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# INTRODUCTION TO MEAN CURVATURE FLOW

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ABSTRACT. — This is a short overview on the most classical results on mean curvature flow as a flow of smooth hypersurfaces. First of all we define the mean curvature flow as a quasilinear parabolic equation and give some easy examples of evolution. Then we consider the M.C.F. on convex surfaces and sketch the proof of the convergence to a round point. Some interesting results on the M.C.F. for entire graphs are also mentioned. In particular when we consider the case of dimension one, we can compute the equation for the translating graph solution to the curve shortening flow and solve it directly.

## 1. Notation and definitions

We consider an $n-$dimensional smooth, orientable manifold $M$ and a smooth immersion in Euclidean space $\mathbf{F} : M \to \mathbb{R}^{n+1}$. Given local coordinates

$$\begin{aligned} \varphi : \qquad & \mathbb{R}^n \to M \\ & (x_1, \dots, x_n) \mapsto \varphi(x_1, \dots, x_n) \end{aligned}$$

we denote by $g$ the metric on $\mathbf{F}(M)$ induced by the standard scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^{n+1}$

$$\forall \mathbf{p} = \varphi(\mathbf{x}) \quad g_{ij}(\mathbf{p}) = \left\langle \frac{\partial \mathbf{F}}{\partial x_i}(\mathbf{p}), \frac{\partial \mathbf{F}}{\partial x_j}(\mathbf{p}) \right\rangle .$$

The elements of the inverse matrix $g^{ij} = \{g_{ij}\}^{-1}$ are also used to arise indices in the Einstein summation convention.

Since $\mathbf{F}(M)$ is orientable, there exists an outer normal vector field $\nu$ on $M$ and we can define the second fundamental form

$$h_{ij}(\mathbf{p}) = \left\langle \frac{\partial \mathbf{F}}{\partial x_i}(\mathbf{p}), \frac{\partial \nu}{\partial x_j}(\mathbf{p}) \right\rangle = -\left\langle \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}(\mathbf{p}), \nu(\mathbf{p}) \right\rangle ,$$

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then the elements of the Weingarten map are $h_i^j = h_{ik}g^{kj}$. The eigenvalues of the Weingarten operator are called principal curvatures and denoted by $\lambda_1, \dots, \lambda_n$. Finally we define the mean curvature as $H = \sum_{i=1}^n \lambda_i$ and the norm of the second fundamental form $|A|^2 = \sum_{i=1}^n \lambda_i^2$.

Actually we are going to consider just two kinds of manifold: closed manifolds (compact and without boundary) and entire graphs (defined on all $\mathbb{R}^n$). In the first case we choose the outward orientation of the normal vector field $\nu$ in such a way that convex surfaces have positive mean curvature, and, to be consistent to this notation, the normal $\nu$ points below in the case of graphs.

Now we are able to define the mean curvature flow: it is the 1-parameter family of immersions satisfying the following initial value problem

$$\text{(M.C.F.)} \qquad \begin{cases} \frac{\partial \mathbf{F}}{\partial t}(\mathbf{p},t) = -H(\mathbf{p},t)\,\nu(\mathbf{p},t) & \text{for } t > 0 \\ \mathbf{F}(\mathbf{p},0) = \mathbf{F}_0(\mathbf{p}) \end{cases}$$

where the initial datum $\mathbf{F}_0$ is a given smooth immersion.

## 2. Examples

A) Minimal surfaces, having $H \equiv 0$, are the stationary solutions for this problem.

B) The first nontrivial example of mean curvature flow is given by the evolution of spheres. Let $\mathbf{F}_0 = \partial B_{r_0}^{n+1}$ be the boundary of a ball of radius $r_0$, by symmetry the evolving surfaces are all spheres of radius $r(t)$. In this case, since the principal curvatures are $\lambda_i(t) = \frac{1}{r}$ for any $i = 1, \dots, n$, and the mean curvature is $H = \frac{n}{r(t)}$ everywhere, the M.C.F. gives an ordinary differential equation for $r(t)$

$$\frac{d}{dt} r(t) = -\frac{n}{r(t)}.$$

The solution $r(t) = \sqrt{r_0^2 - 2nt}$ does not exist for times larger than $T = \frac{r_0^2}{2n}$: at time $T$ the evolving spheres shrink to a point and the flow develops a singularity.

C) Let us consider the cylinder $\mathbf{F}_0 = \partial B_{r_0}^k \times \mathbb{R}^{n+1-k}$ with $1 \leqslant k \leqslant n$. In this case the flat component is stationary under the flow, whereas the round one tends to shrink as we have seen with spheres. Hence we have again the formation of a singularity in finite time when the cylinder shrinks to a line.

## 3. Mean curvature flow as a nonlinear heat equation

Mean curvature flow can be written as

$$\frac{\partial \mathbf{F}}{\partial t} = \Delta \mathbf{F}.$$

This expression looks very much like a heat equation, but what we mean here with $\Delta$ is, instead, a nonlinear operator: it is the contraction of two covariant derivatives of the vector $\mathbf{F}$. Component by component we compute

$$\begin{aligned} \Delta \mathbf{F} &= g^{ij}\nabla_i\left(\nabla_j \mathbf{F}\right) = g^{ij}\nabla_i\left(\partial_j \mathbf{F}\right) \\ &= g^{ij}\partial_i\left(\partial_j \mathbf{F}\right) - g^{ij}\Gamma^k_{ij}\partial_k \mathbf{F}, \end{aligned} \tag{3.1}$$

and, since $\partial_i\partial_j \mathbf{F} = \Gamma^k_{ij}\partial_k \mathbf{F} - h_{ij}\nu$, we have

$$\Delta \mathbf{F} = -g^{ij}h_{ij}\nu = -H\nu.$$

Looking at (3.1) one can recognize the standard form of a quasilinear parabolic operator. Hence the short time existence and uniqueness of a solution to M.C.F. is a consequence of well known results about quasilinear parabolic equations.

An important tool in studying the mean curvature flow is the following avoidance principle.

Proposition 3.1 (Comparison principle). — *Any two smooth solutions of M.C.F. which are initially disjoint, stay disjoint.*

The proof relies on the use of maximum principle for parabolic equations.

As a consequence of this statement we can easily see that

*Remark 3.2.* — All compact manifold develop a singularity in finite time under M.C.F.

*Proof.* — For any compact hypersurface $N$ in Euclidian space $\mathbb{R}^{n+1}$, there exist a ball $B^{n+1}_R$ s. t. $N \subset B^{n+1}_R$ for a sufficiently large $R$, hence the sphere $\partial B^{n+1}_R$ encloses $N$. Now we can apply the mean curvature flow to both $\partial B^{n+1}_R$ and $N$. The comparison principle assures that smooth solutions of the flow never touch each other, i. e. as long as the evolution of $N$ is smooth, it is enclosed in the shrinking sphere. Finally we have two possible situation: or $N$ shrinks to a point before the extinguish time of the enclosing sphere, or it develops a singularity before shrinking to a point. □

## 4. Convex surfaces

The first important result on mean curvature flow, at least in the approach we consider here, as a flow of smooth surfaces, is a quite complete description of the behavior of convex surfaces.

THEOREM 4.1 (Huisken). — *If $M_0$ is a smooth, embedded, compact and convex hypersurface of $\mathbb{R}^{n+1}$, then the solution of the M.C.F.*

$$\begin{cases} \frac{\partial \mathbf{F}}{\partial t}(\mathbf{p},t) = -H(\mathbf{p},t)\,\nu(\mathbf{p},t) & \textit{for } t>0 \\ \mathbf{F}(\cdot,0) = M_0 \end{cases}$$

*is smoothly embedded, compact and convex until it converges to a point in finite time.*

*After a suitable rescaling it converges smoothly to a round sphere.*

*Proof.* — We have already seen that a solution for this flow exists and it is unique. Now we consider the evolution of some important geometric quantities such as the metric, the measure $\mu = \sqrt{\det\{g_{ij}\}}$ and the mean curvature

$$\frac{\partial g_{ij}}{\partial t} = -2Hh_{ij}$$
$$\frac{\partial \mu}{\partial t} = -H^2\mu$$
$$\frac{\partial H}{\partial t} = \Delta H + |A|^2 H.$$

As an immediate consequence of the second equation we see that the area of the evolving surfaces is monotonically decreasing, in fact the mean curvature flow is the gradient flow of the area functional, i.e. it gives the steepest decay of the area functional.

Looking at the evolution equation for the mean curvature and applying the maximum principle for parabolic equations we can deduce that if $H>0$ on the initial manifold $M_0$, then it stays positive for all times. Actually something stronger can be proved using the maximum principle for tensors (see [6]).

*Remark 4.2.* — Convexity is preserved along the M.C.F.: if $\varepsilon$ is a sufficiently small positive number, then the positive definiteness of the matrix $h_{ij} - \varepsilon H g_{ij}$ is preserved.

The following step of the proof consists in showing that the pinching of curvatures is also preserved along the flow.

DEFINITION 4.3. — *On every hypersurface $M_t = \mathbf{F}_t(M)$ of the flow we define the function $f$ as*

$$f := |A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}(\lambda_i - \lambda_j)^2 .$$

*If on $M_t$ we have $f < \gamma$ for a given constant $\gamma \in \mathbb{R}$, then we say that the curvatures of $M_t$ are $\gamma$-pinched.*

Note that $f \geqslant 0$ and if $f \equiv 0$ on a compact surface, then it is a sphere. Hence the function $f$ gives a characterization of spheres and its value is measure of how much a surface differ from being a sphere.

Applying the maximum principle to the evolution equation of $f$

$$\frac{\partial f}{\partial t} = \Delta f + \frac{2}{H}\langle \nabla f, \nabla H\rangle - \frac{2}{H^4}|H\nabla_i h_{lk} - h_{lk}\nabla_i H|^2$$

we deduce that $f$ is monotonically nonincreasing in time. Although, this is not sufficient to prove that it converges to zero: to do this we need to introduce a new function $f_\sigma := fH^\sigma$ where $\sigma$ is a small positive constant.

An estimate on the high $L^p$-norm of $f_\sigma$, combined with Sobolev inequalities and interpolation inequalities allows to deduce that also $f_\sigma$ is a nonincreasing function. Hence we have

$$f = H^{-\sigma} f_\sigma \leqslant H^{-\sigma}\max_{M_0} f_\sigma \to 0 \quad \text{as} \quad H \to \infty,$$

i.e. $f$ converges to zero at those points where the mean curvature diverges.

Finally, long time existence of the flow is proved by an iterative process showing that the curvatures have bounded derivatives of any order: for any $m \in \mathbb{N}$, there exist $c(m)$ s. t. $|\nabla^m A| \leqslant c(m)$.

Now we know that as long as $|A|^2$ is bounded, the flow can be continued as a smooth flow. On the other side, since our manifold is compact, it develops a singularity in finite time. This implies that there exist a time $T < \infty$ such that $\max|A|^2$ diverges as $t \to T$. Moreover one can show that the ratio $\max H/\min H$ converges to one as time $T$ is approached, hence the mean curvature diverges everywhere. As consequences we have:

- $\Longrightarrow$: The evolving surfaces converge to a point in finite time
- $\Longrightarrow$: The pinching function $f \to 0$ everywhere.

The first part of the theorem is then proved.

As for the second part, we need to rescale the evolving surfaces:

$$\tilde{\mathbf{F}} := \psi(t)\,\mathbf{F},$$

for instance we can choose the to rescale them keeping the area fixed. It can be proved that the rescaled surfaces $\tilde{M}_\tau$ converge exponentially fast to a sphere in the $C^\infty$-topology. □

## 5. Nonconvex surfaces

$n = 1$. — If we consider the M.C.F. in dimension one we obtain a flow of planar curves driven by their curvature, it is also called "curve shortening flow". A quite strong and surprising result is known for this flow.

THEOREM 5.1 (Grayson). — *Any closed, smoothly embedded planar curve retains these properties under the curve shortening flow and becomes convex in finite time.*

At this point we are dealing with a convex curve and we can apply an analog theorem to the one in the previous section (see [4]) deducing that any closed and smoothly embedded planar curve converges to a point under the curve shortening flow.

$n > 1$. — In dimension greater than one, instead, the theorem holding for convex surfaces can not be extended to nonconvex ones. There exist in fact compact surfaces developing singularities before contracting to a point. The most important example of this behavior is the "neck pinching" singularity formation. It can be obtained letting evolve by M.C.F. a surface formed by two big spheres connected by a neck: if the neck is long enough and thin enough, it becomes singular (as a cylinder does) while the spheres have still positive radius.

## 6. Graphs

Now we want to study the evolution by M.C.F. of surfaces written as graphs.

Let us assume that our initial surface is an entire graph and it can be written on all $\mathbb{R}^n$ as the graph of a function $u_0$. Then for at least small times $t > 0$ the evolution can be described as

$$\begin{cases} \mathbf{F}_t(\mathbf{x}) & = (x_1, \ldots, x_n, u(x_1, \ldots, x_n, t)) \\ \mathbf{F}_0(\mathbf{x}) & = (\mathbf{x}, u_0(\mathbf{x})) . \end{cases}$$

We choose, as said, the unit normal vector field $\nu$ pointing below, so we define

$$\nu(\mathbf{x},t)=\frac{(Du(\mathbf{x},t),-1)}{\sqrt{1+|Du(\mathbf{x},t)|^2}}.$$

The differential equation M.C.F. becomes then $\frac{\partial \mathbf{F}}{\partial t}=\frac{du}{dt}\mathbf{e}_{n+1}$. Now we observe that

$$-H=\left\langle \frac{\partial \mathbf{F}}{\partial t},\nu\right\rangle=\frac{du}{dt}\langle \mathbf{e}_{n+1},\nu\rangle=-\frac{1}{\sqrt{1+|Du|^2}}\frac{du}{dt}$$

hence $\frac{du}{dt}=\sqrt{1+|Du|^2}H$, and on the other side, the mean curvature can be considered as the divergence of the unit normal vector $\nu$:

$$H=g^{ij}h_{ij}=g^{ij}\left\langle \frac{\partial \mathbf{F}}{\partial x_i},\frac{\partial \nu}{\partial x_j}\right\rangle=g^{ij}\mathbf{e}_i\frac{\partial \nu}{\partial x_j}=\operatorname{div}(\nu).$$

Finally the parabolic equation defining the flow can be written as a scalar equation for the function $u:\mathbb{R}^n\times[0,T)\to\mathbb{R}^{n+1}$

$$\frac{du}{dt}=\sqrt{1+|Du|^2}\operatorname{div}\left(\frac{(Du,-1)}{\sqrt{1+|Du|^2}}\right)=\left(\delta_{ij}-\frac{D_iuD_ju}{1+|Du|^2}\right)D_iD_ju$$

where sum over repeated indices is assumed.

Lemma 6.1. — *A graph evolving by M.C.F. stays a graph as long as the flow exits.*

*Proof.* — Notice that a graph satisfies the property $\left\langle -\nu(\mathbf{x}),\mathbf{e}_{n+1}\right\rangle>0$ for all point $\mathbf{x}$ in the domain. In order to prove the lemma it is sufficient to show that this property is preserved along the flow. We then consider the function $v:=\left\langle -\nu(\mathbf{x}),\mathbf{e}_{n+1}\right\rangle^{-1}$ and compute its evolution equation

$$\frac{\partial v}{\partial t}=\Delta v-2\frac{|\nabla v|^2}{v}-|A|^2v.$$

Applying the maximum principle we deduce that $v$ is monotonically nonincreasing and then $\left\langle -\nu(\mathbf{x}),\mathbf{e}_{n+1}\right\rangle$ is kept bounded away from zero as long as the flow exists. □

The following result was initially proved for graphs with controlled growth at infinity and then was extended to entire graphs of arbitrary growth.

Theorem 6.2 (Ecker-Huisken). — *If the initial surface $M_0$ is a locally Lipschitz continuos entire graph, then the M.C.F. has a smooth solution for all times.*

*Remark 6.3.* — The short time existence of a smooth solution for M.C.F. was also proved by a localization process for graphs defined on a bounded domain.

An other interesting issue about mean curvature flow is to check if there exist translating graph solutions and what is their shape. Without loss of generality we can write the equation describing translating solution as $u(\mathbf{x}, t) = u_0(\mathbf{x}) + t$, it means

$$\sqrt{1+|Du_0|^2}\,\mathrm{div}\left(\frac{Du_0}{\sqrt{1+|Du|^2}}\right) = \frac{du}{dt} = 1.$$

*Example 6.4.* — $n = 1$

In dimension one the above equation can be written as

$$1 = \left(1 - \frac{u_0'^2}{1+u_0'^2}\right) u_0'' = \frac{u_0''}{1+u_0'^2} = (\arctan u_0')'$$

where $u_0' = Du_0$. This can be easily solved: $\arctan u_0' = x$ and $u_0' = \tan x$. This means that, modulo additive constants, for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we have

$$u_0(x) = -\log\cos x$$
$$u_t(x) = u_0(x) + t.$$

This function is know with the name of "grim reaper".

## 7. Applications

The mean curvature flow is a quite natural flow coming out also in the study of some physical phenomena, for instance evolutionary surfaces with prescribed mean curvature model the behavior of grain boundaries in annealing pure metal.

Here we mention just two mathematical applications.

*Isoperimetric inequality.* — A proof of the isoperimetric inequality can be provided using the M.C.F. and exploiting the fact that the area of the evolving surfaces decreases according to

$$\frac{d}{dt}A(M_t) = -n\int |\mathbf{H}|^2\,d\mu$$

where at the right hand side we have the Willmore energy.

*Classification by surgery.* —

Theorem 7.1 (Huisken-Sinestrari). — *Any smooth, closed hypersurface of* $\mathbb{R}^{n+1}$, *with* $n \geqslant 3$, *s. t. the first two smallest principal curvatures satisfy* $\lambda_1 + \lambda_2 \geqslant 0$ *is diffeomorphic to the sphere* $S^n$ *or to a finite number of connected sum* $S^{n-1} \times S^1 \# ... S^{n-1} \times S^1$.

This result has been recently proved letting surfaces evolve by M.C.F. and applying a surgery procedure (the converse of a connected sum) just before a singularity is formed. In this way one can keep track of the topology of the surface while the flow can be continued as a smooth flow.

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