$. In what follows this Γ-module will be denoted by $\mathfrak{sl}(n)_{\mathrm{Ad}\,\rho}$. We obtain an inclusion $T^{Zar} R_n(\Gamma) \hookrightarrow Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$: for a smooth family of representations ρ_t with $\rho_0 = \rho$ we obtain a map $u : \Gamma \to \mathfrak{sl}(n)$ given by

$$(2.1) \qquad u(\gamma) = \frac{d\rho_t(\gamma)}{dt}\Big|_{t=0} \rho(\gamma)^{-1} \,.$$

The map u verifies: $u(\gamma_1 \gamma_2) = u(\gamma_1) + \gamma_1 \cdot u(\gamma_2)$ i.e. $u \in Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ is a *cocycle* or *derivation* in group cohomology. If $\rho_t = \mathrm{Ad}_{A_t} \circ \rho$ is contained in $O(\rho)$ where A_t, $A_0 = I_n$, is a path of matrices, then the corresponding cocycle is a coboundary i.e. there exists $X \in \mathfrak{sl}(n)$ such that $u(\gamma) = (1 - \gamma) \cdot X = X - \mathrm{Ad}_{\rho(\gamma)}(X)$.

In general, the inclusion $T^{Zar} R_n(\Gamma) \hookrightarrow Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ might be strict. More precisely, the space $Z^1(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ is the Zariski tangent space to the scheme $\mathcal{R}(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ at ρ. For more details see [46] and [30, 2.3].

DEFINITION 2.6. — *Let $\rho : \Gamma \to \mathrm{SL}(n)$ be a representation. A derivation $u \in Z^1(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ is called* integrable *if there exists a family of representations $\rho_t : \Gamma \to \mathrm{SL}(n)$ such that $\rho_0 = \rho$ and (2.1) holds.*

The following is a quite useful observation [46, p. iv] for detecting smooth points of the representation variety. In general not every cocycle is integrable and there are different reasons for this (see Remark 2.8). We have the following inequalities

$$(2.2) \qquad \dim_\rho R_n(\Gamma) \leqslant \dim T_\rho^{Zar} R_n(\Gamma) \leqslant \dim Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$$

where $\dim_\rho R_n(\Gamma)$ denotes the local dimension of $R_n(\Gamma)$ at ρ i.e. the maximum of the dimensions of the irreducible components of $R_n(\Gamma)$ containing ρ.

In what follows, will call ρ a *regular* or *scheme smooth* point of $R_n(\Gamma)$ if the equality $\dim_\rho R_n(\Gamma) = \dim Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ holds. In this case every derivation is integrable, and we obtain the following:

LEMMA 2.7 (see [36, Lemma 2.6]). — *Let $\rho \in R_n(\Gamma)$ be a representation. If ρ is regular, then ρ is a smooth point of the representation variety $R_n(\Gamma)$, and ρ is contained in a unique component of $R_n(\Gamma)$ of dimension $\dim Z^1(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$.*

In general the relation between the cohomology group $H^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ and the tangent space $T^{Zar}_{\chi_\rho} X_n(\Gamma)$ is more complicate. However, if ρ is an irreducible regular representation then we have for the character variety

$$T^{Zar}_{\chi_\rho} X_n(\Gamma) \cong H^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho}).$$

(See [46, Lemma 2.18], and [55, Section 13] for a generalisation to completely reducible regular representations).

Remark 2.8. — There are examples such that

$$\dim_\rho R_n(\Gamma) < \dim Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$$

holds. In Example 2.13 of [30] a reducible representation $\rho \colon D(3,3,3) \to \mathrm{SL}(2)$ of the van Dyck group $D(3,3,3)$ is given such that

$$\dim_\rho R_2(D(3,3,3)) < \dim T^{Zar}_\rho R_2(D(3,3,3)) = \dim Z^1(\Gamma, \mathfrak{sl}(2)_{\mathrm{Ad}\,\rho}).$$

Following Lubotzky and Magid [46, pp. 40–43] an example of a finitely presented group Γ and a representation $\rho \colon \Gamma \to \mathrm{SL}(2)$ is given in [30, Example 2.18] such that

$$\dim_\rho R_2(\Gamma) = \dim T^{Zar}_\rho R_2(\Gamma) < \dim Z^1(\Gamma, \mathfrak{sl}(2)_{\mathrm{Ad}\,\rho})$$

holds.

Remark 2.9. — M. Kapovich and J. Millson proved in [39] that there are essentially no restrictions on the *local* geometry of representation schemes of 3-manifold groups to $\mathrm{SL}_2(\mathbb{C})$.

3. The distinguished components and some examples

In general, not much is known about the global structure of the character varieties of knot groups. In this section we will present some facts and some examples.

Example 3.1 (Diagonal representations). — The characters of diagonal representations of a knot group Γ_k form an algebraic component of $X_n(\Gamma_k)$. A representation $\rho \colon \Gamma_k \to \mathrm{SL}(n)$ which is the direct sum of one-dimensional representations is equivalent to a diagonal representation. The image of a diagonal representation is abelian. Hence it factors through $\varphi \colon \Gamma_k \to \mathbb{Z}$. Therefore, the characters of diagonal representations coincide with the characters $X_n(\mathbb{Z}) \hookrightarrow X_n(\Gamma_k)$. Recall that $X_n(\mathbb{Z}) \cong \mathbb{C}^{n-1}$.

3.1. The distinguished components for hyperbolic knots

Let $k \subset S^3$ be a hyperbolic knot i.e. $S^3 \smallsetminus k$ has a hyperbolic metric of finite volume. There exists a unique one-dimensional component $X_0 \subset X(\Gamma_k, \mathrm{PSL}(2, \mathbb{C}))$, up to complex conjugation, which contains the character of the holonomy representation (see [38, Theorem 8.44]). Complex conjugation corresponds to changing the orientation of the three manifold, thus there is a unique $\mathrm{PSL}(2, \mathbb{C})$-character of the holonomy of an oriented knot exterior. The holonomy representation lifts to a representation $\rho \colon \Gamma_k \to \mathrm{SL}(2)$ since $H^2(\Gamma_k; \mathbb{Z}/2\mathbb{Z}) = 0$. The lift is not unique since $H^1(\Gamma_k; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. By composing any lift of the holonomy representation with the rational, irreducible, n-dimensional representation $r_n \colon \mathrm{SL}(2) \to \mathrm{SL}(n)$ we obtain an irreducible representation $\rho_n \colon \Gamma_k \to \mathrm{SL}(n)$. It follows from work of P. Menal-Ferrer and J. Porti [47] that $\chi_{\rho_n} \in X_n(\Gamma_k)$ is a scheme smooth point contained in a unique $(n-1)$-dimensional component of $X_n(\Gamma_k)$. We will call such a component a *distinguished* component of $X_n(\Gamma_k)$. For *odd* n, as $r_n \cong \mathrm{Sym}^{n-1} \colon \mathrm{SL}(2) \to \mathrm{SL}(r)$ factors through $\mathrm{PSL}(2)$, there is a unique distinguished component in $X_n(\Gamma)$.

3.2. Examples

The aim of this subsection is to describe the components of the $\mathrm{SL}(3)$-character varieties of the trefoil knot and the figure eight knot, see [33, 35].

3.2.1. Irreducible SL(3)-representations of the trefoil knot group

Let $k \subset S^3$ be the trefoil knot and $\Gamma = \Gamma_{3_1}$. We use the presentation

$$\Gamma \cong \langle x, y \mid x^2 = y^3 \rangle.$$

The center of Γ is the cyclic group generated by $z = x^2 = y^3$. The abelianization map $\varphi \colon \Gamma \to \mathbb{Z}$ satisfies $\varphi(x) = 3$, $\varphi(y) = 2$, and a meridian of the trefoil is given by $m = xy^{-1}$. Let ω denote a primitive third root of unity, $\omega^2 + \omega + 1 = 0$.

For a given representation $\rho \in R_3(\Gamma)$ we put

$$\rho(x) = A \text{ and } \rho(y) = B.$$

If ρ is irreducible then it follows from Schur's Lemma that the matrix $A^2 = B^3 \in \{\mathrm{id}_3, \omega \,\mathrm{id}_3, \omega^2 \,\mathrm{id}_3\}$ is a central element of $\mathrm{SL}(3)$.

LEMMA 3.2. — *If $\rho \colon \Gamma \to \mathrm{SL}(3)$ is irreducible then $A^2 = B^3 = \mathrm{id}_3$.*

Proof. — The matrix A has an eigenvalue of multiplicity two, and hence A has a two-dimensional eigenspace. Therefore, B has only one-dimensional eigenspaces, otherwise ρ would not be irreducible. This implies that B has three different eigenvalues: λ, $\lambda\omega$, $\lambda\omega^2$ where $\lambda^3 \in \{1, \omega, \omega^2\}$. We obtain $\det(B) = 1 = \lambda^3$. Therefore $B^3 = \mathrm{Id}_3$. $\qquad\square$

Lemma 3.2 implies that the matrices A and B are conjugate to

$$A \sim \left(\begin{smallmatrix} 1 & & \\ & -1 & \\ & & -1 \end{smallmatrix}\right) \quad \text{and} \quad B \sim \left(\begin{smallmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{smallmatrix}\right).$$

The corresponding eigenspaces are the plane $E_A(-1)$, and the lines $E_A(1)$, $E_B(1)$, $E_B(\omega)$, and $E_B(\omega^2)$.

Now, these eigenspaces determine the representation completely, as they determine the matrices A and B, that have fixed eigenvalues. Of course we have $E_A(1) \cap E_A(-1) = 0$ and $E_B(1)$, $E_B(\omega)$, and $E_B(\omega^2)$ are also in general position. Since ρ is irreducible, the five eigenspaces are in general position. For instance $E_A(1) \cap (E_B(1) \oplus E_B(\omega)) = 0$, because otherwise $E_B(1) \oplus E_B(\omega) = E_A(1) \oplus (E_A(-1) \cap (E_B(1) \oplus E_B(\omega)))$ would be a proper invariant subspace.

We now give a parametrization of the conjugacy classes of the irreducible representations. The invariant lines correspond to fixed points in the projective plane $\mathbb{P}^2$, and $E_A(-1)$ determines a projective line.

- The first normalization: the line $E_A(-1)$ corresponds to the line at infinity:

$$\mathbb{P}^1 = E_A(-1) = \langle [0:1:0], [0:0:1] \rangle$$

The four invariant lines $E_A(1)$, $E_B(1)$, $E_B(\omega)$, and $E_B(\omega^2)$ are points in the affine plane $\mathbb{C}^2 = \mathbb{P}^2 \setminus \mathbb{P}^1$. They are in general position.

- We fix the three fixed points of B, corresponding to the following affine frame.

$$E_B(1) = [1:0:0], \ E_B(\omega) = [1:1:0], \text{ and } E_B(\omega^2) = [1:0:1].$$

- The fourth point (the line $E_A(1)$) is a point in $\mathbb{C}^2$ which does not lie in the affine lines spanned by any two of the fixed points of B:
  $E_A(1) = [2:s:t]$ where $s \neq 0$, $t \neq 0$, or $s + t \neq 2$

This gives rise to the subvariety $\{\rho_{s,t} \in R(\Gamma, \mathrm{SL}(3)) \mid (s,t) \in \mathbb{C}^2\}$, where

$$\rho_{s,t}(x) = \begin{pmatrix} 1 & 0 & 0 \\ s & -1 & 0 \\ t & 0 & -1 \end{pmatrix} \quad \text{and} \quad \rho_{s,t}(y) = \begin{pmatrix} 1 & \omega - 1 & \omega^2 - 1 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

We obtain the following lemma:

LEMMA 3.3. — *Every irreducible representation* $\rho\colon \Gamma_{3_1} \to \mathrm{SL}(3)$ *is equivalent to exactly one representation* $\rho_{s,t}$. *Moreover,* $\rho_{s,t}$ *is reducible if and only if* (s,t) *is contained in one of the three affine lines given by* $s = 0$, $t = 0$, *and* $s + t = 2$. *If* $(s,t) \in \{(0,0),(0,2),(2,0)\}$ *is the intersection point of two of those lines then* $\rho_{s,t}$ *fixes a complete flag, and has the character of a diagonal representation.*

The following theorem follows from the above considerations (see [35, Theorem 9.10] for more details). We let $R_n^{irr}(\Gamma) \subset R_n(\Gamma)$ denote the Zariski-open subset of irreducible representation

THEOREM 3.4. — *The GIT quotient* $X = \overline{R_3^{irr}(\Gamma)} /\!\!/ \mathrm{SL}(3)$ *of the trefoil knot group* Γ *is isomorphic to* $\mathbb{C}^2$. *Moreover, the Zariski open subset* $R_3^{irr}(\Gamma)$ *is* $\mathrm{SL}(3)$-*invariant and its GIT quotient is isomorphic to the complement of three affine lines in general position in* $\mathbb{C}^2$.

The affine algebraic set $X_3(\Gamma_{3_1})$ has the following components:

- the component containing the characters of abelian representations;
- one component containing the characters of *partial reducible* representations i.e. representations of the form

$$\rho_\lambda = (\alpha \otimes \lambda^\varphi) \oplus \lambda^{-2\varphi}$$

  where $\alpha \in R_2(\Gamma_{4_1})$ is irreducible (compare Equation (5.1) with β trivial);
- one component containing characters of irreducible representations. This component is isomorphic to $\mathbb{C}^2$.

Remark 3.5. — The same arguments as above apply to torus knots $T(p,2)$, p odd, to prove that the variety of irreducible SL$_3$(ℂ)-characters consist of $(p-1)(p-2)/2$ disjoint components isomorphic to $\mathbb{C}^2$, $(p-1)/2$ components of characters of partial reducible representations, and the component of characters of diagonal representations.

The SL(3)-character variety for torus knots was studied in detail by V. Muñoz and J. Porti [50]. In the case $T(p,q)$, $p,q > 2$, there are 4-dimensional components in $X_3(\Gamma_{T(p,q)})$. These 4-dimensional components correspond to the configuration of 6 points in the projective plane.

3.2.2. The SL(3)-character variety of the figure eight knot

The figure eight knot $k = 4_1$ has genus one, and its complement fibres over the circle [11]. Hence the commutator group of Γ_{4_1} is a free group of

rank two, and a presentation is given by

$$(3.1) \qquad \Gamma_{4_1} \cong \langle t, a, b \mid tat^{-1} = ab, \ tbt^{-1} = bab \rangle \cong F_2 \rtimes \mathbb{Z}$$

where $F_2 = F(a, b)$ is a free group of rank two. A peripheral system is given by $(m, \ell) = (t, [a, b])$. The amphicheirality of the figure eight knot implies that its group has an automorphism $h : \Gamma \to \Gamma$ which maps the peripheral system (m, ℓ) to (m^{-1}, ℓ) up to conjugation. Such an automorphism is explicitly given by

$$h(t) = ta^{-1}t^{-1}at^{-1} \sim t^{-1},$$
$$h(a) = a^{-1}tab^{-1}a^{-1}t^{-1}a \sim b^{-1},$$
$$h(b) = a^{-1}tat^{-1}a \sim a.$$

Notice that we obtain

$$h(m) = ta^{-1}m^{-1}t^{-1}a$$
$$h(\ell) = h([a, b]) = a^{-1}ta[b^{-1}, a]a^{-1}t^{-1}a.$$

The relation $t^{-1}a^{-1}t = ba^{-2}$ gives that the peripheral system $(h(m), h(\ell))$ is conjugated to (m^{-1}, ℓ) as desired.

The structure of the SL(3)-character variety of the figure eight knot had been studied in detail in [33], see also [19]. Here we outline the main steps of the article [33].

To compute $X_3^{irr}(\Gamma_{4_1})$ we look at the restriction map induced by the inclusion $F_2 := F(a, b) \to \Gamma_{4_1}$:

$$\mathrm{res} : \ X_3(\Gamma_{4_1}) \to X_3(F_2) \,.$$

For $X_3(F_2)$, we use Lawton's and Will's coordinates (see [45, 62]). There is a two fold branched covering

$$\pi : \ X_3(F_2) \to \mathbb{C}^8,$$

where the coordinates of $\mathbb{C}^8$ are the traces of

$$(3.2) \qquad a, \ a^{-1}, \ b, \ b^{-1}, \ ab, \ b^{-1}a^{-1}, \ ab^{-1}, \ a^{-1}b.$$

The branched covering comes from a ninth coordinate, which is the trace of the commutator $[a, b] = aba^{-1}b^{-1}$. This trace satisfies a polynomial equation

$$x^2 - Px + Q = 0,$$

where P and Q are polynomials on the first eight variables (see [45] for the expression of P and Q). The solutions are precisely the trace of $[a, b]$ and the trace of its inverse.

The computation of $X_3^{irr}(\Gamma_{4_1})$ takes several steps (see [33]): first we compute $\pi(\mathrm{res}(X(\Gamma, G))) \subset \mathbb{C}^8$. From this we describe the image $\mathrm{res}(X(\Gamma, G)) \subset X_3(F_2)$ as a 2:1 ramified covering. Finally, we proved that $X_3^{irr}(\Gamma_{4_1})$ is a 3:1 ramified covering of $\mathrm{res}(X_3(\Gamma_{4_1}))$.

The computation of $\pi(\mathrm{res}(X(\Gamma, G))) \subset \mathbb{C}^8$ is based on a reduction of eight coordinates to four by using conjugation identities:

$$\alpha = \chi(a) = \chi(ab), \qquad \bar{\alpha} = \chi(a^{-1}) = \chi(b^{-1}a^{-1}),$$
$$\beta = \chi(b) = \chi(a^{-1}b), \quad \bar{\beta} = \chi(b^{-1}) = \chi(ab^{-1}).$$

LEMMA 3.6. — *The projection $\pi(\mathrm{res}(X(\Gamma, G)))$ has three components:*

$$U_0 = \{(\alpha, \bar{\alpha}, \beta, \bar{\beta}) \in \mathbb{C}^4 \mid \alpha = \bar{\alpha}, \beta = \bar{\beta}\},$$
$$U_1 = \{(\alpha, \bar{\alpha}, \beta, \bar{\beta}) \in \mathbb{C}^4 \mid \alpha = \bar{\alpha} = 1\},$$
$$U_2 = \{(\alpha, \bar{\alpha}, \beta, \bar{\beta}) \in \mathbb{C}^4 \mid \beta = \bar{\beta} = 1\}.$$

The lemma is proved by elementary trace calculation in Lawton's and Will's coordiantes (see [33, Lemma 5.1]).

To get all the ambient coordinates we need a new variable:

$$\eta(\chi) = \chi([a, b]).$$

We know by [45] that

(3.3) $$\eta^2 - P\eta + Q = 0,$$

for some polynomials $P, Q \in \mathbb{Z}[\alpha, \beta, \bar{\alpha}, \bar{\beta}]$. Using Lemma 3.6 and by replacing the values of P and Q in [45], we obtain:

LEMMA 3.7. — *$W = \mathrm{res}(X(\Gamma, G))$ has three components W_0, W_1 and W_2, each W_i being a two-fold ramified covering of U_i according to (3.3).*

Now, it is proved in [33] that the ramification points of $\mathrm{res} : X_3^{irr}(\Gamma_{4_1}) \to W$ are the characters of the five irreducible metabelian representations $\Gamma_{4_1} \to \mathrm{SL}(3)$. In [8] H. Boden and S. Friedl prove that these representations are smooth points of $X_3(\Gamma_{4_1})$, and so $\overline{X_3^{irr}(\Gamma_{4_1})}$ has the same number of components as W. Summarizing, we have:

PROPOSITION 3.8. — *The set $\overline{X_3^{irr}(\Gamma_{4_1})}$ has three components V_0, V_1 and V_2, that are respective 3:1 branched covers of W_0, W_1 and W_2.*

The branching points in $X_3^{irr}(\Gamma_{4_1})$ are the five metabelian irreducible characters.

The character variety $X_3(\Gamma_{4_1})$ has 5 components:

- the component containing the characters of abelian representations;

- one component containing the characters of the representations $\rho_\lambda = \alpha \otimes \lambda^\varphi \oplus \lambda^{-2\varphi}$ where $\alpha \in R_2(\Gamma_{4_1})$ is irreducible (compare equation (5.1) with β trivial);
- three components V_0, V_1 and V_2 containing characters of irreducible representations. The component V_0 is the distinguished component (see Section 3.1). The two other components which come from a surjection $\Gamma_{4_1} \twoheadrightarrow D(3,3,4)$ onto a triangle group.

Let us describe the components V_1 and V_2 without going too much into the technical details. An epimorphism

$$\phi : \Gamma \twoheadrightarrow D(3,3,4) = \langle k, l \mid l^3, \ k^3, \ (kl)^4 \rangle$$

is given by

$$\phi(a) = k^{-1}l^{-1}kl, \quad \phi(b) = kl \quad \text{and} \quad \phi(t) = klk \,.$$

It satisfies $\phi(b)^4 = 1$ and $\phi(m^3\ell) = 1$. Notice that the surjection ϕ induces an injection

$$\phi^* : \ X_3(D(3,3,4)) \hookrightarrow X_3(\Gamma) \,.$$

Remark 3.9. — The surjection $\phi : \Gamma \twoheadrightarrow D(3,3,4)$ is related to an exceptional Dehn filling on the figure-eight knot K (see [28]). In particular, the Dehn filling manifold $K(\pm 3)$ is a *small* Seifert fibered manifold, and $K(\pm 3)$ fibers over $S^2(3,3,4)$. The orbifold fundamental group $\pi_1^{\mathcal{O}}(S^2(3,3,4))$ is isomorphic to the von Dyck group $\pi_1^{\mathcal{O}}(S^2(3,3,4)) \cong D(3,3,4)$. Hence there is a surjection

$$\Gamma \to \pi_1(K(\pm 3)) \twoheadrightarrow \pi_1(K(\pm 3))/\text{center} \cong D(3,3,4) \,.$$

The center of $\pi_1(K(\pm 3))$ is generated by a regular fibre. Any irreducible representation of $\pi_1(K(\pm 3)) \to \mathrm{SL}(3)$ maps the fibre to the center of $\mathrm{SL}(3)$. By using the description of $X_3(F_2)$ given by Lawton [45]) it quite elementary to determine $X_3(D(3,3,4))$ explicitly. The proof of the next lemma can be found in [33, Lemma 10.1]:

LEMMA 3.10. — *The variety $\overline{X^{irr}(D(3,3,4), \mathrm{SL}(3,\mathbb{C}))}$ has a component $\mathcal{W}$ of dimension 2 and three isolated points. The variety $\mathcal{W}$ is isomorphic to the hypersurface in $\mathbb{C}^3$ given by the equation*

$$\zeta^2 - (\nu\bar\nu - 2)\zeta + \nu^3 + \bar\nu^3 - 5\nu\bar\nu + 5 = 0 \,.$$

Here, the parameters are $\nu = \chi(k^{-1}l)$, $\bar\nu = \chi(kl^{-1})$ and $\zeta = \chi([k,l])$. For every $\chi \in \mathcal{W}$, $\chi(k^{\pm 1}) = \chi(l^{\pm 1}) = 0$ and $\chi((kl)^{\pm 1}) = 1$.

Moreover, all characters in $\mathcal{W}$ are characters of irreducible representations except for the three points $(\nu, \bar\nu, \zeta) = (2,2,1)$, $(2\varpi, 2\varpi^2, 1)$, $(2\varpi^2, 2\varpi, 1)$, $\varpi = e^{2\pi i/3}$.

Now, the components V_1 and V_2 are given by

$$V_1 = \phi^*(\mathcal{W}) \subset X_3(\Gamma_{4_1}) \quad \text{and} \quad V_2 = (\phi \circ h)^*(\mathcal{W}).$$

The components V_1 and V_2 are swapped by $h^*: X_3(\Gamma_{4_1}) \to X_3(\Gamma_{4_1})$, and V_0 is preserved.

Further details in the proof of Lemma 3.10 allow to describe those three isolated points. Composing with ϕ^*, they correspond to the three characters of irreducible metabelian representations in $X_3(\Gamma_{4_1})$ that do not lie in V_2. As already mentioned, there are five characters of irreducible metabelian representations (see [7]), and the corresponding irreducible metabelian representations are scheme smooth (see [8]). The character corresponding to a point of V_0 comes from a surjection $\Gamma_{4_1} \twoheadrightarrow A_4$ composed with the irreducible representation $A_4 \to \mathrm{SL}(3)$.

PROPOSITION 3.11. — *The components V_1 and V_2 are characters of representations which factor through the surjections $\Gamma \twoheadrightarrow \pi_1(K(\pm 3))$ respectively. These components are isomorphic to the hypersurface*

$$\zeta^2 - (\nu\bar{\nu} - 2)\zeta + \nu^3 + \bar{\nu}^3 - 5\nu\bar{\nu} + 5 = 0.$$

Here, the parameters are

$$\nu = \begin{cases} \chi(t) & \text{for } V_2, \\ \chi(t^{-1}) & \text{for } V_1, \end{cases} \quad \bar{\nu} = \begin{cases} \chi(t^{-1}) & \text{for } V_2, \\ \chi(t) & \text{for } V_1, \end{cases} \quad \zeta = \begin{cases} \chi(a) & \text{for } V_2, \\ \chi(b^{-1}) & \text{for } V_1. \end{cases}$$

All characters are irreducible except for the three points$(\nu, \bar{\nu}, \zeta) = (2, 2, 1)$, $(2\varpi, 2\varpi^2, 1)$, $(2\varpi^2, 2\varpi, 1)$, with $\varpi = e^{2\pi i/3}$, that correspond to the intersection $V_1 \cap V_2 = V_0 \cap V_1 \cap V_2$.

4. Deformations of representations

One way to prove that a certain representation $\rho \in R_n(\Gamma)$ is a smooth point of the representation variety is to show that every cocycle $u \in Z^1(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ is integrable (see Lemma 2.7). In order to do this, we use the classical approach, i.e. we first solve the corresponding formal problem, and then apply a theorem of Artin [1].

The formal deformations of a representation $\rho: \Gamma \to \mathrm{SL}_n(\mathbb{C})$ are in general determined by an infinite sequence of obstructions (see [26, 2, 36]). The following result streamlines the arguments given in [34] and [5]. It is a slight generalization of Proposition 3.3 in [32]. For a proof see [30].

Proposition 4.1. — *Let M be a connected, compact, orientable 3-manifold with toroidal boundary $\partial M = T_1 \cup \cdots \cup T_k$, and let $\rho: \pi_1 M \to \mathrm{SL}(n)$ be a representation.*

If $\dim H^1(\pi_1 M; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho}) = k(n-1)$ then ρ is a smooth point of the $\mathrm{SL}(n)$-representation variety $R_n(\pi_1 M)$. Moreover, ρ is contained in a unique component of dimension $n^2 - 1 + k(n-1) - \dim H^0(\pi_1 M; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$.

Definition 4.2. — *Let M, $\partial M = T_1 \cup \cdots \cup T_k$, be a connected, compact, and orientable 3-manifold with toroidal boundary. A representation $\rho: \pi_1 M \to \mathrm{SL}_n(\mathbb{C})$ is called* infinitesimally regular *if $\dim H^1(\pi_1 M; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho}) = k(n-1)$.*

Remark 4.3. — It follows from Proposition 4.1 that infinitesimally regular representations are regular points on the representation variety.

Example 4.4. — Let Γ_k be a knot group and let $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathrm{SL}(n)$ be a diagonal matrix. We define the diagonal representation ρ_D by $\rho_D(\gamma) = D^{\varphi(\gamma)}$. Now, ρ_D is the direct sum of the one-dimensional representations λ_i^φ, and the Γ_k-module $\mathfrak{sl}(n)_{\mathrm{Ad}\,\rho_D}$ decomposes as:

$$\mathfrak{sl}(n)_{\mathrm{Ad}\,\rho_A} = \bigoplus_{i \neq j} \mathbb{C}_{\lambda_i/\lambda_j} \oplus \mathbb{C}^{n-1}.$$

Now, for all $\alpha \in \mathbb{C}^*$ we have $H^1(\Gamma_k; \mathbb{C}_\alpha) = 0$ if and only if $\alpha \neq 1$ and $\Delta_k(\alpha) \neq 0$ (see [4, Lemma 2.3]). Here, $\Delta_k(t)$ denotes the Alexander polynomial of the knot k. Hence, ρ_D is infinitesimally regular if and only if $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\Delta_k(\lambda_i/\lambda_j) \neq 0$ for $1 \leqslant i, j \leqslant n$. In this case it follows that $\dim H^1(\Gamma_k; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho_D}) = n - 1$, and $\rho_D \in R_n(\Gamma_k)$ is a regular point. The representation ρ_D is contained in an unique component of dimension $n^2 - 1$. This component is exactly the component of abelian representations $\varphi^*: R_n(\mathbb{Z}) \hookrightarrow R_n(\Gamma_k)$.

5. Existence of irreducible representations of knot groups

Let $k \subset S^3$ be a knot, and let Γ_k be the knot group. Given representations of Γ_k into $\mathrm{SL}(2)$ there are several constructions which give higher dimensional representations. The most obvious is probably the direct sum of two representatios.

5.1. Deformations of the direct sum of two representations

Starting from two representations $\alpha: \Gamma_k \to \mathrm{SL}_a(\mathbb{C})$ and $\beta: \Gamma_k \to \mathrm{SL}_b(\mathbb{C})$ such that $a + b = n$, we obtain a family of representations $\rho_\lambda \in$

$R_n(\Gamma_k)$, $\lambda \in \mathbb{C}^*$, by $\rho_\lambda = (\lambda^{b\varphi} \otimes \alpha) \oplus (\lambda^{-a\varphi} \otimes \beta) \in R_n(\Gamma_k)$ i.e. for all $\gamma \in \Gamma_k$

$$(5.1) \qquad \rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{b\varphi(\gamma)}\alpha(\gamma) & \mathbf{0} \\ \mathbf{0} & \lambda^{-a\varphi(\gamma)}\beta(\gamma) \end{pmatrix} .$$

Recall that $\lambda^\varphi : \Gamma_k \to \mathbb{C}^*$ is given by $\gamma \mapsto \lambda^{\varphi(\gamma)}$.

Throughout this section we will assume that α and β are both irreducible and infinitesimal regular.

The natural question which arises is if ρ_λ can be deformed to irreducible representations, and if this would be possible what could we say about the local structure of $X_n(\Gamma_k)$ at χ_{ρ_λ}?

5.1.1. The easiest case

A very special case is $\alpha = \beta : \Gamma_k \to \mathrm{SL}_1(\mathbb{C}) = \{1\}$ are trivial. Then $\rho_\lambda = \lambda^\varphi \oplus \lambda^{-\varphi} \in R_2(\Gamma_k)$ i.e. for all $\gamma \in \Gamma_k$

$$(5.2) \qquad \rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix} .$$

The following result goes back to work of E. Klassen [42].

THEOREM 5.1. — *If the diagonal representation $\rho_\lambda \in R_2(\Gamma_k)$ can be deformed to irreducible representations then $\Delta_k(\lambda^2) = 0$.*

Proof. — The function $R_n(\Gamma) \to \mathbb{Z}$ given by $\rho \mapsto \dim Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ is upper-semi continuous. This means that for every $m \in \mathbb{Z}$ the set $\{\rho \in R_n(\Gamma) \mid \dim Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho}) \geqslant m\}$ is closed. Notice that $Z^1(\Gamma, \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho})$ is the kernel of a linear map which depends algebraically on ρ.

Moreover, if the representation $\rho_\lambda \in R_2(\Gamma_k)$ can be deformed into irreducible representations then $\dim Z^1(\Gamma_k, \mathfrak{sl}(2)_{\mathrm{Ad}\,\rho_\lambda}) \geqslant 4$ (see [35, Lemma 5.1]). The Γ_k-module $\mathfrak{sl}(2)_{\mathrm{Ad}\,\rho_\lambda} \cong \mathbb{C} \oplus \mathbb{C}_{\lambda^2} \oplus \mathbb{C}_{\lambda^{-2}}$ decomposes into one-dimensional modules (see Example 4.4). Now, $H^1(\Gamma_k, \mathbb{C}) \cong \mathbb{C}$ and for $\lambda^2 \neq 1$ we have $B^1(\Gamma_k, \mathbb{C}_{\lambda^2}) \cong \mathbb{C}$. Hence, $\dim Z^1(\Gamma_k, \mathfrak{sl}(2)_{\mathrm{Ad}\,\rho_\lambda}) \geqslant 4$ implies that $H^1(\Gamma_k, \mathbb{C}_{\lambda^{\pm 2}}) \neq 0$ or $H^1(\Gamma_k, \mathbb{C}_{\lambda^{-2}}) \neq 0$.

Finally, $H^1(\Gamma_k, \mathbb{C}_{\lambda^{\pm 2}}) \neq 0$ and $\lambda^{\pm 2} \neq 1$ implies that $\Delta_k(\lambda^{\pm 2}) = 0$ (see Example 4.4). $\qquad \square$

Remark 5.2. — Notice that $\Delta_k(t) \doteq \Delta_k(t^{-1})$ is symmetric and hence $H^1(\Gamma_k, \mathbb{C}_{\lambda^{-2}}) \neq 0$ if and only if $H^1(\Gamma_k, \mathbb{C}_{\lambda^2}) \neq 0$. Here $p \doteq q$ means that $p, q \in \mathbb{C}[t^{\pm 1}]$ are *associated elements*, i.e. there exists some unit $c\,t^k \in \mathbb{C}[t^{\pm 1}]$, with $c \in \mathbb{C}^*$ and $k \in \mathbb{Z}$, such that $p = c\,t^k\,q$.

In general, it is still a conjecture that the necessary condition in Theorem 5.1 is also sufficient i.e. infinitesimal deformation implies deformation. Nevertheless, we have the following result [36]:

THEOREM 5.3. — *Let $k \subset S^3$ be a knot and let $\lambda \in \mathbb{C}^*$. If λ^2 is a simple root of $\Delta_k(t)$ then ρ_λ is the limit of irreducible representation.*

More precisely, the character χ_λ of ρ_λ is contained in exactly two components. One component $Y_2 \cong \mathbb{C}$ only contains characters of abelian (diagonal representations), and the second component X_λ contains characters of irreducible representations. Moreover, we have Y_2 and X_λ intersect transversally at χ_ρ, and χ_λ is a smooth point on Y_2 and X_λ.

Remark 5.4. — Related results, also for other Lie groups are: Shors [54], Frohman–Klassen [21], Herald [29], Heusener–Kroll [31], Ben Abdelghani [2, 3], Heusener–Porti [34].

5.1.2. The general case.

Let us go back to the representation $\rho_\lambda = (\lambda^{b\varphi} \otimes \alpha) \oplus (\lambda^{-a\varphi} \otimes \beta) \in R_n(\Gamma_k)$ given by Equation (5.1):

$$\rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{b\varphi(\gamma)}\alpha(\gamma) & \mathbf{0} \\ \mathbf{0} & \lambda^{-a\varphi(\gamma)}\beta(\gamma) \end{pmatrix} .$$

The following generalization of Theorem 5.1 was proved in [35]:

THEOREM 5.5. — *Let $\alpha : \Gamma_k \to \mathrm{SL}_a(\mathbb{C})$ and $\beta : \Gamma_k \to \mathrm{SL}_b(\mathbb{C})$ be irreducible, $a + b = n$, and assume that α and β are infinitesimal regular. If $\rho_\lambda \in R_n(\Gamma_k)$ is a limit of irreducible representations then $\Delta_1^{\alpha \otimes \beta^*}(\lambda^n) = \Delta_1^{\beta \otimes \alpha^*}(\lambda^{-n}) = 0$.*

The main steps in the proof of Theorem 5.5 agree with the main steps in the proof of Theorem 5.1. Here we will only present the setup. First, let us recall some facts about the *twisted Alexander polynomial*. For more details see [35, 40, 41, 59, 60]. Let V be a complex vector space, and $\rho : \Gamma_k \to \mathrm{GL}(V)$ a representation. We let $C_\infty \to C$ denote the infinite cyclic covering of the knot exterior. The *twisted Alexander module* is the $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[t^{\pm 1}]$-module $H_i(C_\infty, V)$. A generator $\Delta_i^\rho(t)$ of its order ideal is called the *twisted Alexander polynomial* $\Delta_i^\rho(t) \in \mathbb{C}[t^{\pm 1}]$. Notice that $H_i(C_\infty, V) \cong H_i(C(k), V[\mathbb{Z}]) \cong H_i(\Gamma_k, V[\mathbb{Z}])$ where $V[\mathbb{Z}] = V \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[\mathbb{Z}]$ is a Γ_k module via $\rho \otimes t^\varphi$.

The dual representation $\rho^* : \Gamma \to \mathrm{GL}(V^*)$ is given by $\rho^*(\gamma)(f) = f \circ \rho(\gamma)^{-1}$ for $f \in V^* = \mathrm{Hom}(V, \mathbb{C})$ and $\gamma \in \Gamma$. In particular, if $\rho : \Gamma \to \mathrm{GL}(n)$ then $\rho^*(\gamma) = {}^t\rho(\gamma)^{-1}$ for all $\gamma \in \Gamma_k$.

LEMMA 5.6. — *The representations ρ and ρ^* are equivalent if and only if there exists a Γ-invariant, non-degenerated bilinear form $V \otimes V \to \mathbb{C}$.*

EXAMPLE 5.7. — If $\rho : \Gamma \to O(n)$ or $\rho : \Gamma \to SL_2(\mathbb{C})$ then ρ and ρ^* are equivalent. For $\lambda \in \mathbb{C}^*$ the dual of the one-dimensional representation λ^φ is $(\lambda^\varphi)^* = \lambda^{-\varphi}$.

The following theorem is proved in [35]:

THEOREM 5.8. — *If $\rho : \Gamma_k \to GL(V)$ is a semisimple representation then $\Delta_i^{\rho^*}(t) \doteq \Delta_i^\rho(t^{-1})$.*

First, we have to understand the Γ_k-module $\mathfrak{sl}(n)_{\mathrm{Ad}\,\rho_\lambda}$. Let $M_{a,b}(\mathbb{C})$ the vector space of $a \times b$ matrices over the complex numbers. The group Γ_k acts on $M_{a,b}(\mathbb{C})$ via $\alpha \otimes \beta^*$ i.e. for all $\gamma \in \Gamma_k$ and $X \in M_{a,b}(\mathbb{C})$ we have

$$\alpha \otimes \beta^*(\gamma)(X) = \alpha(\gamma)X\beta(\gamma^{-1}).$$

Similarly, we obtain a representation $\beta \otimes \alpha^* : \Gamma_k \to M_{b,a}(\mathbb{C})$. The proof of the following lemma is given in [35]:

LEMMA 5.9. — *If $\alpha : \Gamma_k \to SL_a(\mathbb{C})$ and $\beta : \Gamma_k \to SL_b(\mathbb{C})$ are irreducible then the representation $\alpha^* : \Gamma_k \to SL_a(\mathbb{C})$ is also irreducible. Moreover, $\alpha \otimes \beta$ and $\beta \otimes \alpha^*$ are semisimple.*

In what follows we let $\mathcal{M}_t^+$ and $\mathcal{M}_t^-$ denote the Γ_k-modules

$$\mathcal{M}_t^+ = M_{a,b}(\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}] \quad \text{and} \quad \mathcal{M}_t^- = M_{b,a}(\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}]$$

where Γ_k acts via $\alpha \otimes \beta^* \oplus t^\varphi$ and $\beta \otimes \alpha^* \otimes t^\varphi$ repectively.

COROLLARY 5.10. — *If $\alpha : \Gamma_k \to SL_a(\mathbb{C})$ and $\beta : \Gamma_k \to SL_b(\mathbb{C})$ are irreducible then*

$$\Delta_i^{\alpha \otimes \beta^*}(t) \doteq \Delta_i^{\beta \otimes \alpha^*}(t^{-1}).$$

Now, we are in the position to prove Theorem 5.13 by following the main steps of the proof of Theorem 5.1. This is carried out in [35] (see also [30]).

Analogously to Theorem 5.3 there is also a partial converse to Theorem 5.5 which was proved in [35] too.

THEOREM 5.11. — *Let $\alpha : \Gamma_k \to SL_a(\mathbb{C})$ and $\beta : \Gamma_k \to SL_b(\mathbb{C})$ be irreducible, $a + b = n$, and assume that α and β are infinitesimal regular.*

Assume that $\Delta_0^{\alpha \otimes \beta^*}(\lambda^n) \neq 0$ and that λ^n is a simple root of $\Delta_1^{\alpha \otimes \beta^*}(t)$. Then $\rho_\lambda \in R_n(\Gamma_k)$ can be deformed to irreducible representations. Moreover, the character $\chi_\lambda \in X_n(\Gamma_k)$ belongs to precisely two irreducible components Y and Z of $X_n(\Gamma)$. Both components Y and Z have dimension $n - 1$ and meet transversally at χ_λ along a subvariety of dimension $n - 2$.

The component Y contains characters of irreducible representations and Z consists only of characters of reducible ones.

To prove this theorem one make use of Luna's Slice Theorem. Also on has to study the quadratic cone of the representation ρ_λ by identifying the second obstruction to integrability. This relies heavily on the hypothesis about the simple root of the Alexander polynomial.

5.2. Deformation of reducible metabelian representations

In this subsection we will consider certain reducible metabelian representations and their deformations. Irreducible metabelian representations and their deformations had been studied by H. Boden and S. Friedl in a series of papers (in particular see [7, 8]).

Here, the general assumption will be that $\alpha \in \mathbb{C}^*$ is a zero of the Alexander polynomial of k, and hence $H_1(C_\infty; \mathbb{C})$ has a direct summand of the form $\mathbb{C}[t^{\pm 1}]/(t-\alpha)^{n-1}$, $n \in \mathbb{Z}$, $n > 1$.

Recall that a knot group Γ is isomorphic to the semi-direct product $\Gamma \cong \Gamma' \rtimes \mathbb{Z}$. Every metabelian representation of Γ factors through the metabelian group $\Gamma/\Gamma'' \cong (\Gamma'/\Gamma'') \rtimes \mathbb{Z}$. Notice that $H_1(C_\infty; \mathbb{C}) \cong \mathbb{C} \otimes \Gamma'/\Gamma''$. Hence we obtain a homomorphism

$$\Gamma \to (\Gamma'/\Gamma'') \rtimes \mathbb{Z} \to (\mathbb{C} \otimes \Gamma'/\Gamma'') \rtimes \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]/(t-\alpha)^{n-1} \rtimes \mathbb{Z} \,.$$

The multiplication on $\mathbb{C}[t^{\pm 1}]/(t-\alpha)^{n-1} \rtimes \mathbb{Z}$ is given by $(p_1, n_1)(p_2, n_2) = (p_1 + t^{n_1} p_2, n_1 + n_2)$.

Let $I_n \in \mathrm{SL}(n)$ and $N_n \in \mathrm{GL}(n)$ denote the identity matrix and the upper triangular Jordan normal form of a nilpotent matrix of degree n respectively. For later use we note the following lemma which follows easily from the Jordan normal form theorem:

LEMMA 5.12. — *Let $\alpha \in \mathbb{C}^*$ be a nonzero complex number and let $\mathbb{C}^n$ be the $\mathbb{C}[t^{\pm 1}]$-module with the action of t^k given by*

$$(5.3) \qquad\qquad\qquad t^k \mathbf{a} = \alpha^k \mathbf{a} J_n^k$$

where $\mathbf{a} \in \mathbb{C}^n$ and $J_n = I_n + N_n$. Then $\mathbb{C}^n \cong \mathbb{C}[t^{\pm 1}]/(t-\alpha)^n$ as $\mathbb{C}[t^{\pm 1}]$-modules.

There is a direct method to construct a reducible metabelian representation of the group $\mathbb{C}[t^{\pm 1}]/(t-\alpha)^{n-1} \rtimes \mathbb{Z}$ into $\mathrm{GL}(n)$ (see [7, Proposition 3.13]). A direct calculation gives that

$$(\mathbf{a}, 0) \mapsto \begin{pmatrix} 1 & \mathbf{a} \\ \mathbf{0} & I_{n-1} \end{pmatrix}, \quad (0, 1) \mapsto \begin{pmatrix} \alpha & \mathbf{0} \\ \mathbf{0} & J_{n-1}^{-1} \end{pmatrix}$$

defines a faithful representation $\mathbb{C}[t^{\pm 1}]/(t-\alpha)^{n-1} \rtimes \mathbb{Z} \to \mathrm{GL}(n)$.

Therefore, we obtain a reducible, metabelian, non-abelian representation $\tilde{\varrho} : \Gamma \to \mathrm{GL}(n)$ if the Alexander module $H_1(C_\infty, \mathbb{C})$ has a direct summand of the form $\mathbb{C}[t^{\pm 1}]/(t-\alpha)^s$ with $s \geqslant n-1 \geqslant 1$:

$$\tilde{\varrho} : \Gamma \to \mathbb{C}[t^{\pm 1}]/(t-\alpha)^s \rtimes \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]/(t-\alpha)^{n-1} \rtimes \mathbb{Z} \to \mathrm{GL}(n)$$

given by

$$(5.4) \qquad \tilde{\varrho}(\gamma) = \begin{pmatrix} 1 & \tilde{\mathbf{z}}(\gamma) \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} \alpha^{\varphi(\gamma)} & 0 \\ 0 & J_{n-1}^{-\varphi(\gamma)} \end{pmatrix}.$$

It is easy to see that a map $\tilde{\varrho} : \Gamma \to \mathrm{GL}(n)$ given by (5.4) is a homomorphism if and only if $\tilde{\mathbf{z}} : \Gamma \to \mathbb{C}^{n-1}$ is a cocycle i.e. for all $\gamma_1, \gamma_2 \in \Gamma$ we have

$$(5.5) \qquad \tilde{\mathbf{z}}(\gamma_1 \gamma_2) = \tilde{\mathbf{z}}(\gamma_1) + \alpha^{\varphi(\gamma_1)} \tilde{\mathbf{z}}(\gamma_2) J_{n-1}^{\varphi(\gamma_1)}.$$

The unipotent matrices J_n and J_n^{-1} are similar: a direct calculation shows that $P_n J_n P_n^{-1} = J_n^{-1}$ where $P_n = (p_{ij})$, $p_{ij} = (-1)^j \binom{j}{i}$. The matrix P_n is upper triangular with ± 1 in the diagonal and P_n^2 is the identity matrix, and therefore $P_n = P_n^{-1}$.

Hence $\tilde{\varrho}$ is conjugate to a representation $\varrho : \Gamma \to \mathrm{GL}(n)$ given by

$$(5.6) \quad \varrho(\gamma) = \begin{pmatrix} \alpha^{h(\gamma)} & z(\gamma) \\ 0 & J_{n-1}^{h(\gamma)} \end{pmatrix} = \begin{pmatrix} \alpha^{h(\gamma)} & z_1(\gamma) & z_2(\gamma) & \dots & z_{n-1}(\gamma) \\ 0 & 1 & h_1(\gamma) & \dots & h_{n-2}(\gamma) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & h_1(\gamma) \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

where $\mathbf{z} = (z_1, \dots, z_{n-1}) : \Gamma \to \mathbb{C}^{n-1}$ satisfies

$$\mathbf{z}(\gamma_1 \gamma_2) = \alpha^{h(\gamma_1)} \mathbf{z}(\gamma_2) + \mathbf{z}(\gamma_1) J_{n-1}^{h(\gamma_2)}.$$

It follows directly that $\mathbf{z}(\gamma) = \tilde{\mathbf{z}}(\gamma) P_{n-1} J_{n-1}^{h(\gamma)}$ and in particular $z_1 = -\tilde{z}_1$.

We choose an n-th root λ of $\alpha = \lambda^n$ and we define a reducible metabelian representation $\varrho_\lambda : \Gamma \to \mathrm{SL}(n)$ by

$$(5.7) \qquad \varrho_\lambda(\gamma) = \lambda^{-\varphi(\gamma)} \varrho(\gamma).$$

The following theorem generalizes the results of [5] where the case $n = 3$ was investigated. It also applies in the case $n = 2$ which was studied in [2] and [36, Theorem 1.1].

THEOREM 5.13. — *Let k be a knot in the 3-sphere S^3. If the $(t-\alpha)$-torsion τ_α of the Alexander module $H_1(C; \mathbb{C}[t^{\pm 1}])$ is cyclic of the form*

$\mathbb{C}[t^{\pm 1}]/(t-\alpha)^{n-1}$, $n \geqslant 2$, then for each $\lambda \in \mathbb{C}^*$ such that $\lambda^n = \alpha$ there exists a reducible metabelian representation $\varrho_\lambda : \Gamma_k \to \mathrm{SL}(n)$. Moreover, the representation ϱ_λ is a smooth point of the representation variety $R_n(\Gamma)$. It is contained in a unique $(n^2 + n - 2)$-dimensional component R_{ϱ_λ} of $R_n(\Gamma)$ which contains irreducible non-metabelian representations which deform ϱ_λ.

The main part of the proof of this theorem is a cohomological calculation [4]: for the representation $\varrho_\lambda : \Gamma \to \mathrm{SL}(n)$ we have $H^0(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad} \circ \varrho_\lambda}) = 0$ and
$$\dim H^1(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad} \circ \varrho_\lambda}) = \dim H^2(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad} \circ \varrho_\lambda}) = n - 1\,.$$
Then we apply Proposition 4.1.

Remark 5.14. — Let $\rho_\lambda : \Gamma \to \mathrm{SL}(n)$ be the diagonal representation given by $\rho_\lambda(m) = \mathrm{diag}(\lambda^{n-1}, \lambda^{-1}, \ldots, \lambda^{-1}) \in \mathrm{SL}(n)$ where m is a meridian of k. The orbit $\mathcal{O}(\rho_\lambda)$ of ρ_λ under the action of conjugation of $\mathrm{SL}(n)$ is contained in the closure $\overline{\mathcal{O}(\varrho_\lambda)}$. Hence ϱ_λ and ρ_λ project to the same point χ_λ of the variety of characters $X_n(\Gamma_k) = R_n(\Gamma_k) /\!/ \mathrm{SL}(n)$.

It would be natural to study the local picture of the variety of characters $X_n(\Gamma_k) = R_n(\Gamma_k) /\!/ \mathrm{SL}(n)$ at χ_λ as done in [34, §8]. Unfortunately, there are much more technical difficulties since in this case the quadratic cone $Q(\rho_\lambda)$ coincides with the Zariski tangent space $Z^1(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad}\,\rho_\lambda})$. Therefore the third obstruction has to be considered.

5.3. The irreducible representation $r_n : \mathrm{SL}(2) \to \mathrm{SL}(n)$

It is interesting to study the behavior of representations $\rho \in R_2(\Gamma)$ under the composition with the n-dimensional, irreducible, rational representation $r_n : \mathrm{SL}(2) \to \mathrm{SL}(n)$. The representation r_n is equivalent to $(n-1)$-fold symmetric power Sym^{n-1} of the standard representation (see [22, 56] and [32] for more details). In particular, r_1 is trivial, r_2 is equivalent to the standard representation, and r_3 is equivalent to $\mathrm{Ad} : \mathrm{SL}(2) \to \mathrm{O}(\mathfrak{sl}(2)) \subset \mathrm{SL}(3)$. If k is odd then r_k is not injective since it factors trough the projection $\mathrm{SL}(2) \to \mathrm{PSL}(2)$. W. Müller [49] studied the case where $\rho : \pi_1(M) \to \mathrm{SL}(2)$ is the lift of the holonomy representation of a compact hyperbolic manifold. This study was extended by P. Menal-Ferrer and J. Porti [47, 48] to the case of non-compact finite volume hyperbolic manifolds. (For more details see Section 3.1.)

In [32] the authors studied the case related to Theorem 5.3. Let Γ_k be a knot group. We define $\rho_{n,\lambda} : \Gamma_k \to \mathrm{SL}(n)$ by $\rho_{n,\lambda} := r_n \circ \rho_\lambda$ where ρ_λ is given by Equation (5.2).

PROPOSITION 5.15. — *Let $k \subset S^3$ be a knot, and assume that $\rho_0 : \Gamma_k \to$ SL(2) is irreducible. Then $R_n(\Gamma_k)$ contains irreducible representations.*

Proof. — It was proved by Thurston that there is at least a 4-dimensional irreducible component $R_0 \subset R_2(\Gamma_k)$ which contains the irreducible representation ρ_0 (see [12, 3.2.1]).

Let Γ be a discrete group and let $\rho : \Gamma \to$ SL(2) be an irreducible representation. By virtue of Burnside's Theorem on matrix algebras, being irreducible is an open property for representations in $R_n(\Gamma)$. If the image $\rho(\Gamma) \subset$ SL(2) is Zariski-dense then the representation $\rho_n := r_n \circ \rho \in R_n(\Gamma)$ is irreducible. In order to prove the proposition we will show that there is a neighborhood $U = U(\rho_0) \subset R_0 \subset R_2(\Gamma_k)$ such that $\rho(\Gamma) \subset$ SL(2) is Zariski-dense for each irreducible $\rho \in U$.

Let now $\rho : \Gamma_k \to$ SL(2) be any irreducible representation and let $G \subset$ SL(2) denote the Zariski-closure of $\rho(\Gamma_k)$. Suppose that $G \neq$ SL(2). Since ρ is irreducible it follows that G is, up to conjugation, not a subgroup of upper-triangular matrices of SL(2). Then by [43, Sec. 1.4] and [37, Theorem 4.12] there are, up to conjugation, only two cases left:

- G is a subgroup of the infinite dihedral group

$$D_\infty = \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix} \right) \mid \alpha \in \mathbb{C}^* \right\} \cup \left\{ \left(\begin{smallmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{smallmatrix} \right) \mid \alpha \in \mathbb{C}^* \right\}.$$

- G is one of the groups $A_4^{\mathrm{SL}(2)}$ (the tetrahedral group), $S_4^{\mathrm{SL}(2)}$ (the octahedral group) or $A_5^{\mathrm{SL}(2)}$ (the icosahedral group). These groups are the preimages in SL(2) of the subgroups $A_4, S_4, A_5 \subset$ PSL(2, ℂ).

By a result of E. Klassen [42, Theorem 10] there are up to conjugation only finitely many irreducible representations of a knot group into D_∞. Moreover, the orbit of each of those irreducible representation is 3-dimensional. Therefore, there exists a Zariski-open subset $U \subset R_0$ which does not contain representations of Γ_k into D_∞.

For the second case there are up to conjugation only finitely many irreducible representations of Γ_k onto the subgroups $A_4^{\mathrm{SL}(2)}$, $S_4^{\mathrm{SL}(2)}$ and $A_5^{\mathrm{SL}(2)}$. As in the dihedral case these finitely many orbits are closed and 3-dimensional. Hence all the irreducible $\rho \in R_0$ such that $r_n \circ \rho$ is reducible are contained in a Zariski-closed subset of R_0. Hence generically $\rho_n = r_n \circ \rho$ is irreducible for $\rho \in R_0$. □

Remark 5.16. — Recall that a finite group has only finitely many irreducible representations (see [22, 53]). Hence, the restriction of r_n to the groups $A_4^{\mathrm{SL}(2)}$, $S_4^{\mathrm{SL}(2)}$ and $A_5^{\mathrm{SL}(2)}$ is reducible, for all but finitely many $n \in \mathbb{N}$.

For non-trivial knots there exist always irreducible representations of the knot group to $\mathrm{SU}(2) \subset \mathrm{SL}(2)$. This is a deep result of P. Kronheimer and T. Mrowka [44]. Therefore, we obtain the following:

COROLLARY 5.17. — *Let $k \subset S^3$ be a non-trivial knot. Then $R_n(\Gamma_k)$ contains irreducible representations.*

Let $k \subset S^3$ be a knot, and let $\lambda^2 \in \mathbb{C}$ a simple root of $\Delta_k(t)$. We let $R_\lambda \subset R_n(\Gamma_k)$ denote the 4-dimensional component which maps onto the component $X_\lambda \subset X_2(\Gamma_k)$ under $t : R_n(\Gamma) \to X_n(\Gamma)$ (see Theorem 5.3). We obtain:

COROLLARY 5.18. — *Let $k \subset S^3$ be a knot, and $\lambda^2 \in \mathbb{C}$ a simple root of $\Delta_k(t)$. Then the diagonal representation $\rho_{\lambda,n} = r_n \circ \rho_\lambda : \Gamma_k \to \mathrm{SL}(n)$ is the limit of irreducible representations in $R_n(\Gamma_k)$. More precisely, generically a representation $\rho_n = r_n \circ \rho$, $\rho \in R_\lambda$, is irreducible.*

Corollary 5.18 can be made more precise (see [32]):

THEOREM 5.19. — *If λ^2 is a simple root of $\Delta_k(t)$ and if $\Delta_k(\lambda^{2i}) \neq 0$ for all $2 \leqslant i \leqslant n-1$ then the reducible diagonal representation $\rho_{\lambda,n} = r_n \circ \rho_\lambda$ is a limit of irreducible representations. More precisely, there is a unique $(n+2)(n-1)$-dimensional component $R_{\lambda,n} \subset R_n(\Gamma_k)$ which contains $\rho_{\lambda,n}$ and irreducible representations.*

Remark 5.20. — Under the assumptions of Corollary 5.18 it is possible to study the tangent cone of $R_n(\Gamma_k)$ at $\rho_{\lambda,n}$, and thereby to determine the local structure of $R_n(\Gamma)$. There are 2^{n-1} branches of various dimensions of $R_n(\Gamma_k)$ passing through ρ_λ. Nevertheless, only the component $R_{\lambda,n}$ contains irreducible representations. This will be studied in a forthcoming paper.

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