

Institut Fourier — Université de Grenoble I

Actes du séminaire de
**Théorie spectrale
et géométrie**

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Volume 26 (2007-2008), p. 139-144.

[<http://tsg.cedram.org/item?id=TSG_2007-2008__26__139_0>](http://tsg.cedram.org/item?id=TSG_2007-2008__26__139_0)

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ISOTROPIC CURVATURE: A SURVEY

Harish Seshadri

ABSTRACT. — We discuss the notion of isotropic curvature of a Riemannian manifold and relations between the sign of this curvature and the geometry and topology of the manifold.

1. Introduction

Let (M, g) be a Riemannian manifold. For any $p \in M$, the Riemann curvature tensor gives rise to the curvature operator

$$\mathcal{R} : \wedge^2 T_p M \rightarrow \wedge^2 T_p M.$$

We can complexify $T_p M$ to get $T_p M \otimes \mathbb{C}$ and consider the \mathbb{C} -linear extension of \mathcal{R} to $\wedge^2 T_p M \otimes \mathbb{C}$.

The Riemannian metric on $T_p M$ extends as a Hermitian metric $\langle \cdot, \cdot \rangle$ or a \mathbb{C} -bilinear form (\cdot, \cdot) on $T_p M \otimes \mathbb{C}$. The former extension gives rise to a Hermitian metric, again denoted by $\langle \cdot, \cdot \rangle$, on $\wedge^2 T_p M \otimes \mathbb{C}$. A vector $v \in T_p M \otimes \mathbb{C}$ is *isotropic* if $\langle v, v \rangle = 0$. A subspace is *isotropic* if every vector in it is isotropic.

(M, g) is said to have *positive isotropic curvature* if

$$\langle \mathcal{R}(v \wedge w), v \wedge w \rangle > 0$$

for every pair of vectors $v, w \in T_p M \otimes \mathbb{C}$ which span an isotropic 2-plane.

This condition can be formulated in purely real terms as follows: First, v and w span an isotropic 2-plane if and only if there are orthonormal vectors e_1, e_2, e_3 and e_4 so that

$$\sqrt{2} \frac{v}{\|v\|} = e_1 + \sqrt{-1} e_2, \quad \sqrt{2} \frac{w}{\|w\|} = e_3 + \sqrt{-1} e_4.$$

Keywords: Weyl Curvature, Euler Characteristic, Chern-Gauss-Bonnet Theorem, Asymptotically Flat Manifolds, Yamabe metric.

Math. classification: 53C21.

One then checks that

$$\frac{4}{\|v\|^2\|w\|^2} \langle R(v \wedge w), v \wedge w \rangle = R_{1313} + R_{1414} + R_{2323} + R_{2424} + 2R_{1234},$$

where $R_{ijkl} = R(e_i, e_j, e_k, e_l)$.

Hence (M, g) has positive isotropic curvature if and only if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} + 2R_{1234} > 0$$

for all orthonormal 4-frames (e_1, e_2, e_3, e_4) .

The notions of nonnegative, negative and nonpositive isotropic curvature are defined similarly.

2. The second variation of energy and isotropic curvature

We first recall the second variation formula for the energy of curves. Let (M, g) be a Riemannian manifold let Ω be the space of smooth closed curves in M . If $\gamma : S^1 \rightarrow M$ is a smooth curve then the sections of the pull-back bundle γ^*TM on S^1 can be regarded as the tangent space to Ω at γ .

Consider the energy functional $E : \Omega \rightarrow \mathbb{R}$ defined by

$$E(\gamma) = \int_{S^1} \|\gamma'(s)\|^2 ds.$$

The critical points of E are precisely the closed geodesics on M . The Hessian of E , being a symmetric bilinear form on $T_\gamma\Omega$, is given by

$$I(X, X) = \int_{S^1} (\|X'(s)\|^2 - R(X, \gamma', X, \gamma')) ds,$$

where X is a section of γ^*TM . The second term, up to a nonnegative multiple, is just the sectional curvature of the 2-plane spanned by $\{\gamma'(s), X(s)\}$.

Similarly, one can consider smooth maps $\phi : S^2 \rightarrow M$ and the energy E of such maps. The critical points of E are now conformal branched minimal immersions of S^2 in M . The Hessian of E is a bilinear form on the space of sections of ϕ^*TM .

Following Micallef and Moore [9], we can complexify the bundle ϕ^*TM to get $V := \phi^*TM \otimes \mathbb{C}$ and consider the complex linear extension of the Hessian to the space of sections of V . As before, the metric on ϕ^*TM can be extended as a Hermitian metric $\langle \cdot, \cdot \rangle$ or a \mathbb{C} -linear form (\cdot, \cdot) on V and the connection can be extended in a \mathbb{C} -linear fashion. Moreover V can be given a holomorphic structure so that a section s is holomorphic if and only if

$$\bar{\partial}s := \nabla_{\frac{\partial}{\partial \bar{z}}} s = 0.$$

We then have

$$(2.1) \quad I(s, s) = \int_{S^2} \left(\|\bar{\partial}s\|^2 - \left\langle \mathcal{R}(s \wedge \frac{\partial f}{\partial z}), s \wedge \frac{\partial f}{\partial z} \right\rangle \right) dx dy,$$

where $z = x + \sqrt{-1}y$ is any local holomorphic coordinate and $\frac{\partial f}{\partial z} = f_*(\frac{\partial}{\partial z})$.

The fact that f is conformal implies that

$$\left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z} \right) = 0,$$

i.e. $\frac{\partial f}{\partial z}$ is an *isotropic* section. If s is a holomorphic section such that s and $\frac{\partial f}{\partial z}$ span an isotropic 2-plane, then the second term in the integral represents an isotropic curvature.

3. Examples

- (1) If (M, g) has positive (nonnegative) curvature operator then the isotropic curvature is positive (nonnegative).

Hence the standard round sphere (S^n, g_0) has PIC and every locally symmetric space of compact type has nonnegative isotropic curvature.

- (2) If the sectional curvatures of M are pointwise strictly quarter-pinched, i.e. if there is a function $a : M \rightarrow \mathbb{R}^+$ such that the sectional curvatures K at p satisfy

$$\frac{a(p)}{4} < K < a(p),$$

then (M, g) has PIC. If the inequalities above are weak one gets that (M, g) has nonnegative isotropic curvature. [9].

- (3) The product metric on $S^n \times S^1$ has PIC.
 (4) The connected sum of manifolds with PIC admits a PIC metric (Micallef-Wang [10]).

4. Topological implications of PIC

Throughout this section (M, g) will be a compact manifold with PIC.

The first major result relating topology to PIC is due to Micallef and Moore [9]: If $\dim M = n$ then the homotopy groups $\pi_i(M) = \{0\}$ for $2 \leq i \leq [\frac{n}{2}]$. If M is also simply-connected, one can combine this with Poincaré duality and the h-cobordism theorem to conclude that M has to be homeomorphic to S^n . To prove the vanishing of homotopy groups, the

essential ingredient is the second variation formula (2.1). By showing the existence of sufficiently many holomorphic sections of the bundle V one can show that the index of the Hessian (of the energy functional) is at least $\frac{n-3}{2}$. The authors develop a Morse theory for the energy functional on the space Ω of maps of S^2 into M and use the bound on index to show vanishing of homotopy groups of Ω . This, in turn, implies the vanishing of the appropriate homotopy groups of M by standard topology.

For even-dimensional manifolds M Micallef-Wang [10] and Seaman [12] independently proved that PIC implies the vanishing of the second Betti number of M . In fact, they show that the curvature term in the Bochner formula for harmonic 2-forms can be expressed in terms of isotropic curvatures. Hodge theory then gives the desired result.

The fundamental group π of a compact manifold with PIC is expected to be very special: Indeed, based on the connected sum result of Micallef-Wang (Example (4) in Section 3), Gromov [7] has conjectured that π must be virtually free. In this direction Fraser [4] and Fraser-Wolfson [5] prove that π cannot contain any subgroup isomorphic to the fundamental group of a closed orientable surface. The proof is again based on the second variation formula of energy of minimal immersions.

The definitive conjecture about the topology of compact n -manifolds with PIC is due to Schoen [11]: A finite cover of such a manifold must be diffeomorphic to a connected sum of copies of $S^{n-1} \times S^1$. Note that by Examples (3) and (4) such connected do admit PIC metrics. In dimension 4, Hamilton [8] outlined a way of proving Schoen's conjecture using Ricci flow with surgery. Recently [2], Schoen and Brendle showed that Ricci flow preserves the PIC condition in all dimensions. Hence a possible approach to Schoen's conjecture is via Ricci flow with surgery in all dimensions.

In [6], it is shown that Gromov's conjecture implies part of Schoen's conjecture: More precisely, it is proved that if (M, g) has PIC and free fundamental group, then M is *homeomorphic* to a connected sum of copies of $S^{n-1} \times S^1$.

5. Positive vs Nonnegative Isotropic Curvature

It turns out that one can classify manifolds with nonnegative isotropic curvature in terms of those with PIC [13], [1] : Let (M^n, g) , $n \geq 4$, be a compact, orientable, locally irreducible Riemannian manifold with nonnegative isotropic curvature, then one of the following holds:

- (1) M admits a metric with positive isotropic curvature,

- (2) (M, g) is locally symmetric or
- (3) M is Kähler and biholomorphic to $\mathbb{C}P^{\frac{n}{2}}$.

The proof of this is based on the results of Brendle and Schoen on Ricci flow [2], [3] and the following very recent result of Brendle [1]: Every compact Einstein manifold with nonnegative isotropic curvature is locally symmetric.

6. Negative Isotropic Curvature Metrics

In some ways, isotropic curvature behaves like scalar curvature, such as the preservation of PIC under connected sums. It seems reasonable to expect that any compact manifold admits a metric with negative isotropic curvature. This is known to be true in dimension 4 [13]. The proof is based on a variational characterization of the negativity condition.

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