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TYPICAL SURFACES AND RANDOM GRAPHS

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In this talk, we describe an approach to the problem: What does a typical Riemann surface of large genus look like geometrically? In large part, this is joint work with Eran Makover.

As various parts of this program have been described elsewhere ([PS], [SGB], [LFE],[RCRS]), we will take the present occasion to describe some of the motivating ideas behind the program. See [FERS] for an announcement of results in this direction.

A central problem, which we have attacked from a number of points of view, is to come to some geometrical understanding of the following theorem, due to Selberg:

THEOREM 1 ([Sel]). — *Let* $\Gamma = PSL(2,\mathbb{Z})$, *and let*

$$\Gamma_k = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mod k) \right\}.$$

Then the first eigenvalue $\lambda_1(\mathbb{H}^2/\Gamma_k)$ *satisfies*

$$\lambda_1(\mathbb{H}^2/\Gamma_k) \geqslant 3/16.$$

The number 3/16 has been improved by Luo, Rudnick, and Sarnak [LRS], but we will not be interested here in precise constants. Rather, we will say that λ_1 of a Riemann surface is large if it is bounded below by a positive constant independent of the genus.

A natural question arising from Selberg's Theorem is whether the phenomenon of large first eigenvalue is something which is special for arithmetically defined surfaces, or whether it is a property enjoyed by "typical" Riemann surfaces, of which such arithmetically defined surfaces just happen to be good examples.

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To understand this question, we may perform the following thought experiment: let R_g be a Riemann surface whose geometric description is like our usual picture of a Riemann surface:

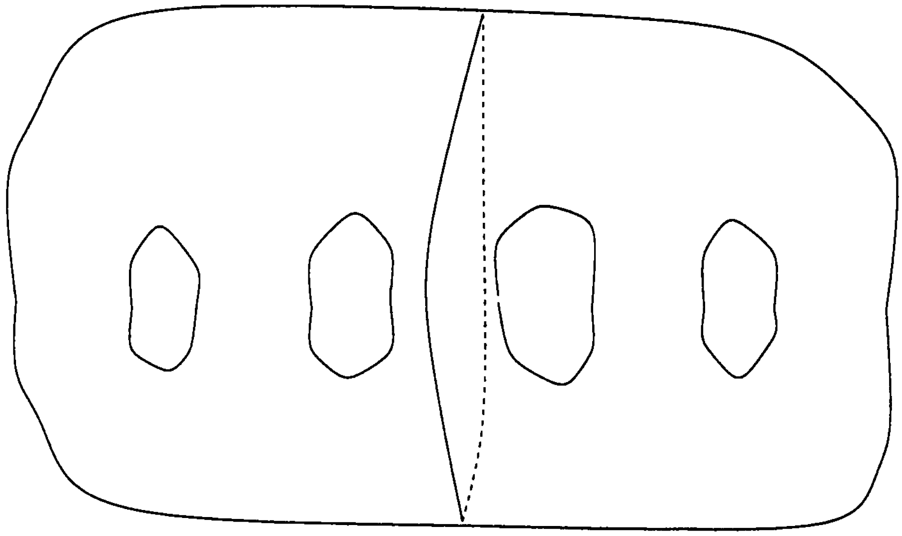

Figure 1: *The surface* R_g

We have drawn on R_g a curve which divides it into two pieces.

Instead of trying to visualize the first eigenvalue, we instead consider the Cheeger constant

$$h(R) = \inf_C \frac{\text{length}(C)}{\min(\text{area}(A), \text{area}(B))},$$

where C is a (possibly disconnected) curve which splits R into two parts A and B.

It is then easy to see that as g gets large, the Cheeger constant $h(R_g)$ tends to 0, as the surface is divided into two pieces of equal size by a curve such as the curve in Figure 1, whose length is fixed independent of the genus.

Now let us divide the surface in half, as in Figure 2 below, and then glue the legs of the top half randomly to legs in the bottom half. It is easy to convince oneself that for a suitably random gluing of the legs, there is no longer any convenient way to divide the surface in half by a relatively short curve.

One would like to believe that a typical Riemann surface looks more like one of the random gluings than like R_g itself. The problem in making this precise is two-fold:

(i) First of all, it would seem to be difficult to describe processes such as the random gluings in terms of, say, Fenchel-Nielsen coordinates. In general, it would seem to be difficult to use Fenchel-Nielsen coordinates to control the spectral geometry of a surface of large genus.

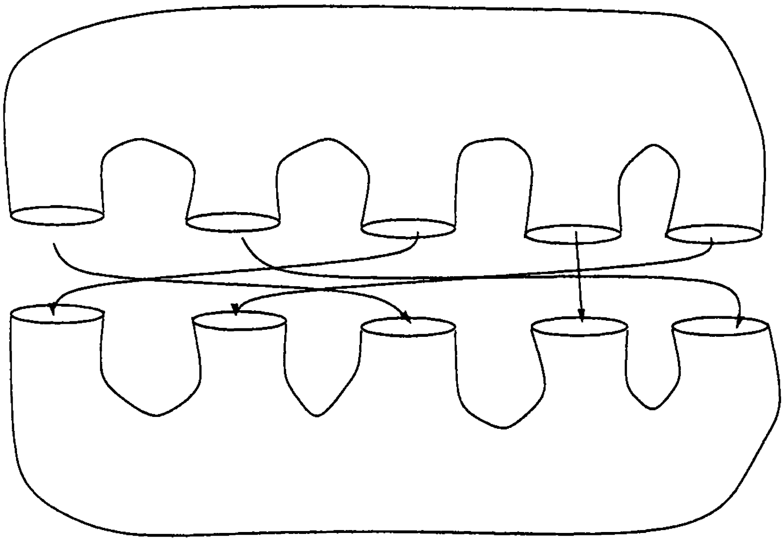

Figure 2: *A random gluing*

(ii) Secondly, the gluing process described above seems to rest on a combinatorial structure which would seem to be absent in a typical Riemann surface. How can one describe a typical Riemann surface in a way which reflects a combinatorial structure analogous to this?

Both of these difficulties are met by the following construction: let G be a finite trivalent graph, and $\mathcal{O}$ an orientation of G– i.e., for each vertex v of G, $\mathcal{O}$ gives a cyclic ordering of the vertices emanating from v.

We may then associate to the pair $(G, \mathcal{O})$ two Riemann surfaces $S^O(G, \mathcal{O})$ and $S^C(G, \mathcal{O})$, as follows: $S^O(G, \mathcal{O})$ is constructed from G by pasting one hyperbolic ideal triangle for each vertex, and gluing triangles together according to the graph and orientation, see [TS] for details. $S^O(G, \mathcal{O})$ is then a finite-area Riemann surface, whose geometry is well-controlled by the pair $(G, \mathcal{O})$. $S^C(G, \mathcal{O})$ is then the conformal compactification of $S^O(G, \mathcal{O})$.

The two problems mentioned above can be rephrased in the following way:

Question 1. — *To what extent can we transfer the good geometric control that we have on the surfaces $S^O(G, \mathcal{O})$ to the surfaces $S^C(G, \mathcal{O})$?*

Question 2. — *To what extent are the surfaces $S^C(G, \mathcal{O})$ typical Riemann surfaces?*

Question 2 is answered by the following theorem, which is an easy consequence of the Belyi Theorem [Be]:

Theorem 2. — *If S is any compact Riemann surface, then for any ε, there is a surface of the form $S^C(G, \mathcal{O})$ is ε-close to S.*

Here, "ε-close" may be taken in any convenient metric on moduli space, for instance the Teichmüller metric. Thus, the surfaces $S^C(G, \mathcal{O})$ are a dense set of surfaces in the moduli space of all surfaces.

The answer to Question 1 is somewhat more complicated. It is not hard to see that the surfaces $S^O(G, \mathcal{O})$ and $S^C(G, \mathcal{O})$ might be quite different geometrically. For instance, $S^O(G, \mathcal{O})$ always carries a complete hyperbolic metric, but $S^C(G, \mathcal{O})$ might be a sphere. However, the theorem of [PS] guarantees that this cannot happen when the cusps are large:

THEOREM 3 ([PS]). — *For any ε, there exists an L with the following property: if S^O is a finite-area Riemann surface, all of whose cusps have length geL, then outside of cusp neighborhoods, depending only on L, the hyperbolic metrics ds_O^2 on S^O and ds_C^2 on its conformal compactification S^C satisfy:*

$$\frac{1}{(1+\varepsilon)} ds_O^2 \leqslant ds_C^2 \leqslant (1+\varepsilon) ds_O^2.$$

The proof is an application of the Ahlfors-Schwarz Lemma [A], together with playing with differential inequalities.

When the condition of large cusps is satisfied, Theorem 3 can be used to show that the geometric control one has over $S^O(G, \mathcal{O})$ transfers to control over $S^C(G, \mathcal{O})$. Furthermore, the large cusps condition has a simple graph-theoretic interpretation which is easily studied.

In [SGB] and [RCRS], we use the Bollobas model of random regular graphs [Bo1], [Bo2] to study the large cusps condition. Let $\mathcal{G}_k$ denote the finite set of 3-regular graphs on $2k$ vertices, and $\mathcal{G}_k^*$ the finite set of oriented 3-regular graphs on $2k$ vertices. Then:

THEOREM 4 ([SGB]). — *With probability $\to 1$ as $k \to \infty$, a graph selected randomly from $\mathcal{G}_k$ carries an orientation $\mathcal{O}$ such that all the cusps of $S^O(G, \mathcal{O})$ are large.*

THEOREM 5 ([RCRS]). — *There is a positive constant C_1 independent of k such that, for a pair $(G, \mathcal{O})$ randomly chosen from $\mathcal{G}_k^*$, $S^O(G, \mathcal{O})$ has large cusps with probablity at least C_1.*

Theorems 4 and 5 can be used to construct compact surfaces which enjoy properties enjoyed by random 3-regular graphs. In particular, Theorem 5 shows that there is a constants C_2 such that a randomly chosen surface $S^C(G, \mathcal{O})$ satisfies

$$\lambda_1(S^C(G, \mathcal{O})) \geqslant C_2$$

with probablity at least C_1.

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