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SHADOW LEMMA ON THE PRODUCT OF HADAMARD MANIFOLDS AND APPLICATIONS

Françoise Dal'Bo & Inkang Kim

ABSTRACT. — In this paper we analyze the limit set of nonelementary subgroups acting by isometries on the product of two pinched Hadamard manifolds. Following M. Burger's and P. Albuquerque's works, we study the properties of Patterson-Sullivan's measures on the limit sets of graph groups associated to convex cocompact groups.

Introduction

Let X_1, X_2 be two pinched Hadamard manifolds. In this paper we are interested in the Riemannian product $Y = X_1 \times X_2$. The geometric boundary of Y, denoted ∂Y, is identified whith $\partial X_1 \times \partial X_2 \times]0, \infty[\cup \partial X_1 \cup \partial X_2$. Consider a subgroup $\Gamma \subset I(Y)$, denote $L(\Gamma)$ its limit set, $L_{\mathrm{reg}}(\Gamma) = L(\Gamma) \cap \partial X_1 \times \partial X_2 \times]0, \infty[$ and $F(\Gamma)$ the projection of $L_{\mathrm{reg}}(\Gamma)$ into $\partial X_1 \times \partial X_2$.

Following the path of Y. Benoist, M. Burger and Y. Guivarc'h ([2], [4], [11]), one proves, in the section 2, the following results. (see also [17]) :

PROPOSITION 2.3. — *Let Γ be a subgroup (not necessary discrete) of $I(Y)$ such that the projections of Γ into $I(X_1)$ and $I(X_2)$ are strongly nonelementary. If $L_{\mathrm{reg}}(\Gamma) \neq \phi$ then $F(\Gamma)$ is a minimal closed non empty Γ- invariant subset of $\partial X_1 \times \partial X_2$ and $L_{\mathrm{reg}}(\Gamma) = F(\Gamma) \times P(\Gamma)$ where $P(\Gamma) \subset$* $]0, \infty[$.

PROPOSITION 2.4. — *Let Γ be a subgroup of $I^+(X_1) \times I^+(X_2)$ such that the projections of Γ into $I^+(X_1)$ and $I^+(X_2)$ are strongly nonelementary*

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and do not contain elliptic isometries then $L_{\mathrm{reg}}(\Gamma) \neq \phi$,

$$P(\Gamma) = \left\{ \overline{\frac{\ell(\gamma_1)}{\ell(\gamma_2)} / (\gamma_1, \gamma_2) \in \Gamma \ \ \textit{and is hyperbolic}} \right\} \cap]0, \infty[$$

and $P(\Gamma)$ *is an interval.*

Let $\Gamma_1,\ \Gamma_2$ be two isomorphic subgroups of $I(X_1)$ and $I(X_2)$. Consider an isomorphism ρ between Γ_1 and Γ_2, and denote Γ_ρ the graph group $\subset I(Y)$ defined by $\Gamma_\rho = \{(\gamma_1, \rho(\gamma_1))/\gamma_1 \in \Gamma_1\}$.

Proposition 2.8. — *Let* Γ *be a subgroup of* $I^+(X_1) \times I^+(X_2)$ *acting freely on* Y*. If* $L(\Gamma) = L_{\mathrm{reg}}(\Gamma)$ *then* Γ *is a graph group* Γ_ρ *and* ρ *preserves the type of isometries.*

In the section 3, one considers only infinite discrete subgroups of $I(Y)$. One says that a measure μ on ∂Y is a (β, p)-conformal measure, for $\beta \geqslant 0$ and $p \in [0, \infty]$, if the support of μ is included in $\partial X_1 \times \partial X_2 \times \{p\}$ if $p \in]0, \infty[$, in ∂X_1 if $p = \infty$ and in ∂X_2 if $p = 0$ and if, for any Borel subset $B \subset \partial Y$, one has:

$$\forall \gamma \in \Gamma, \quad \mu(\gamma(B)) = \int_B |\gamma'(\xi)|^\beta d\mu(\xi)$$

(see section 3 for definition of $|\gamma'(\xi)|$).

Adapting the works of M. Burger [4] and P. Albuquerque [1], one proves the following results.

Theorem 3.2 (Shadow Lemma). — *Let* Γ *be an infinite discrete subgroup of* $I(Y)$*. Suppose that* $L(\Gamma)$ *is not included in a small cell defined by* $\{\xi_1\} \times \partial X_2 \times [0, \infty] \cup \partial X_1 \times \{\xi_2\} \times [0, \infty]$*. If* μ *is a* (β, p)*- conformal measure then there exists* $d_0 > 0$ *such that for every* $r \geqslant d_0$*, there exists* $c(r) \geqslant 1$ *such that*

$$\frac{1}{c(r)} e^{-\beta d(0, \pi^p(\gamma(0))} \leqslant \mu(S(0, B(\gamma(0), r))) \leqslant c(r) e^{-\beta d(0, \pi^p(\gamma(0))} \text{ for every } \gamma \in \Gamma$$

(see section 3 for the definitions).

Using this theorem, one generalizes a result due to M. Burger [4], P. Albuquerque [1] and J-F Quint [19] in the case where X_1 and X_2 are symmetric spaces of Rank 1.

Corollary 3.4 and 3.5. — *Let* Γ_ρ *be a graph group associated to isomorphic nonelementary convex cocompact subgroups* $\Gamma_1,\ \Gamma_2$ *of* $I^+(X_1)$

and $I^+(X_2)$, without elliptic elements. If σ is a Patterson-Sullivan measure associated to Γ_ρ then there exists an unique $\lambda \in P(\Gamma_\rho)$ such that the support of σ is $F(\Gamma_\rho) \times \lambda$.

Using Patterson-Sullivan's measures, we construct measures on

$$\Gamma \backslash^{T^1 X_1 \times T^1 X_2}$$

invariant by the Weyl chamber flow.

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1. Preliminaries on pinched Hadamard manifolds

The goal of this section is to recall well known notions and results which will be used in the next sections. Let X be a simply connected, negatively curved Riemannian manifold. Such a manifold is called a Hadamard manifold. Let K be the sectionnal curvature of X. One supposes that there exist $a > 0$ and $b > 0$ such that $-b^2 \leqslant K \leqslant -a^2$. In this case, X is called a pinched Hadamard manifold. In this paper one always normalizes K by $\sup K = -1$. Denote ∂X the geometric boundary of X and $I(X)$ the group of isometries. The action of $I(X)$ on X induces an action on ∂X by homeomorphism. Let $I^+(X)$ be the subgroup of orientation preserving isometries ; $g \in I^+(X)$ is *elliptic* if it fixes at least one point in X . If g is not elliptic then g fixes exactly 1 or 2 points in ∂X. In the first case g is *parabolic*, in the second one, g is *hyperbolic*. In both cases, $(g^n(x))_{n\geqslant 1}$ and $(g^{-n}(x))_{n\geqslant 1}$ converge, the limits do not depend on $x \in X$ and belong to ∂X. Set $g^+ = \lim_{n\to+\infty} g^n(x)$ and $g^- = \lim\limits_{n\to+\infty} g^{-n}(x)$. The isometry g is parabolic if and only if $g^+ = g^-$. Define the length of $g \in I(X)$ by $\ell(g) = \inf\limits_{x\in X} d(x, g(x))$. If g is hyperbolic $\ell(g) = d(z, g(z))$ for any z in the geodesic $(g^- g^+)$, if g is parabolic or elliptic $\ell(g) = 0$.

Lemma 1.1 ([9]). — *Fix $0 \in X$. Let $g \in I^+(X)$.*

(1) *If g is hyperbolic, there exists $K_1 > 0$ such that :*

$$\forall n \in \mathbb{Z}, |n|\ell(g) \leqslant d(0, g^n(0)) \leqslant K_1 + |n|\ell(g)$$

(2) *If g is parabolic, there exists $K_2 > 0$ such that:*

$$\forall n \in \mathbb{Z}, d(0, g^n(0)) \leqslant K_2 Log(|n| + 1).$$

The first inequality is obvious, the second one is proved in [9] and uses the fact that the curvature is pinched (see also [12]).

Take $\xi \in \partial X$ and $x, y \in X$. Consider a geodesic ray $r(t)$ with extremity ξ. The limit of $d(x, r(t)) - d(y, r(t))$ when $t \to +\infty$, exists, and it does not depend on the origin of r and is denoted by $B_\xi(x, y)$. Fix an origine $0 \in X$ and consider the map $D : \partial X \times \partial X \to \mathbb{R}^+$ defined by $D(\xi, \xi) = 0$ and, for $\xi \neq \eta$, by $D(\xi, \eta) = e^{-1/2(B_\xi(0,z)+B_\eta(0,z))} \leqslant 1$ where z is any point in $(\xi\eta)$. For $g \in I(X)$, set $|g'(\xi)| = e^{B_\xi(0,g^{-1}(0))}$. If g is hyperbolic or parabolic, remark that $|g'(g^\pm)| = e^{\mp\ell(g)}$.

Theorem 1.2 ([3]). — *The map D is a distance (called Gromov-Bourdon distance) on ∂X satisfying:*

(1) *There exist $C \geqslant 1$ and $t > 1$ such that for any $\xi, \eta \in \partial X$*

$$\frac{1}{C} t^{-d(0,(\xi\eta))} \leqslant D(\xi, \eta) \leqslant C t^{-d(0,(\xi\eta))}$$

(2) *For any $g \in I(X)$ and $\xi, \eta \in \partial X$:*

$$D(g(\xi), g(\eta)) = |g'(\xi)|^{1/2} |g'(\eta)|^{1/2} D(\xi, \eta).$$

Let us prove now two lemmas which will be used in the next section.

Lemma 1.3. — *Let $(g_n)_{n\geqslant 1}$ be a sequence of $I(X)$. Suppose that*

$$(g_n(0))_{n\geqslant 1} \quad \text{and} \quad (g_n^{-1}(0))_{n\geqslant 1}$$

converge respectively to $\xi \in \partial X$ and $\xi' \in \partial X$. Then for any $x \in XU\partial X - \{\xi'\}$, one has $\lim_{n\to+\infty} g_n(x) = \xi$. Moreover if g_n are hyperbolic or parabolic then $\lim_{n\to+\infty} g_n^+ = \xi$ and $\lim_{n\to+\infty} g_n^- = \xi'$.

Proof. — Since $d(g_n(0), g_n(x)) = d(0, x)$ for any $x \in X$, $\lim_{n\to+\infty} g_n(x) = \xi$. Let η, η' in $\partial X - \{\xi'\}$, using the theorem 1.2 (1), one obtains

$$\lim_{n\to+\infty} D(g_n(\eta), g_n(\eta')) = 0.$$

Take $x \in (\eta\eta')$, since $\lim_{n\to+\infty} g_n(x) = \xi$ then $\lim_{n\to+\infty} g_n(\eta) = \xi$. Suppose now that g_n is hyperbolic or parabolic, then, according to the theorem 1.2 (2), one has :

$$D(g_n(\eta), g_n^+) = |g_n'(\eta)|^{1/2} e^{-1/2\ell(g_n)} D(\eta, g_n^+)$$

hence $D(g_n(\eta), g_n^+) \leqslant |g_n'(\eta)|^{1/2}$. Since $\eta \neq \xi'$, $\lim_{n\to+\infty} B_\eta(0, g_n^{-1}(0)) = -\infty$ and hence $\lim_{n\to+\infty} g_n^+ = \lim_{n\to+\infty} g_n(\eta) = \xi$. The same argument holds for g_n^-. $\square$

LEMMA 1.4 ([6]). — *Let g and h be hyperbolic isometries*

(1) *If $\{g^+, g^-\} \cap \{h^+, h^-\} = \phi$ then there exist $N > 0$ and $d > 0$ such that for any $n, m \in \mathbb{Z}$:*

$$|\ell(g^{Nn}h^{Nm}) - |n|\ell(g^N) - |m|\ell(h^N)| \leqslant d$$

(2) *If $g^+ = h^+$ then for any $n, m \in \mathbb{Z}^*$:*

$$\ell(g^n h^m) = |n\ell(g) + m\ell(h)|.$$

Proof. —

(1) It is proved in [6] (Lemma 4.1).
(2) One has $g^n h^m(g^+) = g^+$ and $|(g^n h^m)'(g^+)| = e^{-n\ell(g)-m\ell(h)}$, hence gh is hyperbolic and $\ell(g^n h^m) = n\ell(g) + m\ell(h)$.

□

2. Limit sets of groups acting on $X_1 \times X_2$

The results proved in this section are inspired by [2], [4] and [5].

For $i = 1, 2$, let X_i be a pinched Hadamard manifold. Denote $Y = X_1 \times X_2$ the Riemannian product of X_1 and X_2. Fix an origine $0 = (0_1, 0_2) \in Y$. Let φ be the bijection from the unitary tangent bundle, $T_0^1 Y$, of Y at 0, into $T_{0_1}^1 X_1 \times T_{0_2}^1 X_2 \times]0, \infty[\cup T_{0_1}^1 X_1 \cup T_{0_2}^1 X_2$ defined by

$$\varphi(U_1, U_2) = \left(\frac{U_1}{\|U_1\|}, \frac{U_2}{\|U_2\|}, \frac{\|U_1\|}{\|U_2\|} \right) \quad \text{if} \quad \|U_1\|\|U_2\| \neq 0$$

and by $\varphi(U_1, U_2) = U_i$ if $\|U_i\| = 1$. The map φ induces a one-to-one correspondance between the geometric boundary, ∂Y, of Y and $\partial X_1 \times \partial X_2 \times]0, \infty[\cup \partial X_1 \cup \partial X_2$. The *regular (resp. singular)* boundary of Y, denoted ∂Y_{reg}(resp. ∂Y_{sing}) is identified with

$$\partial X_1 \times \partial X_2 \times]0, \infty[(\text{resp. } \partial X_1 \cup \partial X_2).$$

A sequence $y_n = (y_{n_1}, y_{n_2})$ of Y converges to $\xi = (\xi_1, \xi_2, p) \in \partial Y_{\text{reg}}$ if $\lim\limits_{n\to+\infty} y_{n_i} = \xi_i$ and $\lim\limits_{n\to+\infty} \dfrac{d(0_1, y_{n_1})}{d(0_2, y_{n_2})} = p$; and converges to $\xi \in \partial X_1$ (resp. ∂X_2) if $\lim\limits_{n\to+\infty} y_{n_1} = \xi_1$ (resp. $\lim\limits_{n\to+\infty} y_{n_2} = \xi_2$) and $\lim\limits_{n\to+\infty} \dfrac{d(0_1, y_{n_1})}{d(0_2, y_{n_2})} = \infty$ (resp. 0). An element $g \in I(X_1) \times I(X_2)$ is called *hyperbolic* if g_1 and g_2 are hyperbolic. The following lemma is a consequence of the lemma 1.1.

Lemma 2.1. — *Let $g = (g_1, g_2) \in I(X_1) \times I(X_2)$.*
If g is hyperbolic then $\lim_{n\to+\infty} g^n(0) = \left(g_1^+, g_2^+, \dfrac{\ell(g_1)}{\ell(g_2)}\right)$.
If g_i is hyperbolic and the other factor is parabolic then

$$\lim_{n\to+\infty} g^n(0) = g_i^+ \in \partial Y_{\text{sing}}.$$

If Γ is a subgroup of $I(X_1) \times I(X_2)$ denote, for $i = 1, 2, \Gamma_i$ the projection of Γ into $I(X_i)$. One says that Γ_i is *strongly nonelementary* if Γ_i is nonelementary (i.e $L(\Gamma_i)$ is infinite) and if Γ_i has no global fixed point in $L(\Gamma_i)$.

Lemma 2.2. — *Let Γ be a subgroup of $I(X_1) \times I(X_2)$ such that Γ_1 and Γ_2 are strongly nonelementary. Take $\xi = (\xi_1, \xi_2) \in L(\Gamma_1) \times L(\Gamma_2)$, for any $(\eta_1, \eta_2) \in L(\Gamma_1) \times L(\Gamma_2)$ there exists $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ such that $\gamma_i(\xi_i) \neq \eta_i$ for $i = 1, 2$.*

Proof. — Suppose $\xi_1 = \eta_1$ and $\xi_2 \neq \eta_2$. Since Γ_1 and Γ_2 are strongly nonelementary there exist $\gamma = (\gamma_1, \gamma_2)$ and $\gamma' = (\gamma_1', \gamma_2')$ in Γ such that $\gamma_1(\xi_1) \neq \xi_1, \gamma_2'(\xi_2) \neq \xi_2$ and $\gamma_2'(\xi_2) \neq \eta_2$. If $\gamma_2(\xi_2) \neq \eta_2$ or $\gamma_1'(\xi_1) \neq \xi_1$ then γ or γ' is the one that we searched for , otherwise $\gamma\gamma'$ satisfies the required property. Suppose now $\xi_1 = \eta_1$ and $\xi_2 = \eta_2$, consider $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ such that $\gamma_1(\xi_1) \neq \xi_1$. If $\gamma_2(\xi_2) \neq \xi_2$ the lemma is proved, otherwise we are in the previous situation. □

Let $g = (g_1, g_2) \in I(X_1) \times I(X_2)$ and $\xi \in \partial Y$. The action of $I(X_1) \times I(X_2)$ on ∂Y is given by $g(\xi) = (g_1(\xi_1), g_2(\xi_2), p)$ if $\xi = (\xi_1, \xi_2, p) \in \partial Y_{\text{reg}}$ and by $g(\xi) = g_i(\xi_i)$ if $\xi = \xi_i \in \partial Y_{\text{sing}}$. Consider a subgroup $\Gamma \subset I(X_1) \times I(X_2)$, in general its limit set, $L(\Gamma) = \overline{\Gamma(0)} \cap \partial Y$, is not minimal. Set $L_{\text{reg}}(\Gamma) = L(\Gamma) \cap \partial Y_{\text{reg}}$, if $L_{\text{reg}}(\Gamma) \neq \phi$ denote $F(\Gamma)$ the projection of $L_{\text{reg}}(\Gamma)$ into $\partial X_1 \times \partial X_2$ and $P(\Gamma)$ its projection into $]0, \infty[$. The following propositions are already proved in the case where X_1 and X_2 are symmetric spaces [2], [4], [11] or if Γ is a graph group associated to Schottky groups [4], [5]).

Proposition 2.3. — *Let Γ be a subgroup of $I(X_1) \times I(X_2)$ such that Γ_1 and Γ_2 are strongly nonelementary. If $L_{\text{reg}}(\Gamma) \neq \phi$ then $F(\Gamma)$ is a minimal closed invariant subset of $\partial X_1 \times \partial X_2$ and $L_{\text{reg}}(\Gamma) = F(\Gamma) \times P(\Gamma)$.*

Proof. — Consider $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ in $F(\Gamma)$ and let us prove that $\xi \in \overline{\Gamma\eta} \subset F(\Gamma)$. Let $\gamma_n = (\gamma_{n_1}, \gamma_{n_2})$ be a sequence of Γ such that $\gamma_n(0)$ converges to a point $(\xi, p) \in L_{\text{reg}}(\Gamma)$. One can suppose that for $i = 1, 2$ the sequence $\gamma_{n_i}^{-1}(0_i)$ converges to a point $\xi_i' \in \partial X_i$. If $\eta_i \neq \xi_i'$ for $i = 1, 2$, then, according to the lemma 1.3, $\lim_{n\to+\infty} \gamma_{n_i}(\eta_i) = \xi_i$ and hence $\lim_{n\to+\infty} \gamma_n(\eta) = \xi$.

Otherwise, applying the lemma 2.2, one obtains $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ such that $\gamma_i(\eta_i) \neq \xi_i'$ for $i = 1, 2$ hence $\lim_{n\to+\infty} \gamma_n\gamma(\eta) = \xi$. In conclusion, $F(\Gamma)$ is minimal.

Take $p \in P(\Gamma)$ and let us prove that for any $\eta \in F(\Gamma)$, $(\eta, p) \in L(\Gamma)$. Since $p \in P(\Gamma)$, there exist $(\gamma_n)_{n\geqslant 1}$ in Γ and $\xi \in F(\Gamma)$ such that $\lim_{n\to+\infty} \gamma_n(0) = (\xi, p)$. Using the fact that $F(\Gamma)$ is minimal and that $\gamma(\xi, p) = (\gamma(\xi), p)$ one obtains that $(\eta, p) \in L(\Gamma)$. □

PROPOSITION 2.4. — *Let Γ be a subgroup of $I^+(X_1) \times I^+(X_2)$ such that Γ_1 and Γ_2 are strongly nonelementary and do not contain elliptic isometries, then $L_{\mathrm{reg}}(\Gamma) \neq \phi$,*

$$P(\Gamma) = \overline{\left\{ \frac{\ell(\gamma_1)}{\ell(\gamma_2)} / (\gamma_1, \gamma_2) \in \Gamma \textit{ and is hyperbolic} \right\}} \cap]0, \infty[$$

and $P(\Gamma)$ is an interval.

Proof. — Let us first prove that Γ contains hyperbolic isometries. Take $\gamma = (\gamma_1, \gamma_2) \in \Gamma$, according to the lemma 2.2 there exists $g = (g_1, g_2) \in \Gamma$ such that $g_1(\gamma_1^+) \neq \gamma_1^+$ and $g_2(\gamma_2^+) \neq \gamma_2^+$. Using the dynamics of parabolic and hyperbolic isometries ([10] section 8, theorems 16 and 17), one obtains that for some $N, M \in \mathbb{Z}$ the isometry $g_i\gamma_i^N g_i^{-1}\gamma_i^M$ is hyperbolic for $i = 1, 2$. One deduces from the lemma 2.1 that $L_{\mathrm{reg}}(\Gamma) \neq \phi$.

Denote $I = \overline{\left\{ \frac{\ell(\gamma_1)}{\ell(\gamma_2)} / (\gamma_1, \gamma_2) \in \Gamma \text{ and is hyperbolic} \right\}} \cap]0, \infty[$.

Let us prove that $P(\Gamma) = I$. According to the lemma 2.1, it is enough to prove that $P(\Gamma) \subset I$.

Let $p \in P(\Gamma)$ and $\gamma_n = (\gamma_{n_1}, \gamma_{n_2}) \in \Gamma$ such that $\lim_{n\to+\infty} \dfrac{d(0_1, \gamma_{n_1}(0_1))}{d(0_2, \gamma_{n_2}(0_2))} = p$. One can suppose that $\lim_{n\to+\infty} \gamma_{n_i}^+ = \xi_i^+$ and $\lim_{n\to+\infty} \gamma_{n_i}^- = \xi_i^-$ for $i = 1, 2$. Using the lemma 2.2, one obtains $g = (g_1, g_2) \in \Gamma$ such that $g_i(\xi_i^+) \neq \xi_i^-$. According to the lemma 1.3, $\lim_{n\to+\infty} (g_i\gamma_{n_i})^+ = g_i(\xi_i^+)$ and $\lim_{n\to+\infty} (g_i\gamma_{n_i})^- = \xi_i^-$, hence since $g_i(\xi_i^+) \neq \xi_i^-$, for n large enough, $g_i\gamma_{n_i}$ is hyperbolic, moreover there exists $K > 0$ such that $d(0_i, ((g_i\gamma_{n_i})^-(g_i\gamma_{n_i})^+)) < K$ for any $n > 0$. Let $z_{n_i} \in ((g_i\gamma_{n_i})^-(g_i\gamma_{n_i})^+)$ one has $\ell(g_i\gamma_{n_i}) = d(z_{n_i}, g_i\gamma_{n_i}(z_{n_i}))$ hence

$$\ell(g_i\gamma_{n_i}) \leqslant d(0_i, g_i\gamma_{n_i}(0_i)) \leqslant 2K + \ell(g_i\gamma_{n_i}).$$

Since $\lim_{n\to+\infty} d(0_i, g_i\gamma_{n_i}(0_i)) = +\infty$, one deduces from the previous inequality that $p = \lim_{n\to+\infty} \dfrac{\ell(g_1\gamma_{n_1})}{\ell(g_2\gamma_{n_2})}$. In conclusion $P(\Gamma) \subset I$.

Let us prove that I is an interval. Take $\ell = \frac{\ell(\gamma_1)}{\ell(\gamma_2)}$ and $\ell' = \frac{\ell(\gamma_1')}{\ell(\gamma_2')}$ in I. Suppose $\ell < \ell'$ and let us prove that $[\ell, \ell'] \subset I$. Since Γ_1 is strongly nonelementary there exists $h = (h_1, h_2) \in \Gamma$ such that

$$\{h_1(\gamma_1^+), h_1(\gamma_1^-)\} \cap \{\gamma_1^{'+}, \gamma_1^{'-}\} = \phi.$$

Set $g = h\gamma h^{-1}$. Remark that $\ell(g_i) = \ell(\gamma_i)$. One can suppose that $\{g_2^+, g_2^-\} \cap \{\gamma_2^{'+}, \gamma_2^{'-}\} = \phi$ or that $g_2^+ = \gamma_2^{'+}$ by taking g^{-1} if necessary. Applying the lemma 1.4, there exist $N \geqslant 1$ and $d \geqslant 0$ such that for any $n, m \in \mathbb{Z}$:

$$|\ell(g_i^{Nn}\gamma_i^{'Nm}) - |n|\ell(g_i^N) - |m|\ell(\gamma_i^{'N})| \leqslant d$$

for $i = 1, 2$. One deduces from this inequality that

$$\lim_{k\to+\infty} \frac{\ell(g_1^{Nnk}\gamma_1^{'Nmk})}{\ell(g_2^{Nnk}\gamma_2^{'Nmk})} = \frac{|n|\ell(\gamma_1) + |m|\ell(\gamma_1')}{|n|\ell(\gamma_2) + |m|\ell(\gamma_2')}.$$

In conclusion, $\frac{\ell(\gamma_1)+q\ell(\gamma_1')}{\ell(\gamma_2)+q\ell(\gamma_2')} \in I$ for any $q \in \mathbb{Q}_+^*$ hence $[\ell, \ell'] \subset I$. □

Let us introduce a family of subgroups of $I^+(X_1) \times I^+(X_2)$ called *graph groups*. Fix a subgroup $\Gamma_1 \subset I^+(X_1)$ and a faithful representation $\rho : \Gamma_1 \to I^+(X_2)$. The graph group denoted by Γ_ρ associated to ρ is defined by $\Gamma_\rho = \{(\gamma_1, \rho(\gamma_1))/\gamma_1 \in \Gamma_1\}$.

Remark 2.5. — In the particular case where Γ_1 and $\Gamma_2 = \rho(\Gamma_1)$ are *convex-co compact*,(i.e., the closure of the set of closed geodesics in ${}_{\Gamma_i}\backslash^{X_i}$ is compact) ρ induces an homeomorphism φ between $L(\Gamma_1)$ and $L(\rho(\Gamma_1)([T])$, and $F(\Gamma_\rho) = \{(\xi, \varphi(\xi))/\xi \in L(\Gamma_1)\}$.

Fix a system of generators $S_1 = \{s_1, \cdots, s_n\}$ of Γ_1 and put $S_2 = \{\rho(s_1), \cdots, \rho(s_n)\}$. Since Γ_i is convex cocompact, there is a quasi-isometry between $(\Gamma_i, |\,|_{s_i})$ and $(\Gamma_i(0_i), d)$ ([Bo]). This property implies that $L(\Gamma_\rho) = L_{\text{reg}}(\Gamma_\rho)$.

Corollary 2.6. — *Let Γ_ρ be a graph group of $I^+(Y)$ associated to $\rho : \Gamma_1 \to \Gamma_2$. Suppose that Γ_1, Γ_2 are strongly nonelementary subgroups of $I^+(X_1)$ and $I^+(X_2)$ without elliptic elements and that $L(\Gamma_\rho) = L_{\text{reg}}(\Gamma_\rho)$. Then there exists $k > 0$ such that $\ell(\gamma_1) = k\ell(\rho(\gamma_1'))$ for any $\gamma_1 \in \Gamma_1$ if and only if $L(\Gamma_\rho)$ is minimal.*

Proof. — According to the proposition 2.4, one has

$$L(\Gamma_\rho) = F(\Gamma_\rho) \times \left\{ \overline{\frac{\ell(\gamma_1)}{\ell(\rho(\gamma_1))}/\gamma_1 \in \Gamma_1 \text{ and is hyperbolic}} \right\}.$$

Since $F(\Gamma_\rho)$ is minimal and $\Gamma_\rho(F(\Gamma_\rho) \times \{p\}) = F(\Gamma_\rho) \times \{p\}$ then $L(\Gamma_\rho)$ is minimal if and only if $L(\Gamma_\rho) = F(\Gamma_\rho) \times \{k\}$. □

Suppose that X_1 and X_2 are symmetric spaces of rank 1 and Γ_1, Γ_2 are Zariski dense. If ρ preserves the length then, according to the corollary 2.6, one has $L(\Gamma_\rho) = F(\Gamma_\rho) \times \{1\}$. This property implies, according to a result proved in [2] (theorem 1.2) that Γ_ρ is not Zariski dense. Applying a criterion proved in [7],[8], one obtains the following already known result :

Corollary 2.7 ([7]). — *Suppose that X_1 and X_2 are symmetric spaces of rank 1 and Γ_1, Γ_2 are Zariski dense. If ρ preserves the length then X_1 and X_2 are isometric and Γ_1, Γ_2 are conjugate by an isometry.* See [14, 13, 15, 16] for similar results.

The following result is partially proved in [5].

Proposition 2.8. — *Let Γ be a subgroup of $I^+(X_1) \times I^+(X_2)$ acting freely on Y. If $L(\Gamma) = L_{\text{reg}}(\Gamma)$ then Γ is a graph group Γ_ρ and ρ preserves the type of isometries .*

Proof. — Let us prove that for $i = 1, 2$ the projection $q_i : \Gamma \to I^+(X_i)$ is injective. Suppose that $\gamma = (Id, \gamma_2) \in \Gamma$, since Γ acts freely, γ_2 is not elliptic, hence $\gamma^n(0)$ converges to a point in ∂Y_{sing} which contradicts the fact that $L(\Gamma) \cap \partial Y_{\text{sing}} = \phi$. In conclusion q_1 and q_2 are injective and hence $q_2 \circ q_1^{-1}$ is an isomorphism. This proves that $\Gamma = \Gamma q_2 \circ q_1^{-1}$. Using the fact that $L(\Gamma) = L_{\text{reg}}(\Gamma)$ and the lemma 1.1, one obtains that $q_2 \circ q_1^{-1}$ preserves the type of isometries. □

3. Shadow lemma for (β, p)-conformal measures

In this section one adapts the work of P. Albuquerque [1] and of M. Burger [4], replacing the symmetric space by the product of pinched Hadamard manifolds $Y = X_1 \times X_2$.

Set $\partial X_1 \times \partial X_2 \times \{\infty\} = \partial X_1$ and $\partial X_1 \times \partial X_2 \times \{0\} = \partial X_2$, using this notation, one identifies ∂Y with $\partial X_1 \times \partial X_2 \times [0, \infty]$ (see section 2). For $p \in [0, \infty]$, denote $\partial Y_p = \partial X_1 \times \partial X_2 \times \{p\}$. Let $\xi = (\xi_1, \xi_2, p) \in \partial Y$ and $a_1, a_2 \geqslant 0$ such that $a_1^2 + a_2^2 = 1$ and $p = \frac{a_1}{a_2}$; for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in Y, set $B_\xi(x, y) = a_1 B_{\xi_1}(x_1, y_1) + a_2 B_{\xi_2}(x_2, y_2)$. If $r(t)$ is a geodesic ray with extremity ξ then $B_\xi(x, y) = \lim_{t \to +\infty} d(x, r(t)) - d(y, r(t))$ ([B-G-S]). Fix an origin $0 = (0_1, 0_2) \in Y$. For $g \in I(Y)$, denote $|g'(\xi)| = e^{B_\xi(0, g-1(0))}$. One says that a subgroup $\Gamma \subset I(Y)$ is *discrete* if it acts properly discontinuously on Y.

Definition 3.1. — *Let* Γ *be an infinite discrete subgroup of* $I(Y)$ *and* $\beta \geqslant 0, p \in [0,\infty]$. *A* β*-conformal measure* μ *on* ∂Y *is a finite Borel measure such that for any Borel subset* $B \subset \partial Y$ *one has :*

$$\forall \gamma \in \Gamma, \mu(\gamma(B)) = \int_B |\gamma'(\xi)|^\beta d\mu(\xi)$$

A (β,p)*- conformal measure is a* β*- conformal measure whose support is included in* ∂Y_p.

Denote $P(s)$ its Poincaré series $\sum_{\gamma\in\Gamma} e^{-sd(0,\gamma(0))}$ and $\delta(\Gamma)$ the critical exponent of $P(s)$. Since the curvature of X is bounded from the below, $\delta(\Gamma)$ is finite, suppose $\delta(\Gamma) > 0$, let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a suitable function such that $Q(s) = \sum_{\gamma\in\Gamma} h(d(0,\gamma(0)))e^{-sd(0,\gamma(0))}$ diverges at $s = \delta(\Gamma)$ [18]. For $s > \delta(\Gamma)$, one defines a measure :

$$\sigma^s = \frac{\sum_{\gamma\in\Gamma} h(d(0,\gamma(0)))e^{-sd(0,\gamma(0))} D_{\gamma(0)}}{\sum_{\gamma\in\Gamma} h(d(0,\gamma(0)))e^{-sd(0,\gamma(0))}}$$

where $D_{\gamma(0)}$ denotes the Dirac mass at $\gamma(0)$.

A *Patterson-Sullivan measure*, σ, is a weak limit as $s \to \delta(\Gamma)$. The support of σ is included in $L(\Gamma)$ and σ satisfies the following $\delta(\Gamma)$-*conformal* property [1] :$\forall\gamma \in \Gamma, \forall B$ Borel subset of ∂X :

$$\sigma(\gamma(B)) = \int_B |\gamma'(\xi)|^{\delta(\Gamma)} d\sigma(\xi).$$

Let q be the map from ∂Y onto $[0,\infty]$ defined by $q(\eta_1,\eta_2,p) = p$. Disintegrating σ along the fibers of q one obtains for some $p \in [0,\infty]$, such that $\sigma(\partial Y_p) \neq 0$, a $(\delta(\Gamma),p)$-conformal measure $\sigma_p([A])$. Let $x = (x_1,x_2) \in Y - \{0\}$ with $x_1 \neq 0_1$ and $x_2 \neq 0_2$, there exists a unique *Weyl chamber* $W = [0_1,\xi_1) \times [0_2,\xi_2)$ centered at 0 passing through x. The projection of x in the direction $p \in [0,\infty]$, denoted $\pi^p(x)$, is by definition the orthogonal projection of x into the geodesic ray $[0,(\xi_1,\xi_2,p))$. Put $p = \frac{a_1}{a_2}$ with $a_i \geqslant 0$ and $a_1^2 + a_2^2 = 1$. If $x_1 = 0_1$ or $x_2 = 0_2$ then $\pi^p(x)$ depends on the choice of W, but $d(0,\pi^p(x)) = a_1 d(0_1,x_1) + a_2 d(0_2,x_2)$ is well defined. Denote $B(x,r)$ the ball centered at x with radius $r > 0$; for $A \subset Y$, the *Shadow*, $S(x,A)$, of A relative to x, is defined by $S(x,A) = \{\xi \in \partial Y/\xi$ belongs to the boundary of a Weyl chamber centered at x meeting $A\}$.

Theorem 3.2 (Shadow lemma). — *Let* Γ *be an infinite discrete subgroup of* $I(Y)$. *Suppose that* $L(\Gamma)$ *is not included in a small cell defined by* $\{\xi_1\} \times \partial X_2 \times [0,\infty] \cup \partial X_1 \times \{\xi_2\} \times [0,\infty]$. *If* μ *is a* (β,p)*- conformal* *measure then there exists* $d_0 > 0$ *such that for every* $r \geqslant d_0$*, there exists* $c(r) \geqslant 1$ *such that:*

$$\frac{1}{c(r)} e^{-\beta d(0,\pi^p(\gamma(0))} \leqslant \mu(S(0, B(\gamma(0), r)) \leqslant c(r) e^{-\beta d(0,\pi^p(\gamma(0))}$$

for all $\gamma \in \Gamma$.

Proof. — Let us equip $\partial X_1 \times \partial X_2$ with the product distance associated to the Gromov-Bourdon distance introduced in section 1 on each factor and with a standard metric with length 1 on $[0, \infty]$. □

Step 1. — There exist $0 < q < \mu(\partial Y)$ and $\varepsilon_0 > 0$ such that for any measurable set F contained in a ε_0-neighborhood of a small cell, $\mu(F) \leqslant q$.

Proof. — Let $d = \sup \{\mu(\text{small cell})\}$. Since a sequence of small cells has an accumulation point, in Hausdorff topology, which is itself a small cell, d is realized and $d < \mu(\partial Y)$ because the support of μ is not included in a small cell. Set $q = \frac{\mu(\partial Y)+d}{2}$, one has $q > d$. Suppose that there exist a sequence of reals $(\varepsilon_n)_{n\geqslant 1}$ with $\lim\limits_{n\to+\infty} \varepsilon_n = 0$ and a sequence $(F_n)_{n=1}$ of measurable sets in ∂Y included in a ε_n-neighborhood of some small cell such that $\mu(F_n) \geqslant q$. One can suppose that F_n converges, in Hausdorff topology, to a small cell S satisfying $\mu(S) \geqslant q > d$. This contradicts the definition of d. □

Step 2. — For all $\varepsilon > 0$ there exists $c_0 > 0$ such that for any $c > c_0$ and $x = (x_1, x_2)$ the set $\partial Y_p - S(x, B(0, c)) \cap \partial Y_p$, is contained in a ε-neighborhood of a small cell.

Proof. — Fix $\varepsilon > 0$, since X_i are pinched Hadamard manifolds, there exists $c > 0$ such that $\partial X_i - S(x_i, B(0_i, c))$ is included in a ε-neighborhood of some $\eta_i \in \partial X_i$, for each $x_i \in X_i$. One has $S(x, B(0_1, c) \times B(0_2, c)) \subset S(x, B(0, \sqrt{2}c))$ hence

$$\partial Y_p - S(x, B(0, \sqrt{2}c)) \cap \partial Y_p \subset \partial Y_p - S(x, B(0_1, c) x B(0_2, c)) \cap \partial Y_p.$$

Let (ξ_1, ξ_2, p) belong to the previous set, then $[x_1, \xi_1) \times [x_2, \xi_2) \cap B(0_1, c) \times B(0_2, c) = \phi$ and hence $\xi_1 \notin S(x_1, B(0_1, c))$ or $\xi_2 \notin S(x_2, B(0_2, c))$. In conclusion, if $c > c_0$ then there exists $(\eta_1, \eta_2) \in \partial X_1 \times \partial > X_2$ such that

$$\partial Y_p - S(x, B(0, \sqrt{2}c)) \cap \partial Y_p \subset B(\eta_1, \varepsilon) \times \partial X_2 \times \{p\} \cup \partial X_1 \times B(\eta_2, \varepsilon) \times \{p\}$$

which is included in a ε-neighborhood of $\{\eta_1\} \times \partial X_2 \times [0, \infty] \cup \partial X_1 \times \{\eta_2\} \times [0, \infty]$.

Step 3. — Let $c > 0$, for all $\gamma \in \Gamma$ and $\xi \in \partial Y$ such that $\xi \in S(0, B(\gamma(0), c)) \cap \partial Y_p$ one has

$$|-B_\xi(0,\gamma(0)) + d(0, \pi^p(\gamma(0))| \leqslant 6c.$$

□

Proof. — Let $\gamma = (\gamma_1, \gamma_2) \in \Gamma$, suppose $\gamma(0) \neq 0$ and consider a Weyl chamber W centered at 0 passing through $\gamma(0)$. Take $\xi' \in \partial W \cap \partial Y_p$, one has $B_\xi(0, \gamma(0)) = d(0, \pi^p(\gamma(0))$.

Choose now $\xi \in S(0, B(\gamma(0), c)) \cap \partial Y_p$, there exists a Weyl chamber W' centered at 0 such that $[0, \xi) \subset W'$ and $W' \cap B(\gamma(0), c) \neq \phi$. Let $x = (x_1, x_2) \in W' \cap B(\gamma(0), c)$. One has

$$|-B_\xi(0,\gamma(0)) + B_\xi(0,x)| = |B_\xi(x,\gamma(0))| \leqslant d(x,\gamma(0)) \leqslant c.$$

To prove the required inequality it is enough to show that $|-B_\xi(0,x) + B_{\xi'}(0,\gamma(0))| \leqslant 5c$. Set $\xi = (\xi_1, \xi_2, p)$ and $L_i = [0, \xi_i')$, one has $W' = L_1 \times L_2$. Remark that $W' \cap B(\gamma_1(0_1), c) \times B(\gamma_2(0_2), c) \neq \phi$ hence $L_i \cap B(\gamma_i(0_i), c) \neq \phi$. Take $y_i \in L_i$ such that $d(0_i, y_i) = d(0_i, \gamma_i(0_i))$, one has $B_{\xi'}(0, \gamma(0)) = B_\xi(0, y)$ hence

$$|-B_\xi(0,x) + B_{\xi'}(0,\gamma(0))| = |B_\xi(x,y)| \leqslant d(x,y).$$

Moreover using the fact that $L_i \cap B(\gamma_i(0_i), c) \neq \phi$ we get $d(0_i, y_i) \leqslant 3c$ by the triangle inequality. This proves that $d(x,y) \leqslant 5c$.

End of the proof of the theorem 3.2. — Pick c_0, ε_0 in steps 1 and 2. Then for $c > c_0$ and for all $\gamma \in \Gamma$ one has :

$$\mu(\partial Y_p) \geqslant \mu(S(\gamma^{-1}(0), B(0,c))) \geqslant \mu(\partial Y_p) - q > 0.$$

Moreover $S(\gamma^{-1}(0), B(0,c)) = \gamma^{-1}(S(0, B(\gamma(0), c))$ hence

$$\mu(S(\gamma^{-1}(0), B(0,c)) = \int_{S(0,B(\gamma(0),c)} e^{\beta B_\xi(0,\gamma(0))} d\mu(\xi) \geqslant \mu(\partial Y_p) - q.$$

By step 3 we get

$$\mu(S(0, B(\gamma(0), c)) = (\mu(\partial Y_p) - q) C_1(c) e^{-\beta d(0, \pi^p(\gamma(0))}$$

and

$$\mu(S(0, B(\gamma(0), c)) \leqslant \mu(\partial Y_p) C_2(c) e^{-\beta d(0, \pi^p(\gamma(0))}.$$

□

For $p \in [0, \infty]$ and $r > 0$ set :

$$B_p(0,r) = \{y \in Y - \{0\} / d(0, \pi^p(y)) \leqslant r\}.$$

Corollary 3.3. — *Let* Γ *be an infinite discrete subgroup of* $I(Y)$. *Suppose that* $L(\Gamma)$ *is not included in a small cell. If* μ *is a* (β, p)*- conformal measure with* $p \in]0, \infty[$ *then there exists* $c > 0$ *such that for any* $\ell > 0$ *:*

$$|\{\gamma(0)/\gamma \in \Gamma, \gamma(0) \in B_p(0, \ell)\}| \leqslant c\ell e^{\beta\ell}.$$

Proof. — Since $p \neq 0$ and $p \neq \infty$ the ball $B_p(0, \ell)$ is compact, hence there exists $C_1 > 0$ such that for any Weyl chamber W centered at 0 and for any $\ell > 0$ the set $\{y \in W - \{0\}/\ell \leqslant d(0, \pi^p(y)) \leqslant \ell + 1\}$ is covered by $C_1\ell$ balls of radius 1. Using the fact that is discrete, for $c > 0$ one obtains $C_2 > 0$ such that for any $\ell > 0$ and any Weyl chamber W centered at 0 one has :

$$|\{\gamma \in \Gamma/\ell \leqslant d(0, \pi^p(\gamma(0)) \leqslant \ell + 1, W \cap B(\gamma(0), c)) \neq \phi\} \leqslant C_2\ell.$$

Therefore any $\xi \in \partial Y_{\text{reg}}$ belongs to at most $C_2\ell$ shadows of balls $B(\gamma(0),c)$ for such $\gamma's$. Thus there exists $C_3 > 0$ such that

$$\sum_{\{\gamma\in\Gamma/\ell\leqslant d(0,\pi^p(\gamma(0)))\leqslant\ell+1\}} \mu(S(0, B(\gamma(0), c)) \leqslant C_3\ell\mu(\partial Y_p).$$

Applying the theorem 3.2, one obtains for $c \geqslant d$ a constant $C_4 > 0$ such that :

$$|\{\gamma \in \Gamma/\ell \leqslant d(0, \pi^p(\gamma(0))) \leqslant \ell + 1\}|e^{-\beta\ell} \leqslant C_4\ell.$$

Finally there exists $C_5 > 0$ such that :

$$\begin{aligned}
&|\{\gamma(0)/\gamma \in \Gamma, \gamma(0) \in B_p(0, \ell)\}| \\
&\qquad \leqslant \sum_{n=0}^{\ell-1} |\{\gamma \in \Gamma/n \leqslant d(0, \pi^p(\gamma(0))) \leqslant n + 1\}| \\
&\qquad \leqslant C_4 \sum_{n=0}^{\ell-1} ne^{\beta n} \leqslant C_5\ell e^{\ell n}.
\end{aligned}$$

□

Let Γ be a subgroup of $I(Y)$ satisfying the hypothesis of corollary 3.3. One defines the Poincaré series relative to $p \in [0, \infty]$ by :

$$P_p(s) = \sum_{\gamma\in\Gamma} e^{-sd(0,\pi^p(\gamma(0))}.$$

Denote $\delta_p(\Gamma)$ the critical exponent of $P_p(s)$. Since

$$d(0, \pi^p(\gamma(0))) \leqslant d(0, \gamma(0)),$$

one has $\delta_p(\Gamma) \geqslant \delta(\Gamma)$. Write $p = \frac{a}{b}$ with $a^2 + b^2 = 1$, if $p \neq 0$ and $p \neq \infty$ then

$$\delta_p(\Gamma) = \underset{s>0}{Inf}\left\{ s / \sum_{(\gamma_1,\gamma_2)\in\Gamma} e^{-s(ad(0_1,\gamma_1(0_1)+bd(0_2,\gamma_2(0_2)))} < \infty \right\}.$$

Let $A = \left\{ \binom{x_1}{x_2} \in \mathbb{R}^2_+ / \sum_{(\gamma_1,\gamma_2)\in\Gamma} e^{-x_1 d(0_1,\gamma_1(0_1)-x_2 d(0_2,\gamma_2(02)))} < \infty \right\}.$

By Hölder inequality, A is convex. Since A is convex and $\delta_p(\Gamma) \geqslant \delta(\Gamma)$, there exists at most one $q \in [0,\infty]$ such that $\delta_q(\Gamma) = \delta(\Gamma)$. Suppose $\delta_p(\Gamma) < \infty$ and consider a Patterson-Sullivan measure σ on ∂Y. Since $\sigma(\partial Y) \neq 0$, disintegrating σ along $[0,\infty]$ we get a $\delta(\Gamma)$-conformal measure σ_p supported by ∂Y_p such that $\sigma_p(\partial Y) \neq 0$. Since

$$\delta_p(\Gamma) = \limsup_{R\to+\infty} \frac{Log|\Gamma 0 \cap B_p(0,R)|}{R},$$

if $p \neq 0$ and $p \neq \infty$, according to the corollary 3.3, one has $\delta_p(\Gamma) \leqslant \delta(\Gamma)$. Moreover $\delta_p(\Gamma) \geqslant \delta(\Gamma)$, hence $\delta_p(\Gamma) = \delta(\Gamma)$. This proves the following corollary :

Corollary 3.4. — *Let Γ be an infinite discrete subgroup of $I(Y)$. Suppose that $L(\Gamma)$ is not included in a small cell. If for some*

$$p \in]0,\infty[, \sigma_p(\partial Y) \neq 0$$

then $\delta_p(\Gamma) = \delta(\Gamma)$ and such p is unique.

Remark 3.5. — In the particular case where $\Gamma = \Gamma_\rho$ and Γ_1 and $\rho(\Gamma_1)$ are convex cocompact, $L(\Gamma_\rho) = L_{\mathrm{reg}}(\Gamma_\rho)$ (Remark 2.5) and hence, there exists a unique $p_0 \in P(\Gamma)$ such that $\sigma = \sigma_{p_0}$

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