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SURFACES IN \mathbb{S}^3 AND \mathbb{H}^3 VIA SPINORS

Bertrand MOREL

Abstract

We generalize the spinorial characterization of isometric immersions of surfaces in \mathbb{R}^3 given by T. Friedrich to surfaces in \mathbb{S}^3 and \mathbb{H}^3 . The main argument is the interpretation of the energy-momentum tensor associated with a special spinor field as a second fundamental form. It turns out that such a characterization of isometric immersions in terms of a special section of the spinor bundle also holds in the case of hypersurfaces in the Euclidean 4-space.

1. Introduction

It is well known that a description of a conformal immersion of an arbitrary surface $M^2 \hookrightarrow \mathbb{R}^3$ by a spinor field φ on M^2 satisfying the inhomogenous Dirac equation

$$D\varphi = H\varphi, \quad (1)$$

(where D stands for the Dirac operator and H for the mean curvature of the surface), is possible. Recently, many authors investigated such a description (see for example [9],[10],[14]).

In fact, it is clear that any oriented immersed surface $M^2 \hookrightarrow \mathbb{R}^3$ inherits from \mathbb{R}^3 a solution of Equation (1), the surface M being endowed with the induced metric and the induced spin structure. Moreover, the solution has constant length. This solution is obtained by the restriction to the surface of a parallel spinor field on \mathbb{R}^3 . In [6], T. Friedrich clarifies the above-mentioned representation of surfaces in \mathbb{R}^3 in a geometrically invariant way by proving the following:

THEOREM 1.1 (FRIEDRICH [6]) *Let (M^2, g) be an oriented, 2-dimensional manifold and $H : M \rightarrow \mathbb{R}$ a smooth function. Then the following data are equivalent:*

1. *An isometric immersion $(\widetilde{M}^2, g) \rightarrow \mathbb{R}^3$ of the universal covering \widetilde{M}^2 into the Euclidean space \mathbb{R}^3 with mean curvature H .*

2. A solution φ of the Dirac equation

$$D\varphi = H\varphi,$$

with constant length $|\varphi| \equiv 1$.

3. A pair (φ, T) consisting of a symmetric endomorphism T of the tangent bundle TM such that $\text{tr}(T) = H$ and a spinor field φ satisfying, for any $X \in \Gamma(TM)$, the equation

$$\nabla_X \varphi + T(X) \cdot \varphi = 0.$$

In this paper, we prove the analogous characterizations for surfaces in \mathbb{S}^3 and \mathbb{H}^3 (Theorems 4.1 and 4.2). They are obtained by studying the equation of restrictions to a surface of real and imaginary Killing spinor fields (compare with [6]).

We note that the involved symmetric endomorphism T is the energy-momentum tensor associated with the restricted Killing spinor which describes the immersion. Finally, the case of the hypersurfaces of \mathbb{R}^4 is treated (Theorem 5.3).

2. Restricting Killing spinor fields to a surface

Let N^3 be a 3-dimensional oriented Riemannian manifold, with a fixed spin structure. Denote by ΣN the spinor bundle associated with this spin structure. If M^2 is an oriented surface isometrically immersed into N^3 , denote by ν its unit normal vector field. Then M^2 is endowed with a spin structure, canonically induced by that of N^3 . Denote by ΣM the corresponding spinor bundle. The following proposition is essential for what follows (see for example [2],[5],[12],[15]):

PROPOSITION 2.1 *There exists an identification of $\Sigma N|_M$ with ΣM , which after restriction to M , sends every spinor field $\psi \in \Gamma(\Sigma N)$ to the spinor field denoted by $\psi^* \in \Gamma(\Sigma M)$. Moreover, if \cdot_N (resp. \cdot) stands for Clifford multiplication on ΣN (resp. ΣM), then one has*

$$(X \cdot_N \nu \cdot_N \psi)^* = X \cdot \psi^*, \tag{2}$$

for any vector field X tangent to M .

Another important formula is the well-known spinorial Gauss formula: if ∇^N and ∇ stand for the covariant derivatives on $\Gamma(\Sigma N)$ and $\Gamma(\Sigma M)$ respectively, then, for all $X \in TM$ and $\psi \in \Gamma(\Sigma N)$

$$(\nabla_X^N \psi)^* = \nabla_X \psi^* + \frac{1}{2} h(X) \cdot \psi^*, \tag{3}$$

where h is the second fundamental form of the immersion $M \hookrightarrow N$ viewed as a symmetric endomorphism of the tangent bundle of M .

Assume now that N^3 admits a non-trivial Killing spinor field of Killing constant $\eta \in \mathbb{C}$, i.e., a spinor field $\Phi \in \Gamma(\Sigma N)$ satisfying

$$\nabla_Y^N \Phi = \eta Y \cdot_N \Phi \quad (4)$$

for all vector field Y on N . Recall that η has to be real or pure imaginary and that Φ never vanishes on N , as a non-trivial parallel section for a modified connection (see [4],[5]). In what follows, we will consider the model spaces, with their standard metrics, \mathbb{R}^3 with $\eta = 0$, \mathbb{S}^3 with $\eta = 1/2$, and \mathbb{H}^3 with $\eta = i/2$ which are characterized by the fact that they admit a maximal number of linearly independent Killing spinor fields with constant η .

Let (e_1, e_2) be a positively oriented local orthonormal basis of $\Gamma(TM)$ such that (e_1, e_2, ν) is a positively oriented local orthonormal basis of $\Gamma(TN)|_M$. Denote by

$$\omega_3 = -e_1 \cdot_N e_2 \cdot_N \nu$$

the complex volume form on the complex Clifford bundle $\mathbb{C}lN$ and $\omega = e_1 \cdot e_2$ the real volume form on $\mathbb{C}lM$. Recall that ω_3 acts by Clifford multiplication as the identity on ΣN . Therefore, denoting $\varphi := \Phi^*$, formula (2) yields

$$(e_1 \cdot_N \Phi)^* = (-e_1 \cdot_N e_1 \cdot_N e_2 \cdot_N \nu \cdot_N \Phi)^* = e_2 \cdot \varphi = -e_1 \cdot \omega \cdot \varphi$$

$$(e_2 \cdot_N \Phi)^* = (-e_2 \cdot_N e_1 \cdot_N e_2 \cdot_N \nu \cdot_N \Phi)^* = -e_1 \cdot \varphi = -e_2 \cdot \omega \cdot \varphi$$

and

$$(\nu \cdot_N \Phi)^* = (-\nu \cdot_N e_1 \cdot_N e_2 \cdot_N \nu \cdot_N \Phi)^* = \omega \cdot \varphi.$$

Then, these last relations with Equations (3) and (4) show that

$$\forall X \in TM, \quad \nabla_X \varphi + \frac{1}{2} h(X) \cdot \varphi + \eta X \cdot \omega \cdot \varphi = 0 \quad (5)$$

Recall that the spinor bundle ΣM splits into

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M$$

where $\Sigma^\pm M$ is the ± 1 -eigenspace for the action of the complex volume forme $\omega_2 = i\omega$. Under this decomposition, we will denote $\varphi = \varphi^+ + \varphi^-$, and define $\bar{\varphi} := \varphi^+ - \varphi^-$. Therefore Equation (5) is equivalent to

$$\nabla_X \varphi + \frac{1}{2} h(X) \cdot \varphi - i\eta X \cdot \bar{\varphi} = 0.$$

The ambient spinor bundle ΣN can be endowed with a Hermitian inner product $(\cdot, \cdot)_N$ for which Clifford multiplication by any vector tangent to N is skew-symmetric. This product induces another Hermitian inner product on ΣM , denoted by (\cdot, \cdot) making the identification of Proposition 2.1 an isometry. Now, relation (2) shows that Clifford multiplication by any vector tangent to M is skew-symmetric with respect to (\cdot, \cdot) .

PROPOSITION 2.2 *If $\eta \in \mathbb{R}$, then φ has constant length. If $\eta \in i\mathbb{R}^*$, then for all vector X tangent to M ,*

$$X|\varphi|^2 = 2\Re(i\eta X \cdot \bar{\varphi}, \varphi) .$$

Proof . — Since Clifford multiplication by any vector tangent to M is skew-symmetric with respect to (\cdot, \cdot) , we have $\Re(Y \cdot \varphi, \varphi) = 0$ for all $Y \in TM$. Taking this fact into account and computing

$$X|\varphi|^2 = 2\Re(\nabla_X \varphi, \varphi)$$

with the help of formula (5), completes the proof. □

Recalling that the Dirac operator D is defined on $\Gamma(\Sigma M)$ by

$$D = e_1 \cdot \nabla_{e_1} + e_2 \cdot \nabla_{e_2} ,$$

we compute directly that

$$D\varphi = H\varphi + 2\eta\omega \cdot \varphi = H\varphi - 2i\eta\bar{\varphi}$$

where H is the mean curvature of the immersion $M \hookrightarrow N$. It is well known that the action of the Dirac operator satisfies $(D\varphi)^\pm = D\varphi^\mp$ (see [11],[5]). Therefore, we note that

$$D(\varphi^\pm) = (H \pm 2i\eta)\varphi^\mp . \tag{6}$$

We have as in [6] the following

PROPOSITION 2.3 *Let M^2 be a minimal surface in N^3 . Then the restriction of any Killing spinor Φ with constant η on N^3 restricts to an eigenspinor φ^* on the surface M :*

$$D\varphi^* = 2\eta\varphi^*$$

Moreover, if η is real, then φ^ has constant length.*

Proof . — Since $H = 0$, we have

$$D(\varphi^\pm) = \pm 2i\eta\varphi^\mp .$$

Therefore, it suffices to define $\varphi^* = \varphi^+ + i\varphi^-$. □

3. Solutions of the restricted Killing spinor equation

Let (M^2, g) be an oriented, 2-dimensional Riemannian manifold with a spin structure. We endow the spinor bundle ΣM with a Hermitian inner product (\cdot, \cdot) for which Clifford multiplication by any vector tangent to M is skew-symmetric.

We study now some properties of a given solution $\varphi \in \Gamma(\Sigma M)$ of the following equation

$$\nabla_X \varphi + T(X) \cdot \varphi - i\eta X \cdot \bar{\varphi} = 0, \quad (7)$$

or equivalently

$$\nabla_X \varphi + T(X) \cdot \varphi + \eta X \cdot \omega \cdot \varphi = 0, \quad (8)$$

where T stand for a symmetric endomorphism of the tangent bundle of M , and $\eta \in \mathbb{R} \cup i\mathbb{R}$.

In view of the preceding section and for reasons which will become clearer later, we will call this equation the *restricted Killing spinor equation*. The following proposition shows the role of solutions of the restricted Killing spinor equation in the theory of surfaces in \mathbb{R}^3 , \mathbb{S}^3 and \mathbb{H}^3 . In fact, we see that the integrability conditions for such sections of the spinor bundle are exactly the Gauß and Codazzi-Mainardi equations.

In the following, (e_1, e_2) denotes a positively oriented local orthonormal basis of $\Gamma(TM)$.

PROPOSITION 3.1 *Assume that (M^2, g) admits a non trivial solution of Equation (7) and let $S = 2T$, then*

$$(\nabla_X S)(Y) = (\nabla_Y S)(X) \quad (\text{Codazzi-Mainardi Equation}),$$

and

$$R_{1212} - \det(S) = 4\eta^2 \quad (\text{Gauß Equation}),$$

where $R_{1212} = g(R(e_1, e_2) e_2, e_1)$, and R is the Riemann tensor of M .

Proof. — Let φ a non-trivial solution of (7). We compute the action of the spinorial curvature tensor \mathcal{R} on φ defined for all $X, Y \in TM$ by

$$\mathcal{R}(X, Y)\varphi = \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X, Y]}\varphi.$$

Since it is skew-symmetric and $\dim M = 2$, with the help of formula (8), we only compute

$$\begin{aligned} \nabla_{e_1} \nabla_{e_2} \varphi &= \nabla_{e_1} (-T(e_2) \cdot \varphi - \eta e_2 \cdot \omega \cdot \varphi) \\ &= \nabla_{e_1} (-T(e_2) \cdot \varphi - \eta e_1 \cdot \varphi) \\ &= -\nabla_{e_1} T(e_2) \cdot \varphi - T(e_2) \cdot \nabla_{e_1} \varphi - \eta \nabla_{e_1} e_1 \cdot \varphi - \eta e_1 \cdot \nabla_{e_1} \varphi \\ &= -\nabla_{e_1} T(e_2) \cdot \varphi + T(e_2) \cdot T(e_1) \cdot \varphi - \eta T(e_2) \cdot e_2 \cdot \varphi \\ &\quad - \eta \nabla_{e_1} e_1 \cdot \varphi + \eta e_1 \cdot T(e_1) \cdot \varphi - \eta^2 e_1 \cdot e_2 \cdot \varphi \end{aligned}$$

as well as

$$\begin{aligned} \nabla_{e_2} \nabla_{e_1} \varphi &= -\nabla_{e_2} T(e_1) \cdot \varphi + T(e_1) \cdot T(e_2) \cdot \varphi + \eta T(e_1) \cdot e_1 \cdot \varphi \\ &\quad + \eta \nabla_{e_2} e_2 \cdot \varphi - \eta e_2 \cdot T(e_2) \cdot \varphi + \eta^2 e_1 \cdot e_2 \cdot \varphi. \end{aligned}$$

So, taking into account that $[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1$, a straightforward computation gives

$$\begin{aligned} \mathcal{R}(e_1, e_2)\varphi &= \left((\nabla_{e_2} T)(e_1) - (\nabla_{e_1} T)(e_2) \right) \cdot \varphi \\ &\quad - \left(T(e_1) \cdot T(e_2) - T(e_2) \cdot T(e_1) \right) \cdot \varphi \\ &\quad - 2\eta^2 e_1 \cdot e_2 \cdot \varphi \end{aligned} \tag{9}$$

On the other hand, it is well known that this spinorial curvature tensor corresponds to the Riemann tensor R of M via the relation

$$\mathcal{R}(e_1, e_2)\varphi = -\frac{1}{2} R_{1212} e_1 \cdot e_2 \cdot \varphi. \tag{10}$$

Now, it is easy to see that

$$T(e_1) \cdot T(e_2) - T(e_2) \cdot T(e_1) = 2 \det(T) e_1 \cdot e_2$$

and therefore, if we put $S = 2T$ and define the *function*

$$G := R_{1212} - \det(S) - 4\eta^2$$

and the *vector field*

$$C := (\nabla_{e_1} S)(e_2) - (\nabla_{e_2} S)(e_1),$$

Equations (9) and (10) yield

$$C \cdot \varphi = G e_1 \cdot e_2 \cdot \varphi.$$

Note that $e_1 \cdot e_2 \cdot \varphi = -i\bar{\varphi}$, hence

$$C \cdot \varphi^\pm = \pm i G \varphi^\mp.$$

Applying two times this relation, it suffices to note that

$$\|C\|^2 \varphi^\pm = -G^2 \varphi^\pm,$$

and so $C = 0$ and $G = 0$. □

Note that up to rescaling, we can take $\eta = 0, 1/2$, or $i/2$. The case $\eta = 0$ is treated in [6] and is the starting point of the proof of Theorem 1.1. We will discuss the cases $\eta = 1/2$ and $\eta = i/2$ separately. We begin by

LEMMA 3.2 *Let φ be a non trivial solution of the restricted Killing spinor equation (7). Then*

- if $\eta = 1/2$, φ has constant norm and the symmetric endomorphism T , viewed as a covariant symmetric 2-tensor, is given by

$$T(X, Y) = \frac{1}{2} \Re(X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \varphi / |\varphi|^2)$$

- if $\eta = i/2$, φ satisfies $X|\varphi|^2 = -\Re(X \cdot \bar{\varphi}, \varphi)$ and one has

$$T(X, Y)|\varphi|^2 = \frac{1}{2}\Re(X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \varphi) + \frac{1}{2}\left(|\varphi^-|^2 - |\varphi^+|^2\right)g(X, Y)$$

Proof. — The first claim of each case is proved in Proposition 2.2. Let $T_{jk} = g(T(e_j), e_k)$, then, for $j = 1, 2$,

$$\nabla_{e_j} \varphi = -\sum_{k=1}^2 T_{jk} e_k \cdot \varphi + i\eta e_j \cdot \bar{\varphi}.$$

Taking Clifford multiplication by e_l and the scalar product with φ , we get

$$\Re(e_l \cdot \nabla_{e_j} \varphi, \varphi) = -\sum_{k=1}^2 T_{jk} \Re(e_l \cdot e_k \cdot \varphi, \varphi) + \Re(i\eta e_l \cdot e_j \cdot \bar{\varphi}, \varphi).$$

Since $\Re(e_l \cdot e_k \cdot \varphi, \varphi) = -\delta_{lk}|\varphi|^2$, it follows, by symmetry of T

$$\Re(e_l \cdot \nabla_{e_j} \varphi + e_j \cdot \nabla_{e_l} \varphi, \varphi) = 2T_{lj}|\varphi|^2 - 2\Re(i\eta \bar{\varphi}, \varphi)\delta_{lj}.$$

This completes the proof by taking $\eta = 1/2$ or $\eta = i/2$. \square

Now, we prove that the necessary conditions on a spinor field $\psi \in \Gamma(\Sigma M)$ obtained in the previous section (i.e. Proposition 2.2 and Equation (6)) are enough to prove that ψ is a solution of the restricted Killing spinor equation.

The case $\eta = 1/2$: Consider a non-trivial spinor field ψ of constant length, which satisfies $D\psi^\pm = (H \pm i)\psi^\mp$. Define the following 2-tensors on (M^2, g)

$$T^\pm(X, Y) = \Re(\nabla_X \psi^\pm, Y \cdot \psi^\mp).$$

First note that

$$\text{tr} T^\pm = -\Re(D\psi^\pm, \psi^\mp) = -\Re((H \pm i)\psi^\mp, \psi^\mp) = -H|\psi^\mp|^2. \quad (11)$$

We have the following relations

$$\begin{aligned} T^\pm(e_1, e_2) &= \Re(\nabla_{e_1} \psi^\pm, e_2 \cdot \psi^\mp) = \Re(e_1 \cdot \nabla_{e_1} \psi^\pm, e_1 \cdot e_2 \cdot \psi^\mp) \\ &= \Re(D\psi^\pm, e_1 \cdot e_2 \cdot \psi^\mp) - \Re(e_2 \cdot \nabla_{e_2} \psi^\pm, e_1 \cdot e_2 \cdot \psi^\mp) \\ &= \Re((H \pm i)\psi^\mp, e_1 \cdot e_2 \cdot \psi^\mp) + \Re(\nabla_{e_2} \psi^\pm, e_1 \cdot \psi^\mp) \\ &= |\psi^\mp|^2 + T^\pm(e_2, e_1). \end{aligned} \quad (12)$$

LEMMA 3.3 *The 2-tensors T^\pm are related by the equation*

$$|\psi^+|^2 T^+ = |\psi^-|^2 T^-$$

Proof. — This relation is trivial at any point $p \in M$ where $|\psi^+|^2$ or $|\psi^-|^2$ vanishes. Therefore we can assume in the following that both spinors ψ^+ and ψ^- are not zero in the neighbourhood of a point in M .

With respect to the scalar product $\Re(\cdot, \cdot)$, the spinors

$$e_1 \cdot \frac{\psi^-}{|\psi^-|} \quad \text{and} \quad e_2 \cdot \frac{\psi^-}{|\psi^-|}$$

form a local orthonormal basis of $\Gamma(\Sigma^+M)$. Hence, in this basis, we can write

$$\begin{aligned} \nabla_X \psi^+ &= \Re(\nabla_X \psi^+, e_1 \cdot \frac{\psi^-}{|\psi^-|}) e_1 \cdot \frac{\psi^-}{|\psi^-|} + \Re(\nabla_X \psi^+, e_2 \cdot \frac{\psi^-}{|\psi^-|}) e_2 \cdot \frac{\psi^-}{|\psi^-|} \\ &= \frac{T^+(X)}{|\psi^-|^2} \cdot \psi^- \end{aligned}$$

where the vector field $T^+(X)$ is defined by

$$g(T^+(X), Y) = T^+(X, Y), \quad \forall Y \in TM.$$

In the same manner, we can show that

$$\nabla_X \psi^- = \frac{T^-(X)}{|\psi^+|^2} \cdot \psi^+.$$

Since ψ has constant length, for all vector X tangent to M , we have

$$\begin{aligned} 0 &= X|\psi|^2 = X(|\psi^+|^2 + |\psi^-|^2) \\ &= 2\Re(\nabla_X \psi^+, \psi^+) + 2\Re(\nabla_X \psi^-, \psi^-) \\ &= 2\Re(W(X) \cdot \psi^-, \psi^+) \end{aligned} \tag{13}$$

with

$$W(X) = \frac{T^+(X)}{|\psi^-|^2} - \frac{T^-(X)}{|\psi^+|^2}.$$

To conclude, it suffices to note that Equations (11) and (12) imply W is traceless and symmetric, and that Equation (13) implies that W has rank less or equal to 1. This obviously implies $W = 0$. \square

PROPOSITION 3.4 *Assume that there exists on (M^2, g) a non-trivial solution ψ of the equation $D\psi = H\psi - i\bar{\psi}$ with constant length. Then such a solution satisfies the restricted Killing spinor equation with $\eta = 1/2$.*

Proof. — Let $F := T^+ + T^-$. Lemma 3.3 and the beginning of its proof imply

$$\frac{F}{|\psi|^2} = \frac{T^+}{|\psi^-|^2} = \frac{T^-}{|\psi^+|^2}.$$

Hence $F/|\psi|^2$ is well defined on the whole surface M , and

$$\nabla_X \psi = \nabla_X \psi^+ + \nabla_X \psi^- = \frac{F(X)}{|\psi|^2} \cdot \psi \quad (14)$$

where the vector field $F(X)$ is defined by $g(F(X), Y) = F(X, Y)$, $\forall Y \in TM$. Note that by Equation (12), the 2-tensor F is not symmetric. Define now the symmetric 2-tensor

$$T(X, Y) = -\frac{1}{2|\psi|^2} (F(X, Y) + F(Y, X)) .$$

Observe that T is defined as in Lemma 3.2. It is straightforward to show that

$$\begin{aligned} T(e_1, e_1) &= -F(e_1, e_1)/|\psi|^2, & T(e_2, e_2) &= -F(e_2, e_2)/|\psi|^2, \\ T(e_1, e_2) &= -F(e_1, e_2)/|\psi|^2 + \frac{1}{2} & \text{and} & \quad T(e_2, e_1) = -F(e_2, e_1)/|\psi|^2 - \frac{1}{2} \end{aligned}$$

once more by Equation (12). Taking into account these last relations in Equation (14), we conclude

$$\nabla_X \psi = -T(X) \cdot \psi - \frac{1}{2} X \cdot \omega \cdot \psi .$$

□

The case $\eta = i/2$:

PROPOSITION 3.5 *Assume that there exists on (M^2, g) a nowhere vanishing solution ψ of the equation $D\psi = H\psi + \bar{\psi}$. Then, if this solution satisfies*

$$X|\psi|^2 = -\Re(X \cdot \bar{\psi}, \psi), \quad \forall X \in \Gamma(TM),$$

then it is solution of the restricted Killing spinor equation with $\eta = i/2$.

Proof . — Defining the 2-tensors T^\pm as in the previous case, we get

$$\text{tr} T^\pm = -(H \mp 1)|\psi^\mp|^2, \quad (15)$$

and

$$T^\pm(e_1, e_2) = T^\pm(e_2, e_1). \quad (16)$$

First note that

$$-\Re(X \cdot \bar{\psi}, \psi) = -\Re(X \cdot \psi^+, \psi^-) + \Re(X \cdot \psi^-, \psi^+) = 2\Re(X \cdot \psi^-, \psi^+).$$

Therefore, following the proof of Lemma 3.3, we get

$$\Re(X \cdot \psi^-, \psi^+) = \Re(W(X) \cdot \psi^-, \psi^+) \quad (17)$$

with

$$W(X) = \frac{T^+(X)}{|\psi^-|^2} - \frac{T^-(X)}{|\psi^+|^2} .$$

As in the previous case, Equations (15), (16) and (17) imply that $W - \text{Id}_{TM}$ is a symmetric, traceless endomorphism of rank not greater than 1, hence $W = \text{Id}_{TM}$ and we have the relation

$$|\psi^+|^2 T^+ - |\psi^-|^2 T^- = |\psi^+|^2 |\psi^-|^2 g .$$

Therefore, if we define the symmetric 2-tensor $F = T^+ + T^- + \frac{1}{2}(|\psi^+|^2 - |\psi^-|^2)g$, we have on the whole surface M

$$\frac{F}{|\psi|^2} = \frac{T^+ + T^- + (|\psi^+|^2 - |\psi^-|^2)g}{|\psi^+|^2 + |\psi^-|^2} = \frac{T^-}{|\psi^+|^2} + \frac{1}{2}g = \frac{T^+}{|\psi^-|^2} - \frac{1}{2}g .$$

On the other hand, we get

$$\nabla_X \psi = \nabla_X \psi^+ + \nabla_X \psi^- = \frac{T^+(X)}{|\psi^-|^2} \cdot \psi^- + \frac{T^-(X)}{|\psi^+|^2} \cdot \psi^+ .$$

These two last equations imply

$$\nabla_X \psi = \frac{F(X)}{|\psi|^2} \cdot (\psi^+ + \psi^-) + \frac{1}{2}X \cdot \psi^- - \frac{1}{2}X \cdot \psi^+ ,$$

which is equivalent to

$$\nabla_X \psi = -T(X) \cdot \psi - \frac{1}{2}X \cdot \bar{\psi} .$$

Naturally, we put $T = -\frac{F}{|\psi|^2}$ and note that T is defined as in Lemma 3.2. □

4. Surfaces in \mathbb{S}^3 or \mathbb{H}^3

We are now able to generalize Theorem 1.1 to surfaces in \mathbb{S}^3 or \mathbb{H}^3 . In section 2, we saw that an oriented, immersed surface $M^2 \hookrightarrow \mathbb{S}^3$ (resp. \mathbb{H}^3) inherits an induced metric g , a spin structure, and a solution φ of

$$D\varphi = H\varphi - i\bar{\varphi} \quad (\text{resp. } D\varphi = H\varphi + \bar{\varphi}) \tag{18}$$

with constant length (resp. with $X|\varphi|^2 = -\Re(X \cdot \bar{\varphi}, \varphi)$ for all vector X tangent to M). This spinor field φ on M^2 is the restriction of a real (resp. imaginary) Killing spinor field in \mathbb{S}^3 (resp. \mathbb{H}^3). Section 3 shows that at least locally the converse is true. Assume that there exists a solution of Equation (18) on an oriented, 2-dimensional Riemannian manifold (M^2, g) endowed with a spin structure, for a given function $H : M \rightarrow \mathbb{R}$. Then this solution satisfies the restricted Killing spinor equation with a well defined endomorphism $T : TM \rightarrow TM$ with $\text{tr}T = H$. Moreover, there exists an isometric immersion $(M^2, g) \hookrightarrow \mathbb{S}^3$ (resp. \mathbb{H}^3) with second fundamental form $S = 2T$.

THEOREM 4.1 *Let (M^2, g) be an oriented, 2-dimensional manifold and $H : M \rightarrow \mathbb{R}$ a smooth function. Then the following data are equivalent:*

1. An isometric immersion $(\tilde{M}^2, g) \rightarrow \mathbb{S}^3$ of the universal covering \tilde{M}^2 into the 3-dimensional round sphere \mathbb{S}^3 with mean curvature H .
2. A solution φ of the Dirac equation

$$D\varphi = H\varphi - i\bar{\varphi}$$

with constant length.

3. A pair (φ, T) consisting of a symmetric endomorphism T such that $\text{tr}(T) = H$ and a spinor field φ satisfying the equation

$$\nabla_X \varphi + T(X) \cdot \varphi - \frac{i}{2} X \cdot \bar{\varphi} = 0.$$

THEOREM 4.2 Let (M^2, g) be an oriented, 2-dimensional manifold and $H : M \rightarrow \mathbb{R}$ a smooth function. Then the following data are equivalent:

1. An isometric immersion $(\tilde{M}^2, g) \rightarrow \mathbb{H}^3$ of the universal covering \tilde{M}^2 into the 3-dimensional hyperbolic space \mathbb{H}^3 with mean curvature H .
2. A nowhere vanishing solution φ of the Dirac equation

$$D\varphi = H\varphi + \bar{\varphi}$$

satisfying

$$X|\varphi|^2 = -\Re(X \cdot \bar{\varphi}, \varphi) \quad \forall X \in \Gamma(TM).$$

3. A pair (φ, T) consisting of a symmetric endomorphism T such that $\text{tr}(T) = H$ and a spinor field φ satisfying the equation

$$\nabla_X \varphi + T(X) \cdot \varphi + \frac{1}{2} X \cdot \bar{\varphi} = 0 \quad \forall X \in \Gamma(TM).$$

REMARK 4.3 It has been pointed out to us that the case of surfaces in \mathbb{S}^3 has already been treated by Leonard Voss (*Diplomarbeit, Humboldt-Universität zu Berlin*, unpublished).

5. Hypersurfaces in \mathbb{R}^4

We conclude by giving a characterization of hypersurfaces in the Euclidean 4-space in terms of a special section of the intrinsic spinor bundle of the hypersurface, in a very similar way to that of Theorem 1.1.

Let M^3 be an oriented hypersurface isometrically immersed into \mathbb{R}^4 , denote by ν its unit normal vector field. Then M^3 is endowed with a spin structure, canonically induced by that of \mathbb{R}^4 . Denote by ΣM the corresponding spinor bundle and $\Sigma^+ \mathbb{R}^4$ the bundle of positive spinors in \mathbb{R}^4 . We then have the analogous result of Proposition 2.1:

PROPOSITION 5.1 *There exists an identification of $\Sigma^+\mathbb{R}^4$ with ΣM , which after restriction to M , sends every spinor field $\psi \in \Gamma(\Sigma^+\mathbb{R}^4)$ to the spinor field denoted by $\psi^* \in \Gamma(\Sigma M)$. Moreover, if \cdot (resp. \cdot) stands for Clifford multiplication on $\Sigma^+\mathbb{R}^4$ (resp. ΣM), then one has*

$$(X \underset{\mathbb{R}^4}{\cdot} \nu \underset{\mathbb{R}^4}{\cdot} \psi)^* = X \cdot \psi^* , \tag{19}$$

for any vector field X tangent to M .

Recall the following definition

DEFINITION 5.2 *A symmetric 2-tensor $T \in S^2(M)$ is called a Codazzi tensor if it satisfies the Codazzi-Mainardi equation, i.e.*

$$(\nabla_X T)(Y) = (\nabla_Y T)(X) \quad \forall X, Y \in \Gamma(TM) ,$$

(T being viewed in this formula via the metric g as a symmetric endomorphism of the tangent bundle).

We now prove the following

THEOREM 5.3 *Let (M^3, g) be an oriented, 3-dimensional Riemannian manifold. Then the following data are equivalent:*

1. *An isometric immersion $(\widetilde{M}^3, g) \rightarrow \mathbb{R}^4$ of the universal covering \widetilde{M}^3 into the Euclidean space \mathbb{R}^4 with second fundamental form h .*
2. *A pair (φ, T) consisting of a Codazzi tensor T such that $2T = h$ and a non trivial spinor field φ satisfying, for any $X \in \Gamma(TM)$, the equation*

$$\nabla_X \varphi + T(X) \cdot \varphi = 0 .$$

Proof . — Let (M^3, g) be an oriented hypersurface isometrically immersed into \mathbb{R}^4 with second fundamental form h . Let ψ be any parallel positive spinor field on \mathbb{R}^4 . Denote by $\varphi := \psi^* \in \Gamma(\Sigma M)$ the restriction of ψ given by Proposition 5.1. Then Gauß formula (3) yields

$$\nabla_X \varphi + \frac{1}{2}h(X) \cdot \varphi = 0 .$$

Since h is a second fundamental form, it is clear that $T = \frac{1}{2}h$ is a Codazzi tensor and that (φ, T) give the desired pair.

Conversely, if (M^3, g) is an oriented, 3-dimensional Riemannian manifold admitting such a pair (φ, T) , then obviously Codazzi-Mainardi equation holds for $h = 2T$.

Therefore, the action of the spinorial curvature tensor on the spinor φ is given by

$$\mathcal{R}(X, Y)\varphi = \left(T(Y) \cdot T(X) - T(X) \cdot T(Y) \right) \cdot \varphi \quad (20)$$

Let (e_1, e_2, e_3) be a positively oriented local orthonormal basis of $\Gamma(TM)$. Then Equation (20) yields

$$\sum_{k \neq l} \left(\mathcal{R}_{ijkl} + 4T_{il}T_{jk} - 4T_{ik}T_{jl} \right) e_k \cdot e_l \cdot \varphi = 0$$

which imply in dimension 3 that each component

$$\mathcal{R}_{ijkl} + 4T_{il}T_{jk} - 4T_{ik}T_{jl}$$

is zero, since for $1 \leq k < l \leq 3$ and $1 \leq k' < l' \leq 3$,

$$\Re(e_k \cdot e_l \cdot \varphi, e_{k'} \cdot e_{l'} \cdot \varphi) = \pm \delta_{kk'} \delta_{ll'} |\varphi|^2 .$$

Therefore $h = 2T$ satisfies the Gauß equation. \square

REMARK 5.4 *Let (φ, T) be a pair as in Theorem 5.3 (2). Then necessarily the Codazzi tensor T has to be defined as the energy-momentum tensor associated with the spinor field φ (see for example [7], [8] or [13]). Such a special spinor field is then called a Codazzi Energy-Momentum spinor, and generalizes the notion of Killing spinors (see [1], [3], [13]).*

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