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EINSTEIN MANIFOLDS, VOLUME RIGIDITY AND SEIBERG-WITTEN THEORY

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1. Introduction.

These notes stem from some talks we gave at the Institut Fourier of Grenoble in 1998, where we compared Besson-Courtois-Gallot's and Lebrun's approaches to uniqueness of Einstein metrics on real and complex hyperbolic 4-manifolds.

The *Ricci tensor* of a Riemannian manifold (X, g) is the symmetric bilinear form defined on each tangent space by taking the trace of the curvature operator R_g, that is: $\mathrm{Ric}_g(u, v) = \mathrm{Tr}_g R_g(u, \cdot, v, \cdot)$. If u is a unit tangent vector, then $\mathrm{Ric}_g(u) = \mathrm{Ric}_g(u, u)$ is the *Ricci curvature* in the direction of u. An Einstein metric on a differentiable n-manifold is a Riemannian metric whose Ricci curvature is constant in every direction, or, equivalently, which satisfies $\mathrm{Ric}_g = \lambda \cdot g$ for some constant λ. By rescaling the metric, one can always assume that $\lambda = \pm(n-1)$ or 0: we shall say, in this case, that g is a *normalized* Einstein metric. The sign of λ is called the *sign* of the Einstein metric.

Ricci curvature and Einstein metrics arise quite naturally in Riemannian geometry as well as in many other different contexts: they are related to topology and to the convergence theory of Riemannian manifolds (cp. [14], [12], [16] and the recent works of Cheeger and Colding [13]), to complex geometry (the main motivation being Yau and Aubin's solution of the Calabi conjecture, and Ahlfors-Schwarz Lemma), to the well-developed theories of symmetric spaces and homogeneous manifolds (see [17], [4]) and, of course, to gravitational field theory (see [28]). Two definitely good references are [4] (even if not up-to-date) and [11] (focusing on dimension 4). There also exist several short surveys such as [3], [22], [1].

A natural, mostly unsettled, problem is trying to classify Einstein metrics on a given space X, up to homotheties: this set is the so-called *Moduli Space* $\mathcal{E}(X)$ of Einstein structures. This is mainly motivated by the fact that equivalence classes of metrics of constant Ricci curvature on a given manifold X are seen to form, at least locally, a "reasonable"

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space (a real analytic subset of a smooth real analytic manifold of finite dimension), by a theorem of Koiso [4]. [1]

The only examples where one knows a good description of the moduli space, in dimension 4, essentially are: the case of tori (where every Einstein metric is easily seen to be flat, and $\mathcal{E}(T^4)$ is the quotient, by a discrete group, of a convex, open subset of a vector space of dimension 9) and the case of $K3$-surfaces (in this case one can prove that any Einstein metric is Kähler with respect to some complex structure, and $\mathcal{E}(K3)$ is the quotient, by a discrete group, of an open set of the symmetric space $SO(3,19)/(SO(3)\times SO(19))$).

In this paper we shall be concerned with two (at present) exceptional cases of rigidity, i.e. where $\mathcal{E}(X) = \{*\}$:

THEOREM 1.1 (G. Besson-G. Courtois-S. Gallot [6]). — *Every Einstein metric on a closed real hyperbolic 4-manifold (X, g_0) is homothetic to the real hyperbolic metric g_0.*

THEOREM 1.2 (C. Lebrun [19]). — *Every Einstein metric on a closed complex hyperbolic 4-manifold (X, g_0) is homothetic to the complex hyperbolic metric g_0.*

Recall that a real hyperbolic manifold (X, g_0) is a Riemannian manifold with constant sectional curvature $k(g_0) = -1$, while a complex hyperbolic manifold is a regular quotient of the complex hyperbolic space form $H^n(\mathbf{C}) = U(n+1)/(U(1)\times U(n))$ (a complex hyperbolic 4-manifold is a special case of complex surface of general type).

In both cases, it turns out that the exceptional Einstein metrics are minima of some Riemannian functional: the volume-entropy functional, in the real case, and the L^2-scalar curvature in the complex one. Actually the philosophy behind Theorems 1.1 and 1.2 can be summarized by the following

ASSERTION 1.3. — *On a real or complex hyperbolic 4-manifold X, locally symmetric metrics are characterized, among all metrics g with $\mathrm{Ric}_g \geqslant -(n-1)g$, by being volume-minimizing.*

This means that a metric g_0 on X, normalized as above, is locally symmetric *if and only if* $\mathrm{Vol}(X, g_0) = \inf\{\mathrm{Vol}(X, g) \mid g \text{ on } X,\ \mathrm{Ric}_g \geqslant -(n-1)g\}$. This holds true in every dimension for real hyperbolic manifolds, while it is known only in dimension 4 in the complex hyperbolic case. Assertion 1.3 will be the object of the next sections.

Admitting 1.3, one then infers Theorems 1.1 and 1.2 by showing that any normalized Einstein metric g on a hyperbolic 4-manifold (X, g_0) necessarily realizes the minimal value $\mathrm{Vol}(X, g_0)$. This is the common point of both methods, and it is achieved by using Gauss-Bonnet's and Hirzebruch's formulas as follows.

[1] This contrast with the analogous classification problem for metrics of constant *sectional* curvature, or of constant *scalar* curvature, which, in dimension greater than 2, generally form sets which are respectively empty and of infinite dimension.

Let V be the tangent space at some point of a Riemannian 4-manifold (X, g). The metric g gives rise to an action of $SO(V, g)$ on all tensor, symmetric and exterior powers of V. In particular, one has an action of $SO(V, g)$ on the vector space $\mathscr{C}V$ of the *algebraic curvature tensors*, that is the space of the symmetric endomorphisms of $\wedge^2 V$ satisfying the formal Bianchi identity (i.e. the tensors of the same type and which satisfy the same algebraic properties of the Riemannian curvature tensor). As a result, one has a decomposition of $\mathscr{C}V = \mathscr{U}V \oplus \mathscr{J}V \oplus \mathscr{U}^+V \oplus \mathscr{U}^-V$ into irreducible representations of the special orthogonal group: $\mathscr{U}V$ coincides with the one-dimensional subspace $\mathbf{R}\ \mathrm{id}_{\wedge^2 V}$, $\mathscr{J}V$ is the subspace of endomorphisms which anticommute with the Hodge operator $*$ (hence, they exchange $\wedge^{2,+}V$ with $\wedge^{2,-}V$), while $\mathscr{U}^+V$ and $\mathscr{U}^-V$ are respectively made up of self-dual and antiself-dual tensors W commuting with $*$ and satisfying $\mathrm{Tr}_g W = \mathrm{Tr}_g(W \circ *) = 0$ (see [5] for details). With respect to this decomposition, the Riemannian curvature R_g splits into a sum $R_g = U_g + Z_g + W_g^+ + W_g^-$, where

- $U_g = \frac{\mathrm{scal}(g)}{12} \cdot \mathrm{id}_{\wedge^2 V}$ is determined by the scalar curvature;
- $Z_g = \frac{1}{2}\left(\mathrm{Ric}_g - \frac{\mathrm{scal}(g)}{2} g\right) \hat{\otimes} g$ involves the trace-free part of the Ricci tensor;
- $W_g = R_g - U_g - Z_g = W_g^+ + W_g^-$ is the *Weyl tensor* of g, and $W_g^\pm \in \mathrm{End}(\wedge^{2,\pm}V)$.

Here, $\hat{\otimes}$ is the natural operation, called Kulkarni-Nomizu product, which let us obtain an algebraic curvature tensor from two symmetric 2-tensors, by antisymmetrizing with respect to the first and second couple. Of course, $g\hat{\otimes}g$ gives $2\ \mathrm{id}_{\wedge^2 V}$ and is, up to a factor, precisely the curvature tensor of the sphere or of the hyperbolic plane: in other words, the metric g has constant sectional curvature if and only if R_g reduces to U_g. On the other hand, Einstein metrics are characterized by the vanishing of the tensor Z_g.

Gauss-Bonnet's and Hirzebruch's formula [5] take then, in dimension 4, the following simple expressions:

$$\chi(X) = \frac{1}{8\pi^2}\int_X (\|U_g\|^2 - \|Z_g\|^2 + \|W_g\|^2)\, dv_g \tag{1}$$

$$\tau(X) = \frac{1}{12\pi^2}\int_X (\|W_g^+\|^2 - \|W_g^-\|^2)\, dv_g. \tag{2}$$

for any metric g on X. Combining these formulas, one finds that if (X, g) is an Einstein 4-manifold, then

$$\begin{aligned} 2\chi(X) - 3\tau(X) &= \frac{1}{4\pi^2}\int_X (\|U_g\|^2 + 2\|W_g^-\|^2)\, dv_g \\ &\geqslant \frac{\mathrm{scal}^2(g) \cdot \|\ \mathrm{id}_{\wedge^2 V}\ \|^2}{576\pi^2} \cdot \mathrm{Vol}(X, g) \end{aligned} \tag{3}$$

by forgetting the negative Weyl tensor. As $\|\ \mathrm{id}_{\wedge^2 V}\ \|^2 = 6$, we see that *the volume of any normalized Einstein 4-manifold (X, g) of nonzero sign satisfies*

$$\mathrm{Vol}(X, g) \leqslant \frac{2}{3}\pi^2(2\chi(X) - 3\tau(X)) \tag{4}$$

and the equality holds if and only if $W_g^- = 0$. In particular, *a (real or complex) hyperbolic 4-manifold* (X, g_0) *precisely satisfies the equality*

$$\mathrm{Vol}(X, g_0) = \frac{2}{3}\pi^2(2\chi(X) - 3\tau(X)) \tag{5}$$

as $W_{g_0}^- = 0$ by self-duality. But now, since by Assertion 1.3 locally symmetric metrics on a hyperbolic 4-manifold (X, g_0) are characterized by being volume minimizing among metrics with $\mathrm{Ric}_g \geqslant -3g$, from (4) and (5) it follows that every normalized Einstein metric g on (X, g_0) necessarily satisfies the equality $\mathrm{Vol}(X, g) = \mathrm{Vol}(X, g_0)$ and, again by 1.3, g is locally symmetric. Thus, g is isometric to g_0 by Mostow's rigidity theorem. This proves Theorems 1.1 and 1.2, assuming Assertion 1.3.

In section 2 we shall describe Besson-Courtois-Gallot's method, which implies Assertion 1.3 when (X, g_0) is real hyperbolic. Actually, Besson-Courtois-Gallot's proof of Theorem 1.1 by-passes Mostow's rigidity theorem, as it shows directly how to construct an isometry between a normalized Einstein metric g on X and g_0. We shall see that their method provides an even stronger version of Theorem 1.1.

In section 4 we shall explain Assertion 1.3 for complex hyperbolic 4-manifolds. It stems from a remarkable L^2 estimate, due to C. LeBrun, of the scalar curvature of a Riemannian 4-manifold admitting some non-zero Seiberg-Witten invariant (Theorem 4.1).

For the convenience of readers unfamiliar with spin geometry and Seiberg-Witten invariants, we decided to include, in section 3, a short review of those facts of Seiberg-Witten theory which are necessary to understand Lebrun's main estimate 4.1.

2. Real and complex Schwarz lemma

The classical Schwarz-Pick lemma, on holomorphic mappings of the unitary disk of $\mathbf{C}$ in itself, has been generalized in several directions by Ahlfors, Chern, Kobayashi, Griffiths, Wu et al. (see [23]). A Kähler version of the Schwarz lemma (due to S.T. Yau [35]; but see [7] for a proof with the constants reported below) says that any holomorphic map $f : (Y, g) \to (X, g_0)$ from a Kähler manifold into a Kähler manifold of negative Ricci curvature contracts volumes, i.e. $|\mathrm{Jac}\, f| = \left|\frac{f^*\omega_{g_0}}{\omega_g}\right| \leqslant 1$ (provided, of course, that the metrics are suitable normalized, that is: $\mathrm{Ric}(g) \geqslant -(n-1)g$ and $\mathrm{Ric}(g_0) \leqslant -(n-1)g_0$). In particular, $\mathrm{Vol}(Y, g) \geqslant \deg(f) \cdot \mathrm{Vol}(X, g_0)$. Moreover, the equality $\mathrm{Vol}(Y, g) = \deg(f) \cdot \mathrm{Vol}(X, g_0)$ holds if and only if f is a locally isometric covering, i.e. $(df)_y$ is an isometry for all $y \in Y$.

An analogous result has recently been proved for maps between *real* Riemannian manifolds, provided that the target space has negative sectional curvature:

REAL SCHWARZ LEMMA 2.1 (G. Besson, G. Courtois, S. Gallot [7]). — *Let $f : (Y, g) \to (X, g_0)$ be a continuous map between closed Riemannian manifolds of dimension $n \geqslant 3$, and assume that (X, g_0) is negatively curved. Let the metrics g, g_0 be normalized so that $k(g_0) \leqslant -1$ and $\mathrm{Ric}(g) \geqslant -(n-1)g$: then, there exists a family of C^1-maps f_ϵ, with $\epsilon \to 0$, homotopic to f and which verify*

$$|\operatorname{Jac} f_\epsilon| \leqslant (1+\epsilon)^n \tag{6}$$

that is, the f_ϵ's tend to contract volumes infinitesimally. Moreover, if volumes are globally preserved (i.e. if $\mathrm{Vol}(Y, g) = |\deg(f)| \cdot \mathrm{Vol}(X, g_0)$) then a subfamily of the f_ϵ's converges, when $\epsilon \to 0$, to a Riemannian covering $f_0 : (Y, g) \to (X, g_0)$, and in this case g and g_0 necessarily have constant sectional curvature $k(g) = k(g_0) = -1$.

The construction of the maps f_ϵ may be summarized as follows. Let $\tilde{Y}, \tilde{X}$, be the universal coverings of Y, X. Lift g and g_0 to metrics $\tilde{g}, \tilde{g}_0$ on $\tilde{Y}, \tilde{X}$ and call d, d_0 the induced Riemannian distances. The map $f : Y \to X$ can be lifted to a map $\tilde{f} : \tilde{Y} \to \tilde{X}$, the groups of deck transformations $\mathrm{Aut}(\tilde{Y})$, $\mathrm{Aut}(\tilde{X})$ may be identified to $\pi_1(Y)$ and $\pi_1(X)$ respectively, and $\tilde{f}$ is equivariant with respect to the representation $f_* : \pi_1(Y) \to \pi_1(X)$ induced by f. Let us consider the spaces $\mathcal{M}(\tilde{Y})$ and $\mathcal{M}(\tilde{X})$ of positive and finite Borel measures on $\tilde{Y}$ and $\tilde{X}$. The groups $\mathrm{Aut}(\tilde{Y})$ and $\mathrm{Aut}(\tilde{X})$ naturally act on $\mathcal{M}(\tilde{Y})$ and $\mathcal{M}(\tilde{X})$ by pushing forward measures. Then, one can embed $\tilde{Y}$ in $\mathcal{M}(\tilde{Y})$ by means of maps μ_ϵ

$$y \to \mu_\epsilon(y) \doteq e^{-(n-1+\epsilon)d(y,y')} dv_g(y').$$

These are *finite* measures, as the assumption $\mathrm{Ric}(g) \geqslant -(n-1)g$ implies, by Bishop's comparison theorem, that the volume of R-balls in $(\tilde{Y}, \tilde{g})$ grows at most as fast as $e^{(n-1)R}$ (the growth function of R-balls in the hyperbolic n-space): this, in turns, implies that the integral $\int_{\tilde{Y}} e^{-(n-1+\epsilon)d(y,y')} dv_g(y')$ converge for all $\epsilon > 0$. One then compose μ_ϵ with the map $\tilde{f}_* : \mathcal{M}(\tilde{Y}) \to \mathcal{M}(\tilde{X})$ obtained by pushing forward measures via $\tilde{f}$; finally, one comes back on $\tilde{X}$ by the barycentre map [2]. One therefore obtains f_*-equivariant maps $\tilde{f}_\epsilon : \tilde{Y} \to \tilde{X}$, defined by $\tilde{f}_\epsilon(y) = \mathrm{bar}(\tilde{f}_* \mu_\epsilon(y))$, which are "more isometric" than the initial map $\tilde{f}$, when $\epsilon \to 0$.

Then, by using the implicit function theorem, it turns out that the jacobian of the maps $\tilde{f}_\epsilon$ can be expressed in terms of the first and second derivatives of the distance function on $(\tilde{X}, g_0)$. By the assumption $k(g_0) \leqslant -1$, these derivatives can be compared with the corresponding tensors in the hyperbolic space, giving the announced estimate.

We shall not give details of the proof of 2.1, since good surveys on Besson-Courtois-Gallot's construction already exist (see [7] and [8] for instance). However, notice that the authors originally used a family of probability measures $\mu_\epsilon(y)$ supported by the geometric boundary $\partial\tilde{X}$ of $\tilde{X}$, which involve the Busemann functions of $\tilde{X}$; the above mentioned, more elementary, method is fully explained in [31], where we also improved (6) by the topology of the map f.

[2] Recall that the *barycentre* of a measure μ on a simply connected Riemannian manifold $\tilde{X}$ of negative curvature may be defined as the unique absolute minimum of the C^∞-function $\mathcal{L}_\mu(x) = \int_{\tilde{X}} d_0(x, x')^2 \mu(x')$.

Proof of Assertion 1.3 for real hyperbolic manifolds. This is a direct application of the Real Schwarz Lemma. In fact, given a normalized Einstein metric g on a real hyperbolic manifold (X, g_0), applying the coarea formula to the identity map $(X, g) \to (X, g_0)$ yields

$$(1+\epsilon)^n \cdot \mathrm{Vol}(X, g) \geqslant \int_X |\, Jac(id) \,|\, dv_g = \mathrm{Vol}(X, g_0);$$

letting $\epsilon \to 0$ one then finds $\mathrm{Vol}(X, g) \geqslant \mathrm{Vol}(X, g_0)$, and if the equality holds then, by Theorem 2.1, g is a hyperbolic metric too. □

REMARK 2.2. — *Rigidity of Kähler-Einstein metrics.*

Since we quoted the Schwarz Lemma in Kähler geometry, it may be pertinent to stress a difference between Einstein metrics and Kähler-Einstein metrics. Much more is known about rigidity of the latter. Already in the 50's E. Calabi [10] proved uniqueness of Kähler-Einstein metrics on Kähler manifolds X of dimension $n \geqslant 2$ such that $c_1(X) \geqslant 0$, whithin any fixed Kähler class $[\omega_g]$ (the complex structure being fixed). In 1976, T. Aubin proved [2] global uniqueness (and existence) of Kähler-Einstein metrics on complex manifolds with negative first Chern class, and in the same year, S.T. Yau [34] obtained, independently, another proof, achieving Calabi conjecture [3]. Uniqueness, in the Kähler case, has to be intended in the strongest sense: *any two normalized Kähler-Einstein metrics on a complex manifold (X, J) with $c_1(X) < 0$ coincide* (whereas, of course, in the real case, Einstein metrics are unique up to diffeomorphisms, since no compatibility with a fixed complex structure is required). This result can also be deduced from the Schwarz Lemma in the following way. Let g_1, g_2 be normalized Kähler-Einstein metrics on some complex manifold X with negative first Chern class. These metrics necessarily have negative Ricci curvature as, for Kähler manifolds, 2π times the Ricci form represents $c_1(X)$. Then, one applies the Schwarz Lemma to the identity map $\mathrm{id} : (X, J, g_1) \to (X, J, g_2)$ (which is holomorphic as J is fixed) to infer that $\mathrm{Vol}(X, g_1) \geqslant \mathrm{Vol}(X, g_2)$, and viceversa. Therefore, the equality $\mathrm{Vol}(X, g_1) = \mathrm{Vol}(X, g_2)$ holds and id necessarily is an isometry, i.e. $g_1 = g_2$ at every point.

We shall now show how Besson-Courtois-Gallot's Real Schwarz Lemma provides in fact a stronger rigidity statement than 1.1.

THEOREM 2.3. — *Let (X, g_0) be a real hyperbolic 4-manifold. Any Einstein 4-manifold (Y, g) with nontrivial simplicial volume, whose fundamental group is an amenable extension of $\pi_1(X)$, and which satisfies $\chi(Y) \leqslant \chi(X)$, is necessarily isometric to $(X, \lambda \cdot g_0)$, for some $\lambda > 0$.*

By *amenable* extension of a group G, we mean a group epimorphism $\rho : G' \to G$ whose kernel is an amenable subgroup (abelian, for instance).

[3] E. Calabi originally conjectured that, a Kähler manifold (X, J_0, g_0) of complex dimension $n \geqslant 2$ being given, then for any real form α of type $(1,1)$ in the class $2\pi c_1(X)$ there exist a unique Kähler metric g on (X, J_0), within the same Kähler class ω_{g_0} of g_0, with Ricci form equal to α.

Recall that the simplicial volume of a n-dimensional manifold X, denoted $\|X\|$, may be defined in terms of the real homology of X as the infimum of $\sum_i |r_i|$, when $\sum_i r_i\sigma_i$ runs over all singular chains which represent the fundamental class of X. The simplicial volume is an invariant supposed to measure the topological complexity of X, and it is non-vanishing, for instance, for negatively curved manifolds. By definition, one has that, if $f : Y \to X$ is a continuous map between closed manifold of same dimension, then $\|Y\| \geqslant |\deg(f)| \cdot \|X\|$. Cases where the equality $\|Y\| = |\deg(f)| \cdot \|X\|$ holds are frequent, e.g. when f is a covering. More generally, one has [15], [31]:

LEMMA 2.4. — *Let $f : Y \to X$ be a continuous map between closed manifolds of same dimension. If the kernel of the homomorphism induced by f between fundamental groups is amenable, then the equality $\|Y\| = |\deg(f)| \cdot \|X\|$ holds.*

Some concrete examples for Lemma 2.4 can be found in [30]. This lemma essentially relies on the fact that an equivalent definition of simplicial volume can be given using bounded cohomology, and on the fact that the bounded cohomology of an amenable group vanishes [15].

Proof of Theorem 2.3. Orient Y so that $\tau(Y) \geqslant 0$, and normalize g so that $\mathrm{Ric}_g = (n-1)g$: this is possible since clearly Y does not admit metrics with non-negative Ricci curvature, otherwise its fundamental group would have at most polynomial growth and its simplicial volume would vanish. As X is a $K(\pi, 1)$-space, there exists a map $f : Y \to X$ which induces the amenable extension $\rho : \pi_1(Y) \to \pi_1(X)$. Since $H = \ker(\rho)$ is amenable and $\|Y\| \neq 0$, then Lemma 2.4 yields $|\deg(f)| = \|Y\|/\|X\| \geqslant 1$. Thus, by the Real Schwarz Lemma, using the coarea formula, one deduces

$$\begin{aligned} |\deg(f)| \cdot \mathrm{Vol}(X, g_0) &\leqslant \int_X \# f_\epsilon^{-1} dv_{g_0}(x) \\ &= \int_Y |\mathrm{Jac}_y f_\epsilon| dv_g(y) \\ &\leqslant (1+\epsilon)^n \mathrm{Vol}(Y, g) \end{aligned} \tag{7}$$

(as the f_ϵ's are homotopic to f). Letting $\epsilon \to 0$, we get $\mathrm{Vol}(Y, g) \geqslant \mathrm{Vol}(X, g_0)$. But we know that, in dimension 4, Gauss-Bonnet's and Hirzebruch's formulas give

$$\mathrm{Vol}(Y, g) \leqslant \frac{2}{3}\pi^2(2\chi(Y) - 3\tau(Y)) \leqslant \frac{2}{3}\pi^2 \cdot 2\chi(X) = \mathrm{Vol}(X, g_0);$$

this, with (7), implies that $|\deg(f)| = 1$ and $\mathrm{Vol}(Y, g) = \mathrm{Vol}(X, g_0)$ necessarily. Therefore, by the rigidity part of the Real Schwarz Lemma, f is homotopic to an isometry. □

The above argument actually shows that, when $\chi(Y) < \chi(X)$, then Y does not admit any Einstein metric. In other words, there are topological obstructions to the existence of Einstein metrics on 4-manifolds Y dominating some hyperbolic manifold (X, g_0). In [29] we used this fact to show that "most" 4-manifolds do not admit Einstein structures at all.

3. A short review of Seiberg-Witten theory

For references, details and full proofs of facts reported below one can see [24], [25], [32], [33].

3.1. *Spin groups and Clifford algebras*

Let $\mathbf{H}$ be the quaternionic space, endowed with its canonical euclidean structure which makes of $\{1, i, j, k\}$ an orthonormal basis of $\mathbf{H}$ over $\mathbf{R}$, and let $\mathbf{H}^{\mathbf{C}} = \mathbf{H} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathbf{H}$, with the induced complex quadratic form. We shall see quaternions as matrices, since the algebras $\mathbf{H}, \mathbf{H}^{\mathbf{C}}$ have standard representations as

$$\mathbf{H} = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mid a, b \in \mathbf{C} \right\} \subset \mathbf{H}^{\mathbf{C}} = \mathbf{H} \oplus i\mathbf{H} = M(2, \mathbf{C})$$

such that $< Q, Q >= \|Q\|^2 = \det Q$.

The Clifford algebras associated to $\mathbf{H}, \mathbf{H}^{\mathbf{C}}$ may be identified with

$$\mathrm{Cl}(\mathbf{H}) = M(2, \mathbf{H}) \subset \mathrm{Cl}(\mathbf{H}^{\mathbf{C}}) = M(4, \mathbf{C})$$

where the Clifford multiplication $\cdot$ simply is the usual product of matrices. These algebras are canonically isomorphic, *as vector spaces*, to the exterior algebras $\wedge\mathbf{H}, \wedge\mathbf{H}^{\mathbf{C}}$ via the map $\varphi : \wedge\mathbf{H}^{\mathbf{C}} \xrightarrow{\sim} \mathrm{Cl}(\mathbf{H}^{\mathbf{C}})$ defined by

$$\varphi(Q) = \begin{pmatrix} 0 & -\overline{Q}^t \\ Q & 0 \end{pmatrix} \text{ and } \varphi(Q_1 \wedge \cdots \wedge Q_i) = \varphi(Q_1) \cdots \varphi(Q_i), \text{ for } Q, Q_i \in \mathbf{H}^{\mathbf{C}}.$$

Therefore, we can see $\mathbf{H}^{\mathbf{C}}$ and its exterior powers as subspaces of $\mathrm{Cl}(\mathbf{H}^{\mathbf{C}})$ (often dropping the symbol φ). The associated spin groups $\mathrm{Spin}(4) \subset \mathrm{Cl}(\mathbf{H})$, $\mathrm{Spin}^c(4) \subset \mathrm{Cl}(\mathbf{H}^{\mathbf{C}})$ are

$$\begin{aligned} \mathrm{Spin}(4) &= SU^+(2) \oplus SU^-(2) \\ &= \left\{ \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \mid A_+, A_- \in SU(2) \right\} \subset M(4, \mathbf{C}) \\ \mathrm{Spin}^c(4) &= U^+(2) \times_{\det} U^-(2) \\ &= \left\{ \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \mid A_+, A_- \in U(2), \det A_+ = \det A_- \in U(1) \right\}. \end{aligned}$$

These groups have the property that, when we see $\mathbf{H}, \mathbf{H}^{\mathbf{C}} \subset \mathrm{Cl}(\mathbf{H}^{\mathbf{C}})$, the conjugation by elements of $\mathrm{Spin}(4)$ and $\mathrm{Spin}^c(4)$ preserves $\mathbf{H}$ and $\mathbf{H}^{\mathbf{C}}$. Notice that $\mathrm{Spin}^c(4)$ can also be seen as $\mathrm{Spin}(4) \times_{\pm 1} U(1)$.

3.2. *Representations of* $\mathbf{Spin}^c(4)$

The group $\mathrm{Spin}^c(4)$ has three fundamental irreducible representations:

1) an orthogonal representation $\rho : \mathrm{Spin}^c(4) \to SO(4)$, which is obtained by letting a couple of matrices (A_+, A_-) act on the 4-dimensional euclidean space $\mathbf{H}$ as $Q \mapsto A_- Q A_+^{-1}$ (i.e. by restriction to $\mathbf{H}$ of the action by conjugation of $\mathrm{Spin}^c(4)$ on $\mathrm{Cl}(\mathbf{H}^{\mathbf{C}})$). It is easily verified that this action preserves the euclidean metric and the orientation of $\mathbf{H}$;

2) two unitary representations $\pi_\pm^c : \mathrm{Spin}^c(4) \to U(2)$, obtained by projecting $\mathrm{Spin}^c(4)$ on one of its factors $U^\pm(2) \cong U(2)$, that is defined by $(A_+, A_-) \cdot \psi = A_\pm \psi$ for $(A_+, A_-) \in \mathrm{Spin}^c(4)$ and $\psi \in \mathbf{C}^2$. The representation space $\mathbf{C}^2$, endowed of its canonical hermitian metric and of one of these $\mathrm{Spin}^c(4)$-actions will be denoted, respectively, by $S^\pm$. We set $S = S^+ \oplus S^-$.

3) a one dimensional unitary representation $\det_+ = \det_- : \mathrm{Spin}^c(4) \to U(1)$ obtained by taking the determinant $\det(A_+) = \det(A_-)$ of one factor of $(A_+, A_-) \in \mathrm{Spin}^c(4)$. The space $\mathbf{C}$, endowed of its canonical hermitian metric and of the $\mathrm{Spin}^c(4)$-action induced by $\det_\pm$ will be denoted by L.

Summarizing, one checks that the following commutative diagram holds:

$$\begin{tikzcd}
 & & U(2) \arrow[dl, "i"'] \arrow[dd, "h"] \arrow[ddr, "\det"] & & \\
 & \mathrm{Spin}(4) \arrow[dl, "\rho^c"'] \arrow[dr, "c"] & & U(1) \arrow[dl, "\Delta"] \arrow[ddr, "2"] & \\
 & & \mathrm{Spin}^c(4) \arrow[dl, "\rho"'] \arrow[d, "\pi_\pm"] \arrow[drr, "\det_\pm"] & & \\
SO(4) & & U(2) & & U(1)
\end{tikzcd}$$

where i and ^{c} are the natural inclusions, det is the determinant, Δ is the diagonal inclusion, ^{2} is the square map and

$$h(A) = \left(\begin{pmatrix} 1 & 0 \\ 0 & \det A \end{pmatrix}, A \right).$$

The map $\mathrm{Spin}(4) \xrightarrow{\rho^c} SO(4)$ in the diagram, which is the composition of ρ with ^{c}, is the canonical double universal covering of $SO(4)$.

The following relations between the above representations are easily verified:

a) $\det : \wedge^2 S^\pm \xrightarrow{\sim} L$ is an isomorphism of $\mathrm{Spin}^c(4)$-representations;

b) by identifying $\mathrm{End}_{\mathbf{C}}(S)$ with $M(4, \mathbf{C}) \cong \mathrm{Cl}(\mathbf{H}^{\mathbf{C}})$, φ induces isomorphisms of

$\mathrm{Spin}^c(4)$-representations

$$\varphi : \begin{cases} \mathbf{H}^{\mathbf{C}} & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{C}}(S^+, S^-) \\ \wedge^{2,\pm}\mathbf{H}^{\mathbf{C}} & \xrightarrow{\sim} & \mathrm{End}^0_{\mathbf{C}}(S^\pm) \\ \wedge^{2,\pm}\mathbf{H} & \xrightarrow{\sim} & su(S^\pm) \end{cases}$$

where $\wedge^{2,\pm}\mathbf{H}$ are the eigenspaces of the Hodge operator of $\wedge^2\mathbf{H}$, $\wedge^{2,\pm}\mathbf{H}^{\mathbf{C}}$ are their complexifications, and where $\mathrm{End}^0_{\mathbf{C}}(S^\pm)$ and $su(S^\pm)$ respectively denote the sets of traceless endomorphisms and of traceless, anti-hermitian, endomorphisms of $S^\pm$;

c) moreover, one has a quadratic, $\mathrm{Spin}^c(4)$-equivariant map between hermitian spaces

$$\sigma : S^+ \to i \wedge^{2,+} \mathbf{H} \subset \wedge^{2,+}\mathbf{H}^{\mathbf{C}}$$

defined by $\sigma(\psi) = \psi \otimes \psi^* - \frac{\|\psi\|^2}{2}\,\mathrm{id} \in S^+ \otimes (S^+)^* \cong \mathrm{End}^0_{\mathbf{C}}(S^+)$; by identifying $\mathbf{H}$ with its dual via the Euclidean metric, one also obtains a quadratic, $\mathrm{Spin}^c(4)$-equivariant map $\sigma^* : S^+ \to i \wedge^{2,+} \mathbf{H}^*$. The map σ^* satisfies $\|\sigma^*(\psi)\|^2 = \|\psi\|^4/8$.

3.3. $\mathbf{Spin}^c$-*structures and spinor bundles*

A Spin^c-structure on an oriented Riemannian 4-manifold (X, g) is a principal $\mathrm{Spin}^c(4)$-bundle $P_g \to X$, with a principal bundle map $P_g \to SO(X, g)$, equivariant with respect to the homomorphism $\rho : \mathrm{Spin}^c(4) \to SO(4)$. One also says that the bundle P_g lifts the orthogonal frame bundle of (X, g) with respect to ρ.

Now, let $GL_+(4, \mathbf{R})$ be the connected component of $GL(4, \mathbf{R})$ containing the identity, let $\widetilde{GL}_+(4, \mathbf{R})$ be its double universal covering, and let

$$\widetilde{GL}^c_+(4, \mathbf{R}) = \widetilde{GL}_+(4, \mathbf{R}) \times_{\pm 1} U(1).$$

We call Spin^c-prestructure of a differentiable 4-manifold X a principal $\widetilde{GL}^c_+(4, \mathbf{R})$-bundle $P \to X$ which lifts the bundle $GL_+(X)$ of the oriented frames of X, with respect to the natural projection $\widetilde{GL}^c_+(4, \mathbf{R}) \to GL_+(4, \mathbf{R})$.

Remark that a Spin^c-structure $P_g \to X$ precisely coincides with the data of a Spin^c-prestructure P and of a metric g on X. In fact, by the covering theory one deduces an injection $\mathrm{Spin}(4) \hookrightarrow \widetilde{GL}_+(4, \mathbf{R})$, which in turn induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}^c(4) & \xrightarrow{\rho} & SO(4) \\ \nu \downarrow & & \downarrow \\ \widetilde{GL}^c_+(4, \mathbf{R}) & \longrightarrow & GL_+(4, \mathbf{R}) \end{array}$$

Therefore, one can associate to each Spin^c-structure P_g on X the Spin^c-prestructure $P_g \times_\nu \widetilde{GL}^c_+(4, \mathbf{R})$. Conversely, given a metric g on a differentiable manifold X and a principal $\widetilde{GL}^c_+(4, \mathbf{R})$-bundle $P \xrightarrow{\mu} GL_+(X)$ which lifts the bundle of oriented frames of X,

then we get a true Spin^c-structure on (X, g) by taking the subbundle $P_g = \mu^* SO(X, g)$ of P.

Even if the notion of Spin^c-prestructure may seem artificious, it is useful to understand what the Seiberg-Witten invariants depend on. The Seiberg-Witten invariants are attached to a Spin^c-structure on a Riemannian manifold (X, g): thus, the datum of a metric g on X is implicitely required (and their very definition makes use of the metric g). However, they do not really depend on the metric, at least when $b_2^+(X) \geqslant 2$. More precisely, P_g and $P'_{g'}$ will have the same Seiberg-Witten invariants provided that they define the same equivalence class of Spin^c-prestructures (where two Spin^c-prestructures P, P' of X are said to be equivalent if there exists a principal bundle isomorphism $P \to P'$ which commutes with the projections of P, P' on $GL_+(X)$).

As a result of obstruction theory (see [25]), every 4-manifold has at least one Spin^c-structure, thus the Seiberg-Witten invariants help to classify 4-manifolds (however, there exist also 4-manifolds X all of whose Spin^c-structures have trivial Seiberg-Witten invariants – if X admits a metric of positive scalar curvature, for instance).

When X is endowed with an almost complex structure J, then X has a canonical Spin^c-prestructure, obtained by considering the bundle $U(X, J, g)$ of unitary frames, with respect to the choice of some hermitian metric g on (X, J), and then by taking the associated bundle $P_g(J) = U(X, J, g) \times_h \mathrm{Spin}^c(4)$ (which clearly lifts $SO(X, g)$ as $i = \rho \circ h$). Then, the underlying Spin^c-prestructure $P(J)$ does not depend on the choice of g. In fact, the natural inclusion $GL_{\mathbb{C}}(2, \mathbf{C}) \subset GL_+(4, \mathbf{R})$ lifts to an inclusion $\eta : GL_{\mathbb{C}}(2, \mathbf{C}) \to \widetilde{GL}_+^c(4, \mathbf{R})$ [4] which gives a commutative diagram of homomorphisms

$$\begin{array}{ccc} U(2) & \xrightarrow{j} & GL_{\mathbb{C}}(2, \mathbf{C}) \\ h \downarrow & & \downarrow \eta \\ \mathrm{Spin}^c(4) & \xrightarrow{\nu} & \widetilde{GL}_+^c(4, \mathbf{R}) \end{array}$$

so $P_g(J) \times_\nu \widetilde{GL}_+^c(4, \mathbf{R}) = (U(X, J, g) \times_j GL(2, \mathbf{C})) \times_\eta \widetilde{GL}_+^c(4, \mathbf{R}) = GL(X, J) \times_\eta \widetilde{GL}_+^c(4, \mathbf{R})$. Thus, the Seiberg-Witten invariants of $P_g(J)$ only depend on J, and they give invariants of the almost complex manifold (X, J).

Given a Spin^c-structure P_g on (X, g), tensoring P_g by the fundamental representation spaces $\mathbf{H}$, $S^\pm$ and L of $\mathrm{Spin}^c(4)$, one gets the following associated euclidean and hermitian bundles on X:

1) the tangent bundle $TX \cong P_g \times_\rho \mathbf{H}$ with its Riemannian metric g, since $SO(X, g) \cong P_g \times_\rho \mathrm{Spin}^c(4)$;

2) two rank 2 hermitian bundles $S^\pm(P_g) = P_g \times_{\pi_\pm} S^\pm$, called the associated complex spinor bundles;

[4] notice that $GL(2, \mathbf{C}) \cong U(2) \times \mathrm{Herm}^+(2, \mathbf{C})$ and $\widetilde{GL}_+^c(4, \mathbf{R}) \cong \mathrm{Spin}(4) \times_{\pm 1} U(1) \times \mathrm{Simm}^+(4, \mathbf{R})$, *as manifolds.*

3) a hermitian linear bundle $L(P_g) = P_g \times_{\det_+} L$, called the determinant line bundle of the Spin^c-structure.

The relations a),b),c) of the previous paragraph in turn yield:

a) isomorphisms of hermitian linear bundles $\det : L(P_g) \cong \wedge^2 S^+(P_g) \cong \wedge^2 S^-(P_g)$;

b) isomorphisms $TX^{\mathbf{C}} \cong \mathrm{Hom}(S^+(P_g), S^-(P_g))$, $\wedge^{2,\pm} TX^{\mathbf{C}} \cong \mathrm{End}^0(S^\pm(P_g))$ and $\wedge^{2,\pm} TX \cong su(S^\pm(P_g))$;

c) a quadratic, $\mathrm{Spin}^c(4)$-equivariant vector bundles map $\sigma^* : S^+(P_g) \to i\wedge^{2,+} TX^*$.

Notice that the determinant bundle $L(P_g) = L(P)$ is defined as soon as one has a Spin^c-prestructure on X, whereas in order to define the spinor bundles $S^\pm(P_g)$ one needs to choose a metric on X.

Remark that the bundle $L(P)$ determines P up to 2-torsion in $H^2(X, \mathbf{Z})$. Actually, the set $\mathrm{Spin}^c(X, g)$ of equivalence classes of Spin^c-structures over (X, g) (as well as the set of equivalence classes of Spin^c-prestructures on X) form a $H^2(X, \mathbf{Z})$ principal space [5]. By identifying, via the Chern class, $H^2(X, \mathbf{Z})$ with the set of isomorphism classes of principal $U(1)$-bundles on X, the action of $H^2(X, \mathbf{Z})$ on $\mathrm{Spin}^c(X, g)$ is given by $E \cdot [P] = [(P \times_X \mathrm{Spin}^c(4))/\lambda]$, where λ is the multiplication $\mathrm{Spin}^c(4) \times U(1) \to \mathrm{Spin}^c(4)$. Thus, an origin $[P_0] \in \mathrm{Spin}^c(X)$ being fixed, if we have two Spin^c-structures P_1, P_2, then $P_i = E_i \cdot P_0$ for some $U(1)$-bundles E_i determined up to isomorphism. If L_1 and L_2 are the determinant bundles of P_1, P_2, then a simple computation shows that $c_1(L_1) - c_1(L_2) = 2 \cdot (c_1(E_1) - c_1(E_2))$: since the Chern class classifies complex linear C^∞-bundles, it follows that $L_1 \cong L_2$ if and only if $c_1(E_1) - c_1(E_2)$ is a torsion element of order 2 (if there is no torsion – on a simply connected 4-manifold, for instance – one has $L_1 \cong L_2$ if and only if $E_1 \cong E_2$, that is $P_1 \cong P_2$).

Finally, let us remark that when $P_g = P_g(J)$ is the canonical Spin^c-structure of an hermitian manifold (X, J, g), then the vector bundles $L(P)$ and $S^\pm(P_g)$ are naturally related with the complex tangent bundle and the canonical bundle of (X, J): namely, it easily follows from definitions that $L(P) = K_X^{-1} \cong \wedge^{0,2} T^*X$, $S^-(P_g) = TX$ and $S^+(P_g) = (X \times \mathbf{C}) \oplus K_X^{-1} \cong \wedge^{0,0} T^*X \oplus \wedge^{0,2} T^*X$ (where $\cong$ are the complex linear isomorphisms induced by the hermitian metric g). In this case, the map $\sigma^* : C^\infty(X, \mathbf{C}) \oplus \mathcal{E}^{0,2}(X) \to i\mathcal{E}^{2,+}(X)$ can be expressed as: $\sigma^*(f, \alpha) = i(\|\alpha\|^2 - \|f\|^2)\omega_g/4 - iIm(\overline{f}\alpha)$.

3.4. *Seiberg-Witten equations*

Let a Spin^c-structure $P_g \xrightarrow{\mu_g} SO(X, g)$ on (X, g) be given. In what follows, we shall write for short L, $S_g^\pm$ instead of $L(P)$, $S^\pm(P_g)$ (thus implicitely assuming that the underlying Spin^c-prestructure has been fixed).

[5] that is, $H^2(X, \mathbf{Z})$ acts on $\mathrm{Spin}^c(X)$ freely and transitively, so these spaces can be identified up to the choice of an origin $P_0 \in \mathrm{Spin}^c(X, g)$.

We shall be interested in the space $C_{comp}(P_g)$ of connections H on P_g which are *compatible* with the Levi Civita connection H^{∇} of $SO(X,g)$: this means that the horizontal, $\mathrm{Spin}^c(4)$-stable distribution H of TP_g is mapped by $d\mu_g$ precisely into H^{∇}. Any $H \in C_{comp}(P_g)$ induces therefore the Levi Civita connection ∇ on TX, and connections on $S_g^{\pm}$ and L. Actually, the connection A induced on L by H completely determines H: in fact, in order to reconstruct H from A, it is enough to consider the product distribution $H^{\nabla\times A} = (H^{\nabla} \times H^A)|_{\Delta}$ on the principal bundle $SO(X,g) \times_X U(L)$ (where Δ is the diagonal inclusion of X in $X \times X$, and H^A is the $U(1)$-connection defined by A on the bundle of complex frames of L), and then verify that H coincides with the pullback of $H^{\nabla\times A}$ via the double covering $P_g \to SO(X,g) \times_X U(L)$.

Therefore, we can restrict our attention to the space $C(L) \cong C_{comp}(P_g)$ of connections on L: as L has complex rank one, this can be identified, up to the choice of an origin $A_0 \in C(L)$, to the vector space $i\mathcal{E}^1(X)$ (because any other connection A on L can be expressed as $A_0 + \omega$, for some pure imaginary 1-form ω on X). We shall denote by ∇_A the connection induced by A on the complex spinor bundles $S_g^{\pm}$. In formulas, compatibility with the Levi Civita connection can be expressed as:

$$\nabla_A(\phi(V)\cdot\psi) = \phi(\nabla V)\cdot\psi + \phi(V)\cdot\nabla_A\psi, \quad \text{for } V \in \Gamma(TX), \psi \in \Gamma(S_g^{\pm}).$$

Also notice that the compatibility with the Levi Civita connection implies that the quadratic map $\sigma^* : S_g^+ \to i\wedge^{2,+} TX^*$ takes parallel sections (with respect to ∇_A) into parallel sections (with respect to the Levi Civita connection), because σ^* is $\mathrm{Spin}^c(4)$-equivariant.

Now, the Seiberg-Witten equations for P_g are a system of partial differential equations for a connection $A \in C(L)$ and a complex spinor $\psi \in S_g^+$, and they can be written:

$$(SW)_g : \begin{cases} D_A\psi &= 0 \\ \sigma^*(\psi) &= \Omega^+(A) \end{cases}$$

The symbol $\Omega^+(A)$ denotes the self-dual part (with respect to g) of the curvature of the connection A (i.e. $\Omega^+(A) \in i\wedge^{2,+} TX$ pointwise), while $D_A : \Gamma(S_g^+) \to \Gamma(S_g^-)$ is the Dirac operator associated with A, that is the composition

$$D_A : \Gamma(S_g^+) \xrightarrow{\nabla_A} \Gamma(TX^* \otimes S_g^+) \xrightarrow{\cdot} \Gamma(S^-)$$

where $\cdot$ is obtained by letting the first factor $TX^* \cong TX \subset \mathrm{Hom}(S_g^+, S_g^-)$ act on $S_g^+ \oplus S_g^-$ by Clifford multiplication (in formulas, $D_A\psi = \Sigma_{i,j} dx_i \cdot \psi_j$ if $\nabla_A\psi = \Sigma_{i,j} dx_i \otimes \psi_j$ locally). For instance, when (X, J, g) is a Kähler manifold endowed with its canonical Spin^c-structure $P = P(J)$ and A is the connection on $L(P) \cong K_X^{-1}$ induced by the Levi Civita connexion, a direct computation shows that the Dirac operator

$$D_A : \Gamma(S_g^+) = C^\infty_{\mathbf{C}}(X) \oplus \mathcal{E}^{0,2}(X) \to \Gamma(S_g^-) \cong \mathcal{E}^{0,1}(X)$$

can be expressed in terms of more familiar differential operators as $D_A(f,\alpha) = \sqrt{2}\cdot\bar\partial f + \bar\partial^*\alpha$ (where $\bar\partial^*$ denotes the formal adjoint of $\bar\partial$).

The first of Seiberg-Witten equations is linear of the first order, and that the second one has order 2, with quadratic terms of order 0. The philosophy behind this system is that the invariants of the space of solutions (dimension, homology etc.) will give subtle invariants of the initial Spin^c-prestructure. However, in order to get a smooth space of solutions, one has generally to consider a generic perturbation of the Seiberg-Witten equations, that is to study the whole family of systems:

$$(SW)_{g,\delta} : \begin{cases} D_A\psi &= 0 \\ \sigma^*(\psi) &= \Omega^+(A) + \delta \end{cases}$$

for $\delta \in i\mathcal{E}^{2,+}(X)$. A solution (ψ, A) will be called *reducible* if $\psi = 0$.

3.5. *Space of solutions and Seiberg-Witten invariants*

Let $\mathcal{G} = \mathcal{C}^\infty(X, U(1))$: this group can be identified with the automorphism group of a Spin^c-prestructure P on X, since $P \to \widetilde{GL}_+(X)$ is a principal bundle with structure group $U(1)$. The group $\mathcal{G}$ acts therefore on the spaces of sections of all vector bundles associated to P, as well as on the relative spaces of connections: for instance, if $A \in C(L)$, then $f \cdot A = A - 2df/f$. It is easily verified that, when (ψ, A) is a solution of $(SW)_{g,\delta}$, then also $(f\psi, f \cdot A)$ is a solution; in fact, $\mathcal{G}$ acts trivially on the terms of the second equation, while $D_{fA} f\psi = f D_A\psi$. What is therefore interesting is the space of solutions modulo the *gauge group* $\mathcal{G}$. Choosing a point $\overline{x} \in X$, one can identify the gauge group to $S^1 \times \overline{\mathcal{G}}$, where $\overline{\mathcal{G}}$ is the subgroup of $\mathcal{G}$ made up of maps u with $u(\overline{x}) = 1$.

So, let $A_g = C(L) \times \Gamma(S_g^+)$ be the ambient space, let $A_g^* = C(L) \times (\Gamma(S_g^+) \setminus \{0\})$, and let $\mathcal{A}_g = A_g/\mathcal{G}$, $\mathcal{A}_g^* = A_g^*/\mathcal{G}$ and $\overline{\mathcal{A}}_g^* = A_g^*/\overline{\mathcal{G}}$. Then, let $Z_{g,\delta}$ be the space of solutions of $(SW)_{g,\delta}$, let $Z_{g,\delta}^* = Z_{g,\delta} \cap A_g^*$ be the space of irreducible solutions, and let $Z_g = \bigcup_\delta Z_{g,\delta}$, $Z_g^* = \bigcup_\delta Z_{g,\delta}^*$. Finally, let us denote by $\widehat{\mathcal{Z}}_{g,\delta}$, $\widehat{\mathcal{Z}}_{g,\delta}^*$, $\widehat{\mathcal{Z}}_g$ and $\widehat{\mathcal{Z}}_g^*$ the corresponding spaces modulo the gauge group.

These are (quotients of) spaces of sections, and one should specify their regularity. The natural framework is that of L_k^2-sections, for k big enough in order that the Seiberg-Witten equations make sense at least as a functional

$$SW_{g,\delta} : L_k^2(iT^*X \oplus S_g^+) \to L_{k-1}^2(S_g^- \oplus i\wedge^{2,+} T^*X)$$

between Hilbert spaces, where $SW_{g,\delta}(A, \psi) = (D_A\psi, \sigma^*(\psi) - \Omega^+(A) - \delta)$. What follows holds when these spaces are endowed with the L_k^2-topology (and the corresponding spaces modulo gauge group endowed with the quotient topology), for all $k \gg 0$.

The space $\mathcal{A}_g^*$ is a smooth Hilbert manifold (since $\mathcal{G}$ acts properly on A_g, and $C(L) \times \{0\} \subset A_g$ is the set of fixed points for this action). Moreover, the space $\overline{\mathcal{A}}_g^*$ is a natural S^1-fibration over $\mathcal{A}_g^*$. On the other hand, about the structure of the space of solutions one knows that:

i) the projection $\pi_g : \mathcal{Z}_g \to i\mathcal{E}^{2,+}(X)$ on the space of perturbation parameters is proper;

ii) $\widetilde{\mathcal{Z}}_g^*$ is a C^∞ Hilbert manifold;

iii) the projection $\pi_g^* : \widetilde{\mathcal{Z}}_g^* \to i\mathcal{E}^{2,+}(X)$ is C^∞ and Fredholm.

Thus, there are two types of singular values for π_g ("singular perturbations"): the perturbations δ for which there exists a reducible solution of $(SW)_{g,\delta}$ (that is, such that $\pi_g^{-1}(\delta)$ meets $\widetilde{\mathcal{Z}}_g \setminus \widetilde{\mathcal{Z}}_g^*$), and the critical values of π_g^*. Since π_g^* is Fredholm, by the Sard-Smale theorem these last points form a closed subset of $i\mathcal{E}^{2,+}(X)$ with empty interior. On the other hand, to investigate the first type of singular values, one has to study the image of the map $A \mapsto \Omega^+(A)$ of $C(L)$ into $i\mathcal{E}^{2,+}(X)$: this is the space of self-dual 2-forms which are orthogonal to the harmonic ones, hence it is a closed subspace of codimension $b_2^+(X) = \dim \mathcal{H}^{2,+}(X)$. So, if $b_2^+(X)$ is at least 1, a generic δ will be nonsingular. By the Sard-Smale theorem one then deduces that:

iv) if $b_2^+(X) \geqslant 1$ then, for δ generic, $\widetilde{\mathcal{Z}}_{g,\delta}$ is a compact smooth submanifold of $\widetilde{\mathcal{Z}}_g^*$.

The dimension of $\widetilde{\mathcal{Z}}_{g,\delta}^*$ is given by the index of the map π_g^*, which is computed by identifying the tangent space $T_{[A,\psi]}\widetilde{\mathcal{Z}}_{g,\delta}$ to a class of solutions $[A, \psi]$ with a subspace of $T_{A,\psi}Z_{g,\delta}$ supplementary to the $\mathcal{G}$-fiber, and then by differentianting the Seiberg-Witten operator. By using the Atiyah-Singer theorem, one finds:

v) the dimension of $\widetilde{\mathcal{Z}}_{g,\delta}$ is $d = (\widetilde{\mathcal{Z}}_{g,\delta}) = \frac{c_1(L)^2 - 2\chi(X) - 3\tau(X)}{4}$.

In addition, one can see that the generic smooth fiber $\widetilde{\mathcal{Z}}_{g,\delta}$ is orientable (a canonical orientation being determined by the choice of orientations of the vector spaces $H^0(X, \mathbf{R})$, $H^1(X, \mathbf{R})$ and $\mathcal{H}^{2,+}(X, \mathbf{R})$).

Now, as soon as $b_2^+(X) \geqslant 2$, for any choice of metrics g_1, g_2 on X and of nonsingular perturbations δ_1, δ_2 of the Seiberg-Witten equations, one can clearly find a path g_t in the space of metrics and a path δ_t in $i\mathcal{E}^{2,+}(X)$ made up of nonsingular perturbations for π_{g_t}: then, the space $\widetilde{\mathcal{Z}}_{g_2,\delta_2}^{g_1,\delta_1} = \cup_t \widetilde{\mathcal{Z}}_{g_t,\delta_t}$ will be a smooth manifold which realizes an oriented cobordism between $\widetilde{\mathcal{Z}}_{g_1,\delta_1}$ and $\widetilde{\mathcal{Z}}_{g_2,\delta_2}$. Therefore, all the homological invariants of a generic fiber $\widetilde{\mathcal{Z}}_{g,\delta}$ do not depend on g, δ, and are invariants of the initial Spin^c-prestructure P. Namely, the Seiberg-Witten invariants of P are defined as

$$SW(P) = \int_{\widetilde{\mathcal{Z}}_{g,\delta}} c_1(\overline{\mathcal{J}}_g^*)^{d/2}$$

where $c_1(\overline{\mathcal{J}}_g^*) \in H^2(\mathcal{J}_g^*, \mathbf{R})$ is the first Chern class of the S^1-bundle $\overline{\mathcal{J}}_g^* \to \mathcal{J}_g^*$. The invariant $SW(P)$ is defined as zero if d is odd.

On the other hand, if $b_2^+(X) = 1$, one cannot generally find paths g_t, δ_t which avoids the singular perturbations. However, notice that there do not exist reducible solutions of $(SW)_g$ (and, consequently, of $(SW)_{g,\delta}$ as well, for δ generic and arbitrarily small) unless the projection $c_1^+(L)$ of $c_1(L)$ on the subspace of self-dual harmonic 2-forms $\mathcal{H}^{2,+}(X)$ is zero: in fact, if $(0, A)$ is a solution, then $\Omega^+(A) = 0$ and $c_1(L) = [\Omega^-(A)/2\pi i] \in \mathcal{H}^{2,+}(X)^\perp$. So, when $b_2^+(X) = 1$, if P_g is a Spin^c-structure on (X, g) such that $c_1^+(L) \neq 0$, one can still define the Seiberg-Witten invariants $SW(P_g)$ as the integral of $c_1(\overline{\mathcal{J}}_g^*)$

over the generic smooth fiber $\mathcal{F}_{g,\delta}$: this number (being an integer) will not depend on δ, if δ is sufficiently small, but it will now depend, a priori, on the metric g. Actually, it can be shown that $SW(P_g) = SW(P_{g'})$ when g and g' define the same *polarization* of X. A polarization of a manifold X is a maximal subspace H^+ of $H^2(X, \mathbf{R})$ where the intersection form is positive definite: any metric g on X determines a polarization H_g^+, given by the subspace $\mathcal{H}^{2,+}(X) \subset \mathcal{H}^2(X) \cong H^2(X, \mathbf{R})$ of self-dual harmonic 2-forms. In conclusion, when $b_2^+(X) = 1$, the Seiberg-Witten invariants depend on the initial Spin^c-prestructure P and on the choice of a polarization H^+ of X, and they are denoted by $SW(P, H^+)$.

Finally, let us notice that in case $P = P(J)$ is the canonical Spin^c-prestructure of a complex manifold (X, J), then the space of solutions of a generic perturbation of the Seiberg-Witten equations (modulo gauge) has dimension zero, as we have $c_1(L)^2 = c_1(K_X)^2 = 2\chi(X) + 3\tau(X)$ precisely, by Hirzebruch's formula. The Seiberg-Witten invariant $SW(P)$ then simply reduces to count a finite number of points with signs ± 1.

Actually, one can see that the Seiberg-Witten equations for a Kähler manifold (X, J, g) (written with respect to the Kähler metric) have only reducible solutions if K_X has negative degree, and have only one solution, modulo gauge, if K_X has positive degree (where the degree of K_X is by definition the number $c_1(K_X) \cdot [\omega_g]$).

4. A general scalar curvature L^2 estimate

A direct consequence of the nonvanishing of the Seiberg-Witten invariants of some Spin^c-prestructure on a differentiable manifold X is the following remarkable estimate of the L^2-norm of the scalar curvature of *any* metric on X.

THEOREM 4.1 (C. LeBrun [19], [20]). — *Let X be a smooth 4-manifold with $b_2^+(X) \geqslant 2$. If X has a Spin^c-prestructure P with nonvanishing Seiberg-Witten invariant $SW(P) \neq 0$ and $L = L(P)$, then*

$$\int_X \mathrm{scal}_g^2 \, dv_g \geqslant 32\pi^2 c_1^+(L)^2 \quad \textit{for all } g \textit{ on } X. \tag{8}$$

If $c_1^+(L) \neq 0$, the equality

$$\int_X \mathrm{scal}_g^2 \, dv_g = 32\pi^2 c_1^+(L)^2 \tag{9}$$

holds if and only if there exists a complex structure J on X which induces $P = P(J)$, and g is a metric of constant scalar curvature which is Kähler with respect to J.

Moreover, the equality $\int_X \mathrm{scal}_g^2 \, dv_g = 32\pi^2 c_1(L)^2$ (which is stronger than (9)) is satisfied if and only if g is, in addition, Einstein.

The same conclusions hold when $b_2^+(X) = 1$ for metrics g such that $SW(P, H_g^+) \neq 0$.

The symbol $c_1^+(L)$ here clearly denotes the projection of $c_1(L)$ on the space of self-dual harmonic 2-forms $\mathscr{H}^{2,+}(X)$. Hence, the right-hand side of (8) depends on g; however, notice that one always has $c_1^+(L)^2 \geqslant c_1(L)^2$ (which does not depend on g).

We need now to recall some basic facts about the curvature of Kähler manifolds (X, J, g). Let $\sim$ denote the isomorphism which transforms real hermitian forms α of TX into real antisymmetric forms of type $(1,1)$, i.e. $\tilde{\alpha}(\cdot,\cdot) = \alpha(J\cdot,\cdot)$. The curvature of g, when seen as a symmetric endomorphism of $\wedge^2 TX$, has matrix

$$\mathrm{R}_g \equiv \begin{array}{c} \begin{array}{cc} \underbrace{\qquad\qquad}_{\omega_g\ \ \wedge^{2,+}TX} & \underbrace{\qquad\qquad}_{\wedge^{2,-}TX} \end{array} \\ \left(\begin{array}{ccc|ccc} \mathrm{scal}_g & 0 & 0 & \rho_1 & \rho_2 & \rho_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \rho_1 & 0 & 0 & & & \\ \rho_2 & 0 & 0 & & \mathrm{A} & \\ \rho_3 & 0 & 0 & & & \end{array}\right) \end{array} \begin{array}{l} \omega_g \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \wedge^{2,+}TX \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \wedge^{2,-}TX \end{array} \tag{10}$$

with respect to any orthonormal basis of the form $\{\alpha_1^+ = \omega_g, \alpha_2^+, \alpha_3^+\} \cup \{\alpha_1^-, \alpha_2^-, \alpha_3^-\}$ of $\wedge^{2,+}TX \oplus \wedge^{2,-}TX$ (cp. [5]). With this notation, one has

$$\widetilde{\mathrm{Ric}}_g = R_g(\omega_g) = \frac{1}{4}\,\mathrm{scal}_g \cdot \omega_g + \sum_{i=1}^{3} \rho_i \alpha_i\,, \qquad \widetilde{\mathrm{Ric}}_g^+ = \mathrm{scal}_g \cdot \omega_g/4\,. \tag{11}$$

Proof of Theorem 4.1. Let g be any metric on X. We may assume that $c_1^+(L)^2 > 0$, since otherwise the inequality is trivial. As $SW(P) \neq 0$ (or $SW(P, H_g^+) \neq 0$, by assumption, when $b_2^+(X) = 1$), there exist solutions (ψ_k, A_k) of the perturbed Seiberg-Witten equations $(SW)_{g,\delta_k}$, for generic, arbitrarily small, parameters δ_k. As π_g is proper, (a subsequence of) the solutions (ψ_k, A_k) converge to an irreducible solution (ψ, A) of $(SW)_{g,0}$ (we already remarked that there exist no reducible solutions when $c_1^+(L) \neq 0$).

Now let ∇_A^*, D_A^* the formal adjoint operators of ∇_A and D_A with respect to the L^2-scalar product of (X, g). By a standard Weitzenböck-Lichnerowicz formula (see [24], [25]), one can control the difference between the Laplacian of Dirac $D_A^2 = D_A^* D_A$ and the Laplace operator $\nabla_A^* \nabla_A$ by the curvature of g and A:

$$D_A^2 = \nabla_A^* \nabla_A + \frac{1}{4}\,\mathrm{scal}_g \cdot \mathrm{id}_{S^+} + \frac{1}{2}\varphi(\Omega^+(A)). \tag{12}$$

Plugging our solution (ψ, A) into this formula gives

$$0 = \nabla_A^* \nabla_A \psi + \frac{1}{4}\,\mathrm{scal}_g \cdot \psi + \frac{1}{4}\|\psi\|^2 \psi$$

since $\varphi(\Omega^+(A)) = \varphi(\sigma^*(\psi)) = \frac{1}{2}\|\psi\|^2 \cdot \psi$. By taking the L^2-scalar product with ψ, one gets

$$0 = \int_X (\|\nabla_A \psi\|^2 + \frac{1}{4}\ \mathrm{scal}_g \cdot \|\psi\|^2 + \frac{1}{4}\|\psi\|^4) d v_g$$

which in turn yields

$$\left(\int_X \|\psi\|^4 d v_g\right)^2 \leqslant \left(\int_X \mathrm{scal}_g \cdot \|\psi\|^2 d v_g\right)^2 \leqslant \int_X \mathrm{scal}_g^2\, d v_g \cdot \int_X \|\psi\|^4 d v_g \tag{13}$$

that is,

$$\int_X \mathrm{scal}_g^2\, d v_g \geqslant \int_X \|\psi\|^4 d v_g = 8 \int_X \|\Omega^+(A)\|^2 d v_g \tag{14}$$

as $\|\Omega^+(A)\|^2 = \|\sigma(\psi)\|^2 = \|\psi\|^4/8$. Remark that this inequality is strict unless $\nabla_A \psi = 0$ and scal_g is constant.

If we knew that $\Omega^+(A)$ represents $2\pi c_1^+(L)$, we would be done; unfortunately, the self-dual form $\Omega^+(A)$ need not to be closed, a priori, so it remains to use the following trick. One considers the harmonic form α in the cohomology class of $\Omega(A)$, so that the self-dual and the antiself-dual components $\alpha^\pm$ of α do represent $2\pi c_1^\pm(L)$ respectively (since they are closed), and then one writes:

$$\begin{aligned}
\int_X \|\Omega^+(A)\|^2 d v_g &= \frac{1}{2}\int_X (\|\Omega^+(A)\|^2 - \|\Omega^-(A)\|^2) d v_g + \frac{1}{2}\int_X (\|\Omega^+(A)\|^2 + \|\Omega^-(A)\|^2) d v_g \\
&= 4\pi^2 c_1(L)^2 + \frac{1}{2}\int_X \|\Omega(A)\|^2 d v_g \\
&\geqslant 4\pi^2 c_1(L)^2 + \frac{1}{2}\int_X \|\alpha\|^2 d v_g \\
&= \frac{1}{2}\int_X (\|\alpha^+\|^2 - \|\alpha^-\|^2) d v_g + \frac{1}{2}\int_X (\|\alpha^+\|^2 + \|\alpha^-\|^2) d v_g \\
&= \int_X \|\alpha^+\|^2 d v_g = 4\pi^2 c_1^+(L)^2
\end{aligned}$$

as the harmonic forms have minimal L^2-norm within their cohomology class. This proves inequality (8).

When (X, J, g) is a Kähler manifold of constant scalar curvature, and $P = P(J)$ is the canonical Spin^c-structure attached to J, then we know that $L \cong K_X^{-1}$ and that the Ricci form over 2π represents $c_1(X) = c_1(L)$. By the above recalled decomposition of the Kähler curvature operator, we have $\widetilde{\mathrm{Ric}}_g = \frac{1}{4}\ \mathrm{scal}_g \cdot \omega_g + \widetilde{\mathrm{Ric}}_g^- \in 2\pi c_1(L)$; as ω_g is parallel, it follows that $\mathrm{scal}_g \cdot \omega_g / 8\pi$ represents $c_1^+(L)$, hence the equality

$$c_1^+(L)^2 = \frac{1}{64\pi^2}\int_X \mathrm{scal}_g^2 \cdot \omega_g \wedge \omega_g = \frac{1}{32\pi^2}\int_X \mathrm{scal}_g^2\, d v_g$$

is satisfied. If, in addition, g is an Einstein metric, then $\widetilde{\mathrm{Ric}}_g^- = 0$ and $c_1(L) = c_1^+(L)$ necessarily.

Conversely, assume that $\int_X \mathrm{scal}_g^2\, dv_g = 32\pi^2 c_1^+(L)^2$. This forces the inequalities (13) and (14) to be equalities, so that ψ is parallel and scal_g is constant. How to find an almost complex structure J for which g is Kähler? We say that the fact that S^+ has a parallel section ψ (which we may assume of norm 1, pointwise) permits to find a subbundle P_0 of P with structure group $U(2)$. In fact, see P as the bundle of " Spin^c-frames" of $S^+(X) \oplus S^-(X)$, i.e. couples of local unitary sections $(\psi_1^+, \psi_2^+), (\psi_1^-, \psi_2^-)$ satisfying $\det_+(\psi_1^+ \wedge \psi_2^+) = \det_-(\psi_1^- \wedge \psi_2^-)$. Then, let P_0 be the subbundle defined by the frames $(\psi, \psi_2^+), (\psi_1^-, \psi_2^-)$: since ψ is parallel, this bundle has holonomy

$$U(2) \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \det(A_-) \end{pmatrix} \oplus A_- \ , \ A_- \in U(2) \right\} \overset{i}{\hookrightarrow} \mathrm{Spin}^c(4),$$

hence its structure group can be reduced to $U(2)$. As P lifts the bundle of direct orthonormal frames of (X, g), by commutativity of diagram (3.2) the bundle P_0 also lifts $SO(X, g)$: this precisely is an almost complex structure J on X for which g is hermitian [6].

Moreover, to see that the metric g is Kähler with respect to this structure, it is enough to verify that the Levi Civita connection of $SO(X, g)$ comes from a connection on P_0 (cp. [18]): but this is clear as the horizontal, $\mathrm{Spin}^c(4)$-invariant subbundle H of TP defining the compatible connection of P can be restricted to a connection on P_0 (it is easily checked that $H \subset TP_0$, as every horizontal curve of P based at any $p_0 \in P_0$ lies in P_0).

Finally, assume that the stronger equality $\int_X \mathrm{scal}_g^2\, dv_g = 32\pi^2 c_1(L)^2$ is verified. This implies that $c_1(L) = c_1^+(L)$. But we just proved that $P = P(J)$, and that g is Kähler, so (11) holds; as ω_g is parallel, we infer that $\widetilde{\mathrm{Ric}}_g^- = \widetilde{\mathrm{Ric}}_g - \mathrm{scal}_g \cdot \omega_g/4$ is closed and therefore it represents $2\pi c_1^-(L) = 0$. Moreover, $\widetilde{\mathrm{Ric}}_g^-$ necessarily vanishes since

$$\begin{aligned} 0 = c_1^-(L)^2 &= \frac{1}{4\pi^2} \int_X \widetilde{\mathrm{Ric}}_g^- \wedge \widetilde{\mathrm{Ric}}_g^- \\ &= -\frac{1}{4\pi^2} \int_X \widetilde{\mathrm{Ric}}_g^- \wedge * \widetilde{\mathrm{Ric}}_g^- = -\frac{1}{4\pi^2} \int_X \| \widetilde{\mathrm{Ric}}_g^- \|^2 dv_g. \end{aligned}$$

Thus, $\widetilde{\mathrm{Ric}}_g$ reduces to $\frac{1}{4}\, \mathrm{scal}_g \cdot \omega_g$, and g is Einstein. □

Finally, let us see how Theorem 4.1 implies Assertion 1.3 in the complex case.

Proof of Assertion 1.3 for a complex hyperbolic surface (X, J_0, g_0). As g_0 is Kähler-Einstein, one has $c_1(K_X) = -c_1(X) = -\widetilde{\mathrm{Ric}}(g_0)/2\pi = -\mathrm{scal}_{g_0} \cdot \omega_{g_0}/2\pi$, which implies that K_X has positive degree since $\mathrm{scal}_{g_0} < 0$. As we saw in the last section, this implies

[6] Of course, we have a natural candidate to define explicitely this almost complex structure: J is the orientation-preserving, antisymmetric isomorphism of TX associated, with respect to the metric g, with the real, self-dual form of norm 1 given by $i\Omega^+(A)/\|\Omega^+(A)\|$. This is clearly a parallel form (hence, (X, g, J) is Kähler) since ψ is parallel and $\sigma^*(\psi) = \Omega^+(A)$ (σ^* sends parallel sections into parallel sections).

that $SW(P(J_0)) \neq 0$. However, this can be checked directly as follows. If ∇ denotes the connection on K_X^{-1} induced by the Levi Civita connection of TX, the couple

$$\begin{cases} \psi_0 = (\sqrt{-\operatorname{scal}_{g_0}}, 0) \\ A_0 = \nabla \end{cases}$$

evidently satisfies the Seiberg-Witten equations $(SW)_{g_0}$. By computing the differential of the Seiberg-Witten operator one sees that this solution is a regular point for π_{g_0}. Moreover, any other solution (ψ, A) is gauge equivalent to (ψ_0, A_0). In fact, formula (14) yields

$$\int_X \| \operatorname{Ric}_{g_0} \|^2 d v_{g_0} = \frac{1}{8} \int_X \operatorname{scal}_{g_0}^2 \, d v_{g_0} \geqslant \int_X \|\Omega^+(A)\|^2 d v_{g_0}$$

but $\widetilde{\operatorname{Ric}}_{g_0} = \operatorname{scal}_{g_0} \cdot \omega_{g_0}/4 = \widetilde{\operatorname{Ric}}_{g_0}^+$ is harmonic (as $\operatorname{scal}_{g_0}$ is constant), hence by minimality one infers that $\int_X \| \operatorname{Ric}_{g_0} \|^2 d v_{g_0} = \int_X \|\Omega^+(A)\|^2 d v_{g_0}$. This in turn implies that $\|\psi\|^2 = \operatorname{scal}_{g_0}$ and that ψ is ∇_A-parallel. Now, as $K_X^{-1} \subset S^+$ has no parallel sections ($c_1(L) = c_1(K_X^{-1})$ being nonzero), ψ reduces to a section of $X \times \mathbf{C}$, so $\psi = u \cdot \operatorname{scal}_{g_0}$ for some $u \in \mathcal{G}$. Moreover, $\Omega(A) = \Omega^+(A)$ (as $\Omega(A)$ and $\widetilde{\operatorname{Ric}}_{g_0}$ define the same cohomology class, and $\widetilde{\operatorname{Ric}}_{g_0}$ is self-dual). Therefore $A = \nabla + \alpha$ is equal to the Chern connection twisted by some flat connection. But as ψ is parallel, it is easily checked that $\nabla_A|_{X \times \mathbf{C}} = d - u^{-1} du$, so that $\alpha = -u^{-1} du$ and $A = u \cdot \nabla$; that is, $(\psi, A) = (\psi_0, A_0)$ modulo the gauge group.

In conclusion, Lebrun's inequality (8) is avalaible. Notice that there actually exist [26] complex hyperbolic surfaces X with $b_2^+(X) = 1$, but the characteristic numbers of X always satisfy, by Hirzebruch's and Gauss-Bonnet formula (cp. [5]):

$$0 < \frac{\chi(X)}{3} = \tau(X) = b_2^+(X) - b_2^-(X)$$

hence $b_2^-(X) = 0$ and there exists only one polarization of X (so, $SW(P(J_0), H^+) = SW(P(J_0))$ in this case). We then deduce that for all normalized Einstein metrics g on X one has

$$144 \cdot \operatorname{Vol}(X, g) = \int_X \operatorname{scal}_g^2 \, d v_g \geqslant 32\pi^2 c_1(L)^2 = 96\pi^2(2\chi(X) - 3\tau(X) \qquad (15)$$

that is,

$$\operatorname{Vol}(X, g) \geqslant \frac{2}{3}\pi^2(2\chi(X) - 3\tau(X)).$$

But we know, by (4), that the opposite inequality always holds for an Einstein 4-manifold, so we get that

$$\operatorname{Vol}(X, g) = \frac{2}{3}\pi^2(2\chi(X) - 3\tau(X)) \text{ and } W_g^- = 0$$

necessarily. Moreover, this equality forces the equality in (15), and one deduces, by Theorem 4.1, that g is Kähler with respect to some complex structure J. Now, since g is Einstein and $W_g^- = 0$, formulas (10) and (11) yield $\rho_i = 0$ for all i, and $A = \frac{1}{12} \operatorname{scal}_g I$ (as

R_g reduces to $U_g + W_g^+$). Therefore, $R_g = \frac{1}{12}\,\mathrm{scal}_g \cdot \mathrm{id}_{\Lambda^2 - TX} + \frac{1}{4}\,\mathrm{scal}_g \cdot \omega_g^* \otimes \omega_g$. It is now evident that R_g is a parallel tensor, hence that g is a locally symmetric metric. □

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