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GENERALIZED CONDENSERS AND CONFORMAL PROPERTIES OF RIEMANNIAN MANIFOLDS WITH AT LEAST TWO ENDS

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Introduction

In spite of important generalizations such as $\mathcal{A}$-potential theory (cf. [HKM], [H1], [HR2], the conformal potential theory remains an essential tool for studying quasiconformal and quasiregular mappings of Riemannian manifolds (cf. [GLM], [H2], [HR1], [HR3]). However, the usual definition of capacities is perhaps not always the most appropriate one for studying the conformal properties of such a manifold M at infinity, as it only deals with condensers defined by a pair (G, C) where G is a domain in M and $C \subset G$ is compact. This last restriction makes often necessary to consider some sequences of domains (G_i) and pass (once or several times) to the limit.

It seems therefore worthwhile to set a theory of capacities for more general condensers. In [F1], we studied condensers defined by a pair (C_0, C_1) of closed but non necessarily compact sets of M, playing the same rôle; if C_1 is compact this definition reduces to the usual one with $G = M \setminus C_0$ and $C = C_1$. We will first extend this theory to the limit case obtained by letting C_0 tend to a set S of ends of M. Then by letting C_1 also tend to infinity and assuming that M has at least two ends, we obtain condensers whose both components C_0, C_1 are sets of ends of M, with domain $G = M$. The extremal functions relative to those condensers are n-harmonic on M ($n = \dim M$). Hence the existence of non-constant n-harmonic functions on M with a prescribed behaviour at infinity. We also

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obtain an obstruction to the existence of a sequence of K-quasiconformal automorphisms converging to infinity for a manifold M with two ends and $\mathrm{Cap}\,\partial M > 0$.

The paper is organized as follows: in sections 1, 2 we present the basic properties of condensers defined by two closed sets C_0, C_1 of M, only assumed to have at most one compact connected component, and we set some non-obvious topological properties such as Lemmas 2.4 and 2.6.

In section 3 we precise the notion of *end* of M and the topology of the (possibly infinite) set ∂M of ends of M. We introduce the notion of *sub-boundary* of M and extend the theory of capacities to "hybrid" condensers $\Gamma(S, C)$ whose first component S is a sub-boundary of M. In section 4 we prove the existence of extremal functions for hybrid condensers and in section 5 we study the behaviour of those functions when the second component C tends to infinity. Then in sections 6, 7 we can define condensers whose both components are at infinity, and by using the same process of normalization of extremal functions as in [H1] and [HR2] we obtain non-constant n-harmonic functions on M of one of the following types; *i)* bounded, *ii)* unbounded and positive, *iii)* two-side unbounded. Section 8 is devoted to some improvements.

The existence of non-constant n-harmonic functions has been proved in [H1] and [HR2] in the special case of a manifold M of the type $M = N \setminus \{a_1, \ldots, a_k\}$ where $a_1, \ldots, a_k$ are some points of a compact differentiable manifold N; and the extension of this result to the general case is considered in [HR3] as obvious. The present proof has however perhaps the interest of being synthetic and complete. It has been presented at the "École de printemps" (Géométrie conforme, Analyse et applications) held in Domaine de Seillac (France) in 1995.

1. Preliminaries

In what follows M will always denote a non-compact connected Riemannian manifold of class C^1 with dimension $n \geq 2$, and $d\tau$ its volume element. As in [F1] and [F2], $H(M) = C(M) \cap L^1_n(M)$ will be the linear space of continuous real-valued functions u on M whose distributional gradient ∇u satisfies

$$I(u, M) = \int_M |\nabla u|^n d\tau < +\infty .$$

(we emphasize that u is not assumed to be in $L_n(M)$).

Then we will denote $H^*(M)$ the set of functions $u \in H(M)$ which are monotone on M. (Let us recall that a function $u \in C(M)$ is called *monotone* if its supremum and infimum on any relatively compact domain D of M are respectively the same as on ∂D).

The space $H(M)$ will be equipped with two topologies:

a) the *compact-open* topology (c-topology for brevity)

b) the *strong topology* (s-topology for brevity) defined by the family of norms

$$\|u\| = \sup_{x \in K} |u(x)| + I(u, M)^{1/n}$$

where K is any compact set of M.

If A is a subset of $H(M)$ we will reserve the notation $\overline{A}$, or Cl A, for its closure in the strong topology. However a subset B of $C(M)$ will simply be called *bounded* on a subset X of M if the set

$\{u(x) \mid u \in B,\ x \in X\}$ is bounded.

In what follows we simply denote $\inf_X u$ [resp. $\sup_X u$] the infimum [resp. supremum] of a function u on a set X of M; and the oscillation of u on X will be denoted $\operatorname{osc}(u, X)$.

As in [F2] a *relative continuum* of M is a non-empty closed subset of M without any compact connected component, and a *compact continuum* is a compact connected set not reduced to a single point.

At last, according to the usual terminology, a function $u \in C(M) \cap \operatorname{loc} L^1_n(M)$ is called *n-harmonic* if it is a weak solution of $\operatorname{div}(|\nabla u|^{n-2} \nabla u) = 0$.

The following results are known (cf. [F1], [F2], [GLM], [H1]):

1.1. — The limit of a c-convergent sequence of monotone [resp. n-harmonic] functions is monotone [resp. n-harmonic].

1.2. — Let C be a relative continuum of M and assume that $u \in C(M)$ is monotone on $M \setminus C$, with $u = \mathrm{C}^{\mathrm{te}} = k$ on C. If $k = \sup_M u$ or $k = \inf_M u$, then u is monotone on all M.

1.3. — Let (u_k) be a c-convergent sequence in $H(M)$ with $m = \liminf I(u_k, M) < +\infty$. Then $u = \lim(u_k)$ belongs to $H(M)$ with $I(u, M) \leq m$. If the sequence (u_k) is s-convergent, then $I(u, M) = m$.

1.4. — For every compact connected set C in M there exists a constant $k(C)$ such that, for any $u \in H^*(M)$:

$$\operatorname{osc}^n(u, C) \leq k(C) I(u, M)\,.$$

1.5. — For any k fixed, the set $\{u \in H^*(M) \mid I(u, M) \leq k\}$ is equicontinuous.

2. Conformal capacities

In view of extensions we first observe that the elementary theory of conformal capacities is in fact based on the two following general Lemmas.

2.1. LEMMA. — *Let A be a convex subset of $H(M)$ and let write*

$$m(A) = \inf_{u \in A} I(u, M) .$$

If (u_k) is a c-convergent sequence in A with $\lim I(u_k) = m(A)$, then the sequence (u_k) is s-convergent, hence $u = \lim(u_k)$ belongs to $\overline{A}$ with $I(u, M) = m(A)$.

Morever if $v \in A$ also satisfies $I(v, M) = m(A)$, then $v - u = C^{te}$.

Both these assertions follow from Clarkson inequality (cf. [M]).

2.2. LEMMA. — *Let A and $m(A)$ be as in Lemma 2.1. Let additionally assume that A is bounded on all M and that there exists a map $p : A \to \overline{A}$ with an open covering (U_α) of M such that for all $u \in A$:*

i) $p(u)$ is monotone on every set U_α.

ii) $I(p(u), M) \leq I(u, M)$ and $\sup_M |p(u)| \leq \sup_M |u|$.

Then there exists a function $v \in \overline{A}$ satisfying $I(v, M) = m(A)$ and monotone on every set U_α.

Proof. — Let (u_k) be a sequence in A with $\lim I(u_k, M) = m(A)$. Then the sequence $I(p(u_k), M)$ also tends to $m(A)$. The restriction of the sequence $(p(u_k))$ to every set U_α is equicontinuous, hence this sequence is equicontinuous on all M. As it is bounded on M, it admits a c-convergent subsequence whose limit v satisfies the claim; and from Lemma 2.1 such a function v is unique except for addition of a constant.

Condensers and capacities.

As in [F2] we deal here with condensers whose both boundary components play the same rôle, *none of them being assumed to be compact.* For that reason we denote $\Gamma(C_0, C_1)$ the condenser whose boundary components C_0, C_1 are any closed disjoint subsets of M, its domain being $M \setminus (C_0 \cup C_1)$. The conformal capacity of $\Gamma(C_0, C_1)$ is

$$\mathrm{Cap}(C_0, C_1) = \inf_u I(u, M)$$

where u runs into the set $A(C_0, C_1)$ of functions $u \in H(M)$, called *admissible* for $\Gamma(C_0, C_1)$, which satisfy $u = 0$ on C_0, $u = 1$ on C_1, and $0 \leq u \leq 1$ everywhere. If $A(C_0, C_1) = \emptyset$ we set $\mathrm{Cap}(C_0, C_1) = +\infty$. If $\mathrm{Cap}(C_0, C_1) < +\infty$ it follows from Lemma 2.1 that there is

at most one function $u \in A(C_0, C_1)$ with $I(u, M) = \mathrm{Cap}(C_0, C_1)$. Such a function will be denoted $\mathrm{extr}(C_0, C_1)$. The following result is known (cf. [F1], [F2]):

2.3. THEOREM. — *Let (C_0, C_1) be a pair of compact or relative continua of M with $\mathrm{Cap}(C_0, C_1) < +\infty$. Then $u = \mathrm{extr}(C_0, C_1)$ exists and this function is n-harmonic on $M \setminus (C_0 \cup C_1)$.*

If C_0, C_1 are relative continua, u is monotone on all M. In the other cases u is monotone on the domain obtained by removing from M a point of C_i if it is compact ($i = 0, 1$).

Precisions. — It is easy to see that, for any open set V of $M \setminus (C_0 \cup C_1)$, the function $u = \mathrm{extr}(C_0, C_1)$ realizes the infimum of $I(v, V)$ among all the functions $v \in H(M)$ which satisfy $v = u$ on ∂V. From this principle we get the following properties.

2.4. LEMMA.

a) For any domain $D \subset M \setminus (C_0 \cup C_1)$ (not assumed to be relatively compact) the extremal function $u = \mathrm{extr}(C_0, C_1)$ satisfies

$$\inf_D u = \inf_{\partial D} u, \quad \sup_D u = \sup_{\partial D} u .$$

In other terms the monotonicity of u still holds if we adopt the more restrictive definition of [M].

b) If (γ_0, γ_1) is another pair of compact or relative continua with $\gamma_0 \subset C_0$ and $\gamma_1 \subset C_1$, we have

$$\mathrm{extr}(\gamma_0, C_1) \geq \mathrm{extr}(C_0, C_1) \geq \mathrm{extr}(C_0, \gamma_1) .$$

Proof. — Assertion *a)* is almost obvious. For proving assertion *b)* let us write $u = \mathrm{extr}(C_0, C_1)$, $v = \mathrm{extr}(\gamma_0, C_1)$ and assume that $V = \{x \in M \mid v(x) < u(x)\}$ is not empty. Then $V \subset M \setminus (C_0 \cup C_1)$ and $v = u$ on ∂V. As $u|_V$ and $v|_V$ both realize the infimum of $I(w, M)$ in $\{w \in H(V) \mid w = u \text{ on } \partial V\}$ we have $v = u$ on V, in contradiction with our assumption. Hence $V = \emptyset$, which proves the first inequality. The second one follows by exchanging C_0 with C_1 and γ_0 with γ_1.

Moreover, from Theorem B in [F3], which generalizes a classical result, we can state:

2.5. PROPOSITION. — *With the same assumptions and notations as in Theorem 2.3 let write $C_t^- = \{x \in M \mid u(x) \leq t\}$ and $C_t^+ = \{x \in M \mid u(x) \geq t\}$, $(0 \leq t \leq 1)$. Then for all $0 \leq \alpha \leq \beta \leq 1$ the function $u_{\alpha\beta} = \mathrm{extr}(C_\alpha^-, C_\beta^+)$ is defined by $u_{\alpha\beta} = 0$ on C_α^-, $u_{\alpha\beta} = 1$ on C_β^+ and $u_{\alpha\beta} = (u - \alpha)/(\beta - \alpha)$ on $D_{\alpha\beta} = \{x \in M \mid \alpha < u(x) < \beta\}$, hence*

$$\mathrm{Cap}(C_\alpha^-, C_\beta^+) = (\beta - \alpha)^{1-n} \mathrm{Cap}(C_0, C_1) .$$

Remark. — A sufficient (but not necessary) condition for having $\mathrm{Cap}(C_0, C_1) < +\infty$ is that one at least of the sets C_0, C_1 is compact.

We complete this result by the following one.

2.6. LEMMA. — *Let C_0, C_1 be two relative continua such that ∂C_0 is compact. If $C_0 \cap C_1 = \emptyset$, $\mathrm{Cap}(C_0, C_1)$ is finite with*

$$\mathrm{Cap}(C_0, C_1) = \mathrm{Cap}(\partial C_0, C_1) \text{ and } \mathrm{extr}(\partial C_0, C_1) = \mathrm{extr}(C_0, C_1).$$

We point out that the existence of $\mathrm{extr}(\partial C_0, C_1)$ is here obtained without assuming that ∂C_0 is connected.

Proof. — We know that $A(\partial C_0, C_1)$ is not empty. If $u \in A(\partial C_0, C_1)$ the function v defined by $v = 0$ on C_0 and $v = u$ on $M \setminus C_0$ is admissible for $\Gamma(\partial C_0, C_1)$ and $\Gamma(C_0, C_1)$, with $I(v, M) < I(u, M)$ if $v \neq u$. From (2.3) the function $w = \mathrm{extr}(C_0, C_1)$ exists and $I(w, M) = \mathrm{Cap}(C_0, C_1) \leq I(v, M)$ hence $I(w, M) \leq I(u, M)$ for all $u \in A(\partial C_0, C_1)$. As $w \in A(\partial C_0, C_1)$, we necessarily have $w = \mathrm{extr}(\partial C_0, C_1)$.

In what follows we shall consider generalized condensers whose one boundary component at least is at infinity.

3. Condensers with one boundary component at infinity

Preliminaries.

Let recall that an end E of M is the projective limit of a family (E_L), where L ranges in the set $\mathcal{K}(M)$ of compact sets of M and E is a connected component of $M \setminus L$ chosen in such a way that $K \subset L$ implies $E_K \supset E_L$. Let ∂M denote the set of ends of M. It is known ([Fr1], [Fr2], [B], [Z]) that the topological structure of M can be extended to $\overline{M} = M \cup \partial M$ in such a way that $\overline{M}$ and ∂M are compact. However for all subset X of M we will go on denoting $\overline{X}$ and ∂X, respectively, its closure and its boundary in M. The traces on M of the neighborhoods of a subset S of ∂M will be called *relative neighborhoods* of S. Particularly, for every end E, the family $(E_L)_{L \in \mathcal{K}(M)}$ is a basis for relative neighborhoods of E. Let observe that for any pair (K, L) of compact sets in M, $E_K \cap E_L \supset E_{K \cup L}$, hence $E_K \cap E_L$ is never empty.

Sub-boundaries.

For brevity the closed, hence compact, subsets of ∂M will be called *sub-boundaries* of M. For all subset S of ∂M and for all $L \in \mathcal{K}(M)$ we will write $S_L = \bigcup_{E \in S} E_L$.

3.1. LEMMA. — *Let S be a sub-boundary of M.*

a) For every $L \in \mathcal{K}(M)$, there exists a finite subset X of S such that $S_L = X_L$; hence S_L has a finite number of components and $\overline{S}_L$ is a relative continuum.

b) The family $(S_L)_{L\in\mathcal{K}(M)}$ is a basis for relative neighborhoods of S.

Proof.

a) From the compacity of S there exists a finite subset X of S such that $X_L = \bigcup_{E\in X} E_L$ is a relative neighborhood of S. Then for every end $E \in S$, we have $E_L \cap X_L \neq \emptyset$, hence there exists an end $F \in X$ such that $E_L \cap F_L \neq \emptyset$. As E_L and F_L are connected components of $M \setminus L$, necessarily $E_L = F_L$. Hence $S_L = X_L$ and $\overline{S}_L = \overline{X}_L = \bigcup_{F\in X} \overline{F}_L = \bigcup_{E\in S} \overline{E}_L$. As every set $\overline{E}_L$ is a non-compact continuum, $\overline{S}_L$ is a relative continuum.

b) Let Ω be a relative neighborhood of S. For every end $E \in S$ there exists a set $C(E) \in \mathcal{K}(M)$ with $E_{C(E)} \subset \Omega$. From the compacity of S there exists a finite subset X of S such that $V = \bigcup_{E\in X} E_{C(E)}$ is a relative neighborhood of S.Then $V \subset \Omega$ and $L = \bigcup_{E\in X} C(E)$ is a compact set.

For every end $E \in S$, we have $E_L \cap V \neq \emptyset$ and there exists an end $F \in X$ such that $E_L \cap F_{C(F)} \neq \emptyset$. As $E_L \subset E_{C(F)}$, the sets $E_{C(F)}$ and $F_{C(F)}$ are two intersecting connected components of $M \setminus C(F)$, which implies $E_{C(F)} = F_{C(F)}$, hence $E_L \subset V \subset \Omega$. Finally $S_L = \bigcup_{L\in S} E_L$ is contained in Ω. The claim follows.

Remark.

a) From Lemma 3.1 b) it appears that every sub-boundary S of M is the projective limit of the family $(S_L)_{L\in\mathcal{K}(M)}$, which could allow to give a direct definition of sub-boundaries, generalizing the definition of ends. We shall keep in mind that two sub-boundaries S, Σ are disjoint if, and only if, there exists a compact set L with $S_L \cap \Sigma_L = \emptyset$. In that case there also exists a compact set K such that $\overline{S}_L \cap \overline{\Sigma}_K = \emptyset$.

b) The same proposition allows us to say that a map f of M into a topological space X admits a point x of X for limit [resp. cluster value] at the sub-boundary S if, for every neighborhood V of x in X there exists a compact set L such that $f(S_L) \subset V$ [resp. $V \cap f(S_L) \neq \emptyset$].

At last we will say that a sequence (B_p) of subsets of M tends to a sub-boundary S if, for all $L \in \mathcal{K}(M)$ there exists $p_L \in \mathbb{N}$ such that $p \geq p_L$ implies $B_p \subset S_L$.

Hybrid condensers.

We will now consider *hybrid condensers i.e.* of the type $\Gamma(S, C)$, where S is a sub-

boundary of M and C a compact set or a relative continuum of M. Then $A(S, C)$ will denote the set of functions $u \in H(M)$ with $u = 1$ on C, $u = 0$ on S_L for some choice of the compact set L and $0 \leq u \leq 1$ everywhere.

3.2. DEFINITION. — *With above notations the capacity of* $\Gamma(S, C)$ *is defined by*

$$\mathrm{Cap}(S, C) = \inf_{u \in A(S,C)} I(u, M) = \inf_{L} \mathrm{Cap}(\overline{S}_L, C).$$

If $S = \partial M$ and if C is a compact set of M we recover the usual capacity of C, simply denoted Cap C.

If M is a domain of $\mathbb{R}^n$ and if S is the union of a set of boundary components of M, we recover a classical definition.

Properties of $\mathrm{Cap}(S, C)$.

In what follows we shall say that a set C is *strongly disjoint* from S if there exists a compact set L with $\overline{S}_L \cap C = \emptyset$. This condition is always satisfied if C is compact and S_L sufficiently close to S. Then we have:

3.3. LEMMA. — *If C is a compact set, or a relative continuum strongly disjoint from S, then* $\mathrm{Cap}(S, C)$ *is finite.*

Proof. — Let L be a compact set with $\overline{S}_L \cap C = \emptyset$. Then from Lemma 2.6 $\mathrm{Cap}(\overline{S}_L, C) = \mathrm{Cap}(\partial S_L, C)$ is finite and $\mathrm{A}(S, C)$ is not empty.

3.4. LEMMA.

a) If S_1 , S_2 are two sub-boundaries of M with $S_1 \subset S_2$ and if each C_i $(i = 1, 2)$ is a compact set or a relative continuum of M with $C_1 \subset C_2$, then

$$\mathrm{Cap}(S_1, C_1) \leq \mathrm{Cap}(S_2, C_2)\,.$$

b) If (S_i) is a family of sub-boundaries of M and (C_j) a family of compact sets or relative continua,

$$\mathrm{Cap}(US_i, UC_j) \leq \sum_{i,j} \mathrm{Cap}(S_i, C_j)\,.$$

3.5. THEOREM. — *Let S be a sub-boundary of M. If there exists a compact continuum K with* $\mathrm{Cap}(S, K) = 0$, *then* $\mathrm{Cap}(S, H) = 0$ *for any compact set H of M and* $\mathrm{Cap}(\overline{S}_L, H)$ *tends to zero when S_L tends to S.*

In that case we shall write Cap $S = 0$. In the opposite case we write Cap $S > 0$

Proof. — If K is not contained in H let us choose $a \in K \setminus H$ and $b \in K \setminus \{a\}$. Then there exists a compact continuum C with $H \cup \{b\} \subset C \subset M \setminus \{a\}$. Hence the existence of a constant k such that the inequality $\operatorname{osc}^n(u, C) \leq kI(u, M)$ holds for any $u \in H(M)$ which is monotone on $M \setminus \{a\}$ (cf. Prop. 1.4).

Now, $\varepsilon > 0$ given, there exists from hypothesis a function $u_\varepsilon \in A(S, K)$ satisfying $I(u_\varepsilon, M) < 2^{-n}\varepsilon$, and from Lebesgue straightening Lemma (cf. [M]) we can assume that u_ε is monotone on $M \setminus K$, hence also on $M \setminus \{a\}$ from (1.2). By choosing $\varepsilon < 1/k$ we have therefore $\operatorname{osc}^n(u_\varepsilon, C) \leq kI(u_\varepsilon, M) \leq 2^{-n}$, hence $u_\varepsilon \geq 1/2$ on H. Then $v = \inf(2u_\varepsilon, 1)$ belongs to $A(S, H)$, hence $\operatorname{Cap}(S, H) \leq 2^n I(u_\varepsilon) \leq \varepsilon$, and $\operatorname{Cap}(S, H) = 0$ by letting ε tend to zero.

If $K \subset H$ we can choose a pair (H_1, H_2) of compact sets not containing K with $H = H_1 \cup H_2$ and from above arguments $\operatorname{Cap}(S, H_1) = \operatorname{Cap}(S, H_2) = 0$ hence $\operatorname{Cap}(S, H) = 0$.

At last, it is obvious that the above notions of sub-boundaries and hybrid capacities are conformally invariant. More precisely:

3.6. Let M, N be two Riemannian n-manifolds. Then every K-quasiconformal map of M onto N can be continuously extended into a homeomorphism of $M \cup \partial M$ onto $N \cup \partial N$, and for any pair (S, C), where S is a sub-boundary of M and C a compact set or a relative continuum of M, we have:

$$K^{-1} \operatorname{Cap}(S, C) \leq \operatorname{Cap}(fS, fC) \leq K \operatorname{Cap}(S, C).$$

4. Extremal functions for hybrid condensers

With the same notations as in § 3, we can state:

4.1. THEOREM. — *For any sub-boundary S of M and any compact or relative continuum C strongly disjoint from S the function $v = \sup_L \operatorname{extr}(\overline{S}_L, C)$ is the only one in $\operatorname{Cl} A(S, C)$ satisfying $I(v, M) = \operatorname{Cap}(S, C)$. This function is the strong projective limit of $\operatorname{extr}(\overline{S}_L, C)$ when S_L tends to S. It satisfies:*

$$\int_M |\nabla v|^{n-2} \nabla v \cdot \nabla w \, d\tau = 0 \tag{4.2}$$

for all $w \in H(M)$ satisfying $w = 0$ on $C \cup S_L$ for some compact L. Hence v is n-harmonic on $M \setminus C$ and monotone on $M \setminus \{a\}$ for all $a \in C$ if C is compact [resp. monotone on all M if C is a relative continuum]. It will be denoted $v = \operatorname{extr}(S, C)$. If $\operatorname{Cap} S = 0$ it reduces to the constant 1. Moreover Proposition 2.5 still holds with v in place of u, namely, for all $0 \leq \alpha < \beta \leq 1$:

$$\operatorname{Cap}(\{x \in M \mid v(x) \leq \alpha\}, \ \{x \in M \mid v(x) \geq \beta\} = (\beta - \alpha)^{1-n} \operatorname{Cap}(S, C)\,. \tag{4.3}$$

Proof. — If S_L is sufficiently close to S for having $C \cap \overline{S}_L = \emptyset$ the function $v_L = \mathrm{extr}(\partial S_L, C)$ exists and from Lemma 2.6 $\mathrm{extr}(\overline{S}_L, C) = v_L$. Now $\varepsilon > 0$ given there is $u_\varepsilon \in A(S, C)$ satisfying $I(u_\varepsilon, M) \leq \mathrm{Cap}(S, C) + \varepsilon$ and vanishing on S_H for some compact set H. For any compact set L with $S_L \subset S_H$ the function $v_L = \mathrm{extr}(\overline{S}_L, C)$ satisfies

$$\mathrm{Cap}(S, C) \leq I(v_L, M) = \mathrm{Cap}(\overline{S}_L, C) \leq \mathrm{Cap}(\overline{S}_H, C) \leq I(u_\varepsilon, M) \leq \mathrm{Cap}(S, C) + \varepsilon$$

hence $I(v, M) = \mathrm{Cap}(\overline{S}_L, C)$ tends to $\mathrm{Cap}(S, C)$ when S_L tends to S. On the other hand v_L is monotone on M or at least on $M \setminus \{a\}$ for any $a \in C$, the set $A(S, C)$ is convex, and from Lemma 2.4 the inclusion $S_L \subset S_H$ implies $v_L \geq v_H$.

Then it follows from Lemmas 2.1 and 2.2 that the family (v_L) is strongly converging to $v = \sup_L v_L$ and that v is the only function in $\mathrm{Cl}\, A(S, C)$ satisfying $I(v, M) = \mathrm{Cap}(S, C)$. Hence the variational condition (4.2) which gives an elementary proof of the fact that v is n-harmonic on $M \setminus C$ and makes the proof of Proposition 2.5 given in [F3] still valid. Hence the claim.

4.4. Proposition. — *Let S, S' be two sub-boundaries of M with $S \subset S'$, and let C, C' be two compact or relative continua with $C \subset C'$. Then*

$$\mathrm{extr}(S', C) \leq \mathrm{extr}(S, C) \leq \mathrm{extr}(S, C') \,. \tag{4.5}$$

Proof. — From definition of sub-boundaries we have $S_L \subset S'_L$ for all compact set L of M, hence from Lemma 2.4:

$$\mathrm{extr}(S'_L, C) \leq \mathrm{extr}(S_L, C) \leq \mathrm{extr}(S_L, C') \leq \mathrm{extr}(S, C').$$

The first inequality (4.5) follows by letting S'_L tend to S' and the second one by letting S_L tend to S.

Behaviour of $\mathrm{extr}(S, C)$ at S and C.

It is first obvious that $v = \mathrm{extr}(S, C)$ always satisfies $v = 1$ on C and $0 \leq v \leq 1$ on all M. If $M \subset\subset N$ is a regular subdomain of a manifold N (cf. [H1] it appears that v tends to zero at S. In the general case we can interpret (4.2) by saying that v is the weak solution of $\mathrm{div}(|\partial v|^{n-1} \partial v) = 0$ on $M \setminus C$ with boundary conditions $v = 1$ on C, $v = 0$ on S and $dv/dn = 0$ on $\partial M \setminus S$, but the two last conditions must be interpreted in a weak sense. In fact we can only state:

4.6. Proposition. — *With the same notations as in Theorem 4.1, the function $v = \mathrm{extr}(S, C)$ admits zero for cluster value at any end $E \in S$ with $\mathrm{Cap}\, E > 0$. Consequently $v(M) =]0, 1]$.*

The first assertion will be proved below as an application of Theorem 5.1. The second one follows from Harnack inequality.

5. Limits of $\mathrm{Cap}(S, C)$ when C tends to infinity and applications

We first assume that C is a compact set of the type $C = \partial S_L$ and that S_L tends to S.

5.1. THEOREM. — *Let S be a sub-boundary of M with* $\mathrm{Cap}\, S > 0$. *Then* $\mathrm{Cap}(S, \partial S_L)$ *tends to* $+\infty$ *when S_L tends to S.*

Proof. — Let suppose that there exists a sequence (L_p) of compact sets of M such that S_{L_p} tends to S, with $\mathrm{Cap}(S, \partial S_{L_p}) \leq k < +\infty$ for all p. Then for all p there exists $q \in \mathbb{N}$ such that $q \geq q_p$ implies $S_{L_q} \subset S_{L_p}$ and also

$$\mathrm{Cap}(\overline{S}_{L_q}, \partial S_{L_p}) \leq \mathrm{Cap}(S, \partial S_{L_p}) + k \leq 2k\,.$$

After extraction of a subsequence we can assume $q_p = p + 1$, hence $\mathrm{Cap}(\overline{S}_{L_{p+1}}, \partial S_{L_p}) \leq 2k$ for all p. Now for all $q > p$ the domain $D_{q,p} = S_{L_p} \setminus \overline{S}_{L_q}$ of the condenser $\Gamma(\overline{S}_{L_q}, \partial S_{L_p})$ contains the $q - p$ disjoint sets $D_{q,q-1}, \ldots, D_{p+1,p}$. For convenience let write $m_{p,q} = \Big(\mathrm{Cap}(\overline{S}_{L_q}, \partial S_{L_p})\Big)^{1/(1-n)}$.

From a classical inequality relative to moduli (cf. [M], 7.2) we have:

$$m_{p,q} \geq \sum_{i=p}^{q-1} m_{i,i+1} \geq (q-p)(2k)^{1/(1-n)}$$

which proves that $\mathrm{Cap}(\overline{S}_{L_q}, \partial S_{L_p}) = m_{p,q}^{1-n}$ tends to zero when, p fixed, q tends to $+\infty$. Hence $\mathrm{Cap}(S, \partial S_{L_p}) = 0$ in contradiction with $\mathrm{Cap}\, S > 0$.

Application. Proof of Proposition 4.6 .

As $E \subset S$ from (4.3),

$$u = \mathrm{extr}(E, C) \geq \mathrm{extr}(S, C) = v$$

while from Theorem 5.1, with E in place of S, $\mathrm{Cap}(E, \partial E_L)$ tends to $+\infty$ when E_L tends to E.

Let write $m_L = \inf_{\partial E_L} u$. For all compact set L with $\overline{E}_L \cap C = \emptyset$, from Lemma 3.4 and (4.3) we have $\mathrm{Cap}(E, \partial E_L) \leq \mathrm{Cap}(E, \{x \in M \mid u(x) \geq m_L\}) = m_L^{1-n}\,\mathrm{Cap}(E, C)$. Hence m_L^{1-n} tends to $+\infty$ when E_L tends to E, and m_L tends to zero. As $v > 0$ the claim follows.

5.2. THEOREM. — *Let S, Σ be two disjoint sub-boundaries of M with* $\mathrm{Cap}\, S > 0$, $\mathrm{Cap}\, \Sigma = 0$, *and let C be a closed set in M.*

Then $\mathrm{Cap}(S, C)$ *tends to zero when C tends to Σ.*

Proof. — Let K be a compact set separating S from Σ, *i.e.* such that $\overline{S}_K \cap \overline{\Sigma}_K = \emptyset$. From hypothesis $\mathrm{Cap}(\Sigma, K) = 0$. Then, $\varepsilon > 0$ given, there exists a compact set L with $\mathrm{Cap}(\overline{\Sigma}_L, K) \leq \varepsilon$. The function $u_L = \mathrm{extr}(\overline{\Sigma}_L, K)$ satisfies $u_L = 0$ on $\overline{\Sigma}_L$ and $u_L = 1$ on $\overline{S}_K$. For all closed set $C \subset \Sigma_L$, the function $1 - u_L$ belongs to $A(\overline{S}_K, C)$, hence

$$\mathrm{Cap}(S, C) \leq \mathrm{Cap}(\overline{S}_K, C) \leq I(u_L, M) = \mathrm{Cap}(\overline{\Sigma}_L, K) \leq \varepsilon.$$

The claim follows.

Remark. — The behaviour of $\mathrm{extr}(S, C)$ and $\mathrm{Cap}(S, C)$ when C tends to some boundary Σ with $\mathrm{Cap}\,\Sigma > 0$ will follow from Theorem 6.1 if we assume that C is of the special type $C = \partial\Sigma_L$ or $C = \overline{\Sigma}_L$.

Application to sequences of quasiconformal automorphisms.

Theorems 5.1 and 5.2 will allow us to prove a result announced in [F4] (Prop. II 7.5).

5.3. THEOREM. — *Assume that M has exactly two ends E, F and that there exists a sequence (f_k) of K-quasiconformal automorphisms of M which c-converges to infinity. Then* $\mathrm{Cap}\,E = \mathrm{Cap}\,F = \mathrm{Cap}\,\partial M = 0$.

Proof. — Let H be a compact continuum separating E from F. After extraction of a subsequence and composition with a fixed map if necessary, we can assume that the maps f_k preserve the ends of M. Then $f_k(H)$ tends to some end of M, let E for precision. As $f_k(H)$ separates E from H, the assumption $\mathrm{Cap}\,E > 0$ would imply

$$\lim \mathrm{Cap}(E, f_k(H)) = +\infty$$

in contradiction with the estimate $\mathrm{Cap}(E, f_k(H)) \leq K\,\mathrm{Cap}(E, H)$ (Proposition 3.6). Consequently we have $\mathrm{Cap}\,E = 0$, hence $\lim \mathrm{Cap}(F, f_k(H)) = 0$, with $\mathrm{Cap}(F, f_k(H)) \geq K^{-1}\,\mathrm{Cap}(F, H)$, hence $\mathrm{Cap}(F, H) = 0$ and $\mathrm{Cap}\,F = 0$.

As $\partial M = E \cup F$ we have $\mathrm{Cap}(\partial M, K) \leq \mathrm{Cap}(E, K) + \mathrm{Cap}(F, K) = 0$ for all compact K, or, in other terms, $\mathrm{Cap}\,\partial M = 0$, as claimed.

This result is particularly interesting for non-compact Lie groups equipped with a left-invariant conformal structure (cf. § 8).

6 Condensers with both boundary components at infinity

We will now consider condensers whose boundary components are two disjoint sub-boundaries S, Σ of M and we must distinguish three cases: *a)* $\mathrm{Cap}\,S > 0$, $\mathrm{Cap}\,\Sigma > 0$, *b)* $\mathrm{Cap}\,S > 0$, $\mathrm{Cap}\,\Sigma = 0$, *c)* $\mathrm{Cap}\,S = \mathrm{Cap}\,\Sigma = 0$.

The first case leads to an easy generalization of rings in the n-space. The two other cases will introduce Green-type functions with poles at infinity. In all cases we will obtain n-harmonic functions globally defined on all M.

6.1. THEOREM. — *Let S, Σ be two disjoint sub-boundaries of M with* $\operatorname{Cap} S > 0$, $\operatorname{Cap}\Sigma > 0$, *and let $A(S,\Sigma)$ be the set of functions $u \in H(M)$ which satisfy $u = 0$ on S_L for some choice of the compact L, $u = 1$ on Σ_K for some choice of the compact K and $0 \leq u \leq 1$ everywhere. Let write*

$$\operatorname{Cap}(S,\Sigma) = \inf_{u \in A(S,\Sigma)} I(u,M) = \inf_{K,L} \operatorname{Cap}(\overline{S}_L, \overline{\Sigma}_K)\,.$$

Then there exists a unique function $v \in \operatorname{Cl} A(S,\Sigma)$ with $I(v,M) = \operatorname{Cap}(S,\Sigma)$. This function is n-harmonic with $0 < v < 1$ on all M and will be denoted $\operatorname{extr}(S,\Sigma)$. It satisfies the variational condition

$$\int_M |\nabla v|^{n-2} \nabla v \cdot \nabla w \, d\tau = 0 \tag{6.2}$$

for all $w \in H(M)$ which vanishes on $S_L \cup \Sigma_K$ for some choice of the compact sets K, L.

Obviously $\operatorname{Cap}(\Sigma, S) = \operatorname{Cap}(S,\Sigma)$ *and* $\operatorname{extr}(\Sigma, S) = 1 - \operatorname{extr}(S,\Sigma)$. *Moreover, as an extension of (4.3), for all $0 \leq \alpha < \beta \leq 1$:*

$$\operatorname{Cap}\left(\{x \in M \mid v(x) \leq \alpha\},\ \{x \in M \mid v(x) \geq (\beta)\}\right) = (\beta - \alpha)^{1-n} \operatorname{Cap}(S,\Sigma)\,. \tag{6.3}$$

Proof. — The arguments are the same as for proving Theorem 4.1. As S, Σ are disjoint there exist some pairs of compact sets (K, L) with $\overline{S}_L \cap \overline{\Sigma}_K = \emptyset$ and $\operatorname{Cap}(S,\Sigma) = \inf_{K,L} \operatorname{Cap}(\overline{S}_L, \overline{\Sigma}_K)$ is finite.

Then the claimed function v is the projective strong limit of $\operatorname{extr}(\overline{S}_L, \overline{\Sigma}_K) = \operatorname{extr}(\partial S_L, \partial \Sigma_K)$ when S_L tends to S and Σ_K tends to Σ. As $\operatorname{extr}(\overline{S}_L, \Sigma)$ is increasing when S_L is decreasing while $\operatorname{extr}(S, \overline{\Sigma}_K)$ is decreasing with Σ_K, we also have

$$v = \operatorname{extr}(S,\Sigma) = \sup_L \operatorname{extr}(\overline{S}_L, \Sigma) = \inf_K \operatorname{extr}(S, \overline{\Sigma}_K)\,.$$

As obviously $0 \leq v \leq 1$, the strict inequality $0 < v < 1$ follows from Harnack inequality. The variational condition (6.2) follows from the strong convergence of $\operatorname{extr}(\overline{S}_L, \overline{\Sigma}_K)$ to v, and makes the proof of Proposition 2.5 given in [F3] still valid with v in place of u, hence (6.3).

Other properties.

Let S, S_0, Σ, Σ_0 be sub-boundaries of M such that $S_0 \subset S$ and $\Sigma_0 \subset \Sigma$. If S, Σ are disjoint, by an easy extension of Lemma 3.4 and Proposition 4.4 we get:

$$\operatorname{Cap}(S_0, \Sigma_0) \leq \operatorname{Cap}(S,\Sigma) \tag{6.4}$$

$$\operatorname{extr}(S, \Sigma_0) \leq \operatorname{extr}(S,\Sigma) \leq \operatorname{extr}(S_0, \Sigma)\,. \tag{6.5}$$

Behaviour of v at S and Σ.

6.6. PROPOSITION. — *Let S, Σ be two disjoint sub-boundaries of M with* $\operatorname{Cap} S > 0$, $\operatorname{Cap} \Sigma > 0$. *The function $v = \operatorname{extr}(S, \Sigma)$ admits 0 for cluster value at any end $E \in S$ with* $\operatorname{Cap} E > 0$, *and 1 for cluster value at any end $F \in \Sigma$ with* $\operatorname{Cap} F > 0$. *Hence $v(M) =]0, 1[$.*

Proof. — Let $E \in S$ be an end with $\operatorname{Cap} E > 0$. Let write $v_E = \operatorname{extr}(E, \Sigma)$ and for any compact set L

$$\alpha_L = \inf_{\partial E_L} v_E, \quad \mu_L = \inf_{\partial E_L} v .$$

From (6.5) we have $v \leq v_E$, hence $\mu_L \leq \alpha_L$. As $\partial E_L \subset \{x \in M \mid v_E(x) \geq \alpha_L\}$ we also have from (6.3)

$$\operatorname{Cap}(E, \partial E_L) \leq \operatorname{Cap}\big(E, \{x \in M \mid v_E(x) \geq \alpha_L\}\big) = \alpha_L^{1-n} \operatorname{Cap}(E, \Sigma) .$$

Now from Theorem 5.1, $\operatorname{Cap}(E, \partial E_L)$ tends to $+\infty$ when E_L tends to E. It follows that α_L and μ_L tend to zero when E_L tends to E, which proves the first assertion. By exchanging S and Σ and using the relation $\operatorname{extr}(\Sigma, S) = 1 - \operatorname{extr}(S, \Sigma)$ we get the second assertion.

7. Other cases. Construction of n-harmonic functions on M

We will enlarge the problem raised in section 6 by considering the limits of condensers of the general type $\Gamma(B, C)$ where B, C are relative continua tending to infinity.

7.1. LEMMA. — *For every pair (B, C) of relative continua in M with* $\operatorname{Cap}(B, C) < +\infty$ *let u_{BC} be the normalized function defined on M by*

$$u_{BC} = \big(\operatorname{Cap}(B, C)\big)^{1/(1-n)} \operatorname{extr}(B, C) .$$

Then for all $0 \leq \alpha < \beta \leq \big(\operatorname{Cap}(B, C)\big)^{1/(1-n)}$ we have:

$$\operatorname{Cap}\Big(\{x \in M \mid u_{BC}(x) \leq \alpha\},\ \{x \in M \mid u_{BC}(x) \geq \beta\}\Big) = (\beta - \alpha)^{1-n} . \tag{7.2}$$

Moreover, if M has at least two ends, for all $(x, y) \in M^2$ we have

$$|u_{BC}(y) - u_{BC}(x)) \leq d_M(x, y) \tag{7.3}$$

in which $d_M = \lambda_M^{1/(1-n)}$ is the λ-distance on M (cf. Appendix).

Proof. — The first assertion immediately follows from Proposition 2.5. Now x, y given, let assume $u_{BC}(y) \geq u_{BC}(x)$ for precision and let write

$$C_x = \{x \in M \mid u_{BC}(z) \leq u_{BC}(x)\}, \quad C_y = \{z \in M \mid u_{BC}(z) \geq u_{BC}(y)\} .$$

Then C_x and C_y are relative continua with $x \in C_x$, $y \in C_y$ (cf. [F1]) and from definition of λ_M we have $\lambda_M(x,y) \leq \mathrm{Cap}(C_x, C_y)$. On the other hand, by taking $\alpha = u_{BC}(x)$ and $\beta = u_{BC}(y)$ in (7.2) we get

$$\mathrm{Cap}(C_x, C_y) = \big(u_{BC}(y) - u_{BC}(x)\big)^{1-n}.$$

The claim follows.

7.4. Corollary. — *Let assume that M has at least two ends and let (B_p), (C_p) be two sequences of relative continua converging to infinity. If the sequence $(v_p) = (u_{B_p,C_p})$ is not c-converging to $+\infty$, it contains a subsequence which is c-converging to a positive n-harmonic function v defined on all M, possibly constant.*

Proof. — The topology associated with the λ_M-distance on any compact set K of M agrees with the topology induced on K by the structure of manifold (cf. [F1]). The functions v_p are therefore equicontinuous and the claim follows from Ascoli Theorem combined with (1.1).

Now we have to look for conditions ensuring that the sequence (v_p) is bounded and that the limit functions v are not constant.

7.5. Lemma. — *Let H be a compact set of M and let (B, C) be a pair of relative continua with* $\mathrm{Cap}(B, C) < +\infty$. *Then*

$$m_H = \inf_H u_{BC} \leq \mathrm{Cap}(B, H)^{1/(1-n)}.$$

Proof. — From definition $H \subset \{x \in M \mid u_{BC}(x) \geq m_H\}$. If $H \cap B \neq \emptyset$, we have $m_H = 0$. If H reduces to a single point $\mathrm{Cap}(B, H) = 0$. In both these cases our assertion is trivial. In all other cases we have

$$\mathrm{Cap}(B, H) \leq \mathrm{Cap}\big(B, \{x \in M \mid u_{BC}(x) \geq m_H\}\big) = m_H^{1-n}.$$

The claim follows.

7.6. Lemma. — *Let S, Σ be two disjoint sub-boundaries of M, (B, C) a pair of relative continua and H, K two compact sets with $B \subset S_H$, $C \subset \Sigma_K$ and $\overline{S}_H \cap \overline{\Sigma}_K = \emptyset$. Then $m_H = \inf_H u_{BC}$ and $\mu_K = \sup_K u_{BC}$ satisfy*

$$\mu_K - m_H \geq \big(\mathrm{Cap}(H, K)\big)^{1/(1-n)}.$$

Proof. — The sets $X = \{x \in M \mid u_{BC}(x) \leq m_H\}$ and $Y = \{x \in M \mid u_{BC}(x) \geq \mu_K\}$ are relative continua. By applying Lemma 2.4 to u_{BC} on $(M \setminus (\overline{S}_H \cup \overline{\Sigma}_K)$ it is easy to see that $X \subset \overline{S}_H$ and $Y \subset \overline{\Sigma}_K$, hence

$$\mathrm{Cap}(H, K) \geq \mathrm{Cap}(\partial S_H, \partial \Sigma_K) = \mathrm{Cap}(\overline{S}_H, \overline{S}_K) \geq \mathrm{Cap}(X, Y) = (\mu_K - m_H)^{1-n}.$$

The claim follows.

Now we can prove the following basic result.

7.7. THEOREM. — *Let S, Σ be two disjoint sub-boundaries of M with $\operatorname{Cap}\Sigma = 0$ and let (B_p), (C_p) be two sequences of relative continua respectively converging to S, Σ.*

a) If $\operatorname{Cap} S > 0$ and if, for all p, B_p is a neighborhood of S, the sequence $(v_p = u_{B_pC_p})$ admits a c-convergent subsequence whose limit v is positive and n-harmonic on M. This function v admits zero for cluster value at any end $E \in S$ with $\operatorname{Cap} E > 0$, and $+\infty$ for cluster value at Σ, hence $v(M) = \mathbb{R}_+^*$.

b) If $\operatorname{Cap} S = 0$ the sequence $(v_p = u_{B_pC_p})$ is c-converging to $+\infty$. However for any fixed point a in M, the sequence $(v_p - v_p(a))$ admits a c-convergent subsequence whose limit v is n-harmonic on M. This function v admits $+\infty$ for cluster value at Σ and $-\infty$ for cluster value at S, hence $v(M) = \mathbb{R}$.

Proof.

First case. We assume that $\operatorname{Cap} S > 0$ and that every B_p is a neighborhood of S. Then for all compact set H and all end $E \in S$ with $\operatorname{Cap} E > 0$, we have, from Definition 3.2, $\operatorname{Cap}(B_p, H) \geq \operatorname{Cap}(E, H) > 0$. Hence from Lemma 7.5

$$\inf_H v_p \leq \big(\operatorname{Cap}(E, H)\big)^{1/(1-n)}.$$

As the functions v_p are 1-lipschitzian in the λ-distance d_M, they also are equally bounded on H and the first assertion in a) follows from Corollary 7.4. The limit function v also satisfies

$$\inf_H v \leq \operatorname{Cap}(E, H)^{1/(1-n)}.$$

Then we can choose $H = \partial E_L$ and let E_L tend to E. From Theorem 5.1 $\operatorname{Cap}(E, \partial E_L)$ tends to $+\infty$, hence $\inf_H v$ tends to zero, which implies that v admits zero for cluster value at E.

Finally let H, K be two compact sets with $\overline{S}_H \cap \overline{\Sigma}_K = \emptyset$. From Lemma 7.6 we easily get

$$\sup_K v - \inf_H v \geq \big(\operatorname{Cap}(H, K)\big)^{1/(1-n)} \tag{7.8}$$

hence v is not constant. If, H fixed, Σ_K tends to Σ, $\operatorname{Cap}(H, K)$ tends to zero, hence $\sup_K v$ tends to $+\infty$, which implies that $+\infty$ is a cluster value of v at Σ.

Second case. $\operatorname{Cap} S = 0$. Let first suppose that the sequence $(v_p = u_{B_pC_p})$ is not c-converging to $+\infty$. Then there must exist a c-convergent subsequence of (v_p) whose limit v satisfies (7.8). As $\operatorname{Cap} S = 0$, $\operatorname{Cap}(H, K)$ tends to zero when, K fixed, H tends to S, hence

$\sup_K v = +\infty$, which leads to a contradiction. The sequence (v_p) is therefore c-converging to $+\infty$.

Now, since the v_p are equicontinuous, the sequence $(v_p - v_p(a))$ admits a c-convergent subsequence whose limit v is n-harmonic on M and satisfies (7.8). Hence easily, as in the first case, $\lim_{K\to\Sigma} \sup_K v = +\infty$. Similarly, as $\operatorname{Cap} S = 0$, $\operatorname{Cap}(H, K)$ tends to zero when, K fixed, H tends to S. Hence $\lim_{H\to S} \inf_H v = -\infty$, which implies that $-\infty$ is a cluster value of v at S.

Remark. — In the first case $\operatorname{Cap} S > 0$ we can also consider the functions $v_{SC} = \big(\operatorname{Cap}(S, C)\big)^{1/(1-n)} \operatorname{extr}(S, C)$ where C is a relative continuum converging to Σ. By the same kind of arguments we obtain:

7.9. THEOREM. — *Let S, Σ be two disjoint sub-boundaries of M with $\operatorname{Cap} S > 0$, $\operatorname{Cap}\Sigma = 0$, and let (C_p) be a sequence of relative continua converging to Σ. Then the sequence v_{S,C_p} admits a c-convergent subsequence whose limit v is a positive n-harmonic function on M, admitting zero for cluster value at any end $E \in S$ with $\operatorname{Cap} E > 0$, and $+\infty$ for cluster value at Σ.*

8. Improvement and conclusion

Theorems 7.7 and 7.8 can be considered as extensions of Theorem 3.27 in [H1], and we can say that the limit function v is a Green-type function with pole at Σ if $\operatorname{Cap} S > 0$ [resp. with poles at S, Σ, if $\operatorname{Cap} S = \operatorname{Cap}\Sigma = 0$]. Our results are however less precise than Theorem 3.27 of [H1] in what concerns the behaviour of v at the poles. For what concerns the behaviour of v at S, it does not seem that we lost any precision since in [H1] it is not proved that the Green function $g(\bullet, y)$ tends to zero at ∂M.

We will now also generalize Lemma 4.1 in [H1]. For brevity we shall say that a neighborhood C of a sub-boundary Σ of M is a r-neighborhood of Σ if it is a relative continuum.

8.1. THEOREM. — *Let S, Σ be two disjoint sub-boundaries of M with $\operatorname{Cap} S > 0$, $\operatorname{Cap}\Sigma = 0$ and assume that $\Sigma = \{F_1, F_2, \ldots\}$ is the union of an enumerable set of ends. Then there exists a positive n-harmonic function v on M admitting zero for cluster value at any end $E \in S$ with $\operatorname{Cap} E > 0$, and $+\infty$ for cluster value at any end F_i.*

Proof. — For simplicity we shall use the functions v_{SC} as in Theorem 7.9. For all $i \in \mathbb{N}^*$ let γ_i be a r-neighborhood of F_i. Then let write $\gamma = \operatorname{Cl}(U\gamma_i)$ and $\operatorname{Cap}(S, \gamma_i) =$

$t_i \operatorname{Cap}(S, \gamma)$. Obviously $0 < t_i \leq 1$ and there exists a sequence (u_i) with $0 < u_i < 1$ and $k = \sum_{i=1}^{\infty} t_i u_i < +\infty$. We can use Theorem 5.2 for constructing a relative continuum $C_i \subset \gamma_i$ such that $\operatorname{Cap}(S, C_i) = u_i \operatorname{Cap}(S, \gamma_i)$. Then $C = \operatorname{Cl}(\bigcup C_i)$ is a relative continuum contained in γ such that for all i:

$$\operatorname{Cap}(S, C_i) \leq \operatorname{Cap}(S, C) \leq \sum_{j=1}^{\infty} \operatorname{Cap}(S, C_j) = k \operatorname{Cap}(S, \gamma) = k \operatorname{Cap}(S, C_i)/(t_i u_i) .$$

As $\operatorname{extr}(S, C) \geq \operatorname{extr}(S, C_i)$ we have easily

$$v_{SC} \geq (t_i u_i / k)^{1/(n-1)} v_{S,C_i} .$$

Now, for all index i, let $\gamma_i^{(p)}$ be a sequence of r-neighborhoods of F_i converging to F_i, and $C_i^{(p)}$ be the associated sequence in the above construction. Obviously $C_i^{(p)}$ tends to F_i, and $C^{(p)} = \operatorname{Cl}(\bigcup C_i^{(p)})$ tends to Σ. By extraction of subsequences we can assume that, for all fixed i, the sequences $(v_{S,C_i^{(p)}})$ are c-convergent, as well as $(v_{S,C^{(p)}})$. The limit functions v_i, v satisfy $v \geq (t_i u_i / k)^{1/(n-1)} v_i$, and from Theorem 6.9, v_i admits $+\infty$ for cluster value at F_i, which proves that v is a desired function.

Summary. — By gathering the results of sections 6, 7, and forgetting the process of construction of the function v we can state:

8.2. THEOREM. — *Let M be a Riemannian n-manifold with at least two ends, and let S, Σ be two disjoint sub-boundaries of M.*

a) If $\operatorname{Cap} S > 0$, $\operatorname{Cap} \Sigma > 0$ *there exists a bounded n-harmonic function v on M which admits zero [resp. $+1$] for cluster value at any end $E \in S$ with* $\operatorname{Cap} E > 0$ *[resp. any end $F \in \Sigma$ with* $\operatorname{Cap} F > 0$*].*

b) If $\operatorname{Cap} S > 0$, $\operatorname{Cap} \Sigma = 0$ *there exists a positive n-harmonic function v on M which admits zero for cluster value at any end $E \in S$ with* $\operatorname{Cap} E > 0$, *and $+\infty$ at any end F of a given closed enumerable subset of Σ.*

c) If $\operatorname{Cap} S = \operatorname{Cap} \Sigma = 0$, *there exists a n-harmonic function v on M which admits $-\infty$ for cluster value at S and $+\infty$ for cluster value at Σ.*

By looking at what happens for domains of $\mathbb{R}^n$ it appears that such a function v is generally not unique. However, Theorem 8.2 seems to have some interest for the classification of Riemannian manifolds in nonlinear potential theory (cf. [HR2]). Moreover by applying this theorem to two-ended Lie groups we obtain:

8.3. THEOREM. — *Let G be a Lie group with two ends E, F, equipped with a left-invariant Riemannian metric, and* $n = \dim G$. *Then there exists a n-harmonic function v on G admitting $-\infty$ for cluster value at E and $+\infty$ for cluster value at F.*

Proof. — From Theorem 5.3, necessarily $\operatorname{Cap} E = \operatorname{Cap} F = 0$ hence we are in the case *b)* of Theorem 8.2.

Finally it is perhaps also convenient to recall the following application of Theorem 5.3 which has been stated in [F4].

8.4. THEOREM. — *Let M be a Riemannian n-manifold with a finite number $p \geq 2$ of ends. If $p \geq 3$, or if $p = 2$ with $\operatorname{Cap} \partial M \neq 0$, the conformal group $C(M)$ of M is compact; and more generally, for all real $K \geq 1$, the set $Q_K(M)$ of K-quasiconformal automorphisms of M is compact.*

APPENDIX: The function λ_M and the associated metric

For all non-compact Riemannian n-manifold M and all $(x, y) \in M^2$ we set

$$\lambda_M(x,y) = \inf_{C_0, C_1} \operatorname{Cap}(C_0, C_1) \tag{1}$$

where C_0, C_1 are relative continua with $x \in C_0$, $y \in C_1$ (cf. [F1], [F2]). We always have $\lambda_M(x,y) > 0$ but a general problem is to decide whether $\lambda_M(x,y)$ is finite when $x \neq y$ (or, equivalently, whether λ_M is not identically zero on M, cf. [F5]).

By using Theorem 3.5 we can here prove (without using [F5]):

THEOREM A. — *If M has at least two ends E, F, $\lambda_M(x,y)$ is finite for all $y \neq x$, and tends to zero when, x fixed, y tends to an end X with $\operatorname{Cap} X = 0$. Hence $d_M = \lambda_M^{1/(1-n)}$ is a distance on M and, if $\operatorname{Cap} \partial M = 0$, the d_M-balls are all compact.*

Proof. — Let x, y be given with $y \neq x$. There exist two compact sets H, L such that $\overline{E}_H \cap \overline{F}_L = \emptyset$ and $x \in M \setminus \overline{F}_L$, $y \in M \setminus \overline{E}_H$. Then we can construct two compact disjoint paths γ_0, γ_1 resp. joining x to $\overline{E}_H$ and y to $\overline{F}_L$ with $\gamma_0 \subset M \setminus \overline{F}_L$ and $\gamma_1 \subset M \setminus \overline{E}_H$. From Theorem 2.3 and Lemma 2.6, we have $\operatorname{Cap}(\gamma_0 \cup \partial E_H, \gamma_1 \cup \partial F_L) = \operatorname{Cap}(\gamma_0 \cup \overline{E}_H, \gamma_1 \cup \overline{F}_L) < +\infty$.

Now $C_0 = \gamma_0 \cup \overline{E}_H$ and $C_1 = \gamma_1 \cup \overline{F}_L$ are relative continua resp. containing x, y. Hence $\lambda_M(x,y) \leq \operatorname{Cap}(C_0, C_1) < +\infty$, which implies that d_M is a distance on M. If $y \in \overline{F}_L$ we can take $C_1 = \overline{F}_L$, hence $\lambda_M(x,y) \leq \operatorname{Cap}(\gamma_O \cup \partial E_H, \overline{F}_L)$. Then if $\operatorname{Cap} F = 0$, $\operatorname{Cap}(\gamma_0 \cup \partial E_H, \overline{F}_L)$ tends to zero when $\overline{F}_L$ tends to F (cf. § 3) hence $\lim_{y \to F} \lambda_M(x,y) = 0$. The last assertions follow.

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