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HARNACK INEQUALITIES ON GRAPHS

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Abstract

We present how one can use discrete Harnack inequalities to characterize the graphs whose Markov kernel satisfies Gaussian estimates or to study the minimal growth of harmonic functions.

1. Parabolic Harnack inequalities and estimates of the heat kernel.

Let us write for an introduction the heat kernel on $\mathbb{R}^D$.

$$p_t^{\mathbb{R}^D}(x,y) = \frac{1}{(4\pi t)^{D/2}} e^{-\frac{\|x-y\|^2}{4t}}. \tag{1.1}$$

Its meaning is a probability density in y to reach y after a Brownian motion of duration t starting at x. It is from the analysis point of view related to the differential operator Laplacian $\triangle$ since it is a fundamental solution (fix x and consider the function of t and y) of the heat equation

$$\partial_t - \triangle = 0.$$

Now we may consider other kernels or fundamental solutions associated to other second order differential operators L instead of $\triangle$. It is well known for instance after the works of E. de Giorgi [8], J. Nash [20] and J. Moser [18, 19] that if L is an uniformly elliptic second order operator in divergence form, the associated kernel satisfies estimates similar to (1.1): $\exists c, C > 0,\ \forall x, y, t,$

$$\frac{c}{V(x,\sqrt{t})} e^{-\frac{Cd(x,y)^2}{t}} \leqslant p_t(x,y) \leqslant \frac{C}{V(x,\sqrt{t})} e^{-\frac{cd(x,y)^2}{t}}. \tag{G}$$

Here we have introduced some notations so that these estimates will make sense on more general geometric backgrounds: $d(x,y)$ stands for the distance and $V(x,\sqrt{t})$ for the volume of the ball $B(x,\sqrt{t})$ centered at x and of radius $\sqrt{t}$.

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Thus, we can wonder when are estimates (G) true for the heat kernel associated to the Laplace-Beltrami operator $\triangle$ on a Riemannian manifold. The previous case of the uniformly elliptic operators in divergence form on $\mathbb{R}^D$ is a particular one since the coefficients of L can be hidden in the Riemannian metric. The estimates (G) are not always true for any manifold and its associated operator $\triangle$ but they may be under certain geometric conditions. For instance P. Li and S.-T. Yau proved them when the Ricci curvature is nowhere negative [15]. In fact the weakest (see (1.2) below) geometric conditions have been given independently by A. Grigor'yan [12] and L. Saloff-Coste [23, 24] in 1992. They proved (G) under the conjunction of two geometric properties (DV) and (P). We give in the comment after Theorem 3.2 an example of manifold with negative curvature somewhere but which satisfies (DV) and (P).
(DV) is a volume regularity (or "doubling volume" property):

$$\exists C > 0,\ \forall x, r,\ \ V(x,2r) \leqslant CV(x,r). \tag{DV}$$

(P) is functional L^2 Poincaré inequality:

$$\exists C > 0,\ \forall x, r, f,\ \int_{B(x,r)} (f - f_B)^2 \mathrm{d}\mu \leqslant Cr^2 \int_{B(x,2r)} \|\nabla f\|^2 \mathrm{d}\mu, \tag{P}$$

where f_B minimizes the left term, that is f_B is the mean value of f in the ball $B(x,r)$ with respect always to the Riemannian measure $\mathrm{d}\mu$.

The proof is based on Moser's iterative method and so goes through a parabolic Harnack inequality. Let us first state the weaker elliptic Harnack inequality [17] which applies to harmonic functions u:
We say that (HE) is satisfied if

$$\exists C > 0,\ \forall x, r, \forall u \geqslant 0,\ \ \triangle u = 0 \text{ in } B(x,2r) \Rightarrow \sup_{B(x,r)} u \leqslant C \inf_{B(x,r)} u. \tag{HE}$$

Whereas the parabolic Harnack inequality [18] applies to solutions of the heat equation:
We say that (HP) is satisfied if

$$\exists C > 0,\ \forall x, r,\ \forall u \geqslant 0,\ \ \partial_t u - \triangle u = 0 \text{ in } [0,4r^2]\times B(x,2r) \Rightarrow \sup_{Q_-} u \leqslant C \inf_{Q_+} u, \tag{HP}$$

where $Q_- = [r^2, 2r^2]\times B(x,r)$ and $Q_+ = [3r^2, 4r^2]\times B(x,r)$. As was said, these geometric conditions are the weakest, L. Saloff-Coste also proved reverse statements:

$$(DV) \text{ and } (P) \iff (HP) \iff (G). \tag{1.2}$$

The following shows how to implement such a scheme on graphs and to obtain this way bounds for random walks. The interest is to avoid any algebraic structure assumption. In case the graph is generated as the Cayley graph of a finitely generated graph of polynomial volume growth, the estimates were proved by W. Hebisch and L. Saloff-Coste [13] in 1993. Other related works tending to a softer algebraic structure are [2] on more general graphs than Cayley graphs and [25] on the graphs $\mathbb{Z}^D$ but with non-uniform transition.

2. Graphs.

We shall first define the geometric setting of graphs. The graph structure will be symmetrically weighted (like a Riemannian geometry) or alternatively we will consider only reversible Markov chains.

Let Γ be an infinite set and $\mu_{xy} = \mu_{yx} \geqslant 0$ a symmetric weight on $\Gamma \times \Gamma$. It induces a graph structure if we call x and y neighbours ($x \sim y$) when $\mu_{xy} \neq 0$ (note that loops are allowed). Vertices are weighted by $m(x) = \sum_{y \sim x} \mu_{xy}$. For clarity we will assume that this graph is connected and satisfies a $\triangle^*(\alpha)$ condition:

DEFINITION 2.1. — *Let $\alpha > 0$, the weighted graph (Γ, μ) satisfies $\triangle^*(\alpha)$ if*

$$x \sim y \Rightarrow \mu_{xy} \geqslant \alpha m(x).$$

Thus, the graph is locally uniformly finite ($\forall x \in \Gamma$, $\#\{y \mid y \sim x\} \leqslant \alpha^{-1}$). The graph is endowed with its natural metric (the smallest number of edges of a path between two points). We define balls (for r real) $B(x,r) = \{y \mid d(x,y) \leqslant r\}$ and the volume of a subset A of Γ, $V(A) = \sum_{x \in A} m(x)$. We will write $V(x,r)$ for $V(B(x,r))$.

To the weighted graph we associate now a discrete-time Markov kernel. Set $p(x,y) = \dfrac{\mu_{xy}}{m(x)}$, the discrete kernel $p_n(x,y)$ is defined by

$$\left\{ \begin{array}{lcl} p_0(x,z) & = & \delta(x,z) \\ p_{n+1}(x,z) & = & \sum_y p(x,y)\, p_n(y,z). \end{array} \right. \tag{2.3}$$

This kernel is not symmetric but $\dfrac{p_n(x,y)}{m(y)} = \dfrac{p_n(y,x)}{m(x)}$. We keep this notation which means the probability to go from x to y in n steps but it may also be interesting to think to the density $h_n(x,y) = \dfrac{p_n(x,y)}{m(y)}$ which is symmetric and is the right analog of the kernel p_t in the previous section.

We will say that u satisfies the (discrete-time) parabolic equation on (n,x) if

$$m(x)u(n+1,x) = \sum_y \mu_{xy} u(n,y). \tag{2.4}$$

It is the case of $p_.(.,y)$. We could also write (2.4) this way:

$$m(x)[u(n+1,x) - u(n,x)] - \sum_y \mu_{xy}[u(n,y) - u(n,x)] = 0,$$

and recognize a form $\partial_t - \triangle = 0$. It is then possible to define a parabolic Harnack inequality property $HP(C_H)$ as on manifolds, see [11] for technical precisions.

DEFINITION 2.2. — *The weighted graph* (Γ, μ) *satisfies the Gaussian estimates* $G(c_l, C_l, C_r, c_r)$ *[all constants are positive] if*

$$d(x,y) \leqslant n \implies \frac{c_l m(y)}{V(x,\sqrt{n})} e^{-\frac{C_l d(x,y)^2}{n}} \leqslant p_n(x,y) \leqslant \frac{C_r m(y)}{V(x,\sqrt{n})} e^{-\frac{c_r d(x,y)^2}{n}}.$$

We have here a first distinctive feature of the discrete case, there is of course no hope for a lower bound if $d(x,y) > n$ since then $p_n(x,y) = 0$.

The second feature is a lot more serious. It appears that $p_n(x,y)$ may be zero, simply because $d(x,y)$ and n do not have the same parity. It is even the case of one of the easiest example, the standard random walk on $\mathbb{Z}$. Take $\mu_{mn} = 1$ if $|m-n| = 1$ and $\mu_{mn} = 0$ otherwise, that is at each step one goes left or right with probability $1/2$ for each. Then if one starts at zero, one is always at an even integer for even times n and at an odd integer for odd times. So there is no hope for a lower bound without some care. Furthermore the behaviour of this standard random walk on $\mathbb{Z}$ illustrates the fact that the discrete diffusion is somehow not so smooth as in the continuous case.

The solution adopted in [10, 11] is to assume that there is a loop on each vertex, more precisely we will complete the previous $\Delta^*(\alpha)$ condition and consider the following.

DEFINITION 2.3. — *Let* $\alpha > 0$, (Γ, μ) *satisfies* $\Delta(\alpha)$ *if*

$$\begin{cases} \forall x \in \Gamma, \ x \sim x \\ x \sim y \Rightarrow \mu_{xy} \geqslant \alpha m(x). \end{cases}$$

So in particular $p(x,x) \geqslant \alpha$. Assuming this property, the next theorem gives an analog of (1.2):

THEOREM 2.4. — *Let* $\alpha > 0$ *and assume* (Γ, μ) *satisfies* $\Delta(\alpha)$, *then the statements below are equivalent:*

(i) $\exists C_1, C_2 > 0$ *for which* (Γ, μ) *satisfies* $DV(C_1)$ *and* $P(C_2)$.

(ii) $\exists C_H > 0$ *for which* (Γ, μ) *satisfies* $HP(C_H)$.

(iii) $\exists c_l, C_l, C_r, c_r > 0$ *for which* (Γ, μ) *satisfies* $G(c_l, C_l, C_r, c_r)$.

We have only given an idea of how to define $HP(C_H)$ and have refered to [11] for technical precisions. Here are now the definitions of the geometric conditions in (i).

DEFINITION 2.5. — *The weighted graph* (Γ, μ) *satisfies the volume regularity (or doubling volume property)* $DV(C_1)$ *if*

$$\forall x \in \Gamma, \forall r \in \mathbb{R}^+, \ V(x, 2r) \leqslant C_1 V(x, r).$$

This implies for $r \geqslant s$ that

$$\begin{aligned} V(x,r) &\leqslant V\left(x, 2^{\left[\frac{\ln(r/s)}{\ln 2}\right]+1} s\right) \\ &\leqslant C_1 C_1^{\frac{\ln(r/s)}{\ln 2}} V(x,s) = C_1 \left(\frac{r}{s}\right)^{\ln C_1/\ln 2} V(x,s). \end{aligned} \tag{2.5}$$

DEFINITION 2.6. — *The weighted graph* (Γ, μ) *satisfies the Poincaré inequality* $P(C_2)$ *if* $\forall f \in \mathbb{R}^\Gamma$, $\forall x_0 \in \Gamma$, $\forall r \in \mathbb{R}^+$,

$$\sum_{x \in B(x_0,r)} m(x) |f(x) - f_B|^2 \leqslant C_2 r^2 \sum_{x,y \in B(x_0,2r)} \mu_{xy} [f(y) - f(x)]^2,$$

where $f_B = \dfrac{1}{V(x_0,r)} \displaystyle\sum_{x \in B(x_0,r)} m(x) f(x)$.

We shall now return to the condition $p(x,x) \geqslant \alpha$. First it may seem frustrating since very often random walks are considered without loops. But the result of Theorem 2.4 can be applied on a sort of twice iterated graph (see [11]). For instance in the case of the standard random walk on $\mathbb{Z}$, after two steps one stays at the same integer with probability $1/2$ (one moves $+2$ with probability $1/4$ and one moves -2 with probability $1/4$). Then it is possible to sort out a result for the original random walk, but this result, especially for the lower bound, is very dependent on the graph and in particular of the presence of odd length cycles.

It should be also noted that the condition $p(x,x) \geqslant \alpha$ is not only a trick to avoid parity problems but it has also turned out to be a very useful technical ingredient. As was said after Definition 2.2, the discrete diffusion is not so nice to deal with and Moser's method (in particular the Cacciopoli inequalities) cannot be directly adapted by translating differential calculations into difference ones. And here the problem is not especially with the discrete geometry but rather with the discrete time. As far as the discrete geometry is concerned it has been possible in three different works [9, 14, 22] to implement Moser's method and obtain elliptic Harnack inequalities (namely $\triangle^* + DV + P \Rightarrow HE$). But when we consider the discrete time we have to use something more, like the condition $p(x,x) \geqslant \alpha$. There are different ways to use it. At the end of the introduction in [11], it was only sketched how it could be used in a proof of a Cacciopoli inequality but this idea was given up. Later in [6] the authors managed to implement one part of Moser's iteration and obtained mean-value inequalities. Perhaps the other part, which is about the behaviour of the logarithm of an harmonic function, could also be implemented. In [11] we have managed to separate the difficulties by applying first the scheme to a continuous-time Markov kernel on the graphs and then to introduce the condition $p(x,x) \geqslant \alpha$ for a comparison between the continuous-time and the discrete-time kernels. We describe briefly this comparison in the following.

The continuous-time Markov kernel may be defined by

$$\mathscr{P}_t(x,z) = e^{-t} \sum_{k=0}^{+\infty} \frac{t^k}{k!} p_k(x,z).$$

One can check that it is the solution for $(t,x) \in \mathbb{R}^+ \times \Gamma$ of

$$\left\{\begin{array}{rcl} \mathscr{P}_0(x,z) & = & \delta(x,z) \\ m(x)\dfrac{\partial}{\partial t}\mathscr{P}_t(x,z) & = & \displaystyle\sum_y \mu_{xy}\left[\mathscr{P}_t(y,z) - \mathscr{P}_t(x,z)\right]. \end{array}\right.$$

Now assume $p(x,x) \geqslant \alpha$ so that we can consider the positive submarkovian kernel $\bar{p} = p - \alpha\delta$ [this means $\bar{p}(x,y) = p(x,y) - \alpha\delta(x,y)$, then $\bar{p}_n(x,y)$ is defined as in (2.3)] and compute $\mathscr{P}_n$ and p_n with $\bar{p}$:

$$\mathscr{P}_n(x,y) = e^{(\alpha-1)n}\sum_{k=0}^{+\infty}\frac{n^k}{k!}\bar{p}_k(x,y) = \sum_{k=0}^{+\infty} a_k\bar{p}_k(x,y). \tag{2.6}$$

$$p_n(x,y) = \sum_{k=0}^{n} C_n^k\alpha^{n-k}\bar{p}_k(x,y) = \sum_{k=0}^{n} b_k\bar{p}_k(x,y).$$

To compare the two sums we should study $c_k = b_k/a_k$ for $0 \leqslant k \leqslant n$,

$$c_k = \frac{n!\alpha^{n-k}}{(n-k)!e^{(\alpha-1)n}n^k}.$$

It appears that

$$0 \leqslant k \leqslant n \implies c_k \leqslant C(\alpha), \tag{2.7}$$

$$n \geqslant \frac{a^2}{\alpha^2} \text{ and } |k-(1-\alpha)n| \leqslant a\sqrt{n} \implies c_k \geqslant C(a,\alpha) > 0. \tag{2.8}$$

The meaning of condition $n \geqslant a^2/\alpha^2$ is only to ensure that $a\sqrt{n} \leqslant \alpha n$ and so that k cannot be bigger than n.

Once we have "good" estimates for the continuous-time kernel (we put quotes because it is known from the works by M. M. H. Pang and E. B. Davies that the large deviation estimates are here more complicated than (G)), we can use directly (2.7) and obtain an upper bound for the discrete-time kernel. To obtain a lower bound with (2.8), we must first find a value of a which ensures that for instance half of the whole sum (2.6) is contained by the terms for which $|k-(1-\alpha)n| \leqslant a\sqrt{n}$. This can be done thanks precisely to discrete-time kernel upper estimates which yield estimates for $\bar{p}_k(x,y)$.

3. Harmonic functions with polynomial growth.

In this section we shall study the following spaces and in particular their dimension.

Definition 3.1. — *Let (Γ,μ) be a weighted graph, $\mathscr{H}^d(\Gamma,\mu)$ is the set of all fonctions u which are harmonic on the whole graph and such that*

$$\exists x_0 \in \Gamma, \exists C > 0, \quad \forall x \in \Gamma \setminus \{x_0\}, |u(x)| \leqslant Cd(x_0,x)^d.$$

Similarly to (2.4), we say that u is harmonic at x when

$$m(x)u(x) = \sum_y \mu_{xy}u(y),$$

that is, when the value of u at x is the mean value of its values at x's neighbours with respect to the weight μ. The constants A and C depend on the function u but the vertex x_0 may be chosen anywhere if one can change C. So x_0 will be fixed hereafter.

We will use the implementation of Moser's iteration scheme, first to show that for small d these spaces are reduced to constant functions, then to bound their dimension for any d. Note that only the elliptic version of the iteration scheme is needed and so only a $\Delta^*(\alpha)$ condition.

THEOREM 3.2. — *If* (Γ, μ) *satisfies* $DV(C_1)$, $P(C_2)$ *and* $\Delta^*(\alpha)$, *then*

$$d < h \implies \dim \mathscr{H}^d(\Gamma, \mu) = 1,$$

where $h > 0$ *depends on* C_1, C_2 *and* α.

Proof: The elliptic Harnack inequality yields a Hölder regularity property (see Proposition 6.2 in [9]) which says that if u is harmonic on $B(x_0, 2r)$ and $x, y \in B(x_0, r)$ then

$$|u(y) - u(x)| \leqslant C\left(\frac{d(x,y)}{r}\right)^h \sup_{B(x_0,r)} |u|.$$

For $u \in \mathscr{H}^d(\Gamma, \mu)$ and two fixed vertices x and y this yields, as soon as r is big enough for $B(x_0, r)$ to contain the two vertices,

$$|u(y) - u(x)| \leqslant \underbrace{C\left(\frac{d(x,y)}{r}\right)^h C' r^d}_{\to 0 \text{ when } r \to +\infty \text{ if } d<h}.$$

So $u(y)=u(x)$, u must be constant. ∎

About the last property it may be worth here to compare on the manifold setting the conjunction of (DV) and (P) with the stronger condition that the Ricci curvature is nowhere negative. In the latest case, S. Cheng and S.-T. Yau proved in [1] a gradient inequality which implies a Lipschitz inequality so that the property above is true with $d = 1$. This is false in the more general case $(DV) + (P)$. For an idea of what can be such a manifold with negative curvature somewhere, one can check on $\mathbb{R}^2$ with polar coordinates that $u(r, \theta) = r^\eta \cos\theta$ is a solution of $\operatorname{div}(A\nabla u) = 0$ for a section A defined by $A_{(r,\theta)}e_r = e_r$ and $A_{(r,\theta)}e_\theta = \eta^2 e_\theta$. Consider so $0 < \eta < 1$, the idea would be to define a metric on $\mathbb{R}^2$ by $g_m(\xi, \zeta) = A_m^{-1}\xi.\zeta$. The section A is not smooth near the center but we can clearly take a regularisation of it there to define a smooth Riemannian metric without changing the behaviour of u for large r and without losing the ellipticity property

$\eta^2\|\xi\|^2 \leqslant A_m\xi.\xi \leqslant \|\xi\|^2$ which ensures that the (DV) and (P) properties of $\mathbb{R}^2$ yield ones for the new metric. As far as curvature for this metric is concerned, it is zero except near the center where there is a concentration of negative curvature.

We turn now to the dimension of the spaces $\mathcal{H}^d$ for bigger values of d.

THEOREM 3.3. — *If* (Γ, μ) *satisfies* $DV(C_1)$, $P(C_2)$ *and* $\Delta^*(\alpha)$, *then for* $d \geqslant 1$,

$$\dim \mathcal{H}^d(\Gamma, \mu) \leqslant Cd^\nu$$

where C *depends on* C_1, C_2 *and* α *and where* $\nu = \ln C_1 / \ln 2$ *has appeared in (2.5).*

This theorem has been proved on Riemannian manifolds by T. H. Colding and W. P. Minicozzi II in 1996 [3, 4, 5]. It is a positive answer to a conjecture by S. T. Yau [26] which said that these spaces should be of finite dimension (when the Ricci curvature is nowhere negative). Their techniques based on Moser's iteration method may also be applied on our graph setting. In the following we use the proof of P. Li in [16].

We shall estimate the dimension of a vectorial subspace $H \subset \mathcal{H}^d(\Gamma, \mu)$ assumed of finite dimension, so that for $r \geqslant r_0$, the symmetric nonnegative bilinear forms

$$A_r(u, v) = \sum_{x \in B(x_0, r)} m(x)u(x)v(x)$$

are not degenerated on H and we may use associated orthonormal bases. Q_r will denote the quadratic forms $Q_r(u) = A_r(u, u)$.

In fact the Poincaré inequality is not needed but only the mean-value inequality below which appears in the proof of the elliptic Harnack inequality by Moser's iteration and which is true for a larger class of graphs (see [6]) than those which satisfy (DV) and (P).

LEMMA 3.4. — *For any* $x \in \Gamma$, *any* $r \geqslant 0$ *and any function* u *harmonic on* $B(x, r)$,

$$u^2(x) \leqslant \frac{C_m}{V(x, r)} \sum_{y \in B(x,r)} m(y)u^2(y) \tag{3.9}$$

where C_m *depends on* C_1, C_2 *and* α.

Theorem 3.3 will follow from the two next lemmas. The first uses the mean-value inequality and the second the polynomial volume growth.

LEMMA 3.5. — *Let* $0 < \epsilon \leqslant 1$ *and* $(u_i)_{1 \leqslant i \leqslant k}$ *harmonic fuctions.*

$$\sum_{i=1}^k Q_r(u_i) \leqslant C\epsilon^{-\nu} \sup_{\sum \alpha_i^2 = 1} Q_{(1+\epsilon)r}\left(\sum_{i=1}^k \alpha_i u_i\right) \tag{3.10}$$

where C *depends on* C_1, C_2 *and* α.

Proof: For all x, we define

$$\alpha_i(x) = \frac{u_i(x)}{\sqrt{\sum_{j=1}^k u_j^2(x)}}.$$

In case the denominator is zero, (3.11) doesn't need any proof. These coefficients are such that $\sum_{i=1}^k \alpha_i^2(x) = 1$. Let us apply now (3.9) to the function $\sum_{i=1}^k \alpha_i(x) u_i$ for x fixed:

$$\begin{aligned}
\sum_{i=1}^k u_i^2(x) &= \left(\sum_{i=1}^k \alpha_i(x) u_i(x)\right)^2 \\
&\leqslant \frac{C_m}{V(x,(1+\epsilon)r - d(x_0,x))} \sum_{y \in B(x,(1+\epsilon)r - d(x_0,x))} m(y) \left(\sum_{i=1}^k \alpha_i(x) u_i(y)\right)^2 \\
&\leqslant \frac{C_m}{V(x,(1+\epsilon)r - d(x_0,x))} \sup_{\sum \alpha_i^2 = 1} Q_{(1+\epsilon)r}\left(\sum_{i=1}^k \alpha_i u_i\right). \qquad (3.11)
\end{aligned}$$

We have forced $d(x,y) \leqslant (1+\epsilon)r - d(x_0,x)$ so that y stays in the ball $B(x_0,(1+\epsilon)r)$.

Now let us sum for $x \in B(x_0, r)$ with respect to m, this yields

$$\sum_{i=1}^k Q_r(u_i) \leqslant C_m.K. \sup_{\sum \alpha_i^2 = 1} Q_{(1+\epsilon)r}\left(\sum_{i=1}^k \alpha_i u_i\right),$$

where

$$K = \sum_{x \in B(x_0,r)} \frac{m(x)}{V(x,(1+\epsilon)r - d(x_0,x))}.$$

A rough estimate of K may be obtained with $(1+\epsilon)r - d(x_0,x) \geqslant \epsilon r$ and (2.5).

$$\begin{aligned}
V(x,\epsilon r) &\geqslant C_1^{-1} V(x,r)\epsilon^{-\nu} \\
&\geqslant C_1^{-2} V(x,2r)\epsilon^{-\nu} \\
&\geqslant C_1^{-2} V(x_0,r)\epsilon^{-\nu}
\end{aligned}$$

So $K \leqslant C_1^{-2}\epsilon^{-\nu}$, and (3.10) follows. ∎

With a more careful manipulation of K, we could replace ν by a slighty smaller exponent, see the ϵ-volume regularity in [4].

LEMMA 3.6. — *Let $k = \dim H$ and $\beta > 1$, $\exists r \geqslant r_0$ such that if $(u_i)_{1 \leqslant i \leqslant k}$ is an $A_{\beta r}$-orthonormal basis of H, then:*

$$\sum_{i=1}^k Q_r(u_i) \geqslant k\beta^{-(2d+\nu+1)}.$$

Proof: The trace and determinant of $A_{r'}$ in an A_r-orthonormal basis will be denoted $\mathrm{tr}_r A_{r'}$ and $\det_r A_{r'}$. In case there would not be any correct value of r, we would have

$$\forall r \geqslant r_0, \ \mathrm{tr}_{\beta r} A_r < k\beta^{-(2d+\nu+1)}.$$

Thus,

$$\det{}_{\beta r} A_r \leqslant \left(\frac{\mathrm{tr}_{\beta r} A_r}{k}\right)^k < \beta^{-k(2d+\nu+1)}.$$

And since $\det_{r''} A_r = \det_{r''} A_{r'} . \det_{r'} A_r$,

$$\det{}_{\beta^j r_0} A_{r_0} < \beta^{-jk(2d+\nu+1)}. \tag{3.12}$$

But if $(u_i)_{1\leqslant i\leqslant k}$ is an A_{r_0}-orthonormal basis, the polynomial growth implies that for any $r \geqslant 1$ and any i, $|u_i|$ is bounded by Cr^d, so

$$Q_r(u_i) \leqslant V(x_0, r).(Cr^d)^2 \leqslant C_1 C^2 m(x_0) r^{2d+\nu}.$$

This way we can estimate all terms in the development of the determinant of A_r in $(u_i)_{1\leqslant i\leqslant k}$:

$$\det{}_{r_0} A_{\beta^j r_0} \leqslant k!(C_1 C^2 m(x_0))^k r_0^{k(2d+\nu)} \beta^{jk(2d+\nu)}.$$

There is a contradiction with (3.12) when $j \to +\infty$. Indeed, $\det_{r_0} A_{\beta^j r_0} . \det_{\beta^j r_0} A_{r_0} = \det_{r_0} A_{r_0} = 1$. ∎

Proof of Theorem 3.3: Let $\epsilon = d^{-1}$, $\beta = 1 + \epsilon$, r given by Lemma 3.6 and $(u_i)_{1\leqslant i\leqslant k}$ an $A_{\beta r}$-orthonormal basis of H,

$$\begin{aligned} k\beta^{-(2d+\nu+1)} &\leqslant \sum_{i=1}^{k} Q_r(u_i) \\ &\leqslant C\epsilon^{-\nu} \sup_{\sum \alpha_i^2 = 1} Q_{(1+\epsilon)r}\left(\sum_{i=1}^{k} \alpha_i u_i\right) \\ &\leqslant C\epsilon^{-\nu} = Cd^{\nu}. \end{aligned}$$

This yields $k \leqslant Cd^\nu$ since thanks to $\epsilon = d^{-1}$ the values of $\beta^{-(2d+\nu+1)}$ may be bounded from below. ∎

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