Actes du séminaire de

Théorie spectrale
et géométrie

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<http://tsg.cedram.org/item?id=TSG_2004-2005__23__105_0>

REAL AND COMPLEX TRANSVERSELY SYMPLECTIC ANOSOV FLOWS OF DIMENSION FIVE

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Résumé

Nous présentons plusieurs résultats de rigidité concernant les flots d'Anosov admettant transversalement des structures symplectiques réelles ou complexes de dimension 5.

Abstract

We present several rigidity results about five-dimensional real or complex transversely symplectic Anosov flows.

1. Real transversely symplectic flows

This exposition contains 4 sections. The first two are devoted to real transversely symplectic Anosov flows, while the other two sections are devoted to complex transversely symplectic Anosov flows and remarks. Let us begin with the real case.

Let ϕ_t be a C^∞ flow defined on a closed and connected manifold M. We denote by X the generator of ϕ_t. Then ϕ_t is said to be *transversely symplectic* if there exists on M a C^∞ closed 2-form ω such that $\operatorname{Ker}\omega = \mathbb{R}X$. Recall that by definition

$$\operatorname{Ker}\omega = \{u \in TM \mid \omega(u,v) = 0, \forall v \in T_{\pi(u)}M\},$$

where π denotes the canonical projection of TM onto M. Then by the Cartan formula, we get

$$\mathcal{L}_X\omega = di_X\omega + i_X d\omega = 0,$$

i.e. ω is ϕ_t-invariant. Firstly, we have the following trivial fact.

LEMMA 1.1 *Let ϕ_t be a transversely symplectic flow as above. Then there are two possibilities:*

1. *Either $X \equiv 0$ and M is of dimension even.*

2. *Or X vanishes nowhere and M is of dimension odd.*

Proof . — If $X \equiv 0$, then ω is non-degenerate. So the dimension of M must be even. If X is not identically zero, then there exists some point $x \in M$ such that $\text{Ker}\,\omega_x = \mathbb{R}X_x \neq 0$. Thus the restriction of ω onto a local transverse passing through x is non-degenerate. So the dimension of M must be odd. Thus for each point $y \in M$, $\text{Ker}\,\omega_y$ is not trivial. We deduce that X vanishes nowhere.   □

If $X \equiv 0$, then the study of (ϕ_t, ω) is just that of classical symplectic geometry, which is not the object of this article. So in the following, we suppose always that X vanishes nowhere, *i.e.* ϕ_t admits no fixed point. The orbits of ϕ_t define a C^∞ foliation on M, which is denoted by Φ. We have the following simple fact.

LEMMA 1.2 *Under the notation above, ϕ_t is transversely symplectic iff Φ admits a transverse symplectic structure.*

Proof . — Suppose firstly that ϕ_t is transversely symplectic. Take a local transverse Σ of Φ and denote by $i : \Sigma \to M$ the natural injection. Since ω is closed and $\text{Ker}\,\omega = \mathbb{R}X$, then $i^*\omega$ gives a symplectic structure on Σ. We need see that these symplectic structures defined on local transverses are invariant under holonomy maps. Take two transverses Σ_1 and Σ_2 related by a holonomy map H. Then there exists a C^∞ map $\alpha : \Sigma_1 \to \mathbb{R}$ such that

$$H(x) = \phi_{\alpha(x)}(x), \ \forall \ x \in \Sigma_1.$$

Thus we get

$$\omega_2(DH(u), DH(v)) = \omega_2(D\alpha(u) \cdot X + D\phi_{\alpha(x)}(u), D\alpha(v) \cdot X + D\phi_{\alpha(x)}(v))$$
$$= \omega(D\phi_{\alpha(x)}(u), D\phi_{\alpha(x)}(v)) = \omega_1(u, v).$$

So Φ admits a transverse symplectic structure.

Inversely, suppose that Φ admits a transverse symplectic structure. Then in each flow box, we can extend its tranverse symplectic form naturally by defining $\omega(X, \cdot) = 0$. It is easily seen that different pieces fit together to give a well-defined close 2-form ω such that $\text{Ker}\,\omega = \mathbb{R}X$, which is in addition ϕ_t-invariant by definition.   □

We deduce that

COROLLARY 1.1 *If ϕ_t is transversely symplectic, then so is any smooth time change of ϕ_t.*

Even though some of the results below are valid in general dimensions, we prefer to consider only the case of dimension 5. So in the following, we always suppose that M is a closed manifold of dimension 5.

Consider the vector field $\frac{\partial}{\partial x_1}$ defined on $\mathbb{R}^5$. It preserves the transverse symplectic form $dx_2 \wedge dx_3 + dx_4 \wedge dx_5$. The flow of $\frac{\partial}{\partial x_1}$ is said to be the *canonical transversely symplectic flow*. Then by the classical Theorem of G. Darboux in symplectic geometry, we get the following

LEMMA 1.3 *Each transversely symplectic flow is locally isomorphic to the canonical one above.*

For any $x \in M$, a subspace E of $T_x M$ is said to be *Lagrangian* if $X_x \notin E$, $\dim E = 2$ and $\omega \mid_E \equiv 0$. A C^1-submanifold Σ of M is said to be *Lagrangian* if for any $x \in \Sigma$, so is $T_x \Sigma$. A C^0-foliation with C^1-leaves $\mathcal{F}$ of M is said to be *Lagrangian* if so is each of its leaf. The set of Lagrangian submanifolds of M is acted on naturally by ϕ_t.

One of the principal ways to construct transversely symplectic flows is to take the restrictions of hamiltonian flows. Let $(\bar{M}, \bar{\omega})$ be a 6-dimensional symplectic manifold. Take an Hamiltonian function $H : \bar{M} \to \mathbb{R}$. Denote by X_H the corresponding field of H, i.e. $i_{X_H} \bar{\omega} = -dH$. Thus the subset $H^{-1}(1)$ is invariant under the flow of X_H denoted by ϕ_t^H. Suppose that $H^{-1}(1)$ is a C^∞ submanifold of $\bar{M}$. Then we get by the definition of X_H that $T(H^{-1}(1)) = \mathrm{Ker}(i_{X_H} \bar{\omega})$. Take at any point $x \in H^{-1}(1)$ a dual basis

$$\{X_H(x), e_2, f_1, f_2, g_1, g_2\}.$$

Then $\mathrm{Ker}(i_{X_H} \bar{\omega})$ must be contained in the vector space generated by

$$\{X_H(x), f_1, f_2, g_1, g_2\}.$$

Since $\bar{\omega}$ is non-denegerate, then we get $T_x(H^{-1}(1)) = \bigvee \{X_H(x), f_1, f_2, g_1, g_2\}$. Thus $\phi_t^H \mid_{H^{-1}(1)}$ is transversely symplectic with respect to $\bar{\omega} \mid_{H^{-1}(1)}$. Here are some examples.

1. Consider $\mathbb{C}^3$. For any α and β in $\mathbb{C}^3$, denote by $\langle \alpha, \beta \rangle$ the natural Hermitian product of these two vectors. Then define

$$\bar{\omega}(\alpha, \beta) = \Im \langle \alpha, \beta \rangle.$$

Thus it is easily seen that $(\mathbb{C}^3, \bar{\omega})$ is a symplectic manifold. Define $H : \mathbb{C}^3 \to \mathbb{R}$ such that $H(\alpha) = \mid \alpha_1 \mid^2 + \mid \alpha_2 \mid^2 + \mid \alpha_3 \mid^2$. Then we have $H^{-1}(1) = \mathbb{S}^5$, $X_H(\alpha) = i\alpha$ and $\phi_t^H(\alpha) = e^{it}\alpha$. ϕ_t^H acts on $\mathbb{S}^5$ properly and freely. Thus the study of $(\phi_t^H, \mathbb{S}^5)$ is equivalent to that of the symplectic geometry of $\mathbb{C}P^2$.

2. Let N be a closed connected manifold and let $\mathbb{L} : TN \to \mathbb{R}$ be a smooth convex superlinear Lagrangian (see [CIPP] for the details). Then the Euler-Lagrange

equation associated to $\mathbb{L}$ defines a complete flow ϕ_t on TN. Using the Legendre transformation, we can see that ϕ_t is Hamiltonian. So the restrictions of ϕ_t to the energy levels are transversely symplectic.

For example, if we fix a Riemannian metric g on N and define $\mathbb{L} : TN \to \mathbb{R}$ such that $\mathbb{L}(u) = g(u, u)$, then the restriction of ϕ_t onto the energy-1 submanifold is just the geodesic flow of g.

3. Let (N, g) be a Riemannian manifold and let Ω be a C^∞ closed 2-form on N, which modelises the effect of a magnetic field. Denote by λ the Liouville form on TN and define
$$\omega = d\lambda + \pi^*\Omega,$$
where π denotes the natural projection of TN onto N. It is easy to see that ω is a symplectic form on TN. Take the Hamiltonian $H : TN \to \mathbb{R}$ such that $H(u) = g(u, u)$. Then the restriction onto T^1N of the Hamiltonian flow of H with respect to ω is transversely symplectic, which is denoted by ϕ_t and is called the magnetic flow of (N, g, Ω). If Ω is exact, then its magnetic flow is Euler-Lagrange.

From the examples above, we see two extremities. On one hand, the flow has no dynamic, as in example 1. So in this case, we return to symplectic geometry. On the other hand, perfectly chaotic transversely symplectic flows exist. For example, if the sectional curvatures of a Riemannian manifold are strictly negative, then its geodesic flow is chaotic and Anosov. In this exposition, we are interested in transversely symplectic Anosov flows, whose definition is recalled below.

2. Rigidity of real transversely symplectic Anosov flows

Let ϕ be a C^∞-flow defined on a closed connected manifold M. ϕ_t is said to be *Anosov* if there exist two continuous and ϕ_t-invariant sub vector bundles E^{su} and E^{ss} of TM and two positive numbers a and b such that the following conditions are satisfied:

1. $TM = E^{su} \oplus \mathbb{R}X \oplus E^{ss}$.

2. $\| D\phi_{-t}(u) \| \le a\mathrm{e}^{-bt} \| u \|$, $\forall\, t > 0$ and $u \in E^{su}$.

3. $\| D\phi_t(u) \| \le a\mathrm{e}^{-bt} \| u \|$, $\forall\, t > 0$ and $u \in E^{ss}$.

It is well-known that E^{su} and E^{ss} integrate both to C^0-foliations with C^∞-leaves, which are called respectively *dilating* and *contracting* foliations (see [HK]). These two C^0-foliations give two geometric invariants of ϕ_t, under flow conjugacies.

If ϕ_t is in addition transversely symplectic, then by the definition of an Anosov flow, it is easy to see that its dilating and contracting foliations are both Lagrangian.

Anosov flows are very chaotic. For example, they admit often plenty of dense orbits and periodic orbits. In addition, orbits of different types mix together in a

random way. We know that the geodesic flows of hyperbolic manifolds are Anosov. The charm is that small Euler-Lagrange perturbations of such flows remain to be Anosov. So in spite of of its internal beauty, the study of Anosov flows may be useful in our understanding of nature.

Based on a series of important works [G], [K], [FK] and [BFL] etc, we believe that dilating and contracting foliations of Anosov flows are rarely smooth. In my ph.D. thesis (see [F]), Anosov flows with smooth foliations are studied in detail. One of our results is the following classification of transversely symplectic Anosov flows with smooth foliations of dimension 5. Recall that in [FK], such 5-dimensional flows have also been studied.

THEOREM 2.1 *Let ϕ_t be a 5-dimensional transversely symplectic Anosov flow. If its dilating and contracting foliations are both C^∞, then up to finite covers and a constant change of time scale, ϕ_t is C^∞ flow equivalent either to a special time change of the geodesic flow of a closed 3-dimensional hyperbolic manifold, or to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^4$.*

Recall that a *special time change* of ϕ_t is just the flow of a vector field of form $\frac{X}{1+\alpha(X)}$, where α is a C^∞ closed 1-form on M such that $1 + \alpha(X) > 0$. The proof of this result is to be published in [F1] (see also my thesis [F]).

COROLLARY 2.1 *Let ϕ_t be a 5-dimensional Euler-Lagrange (or magnetic) flow. If it is Anosov with C^∞-foliations, then up to finite covers and a constant change of time scale, ϕ_t is C^∞ flow equivalent to a special time change of the geodesic flow of a closed 3-dimensional hyperbolic manifold.*

Proof . — We have seen that Euler-Lagrange (or magnetic) flows are transversely symplectic. So by Theorem 2.1, we get two possibilities. Since ϕ_t is Anosov, then by [CIPP], ϕ_t is defined on a manifold diffeomorphic to T^1N. The topology of T^1N is different from that of the suspension manifold of $\mathbb{T}^4$. So we deduce the conclusion.   □

In [HuK], the following elegant result is proved.

THEOREM 2.2 (S. HURDER-A. KATOK) *Let ϕ_t be a 3-dimensional volume-preserving flow. If ϕ_t is Anosov with C^2-foliations, then ϕ_t must have C^∞-foliations.*

By combining the results above with our results concerning quasi-conformal Anosov flows, we get the following result of similar type. Even though this result is not optimal, our idea is different from that of Hurder and Katok.

PROPOSITION 2.1 *Let ϕ_t be a 5-dimensional Euler-Lagrange flow. If ϕ_t is Anosov with C^{32}-foliations, then up to finite covers and a constant change of time*

scale, ϕ_t is C^∞ flow equivalent to a special time change of the geodesic flow of a closed 3-dimensional hyperbolic manifold.

In particular, ϕ_t must have C^∞-foliations.

Proof . — Since the dilating and contracting foliations of ϕ_t are supposed to be C^{32}, then it is enough for the arguments of Theorem 2.1 to go through (see also [BFL] and [BFL1]). However we can only conclude that up to finite covers, ϕ_t is C^{31}-orbit equivalent either to the geodesic flow of a closed 3-dimensional hyperbolic manifold, or to the suspension of a hyperbolic automorphism of $\mathbb{T}^4$.

By simple topological considerations, we can see that the second case is impossible for Euler-Lagrange flows. So up to finite covers, ϕ_t is C^{31}-orbit equivalent to the geodesic flow of a closed 3-dimensional hyperbolic manifold.

We deduce that ϕ_t is quasi-conformal and has the sphere-extension property (see Section 3 below and [F3] for details). Since in addition, ϕ_t is with C^{32}-foliations, then by [F3], we deduce that ϕ_t is, up to finite covers and a constant change of time scale, C^∞ flow equivalent to a special time change of the geodesic flow of a 3-dimensional hyperbolic manifold. So ϕ_t is with C^∞-foliations (see Chapter 4 of my thesis [F]).   □

3. Complex transversely symplectic Anosov flows

In this section, we consider complex transversely symplectic Anosov flows defined on closed connected manifolds of real dimension 5. An Anosov flow is said to be *complex transversely symplectic* if its orbit foliation admits a transversal (holonomy-invariant) holomorphic symplectic structure. By the classical theorem of Darboux, this transversal structure is locally isomorphic to $(\mathbb{C}^2, dz_1 \wedge dz_2)$. Recall also that an holomorphic Riemannian metric is by definition a non-degenerate $\mathbb{C}$-bilinear symmetric holomorphic tensor. For example, $(\mathbb{C}^2, dz_1 \cdot dz_2)$.

We know that there are rarely holomorphic geometric structures on compact complex manifolds (see [IKO] and [D]). The central, though quite simplified, reason is that there exist only trivial holomorphic functions on compact complex manifolds, *i.e.* the maximum principle.

On the other hand, we know that transversely holomorphic foliations could be rigid. For example, 1-dimensional transversely holomorphic foliations defined on closed 3-dimensional manifolds are classified in [B] and [G2].

By combing these two kinds of rigidity ideas, we study chaotic (more precisely, Anosov) 1-dimensional transversely holomorphic symplectic foliations defined on closed 5-dimensional manifolds. Our central, though trivial, idea is to replace maximum principle by dynamics and our result is the following.

THEOREM 3.1 *Let ϕ_t be a 5-dimensional Anosov flows. Then the following statements are equivalent.*

1. ϕ_t *admits transversally an holomorphic symplectic structure.*

2. ϕ_t *admits transversally an holomorphic Riemannian metric.*

3. *Up to finite covers and a constant change of time scale, ϕ_t is C^∞-orbit equivalent either to the geodesic flow of a closed 3-dimensional hyperbolic manifold, or to the suspension of a hyperbolic automorphism of the complex tori.*

Although general Anosov flows are very soft, under certain conditions, they may be rigid. For example, if they have smooth foliations, then they should be rigid (*i.e.* classifiable). Another way to produce rigidity is to suppose quasi-conformity (see [G1], [S], [F2] and [F3]). Let us recall firstly the definition.

An Anosov flow ϕ_t is said to be *quasi-conformal* if the following two functions are bounded, $K^\pm : M \times \mathbb{R} \to \mathbb{R}$, such that

$$K^\pm(x,t) = \frac{\text{Sup}\{\parallel D\phi_t(u) \parallel \mid \parallel u \parallel = 1, u \in E_x^{su(ss)}\}}{\text{Inf}\{\parallel D\phi_t(u) \parallel \mid \parallel u \parallel = 1, u \in E_x^{su(ss)}\}}.$$

For example, holomorphic functions defined on $\mathbb{C}$ are quasi-conformal. We deduce easily from this fact the following lemma.

LEMMA 3.1 *Let ϕ_t be a 5-dimensional transversely holomorphic Anosov flow. Then ϕ_t is quasi-conformal.*

Since ϕ_t is transversely holomorphically symplectic, then it is easy to see that ϕ_t preserves a volume form. Thus by [S] and [F2], we know that $E^{su} \oplus \mathbb{R}X$ and $E^{ss} \oplus \mathbb{R}X$ are both C^∞. These two distributions are known to be also integrable to C^0-foliations with C^∞-leaves, which are called respectively *weak dilating* and *weak contracting* foliations.

On each local transverse Σ, by intersecting weak dilating and weak contracting foliations with Σ, we obtain two C^∞-foliations on Σ, which are called respectively Σ-*dilating* and Σ-*contracting* foliations. The following simple lemma makes the situation much more comfortable.

LEMMA 3.2 *Let ϕ_t be an Anosov flow admitting a transverse holomorphic symplectic structure. Then on each local transverse Σ, the Σ-dilating and Σ-contracting foliations are both holomorphic.*

Proof . — Since ϕ_t is transversally holomorphic symplectic, then each local transverse is locally equivalent to $\mathbb{C}^2$. It is easy to see that the leaves of Σ-dilating and Σ-contracting foliations are all complex. In this way, we get a ϕ_t-invariant C^0-conformal structure along the leaves of dilating foliation of ϕ_t, which is denoted by τ^+. It is evident that τ^+ is C^∞ along the dilating leaves.

By [S], we can see that Σ-*contracting* holonomy maps are holomorphic. Thus we can define naturally on local transverses a complex structure using intersecting Σ-dilating and Σ-contracting leaves.

This complex structure coincides evidently with the initial complex structure because they have the same underlying almost complex structure. In addition, Σ-dilating and Σ-contracting foliations are certainly holomorphic by definition with respect to this complex structure. $\square$

Proof of the equivalence between (1) and (2). — Suppose firstly that ϕ_t admits a transverse holomorphic symplectic structure ω. Since ϕ_t is Anosov, then Σ-dilating and Σ-contracting leaves are all Lagrangian with respect to ω. Thus by Lemma 3.2, we can define a transverse holomorphic Riemannian structure as following:

$$g(u^+, v^+) = g(u^-, v^-) = 0, \; g(u^+, u^-) = g(u^-, u^+) = \omega(u^+, u^-).$$

Inversely, we can define as following a transverse holomorphic structure ω from a transverse holomorphic Riemannian metric g:

$$\omega(u^+, v^+) = \omega(u^-, v^-) = 0, \; \omega(u^+, u^-) = -\omega(u^-, u^+) = g(u^+, u^-).$$

$\square$

Proof of the equivalence between (2) and (3). — Suppose that ϕ_t admits a transverse holomorphic Riemannian metric g. Since ϕ_t is Anosov and volume-preserving, then ϕ admits a dense orbit. So the complex sectional curvature of g is constant. Thus locally g is, up to a dilatation, isomorphic to one of the following two models:

1. $(\mathbb{C}^2, dz_1 \cdot dz_2)$. The connected component of the group of holomorphic isometries is $\mathbb{C}^2 \rtimes \mathbb{C}^*$.

2. $((\mathbb{C}P^2 \times \mathbb{C}P^2) \backslash \Delta, \frac{dz_1 \cdot dz_2}{(z_1 - z_2)^2})$, where Δ denotes the diagonal. The connected component of the group of holomorphic isometries is $PSL(2, \mathbb{C})$, which acts by fractional maps.

So ϕ_t admits a transverse (G, X)-structure. By [S], we can see that the conformal structure τ^+ defined in Lemma 3.2 is complete along the leaves of dilating foliation, *i.e.* each leaf is isomorphic to $\mathbb{C}$. So the developing map sends each dilating leaf onto the corresponding isotropy leaf of the model space. Now we can use exactly the same argument as in [F3] to prove (3).

Inversely, if (3) is true, then it is easily checked that ϕ_t is transversely holomorphically Riemannian. Recall just that $(\mathbb{C}P^2 \times \mathbb{C}P^2) \backslash \Delta$ is the lifted geodesic space of $\mathbb{H}^3$ and $PSL(2, \mathbb{C})$ is the connected component of the isometry group of $\mathbb{H}^3$. $\square$

4. Remarks

This exposition comes from my desire to understand the interactions between certain geometries and dynamics. In Sections 1. and 2., we tried to compare 4-dimensional symplectic geometry with 5-dimensional transversely symplectic flows.

If the flow is chaotic, then what we can view is a dynamical 4-dimensional symplectic manifold.

We have several meaningful examples of such transversely symplectic flows and global invariants are needed to push further the understanding of such flows.

In section 3, we tried to compare holomorphic symplectic surfaces with chaotic 5-dimensional transversely holomorphic symplectic flows. We proved the rigidity of such flows in the Anosov case. In addition, we believe that 5-dimensional transversely holomorphic Anosov flows are also rigid.

Because of our personal preference, we have worked exclusively on the Anosov case. However, in this case, dynamic seems to be too strong and dominates completely the transverse geometry. So rigidity is often obtain, which is, though elegant, rather dispointing from geometric point of view.

So it will be interesting for me to find and understand meaningful situations where (symplectic or complex) geometry and dynamics live together and no one dominates the other. For example, holomorphic automorphisms of $K3$-surfaces studied by S. Cantat seem to be rather satisfactory and pleasant with respect to the point of view of this remark.

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