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Hiroyasu Izeki
Shin Nayatani

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CANONICAL METRIC ON THE DOMAIN OF DISCONTINUITY OF A KLEINIAN GROUP

Hiroyasu IZEKI & Shin NAYATANI

Introduction

A Kleinian group is a discrete subgroup of the conformal automorphism group of the round sphere. Its domain of discontinuity is by definition the largest open subset of the sphere on which the group acts properly discontinuously. The quotient of the domain by the group inherits the flat conformal structure of the sphere. In [17] the second author introduced a canonical Riemannian metric on such a manifold which is compatible with the conformal structure (see §2). He observed that the curvature of this metric well reflects the Hausdorff dimension of the limit set of the Kleinian group. This recovers R. Schoen and S.-T. Yau's earlier result [22] on the relation between the Yamabe conformal invariant of the quotient manifold and the Hausdorff dimension of the limit set. Our result roughly states that the smaller the dimension of the limit set, the stronger the positivity of curvature. Via the classical Bochner technique, this leads to a vanishing theorem for the cohomology of the quotient manifold. The first author [11] then used this vanishing result to generalize R. Bowen's theorem [8] on the Hausdorff dimension of the limit set of a quasi-Fuchsian group to higher dimensions (see §3, §5).

This article surveys various aspects of the canonical metric, and as such it is partly expository. It, however, also contains new results which we have obtained after the writing of [11], [17], [18].

This paper is organized as follows. In §1 we review basic definitions and facts about Kleinian groups and Patterson-Sullivan measures. In §2 we give an interpretation of the canonical metric from the viewpoint of hyperbolic geometry. We also discuss the normalization of metric, as well as the behavior of the metric as the Kleinian group is continuously deformed. In §3 we study the total scalar curvature (precisely, the integral of the (dimension)/2-th power of the absolute value of scalar curvature) of the canonical metric when the Kleinian group is a quasi-Fuchsian group, and prove that this invariant has a sharp lower bound, which is attained by the hyperbolic metric (= the canonical

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metric associated with a cocompact Fuchsian group). In §4 we give a vanishing theorem for the cohomology group and the space of L^2-harmonic forms of a convex-cocompact hyperbolic manifold. In §5 we generalize the vanishing theorem in [17] to an arbitrary flat Hilbert space bundle, and give an application. In §6 we construct distinguished metrics on certain quasi-balls, following the idea which we used to construct the canonical metric.

1. Preliminaries

Let (B^{n+1}, h) denote the Poincaré ball model of hyperbolic $(n+1)$-space, where $B^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| < 1\}$ and h is the Poincaré hyperbolic metric

$$h = \left(\frac{2}{1-|x|^2}\right)^2 \sum_{i=1}^{n+1} (dx^i)^2.$$

Let $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ and let g_0 be the standard induced metric on S^n. As is well-known, each isometry of (B^{n+1}, h) extends to a diffeomorphism of $\overline{B^{n+1}} (= B^{n+1} \cup S^n)$, and, restricted to S^n, gives a conformal automorphism of (S^n, g_0). In this way, the isometry group of (B^{n+1}, h) may be identified with the conformal automorphism group of (S^n, g_0), and we denote both of these groups by the common notation $\mathrm{Möb}(n)$.

Let Γ be a Kleinian group, that is, a discrete subgroup of $\mathrm{Möb}(n)$. For the sake of simplicity, we assume Γ is torsion-free throughout this paper. Its *limit set* $\Lambda(\Gamma)$ is defined as the set of accumulation points in $\overline{B^{n+1}}$ of Γ-orbit of any point in B^{n+1}. Since Γ acts properly discontinuously on B^{n+1}, $\Lambda(\Gamma)$ lies in S^n. Let $d(\Gamma)$ denote the Hausdorff dimension of $\Lambda(\Gamma)$. The complement $\Omega(\Gamma) = S^n \setminus \Lambda(\Gamma)$ is called the *domain of discontinuity* of Γ, which is possibly disconnected. Γ acts on $B^{n+1} \cup \Omega(\Gamma)$ properly discontinuously, and freely since Γ is torsion-free. Hence the quotient $Y = [B^{n+1} \cup \Omega(\Gamma)]/\Gamma$ is a smooth manifold-with-boundary. Its boundary $X = \Omega(\Gamma)/\Gamma$ inherits the flat conformal structure of S^n.

Definition. — We say that a Kleinian group Γ is *convex-cocompact* if the quotient $C(\Lambda(\Gamma))/\Gamma$ is compact, where $C(\Lambda(\Gamma))$ is the hyperbolic convex hull of $\Lambda(\Gamma)$ in B^{n+1}.

It is known that a Kleinian group Γ is convex-cocompact if and only if the associated manifold Y is compact. It is also worth noting that Γ is not necessarily convex-cocompact even though X is compact.

The *critical exponent* $\delta(\Gamma)$ of a Kleinian group Γ is defined by

$$\delta(\Gamma) = \inf\left\{ s > 0 \;\middle|\; \sum_{\gamma \in \Gamma} e^{-s\, d(x, \gamma y)} < \infty \right\},$$

where $x, y \in B^{n+1}$ and d is the hyperbolic distance function on B^{n+1}. Notice that $\delta(\Gamma)$ is independent of the particular choice of the points x, y. It is known that $0 \leqslant \delta(\Gamma) \leqslant n$ and if Γ is *non-elementary*, that is, $\Lambda(\Gamma)$ contains at least three points, then $\delta(\Gamma) > 0$.

S. J. Patterson [21] and D. Sullivan [23] introduced a distinguished family of measures supported on the limit set of a Kleinian group. We review Patterson-Sullivan's construction. For $x \in B^{n+1}$ and $s > \delta = \delta(\Gamma)$, we define a measure $\mu_{x,s}$ on $\overline{B^{n+1}}$ by

$$\mu_{x,s} = \frac{1}{\sum_{\gamma\in\Gamma} e^{-s\,d(0,\gamma 0)}} \sum_{\gamma\in\Gamma} e^{-s\,d(x,\gamma 0)}\delta_{\gamma 0},$$

where $\delta_{\gamma 0}$ denotes the Dirac measure at $\gamma 0$. By the triangle inequality, we have

$$e^{-s\,d(0,x)} \leqslant \mu_{x,s}(\overline{B^{n+1}}) \leqslant e^{s\,d(0,x)}.$$

In particular, $\mu_{x,s}(\overline{B^{n+1}})$ is bounded independently of s in the range $\delta < s < \delta + 1$. Hence there exists a sequence s_j approaching δ^+ such that the measures μ_{x,s_j} converge weakly to a measure μ_x on $\overline{B^{n+1}}$. In fact, it can be shown that for any $y \in B^{n+1}$, μ_{y,s_j} also converge weakly, whose limit we denote by μ_y.

We summerize the properties of the measures μ_x for convex-cocompact Γ as

PROPOSITION 1.1. — *Suppose that Γ is convex-cocompact. Then the measures μ_x, $x \in B^{n+1}$, have the following properties:*

(a) *Each μ_x is supported on $\Lambda(\Gamma)$.*

(b)
$$\mu_x = e^{-\delta(\Gamma)\,b_0(x,\cdot)}\mu_0, \quad x \in B^{n+1},$$

where $b_0(x,\cdot)$ is the Busemann function of hyperbolic space with respect to the reference point 0.

(c)
$$\gamma_*\mu_x = \mu_{\gamma x}, \quad x \in B^{n+1},\ \gamma \in \Gamma.$$

(d) *μ_0 coincides, up to a constant multiple, with the restriction of the $\delta(\Gamma)$-dimensional Hausdorff measure to $\Lambda(\Gamma)$. In particular, $d(\Gamma) = \delta(\Gamma)$.*

It is known that the assertions (a), (b), (c) and the equality $d(\Gamma) = \delta(\Gamma)$ in (d) hold more generally if Γ is non-elementary and geometrically finite. Here we call Γ *geometrically finite* if it has a fundamental polyhedron in B^{n+1} with finitely many faces. A convex-cocompact group is characterized as a geometrically finite group without parabolic elements. When Γ is not geometrically finite, it may happen that $\sum_{\gamma\in\Gamma} e^{-\delta\,d(0,\gamma 0)} < \infty$. Then μ_x will be again a sum of weighted Dirac measures placed on the Γ-orbit of 0. To avoid this and obtain a measure supported on $\Lambda(\Gamma)$, a certain modification is necessary in the above construction of μ_x. With this modification made, the assertions of Proposition 1.1 hold except (d). We do not go into the details of this point and refer the reader to Nicholls' book [19] or the papers cited above.

DEFINITION. — The family of measures $\{\mu_x \mid x \in B^{n+1}\}$ is called a *Patterson-Sullivan density* (with respect to the reference point 0), and a family of measures with

the properties (a), (b), (c) of Proposition 1.1 and with μ_0 a probability measure is called a *conformal density* (with respect to the reference point 0).

It is known that a conformal density (exists and) is unique for geometrically finite Γ [24]. Moreover, for geometrically finite Γ, a family of measures with the properties (b) and (c) of Proposition 1.1 must satisfy (a); such a measure family turns out to be the unique conformal density.

We now let x approach a point $\zeta \in \Omega(\Gamma)$. Since

$$b_0(x,\xi) = -\log \frac{1-|x|^2}{|x-\xi|^2}, \quad x \in B^{n+1},\ \xi \in S^n,$$

we have

$$\mu_x = \left(\frac{1-|x|^2}{|x-\xi|^2}\right)^{\delta} \mu_0$$

by Proposition 1.1 (b), and hence μ_x converges to a zero measure as $x \to \zeta$. However, if we divide μ_x by $\left[(1-|x|^2)/2\right]^{\delta}$, the resulting measures $\tilde{\mu}_x = \varphi(x,\xi)^{-\delta}\mu_0$ converge to a positive finite measure $\tilde{\mu}_\zeta = \varphi(\zeta,\xi)^{-\delta}\mu_0$ as $x \to \zeta$, where

$$\varphi(x,y) = \frac{1}{2}|x-y|^2, \quad x,y \in \overline{B^{n+1}}.$$

We shall refer to the new family of measures $\{\tilde{\mu}_x \mid x \in B^{n+1} \cup \Omega(\Gamma)\}$ as a *modified Patterson-Sullivan density.*

2. Canonical Metrics

Let Γ be a torsion-free discrete subgroup of Möb(n) with $\delta = \delta(\Gamma) > 0$, and let $\Omega(\Gamma)$ denote the domain of discontinuity of Γ. In [17] we introduced a canonical metric g on $\Omega(\Gamma)$, given by

$$\begin{aligned} g_x &= \left(\int_{\Lambda(\Gamma)} \varphi(x,\xi)^{-\delta} d\mu_0(\xi)\right)^{2/\delta} (g_0)_x \\ &= \left[\tilde{\mu}_x(\Lambda(\Gamma))\right]^{2/\delta} (g_0)_x, \quad x \in \Omega(\Gamma), \end{aligned}$$

where μ_0 and $\tilde{\mu}_x$ are the measures as in the previous section. The metric g is Γ-invariant. In fact, for any $\gamma \in \Gamma$

$$\begin{aligned} \int_{\Lambda(\Gamma)} \varphi(\gamma x,\xi)^{-\delta} d\mu_0(\xi) &= \int_{\Lambda(\Gamma)} \varphi(\gamma x,\gamma\xi)^{-\delta} d(\gamma^*\mu_0)(\xi) \\ &= \int_{\Lambda(\Gamma)} \left(j_\gamma(x)\, j_\gamma(\xi)\varphi(x,\xi)\right)^{-\delta} j_\gamma(\xi)^{\delta} d\mu_0(\xi) \\ &= j_\gamma(x)^{-\delta} \int_{\Lambda(\Gamma)} \varphi(x,\xi)^{-\delta} d\mu_0(\xi), \end{aligned}$$

where $j_\gamma = \exp(-b_0(\gamma^{-1}0, \cdot))$. Since $\gamma^* g_0 = j_\gamma^2 g_0$, we get $\gamma^* g = g$. Hence g projects to a metric on the quotient $X = \Omega(\Gamma)/\Gamma$, compatible with the conformal structure. We denote this metric on X by the same symbol g. Recently J. Maubon [16] proved that for geometrically finite Γ, the metric g is complete if and only if Γ contains no parabolic elements of rank less than δ.

Since the argument in [17] to deduce the Γ-invariance of g is valid on $B^{n+1} \cup \Omega(\Gamma)$, g naturally extends to a metric on $Y = [B^{n+1} \cup \Omega(\Gamma)]/\Gamma$, which we continue to denote by g. By using the hyperbolic metric, this metric may be rewritten on B^{n+1} as

$$\begin{aligned} g_x &= \left(\int_{\Lambda(\Gamma)} e^{-\delta\, b_0(x,\xi)}\, d\mu_0(\xi) \right)^{2/\delta} h_x \\ &= [\mu_x(\Lambda(\Gamma))]^{2/\delta}\, h_x, \quad x \in B^{n+1}. \end{aligned}$$

It is worth mentioning that, when Γ is convex-cocompact, (Y, g) gives a natural conformal compactification of the hyperbolic manifold $(B^{n+1}, h)/\Gamma$. It should also be mentioned, however, that this metric on Y is not really a new product. Indeed, if we identify B^{n+1} with a hemisphere in S^{n+1} and extend the action of Γ to that on S^{n+1} in the standard way, the metric is nothing but the restriction of the canonical metric of $\Omega^{n+1}(\Gamma)/\Gamma$, where $\Omega^{n+1}(\Gamma)$ is the domain of discontinuity of Γ viewed as acting on S^{n+1}.

Since g is conformally flat, its curvature is completely determined by the Ricci tensor Ric. To write down the formula for Ric, we introduce a symmetric bilinear form B, defined by

$$\begin{aligned} B_x &= \int_{\Lambda(\Gamma)} db_{0(x,\xi)} \otimes db_{0(x,\xi)}\, d\nu_x(\xi) \\ &\quad - \int_{\Lambda(\Gamma)} db_{0(x,\xi)}\, d\nu_x(\xi) \otimes \int_{\Lambda(\Gamma)} db_{0(x,\xi)}\, d\nu_x(\xi), \quad x \in B^{n+1}, \end{aligned}$$

where $\nu_x = \mu_x / \|\mu_x\|$ with $\| \cdot \|$ denoting the total mass. B is nonnegative by Schwarz' inequality, and

$$\operatorname{tr}_h B_x = 1 - \left| \int_{\Lambda(\Gamma)} db_{0(x,\xi)}\, d\nu_x(\xi) \right|^2.$$

The Ricci tensor and scalar curvature of the canonical metric g (on B^{n+1}) are given by

$$\mathrm{Ric} = -(n-1)(\delta+1)B + (n-1-\delta)\left(\operatorname{tr}_g B\right) g,$$

$$S = n(n-1-2\delta)\operatorname{tr}_g B$$

respectively. Letting $x \in B^{n+1}$ approach a point of $\Omega(\Gamma)$ and restricting to the tangent space of the sphere, we recover the formula for the Ricci tensor of g (on $\Omega(\Gamma)$) given in [17]:

$$\mathrm{Ric} = -(n-2)(\delta+1)A + (n-2-\delta)\left(\operatorname{tr}_g A\right) g,$$

$$S = (n-1)(n-2-2\delta)\operatorname{tr}_g A, \tag{2.1}$$

where

$$A_x = \int_{\Lambda(\Gamma)} \left(\varphi^{-2} d\varphi \otimes d\varphi\right)_{(x,\xi)} d\nu_x(\xi) - \int_{\Lambda(\Gamma)} \left(\varphi^{-1} d\varphi\right)_{(x,\xi)} d\nu_x(\xi) \otimes \int_{\Lambda(\Gamma)} \left(\varphi^{-1} d\varphi\right)_{(x,\xi)} d\nu_x(\xi), \quad x \in \Omega(\Gamma).$$

Note that since $\nu_x = \tilde{\mu}_x / \| \tilde{\mu}_x \|$, it is defined for $x \in \Omega(\Gamma)$ also.

The above construction of the canonical metric actually depends on the choice of reference point of hyperbolic space, as which we have used the center 0 of B^{n+1} so far. Assuming that the uniqueness of conformal density (with respect to a fixed reference point) holds for Γ, we now vary the reference point; for each $a \in B^{n+1}$, we have the metric

$$g_x^{(a)} = \mu_x^{(a)}(\Lambda(\Gamma))^{2/\delta} h_x, \quad x \in B^{n+1},$$

where $\{\mu_x^{(a)} \mid x \in B^{n+1}\}$ is the unique conformal density with the reference point a. Then we have

$$g^{(a)} = \left(\mu_a^{(0)}(\Lambda(\Gamma))\right)^{-2/\delta} g^{(0)}.$$

Indeed, this identity is an immediate consequence of

$$\mu_x^{(a)} = \left(\mu_a^{(0)}(\Lambda(\Gamma))\right)^{-1} \mu_x^{(0)}, \tag{2.2}$$

which we now prove. Using

$$b_0(x,\xi) = b_a(x,\xi) - b_a(0,\xi) = b_a(x,\xi) + b_0(a,\xi),$$

we compute

$$\begin{aligned} \mu_x^{(0)} &= e^{-\delta\, b_0(x,\xi)} \mu_0^{(0)} \\ &= e^{-\delta\, b_a(x,\xi)} e^{-\delta\, b_0(a,\xi)} \mu_0^{(0)} \\ &= e^{-\delta\, b_a(x,\xi)} \mu_a^{(0)}. \end{aligned}$$

By the uniqueness of conformal density, we must have $\mu_x^{(0)} = \underline{\text{const}}\, \mu_x^{(a)}$. In particular, $\mu_a^{(0)} = \underline{\text{const}}\, \mu_a^{(a)}$, and hence $\underline{\text{const}} = \mu_a^{(0)}(\Lambda(\Gamma))$. This proves (2.2).

We now set

$$F(a) = \mu_a^{(0)}(\Lambda(\Gamma)) = \int_{\Lambda(\Gamma)} e^{-\delta\, b_0(a,\xi)} d\mu_0^{(0)}(\xi), \quad a \in B^{n+1},$$

so that $g^{(a)} = F(a)^{-2/\delta} g^{(0)}$. Note that F is Γ-invariant and tends to zero as a approaches a point of $\Omega(\Gamma)$. Hence F can be viewed as a continuous function on the manifold-with-boundary $Y = [B^{n+1} \cup \Omega(\Gamma)]/\Gamma$ which is positive in the interior and vanishes on the boundary. We now suppose that Γ is convex-cocompact, so that Y is compact. Then F

attains its maximum at some point $a \in B^{n+1}$, and $g^{(a)}$ is minimal among all its companion metrics. We shall refer to $g^{(a)}$ as the *minimal canonical metric*. Note that F does not always attain maximum without the assumption of convex-cocompactness; consider Γ with single limit point, for example. It is also worth mentioning that since

$$dF_a = -\delta \int_{\Lambda(\Gamma)} db_{0(a,\xi)} d\mu_a^{(0)}(\xi),$$

a point $a \in B^{n+1}$ where F attains its maximum (if exists) is the barycenter of the measure $\mu_a^{(0)}$, which coinsides with that of $\mu_a^{(a)}$; $\mathrm{bar}(\mu_a^{(0)}) = \mathrm{bar}(\mu_a^{(a)}) = a$ (see §3 for the definition of barycenter).

For $\alpha \in \mathrm{M\ddot{o}b}(n)$, let $\Gamma' = \alpha\Gamma\alpha^{-1}$ (the "push-forward" of Γ by α) and label the corresponding objects by $'$. Then we have

$$\alpha^* \mu_x'^{(0)} = \mu_{\alpha^{-1}x}^{(\alpha^{-1}0)}. \tag{2.3}$$

Indeed, for $s > \delta$

$$\begin{aligned} \alpha^* \mu_{x,s}'^{(0)} &= \frac{1}{\sum_{\gamma\in\Gamma} e^{-s\,d(0,\alpha\gamma\alpha^{-1}0)}} \sum_{\gamma\in\Gamma} e^{-s\,d(x,\alpha\gamma\alpha^{-1}0)} \delta_{\gamma\alpha^{-1}0} \\ &= \frac{1}{\sum_{\gamma\in\Gamma} e^{-s\,d(\alpha^{-1}0,\gamma\alpha^{-1}0)}} \sum_{\gamma\in\Gamma} e^{-s\,d(\alpha^{-1}x,\gamma\alpha^{-1}0)} \delta_{\gamma\alpha^{-1}0}. \end{aligned}$$

Letting $s \to \delta$, we obtain (2.3).

By (2.3)

$$\begin{aligned} g_x'^{(0)} &= \mu_x'^{(0)}(\Lambda(\Gamma'))^{2/\delta} h_x \\ &= \mu_{\alpha^{-1}x}^{(\alpha^{-1}0)}(\Lambda(\Gamma))^{2/\delta} h_x \end{aligned}$$

since $\Lambda(\Gamma') = \alpha(\Lambda(\Gamma))$. It follows that

$$(\alpha^* g'^{(0)})_x = \mu_x^{(\alpha^{-1}0)}(\Lambda(\Gamma))^{2/\delta} h_x = g_x^{(\alpha^{-1}0)}.$$

This formula means that the conjugation of Γ has the same effect on the metric as the change of reference point. Clearly, α pulls back the minimal canonical metric for Γ (if exists) to that for Γ'.

Next we show that the canonical metrics vary continuously on the set of convex-cocompact Kleinian groups. Here we only consider the behaviour of $g^{(0)}$, which we drop the superscript and denote simply by g. It is not, however, so difficult to see that $g^{(a)}$ for fixed a also varies continuously along deformations of convex-cocompact Kleinian groups.

Let $\mathrm{Hom}(G, \mathrm{M\ddot{o}b}(n))$ be the set of representations of a torsion-free group G into $\mathrm{M\ddot{o}b}(n)$. The topology on this set is given by the pointwise convergence of representations as maps into $\mathrm{M\ddot{o}b}(n)$. If G is finitely generated, this topology is equivalent to that

given by the uniform convergence of representations on a finite set of generators. We denote by $C(G, \mathrm{M\ddot{o}b}(n))$ the set of faithful representations whose images are discrete and convex-cocompact.

THEOREM 2.1. — *On the set $C(G, \mathrm{M\ddot{o}b}(n))$, the canonical metrics vary continuously. More precisely speaking, if $\rho_j \to \rho$ in $C(G, \mathrm{M\ddot{o}b}(n))$, then there is a sequence of diffeomorphisms $\psi_j : (\Omega(\Gamma)/\Gamma, g) \longrightarrow (\Omega(\Gamma_j)/\Gamma_j, g_j)$ such that $\psi_j^* g_j \to g$ in C^∞-topology, where $\Gamma_j = \rho_j(G), \Gamma = \rho(G)$, g_j and g are the canonical metrics for Γ_j and Γ respectively.*

The basic ingredient of the proof is the following lemma.

LEMMA 2.2. — *Assume G is torsion-free.*

(a) $C(G, \mathrm{M\ddot{o}b}(n))$ *is open in* $\mathrm{Hom}(G, \mathrm{M\ddot{o}b}(n))$.

(b) *The critical exponent regarded as a function on $C(G, \mathrm{M\ddot{o}b}(n))$ is continuous.*

(c) *The map $\rho \mapsto \mu_x(\rho(G))$ for fixed x is continuous on $C(G, \mathrm{M\ddot{o}b}(n))$, where $\mu_x(\rho(G))$ denotes the measure μ_x for $\rho(G)$.*

The first and the second parts have been shown in [7]. A different proof in terms of conformal geometry on $\Omega(\Gamma)$ can be found in [12]. We give the proof of the third part here.

Proof of (c). First recall that, for any Kleinian group Γ, the Patterson-Sullivan measure μ_0 associated to Γ is a probability measure on S^n. Suppose $\rho_k \to \rho$ in $C(G, \mathrm{M\ddot{o}b}(n))$. We denote $\rho(G)$ by Γ. And let μ_x^k and δ_k be the Patterson-Sullivan measure at x and the critical exponent of $\rho_k(G)$ respectively. By (b), $\delta_k \to \delta = \delta(\Gamma)$. Since the space of probability measures on a compact metric space is compact with respect to weak-$*$ topology, there is a subsequence of $\{\mu_0^k\}$, which we denote by $\{\mu_0^j\}$, and the limit μ_0 of the subsequence. Let f be a continuous function on S^n. Then, by Proposition 1.1 (b), we have

$$\int_{S^n} f(\xi)\, d\mu_x^j(\xi) = \int_{S^n} f(\xi) e^{-\delta_j b_0(x,\xi)} d\mu_0^j(\xi).$$

By (b) and the continuity of $b_0(x, \xi)$, the function $f(\xi)\exp(-\delta_j b_0(x, \xi))$ converges to $f(\xi)\exp(-\delta\, b_0(x, \xi))$ uniformly on S^n, namely

$$\varepsilon_j = \max_{\xi \in S^n} \left| f(\xi) e^{-\delta\, b_0(x,\xi)} - f(\xi) e^{-\delta_j b_0(x,\xi)} \right| \to 0 \quad \text{as} \quad j \to \infty. \tag{2.4}$$

On the other hand,

$$\begin{aligned} &\left| \int_{S^n} f(\xi) e^{-\delta\, b_0(x,\xi)} d\mu_0(\xi) - \int_{S^n} f(\xi)\, d\mu_x^j(\xi) \right| \\ &\leqslant \left| \int_{S^n} f(\xi) e^{-\delta\, b_0(x,\xi)} d\mu_0(\xi) - \int_{S^n} f(\xi) e^{-\delta\, b_0(x,\xi)} d\mu_0^j(\xi) \right| \\ &\quad + \left| \int_{S^n} f(\xi) e^{-\delta\, b_0(x,\xi)} d\mu_0^j(\xi) - \int_{S^n} f(\xi) e^{-\delta_j b_0(x,\xi)} d\mu_0^j(\xi) \right|, \end{aligned}$$

and the uniform convergence (2.4) implies that the second term in the right-hand side is less than ε_j since μ_0^j's are probability measures. Together with $\mu_0^j \to \mu_0$, we see $\mu_x^j \to \mu_x = e^{-\delta b_0(x,\xi)}\mu_0$. Therefore $\{\mu_x\}_{x\in H^{n+1}}$ satisfies Proposition 1.1 (b). For any $h \in G$, $\gamma_j = \rho_j(h) \to \rho(h) = \gamma$ uniformly on S^n. Then it is easy to see $\gamma_j^* \mu_x^j \to \gamma^* \mu_x$ by an argument similar to the above. On the other hand, again by Proposition 1.1 (b), for any continuous function f on S^n

$$\int_{S^n} f(\xi)\, d\mu^j_{\gamma_j^{-1}x}(\xi) = \int_{S^n} f(\xi) e^{-\delta_j b_0(\gamma_j^{-1}x,\xi)}\, d\mu_0^j(\xi).$$

Since the Busemann function $b_0(x,\xi)$ is Lipschitz continuous with respect to x for fixed ξ, $b_0(\gamma_j^{-1}x,\xi) \to b_0(\gamma^{-1}x,\xi)$ uniformly on S^n. This implies the uniform convergence

$$f(\xi)e^{-\delta_j b_0(\gamma_j^{-1}x,\xi)} \to f(\xi)e^{-\delta b_0(\gamma^{-1}x,\xi)}$$

for any continuous function f on S^n. Combining with $\mu_0^j \to \mu_0$, we see

$$\int_{S^n} f(\xi)e^{-\delta_j b_0(\gamma_j^{-1}x,\xi)}\, d\mu_0^j(\xi) \to \int_{S^n} f(\xi)e^{-\delta b_0(\gamma^{-1}x,\xi)}\, d\mu_0(\xi)$$

for any continuous function f by the same argument as above, which means $\mu^j_{\gamma_j^{-1}x} \to \mu_{\gamma^{-1}x}$. Thus we have $\gamma^*\mu_x = \mu_{\gamma^{-1}x}$ by the uniqueness of the limit, which is equivalent to say $\gamma_*\mu_x = \mu_{\gamma x}$. Therefore $\{\mu_x\}$ satisfies Proposition 1.1 (c). By the uniqueness of the conformal density for convex-cocompact Kleinian groups (cf. §1), $\{\mu_x\}_{x\in H^{n+1}}$ must be the Patterson-Sullivan density of Γ. Note that our proof shows any convergent subsequence of $\{\mu_0^k\}$ has μ_0 as its limit. Suppose our original sequence $\{\mu_0^k\}$ itself does not converge to μ_0. Then there is a subsequence of $\{\mu_0^k\}$ which does not converge to μ_0. On the other hand, because of the compactness of the set of probability measures on S^n, this subsequence must contain a convergent subsequence whose limit is not μ_0. This contradicts what we have seen above. Thus $\{\mu_0^k\}$ itself must converge to μ_0. By the transformation law (b) in Proposition 1.1 and (b), the same is true for $\{\mu_x^k\}$ with any fixed x. This completes the proof. □

Proof of Theorem 2.1. As was shown in [12], for large j, there is a quasiconformal mapping ψ_j which conjugates ρ_j to ρ. By the construction of ψ_j in the proof of Theorem 1 in [12], it is an equivariant diffeomorphism between $\Omega(\Gamma)$ and $\Omega(\Gamma_j)$ and $\psi_j \to \mathrm{id}$ with respect to C^∞-topology on each compact subset of $\Omega(\Gamma)$. Also ψ_j is an equivariant homeomorphism between $\Lambda(\Gamma)$ and $\Lambda(\Gamma_j)$, and $\psi_j \to \mathrm{id}$ uniformly on S^n. We denote the critical exponent and the Patterson-Sullivan measure μ_0 of Γ_j by δ_j and μ_0^j respectively as in the proof of Lemma 2.2. Then for $\xi \in \Omega(\Gamma)$

$$
\begin{aligned}
(\psi_j^* g_j)_\xi &= \left(\int_{\Lambda(\Gamma_j)} \left(\frac{1}{2}\left|\psi_j(\xi)-\eta\right|^2\right)^{-\delta_j} d\mu_0^j(\eta)\right)^{2/\delta_j} (\psi_j^* g_0)_\xi \\
&= \left(\int_{\Lambda(\Gamma)} \left(\frac{1}{2}\left|\psi_j(\xi)-\psi_j(\eta_j')\right|^2\right)^{-\delta_j} d\mu_0^j(\psi_j(\eta_j'))\right)^{2/\delta_j} (\psi_j^* g_0)_\xi \\
&= \left(\int_{\Lambda(\Gamma)} \left(\frac{1}{2}\left|\psi_j(\xi)-\psi_j(\eta_j')\right|^2\right)^{-\delta_j} d(\psi_j^*\mu_0^j)(\eta')\right)^{2/\delta_j} (\psi_j^* g_0)_\xi,
\end{aligned}
$$

where $\psi_j(\eta_j') = \eta$. Recall that $\psi_j \to \mathrm{id}$ uniformly on S^n, and $\psi_j \to \mathrm{id}$ with respect to C^∞-topology on each compact subset of $\Omega(\Gamma)$. This, together with $\delta_j \to \delta$ and $\mu_0^j \to \mu_0$, implies that $g_j \to g$ uniformly on each compact subset of $\Omega(\Gamma)$ by a slight modification of the argument in the proof of Lemma 2.2

This is also true for the derivatives of $\psi_j^* g_j$'s since these derivatives come from those of $|\xi-\eta|^{-\delta_j}$, ψ_j and g_0 with respect to ξ. Therefore $\psi_j^* g_j \to g$ with respect to C^∞-topology on each compact subset of $\Omega(\Gamma)$. Since $\Omega(\Gamma)/\Gamma$ is compact and ψ_j's are equivariant maps, this gives the desired sequence of diffeomorphisms. □

In [12], the first author proved that the Teichmüller space of flat conformal structures on a manifold satisfying certain conditions can be embedded into the space $C(\pi_1(M), \mathrm{Möb}(\dim M))$. We can compactify the Teichmüller space by compactifying its image in $\mathrm{Hom}(\pi_1(M), \mathrm{Möb}(\dim M))$. Moreover, under certain assumptions, it is possible to associate a Riemannian metric to each point of the boundary. These metrics are constructed in the same manner as our canonical metrics, and the metrics vary continuously up to the boundary. This will be useful for investigating the Teichmüller space of flat conformal structures and topology of conformally flat manifolds.

3. Quasi-Fuchsian Groups

Let $\mathrm{Möb}_0(n)$ denote the identity component of $\mathrm{Möb}(n)$. Let Γ_0 be a convex-cocompact discrete subgroup of $\mathrm{Möb}_0(n)$ whose limit set is a round p-sphere S^p for some $p < n$. Then $B^{p+1} = C(S^p)$ is invariant under Γ, and the quotient B^{p+1}/Γ is compact. In other words, Γ_0 is an extension of a cocompact lattice in $\mathrm{Möb}(p)$. We assume this lattice lies in $\mathrm{Möb}_0(p)$. Let $\rho : \Gamma_0 \to \mathrm{Möb}_0(n)$ be a faithful representation whose image $\Gamma = \rho(\Gamma_0)$ is discrete and convex-cocompact. In this paper, we call such a representation *a quasi-Fuchsian representation* and its image a *quasi-Fuchsian group*. By the result of R. Bowen [8], H. Izeki [11], M. Bourdon [6] and C.-B. Yue [25], we have $d(\Gamma) \geqslant p\,(= d(\Gamma_0))$ with the equality sign holding if and only if $\Lambda(\Gamma)$ is a round p-sphere. If $p = n-1 \geqslant 2$, this last condition implies, by the Mostow rigidity, that Γ is conjugate to Γ_0 in $\mathrm{Möb}(n)$.

More recently, G. Besson, G. Courtois and S. Gallot [4] have given a new proof of this result by studying the Jacobian of two maps defined in terms of the Patterson-Sullivan measures and the barycenter map. To describe one of these maps, we first recall [10] that

for a measure μ on S^n without atom, its barycenter, denoted by $\mathrm{bar}(\mu)$, is defined as the unique critical point of the function

$$\mathcal{B}_\mu(x) = \int_{S^n} b_0(x,\xi)\, d\mu(\xi), \quad x \in B^{n+1}.$$

The barycenter map bar has the equivariance property

$$\mathrm{bar}(\gamma_*\mu) = \gamma(\mathrm{bar}(\mu)), \quad \gamma \in \mathrm{M\ddot{o}b}(n). \tag{3.1}$$

We also recall the existence of a canonical homeomorphism $f : \Lambda(\Gamma) \to \Lambda(\Gamma_0) = S^p$ satisfying

$$f(\gamma\xi) = \rho^{-1}(\gamma) f(\xi), \quad \gamma \in \Gamma,\ \xi \in \Lambda(\Gamma). \tag{3.2}$$

We then define the map $F : B^{n+1} \to B^{p+1}$ by

$$F(x) = \mathrm{bar}(f_*\mu_x), \quad x \in B^{n+1},$$

where μ_x are the Patterson-Sullivan measures and f_* represents the push-forward of measures by f. By Proposition 1.1 (c), (3.1) and (3.2), F satisfies

$$F(\gamma x) = \rho^{-1}(\gamma)F(x), \quad \gamma \in \Gamma,\ x \in B^{n+1}.$$

It is proved, moreover, that F is a smooth map.

We now observe that F naturally extends to a map defined on $B^{n+1} \cup \Omega(\Gamma)$. Indeed, the map F remains unchanged if we use the modified Patterson-Sullivan measures $\tilde{\mu}_x$ to define it instead of μ_x, and $\tilde{\mu}_x$ are defined for $x \in \Omega(\Gamma)$ also. One can show that the extended map $\overline{F} : B^{n+1} \cup \Omega(\Gamma) \to B^{p+1}$ is also smooth. Since both F and the action of Γ extend smoothly up to $\Omega(\Gamma)$, $\overline{F}$ satisfies

$$\overline{F}(\gamma x) = \rho^{-1}(\gamma)\overline{F}(x), \quad \gamma \in \Gamma,\ x \in B^{n+1} \cup \Omega(\Gamma). \tag{3.3}$$

We now suppose $p = n-1$. Then $\Omega(\Gamma)$ consists of two contractible connencted components, both of which are invariant under the action of Γ. Take one of these connected components and denote it by Ω_0. We shall denote the restriction of $\overline{F}$ to Ω_0 by G, and study the Jacobian of the mapping $G : \Omega_0 \to B^n$. We may use $\nu_x = \tilde{\mu}_x / \| \tilde{\mu}_x \|$ instead of $\tilde{\mu}_x$ to define G. Since

$$\begin{aligned} d\mathcal{B}_{f_*\nu_x}(y) &= \int_{S^{n-1}} db_{0(y,\xi)}\, d(f_*\nu_x)(\xi) \\ &= \int_{\Lambda(\Gamma)} db_{0(y,f(\xi))}\, d\nu_x(\xi), \end{aligned}$$

setting $y = G(x)$, we have

$$\frac{1}{\int \varphi(x,\xi)^{-\delta} d\mu_0(\xi)} \int db_{0(G(x),f(\xi))}\, \varphi(x,\xi)^{-\delta} d\mu_0(\xi) = 0.$$

Taking covariant derivative in the direction $u \in T_xS^n$, substituting $v \in T_{G(x)}B^n$ and rearranging, we obtain

$$\begin{aligned}
&\int \nabla db_{0(G(x),f(\xi))}(dG(u), v)\, d\nu_x(\xi) \\
&\quad = \delta \int \left(\left(\varphi^{-1} d\varphi\right)_{(x,\xi)} - \int \left(\varphi^{-1} d\varphi\right)_{(x,\xi)} d\nu_x(\xi) \right) (u)\, db_{0(G(x),f(\xi))}(v)\, d\nu_x(\xi) \\
&\quad \leqslant \delta \left[\int \left(\left(\varphi^{-1} d\varphi\right)_{(x,\xi)} - \int \left(\varphi^{-1} d\varphi\right)_{(x,\xi)} d\nu_x(\xi) \right) (u)^2\, d\nu_x(\xi) \right]^{1/2} \\
&\qquad \times \left[\int db_{0(G(x),f(\xi))}(v)^2\, d\nu_x(\xi) \right]^{1/2}.
\end{aligned}$$

If we define linear endomorphisms H and K of $T_{G(x)}B^n$ by

$$h(H\cdot, \cdot) = \int db_{0(G(x),f(\xi))}(\cdot)^2\, d\nu_x(\xi),$$

$$h(K\cdot, \cdot) = \int \nabla db_{0(G(x),f(\xi))}(\cdot, \cdot)\, d\nu_x(\xi),$$

where h is the hyperbolic metric, the above estimate may be rewritten as

$$|h(K \circ dG(u), v)| \leqslant \delta\, g(Au, u)^{1/2} h(Hv, v)^{1/2},$$

where g is the canonical metric on Ω_0 ($\subset \Omega(\Gamma)$). By elementary linear algebra, we obtain the estimate

$$\det K |\operatorname{Jac} G| \leqslant \delta^n (\det A)^{1/2} (\det H)^{1/2}.$$

Since

$$\nabla db_{0(x,\xi)} = h_x - db_{0(x,\xi)} \otimes db_{0(x,\xi)}$$

for the Busemann function of hyperbolic space, we have $K = I - H$. Hence

$$|\operatorname{Jac} G| \leqslant \delta^n (\det A)^{1/2} \frac{(\det H)^{1/2}}{\det(I - H)}.$$

We now assume $n \geqslant 3$. It has been proved by G. Besson et al. [3] that the inequality

$$\frac{(\det H)^{1/2}}{\det(I - H)} \leqslant \frac{\left(\det \frac{1}{n} I\right)^{1/2}}{\det\left(I - \frac{1}{n} I\right)} = \frac{n^{n/2}}{(n-1)^n}$$

holds for any positive symmetric matrix H such that $\operatorname{tr} H = 1$. Using this together with $\det A \leqslant (\operatorname{tr} A/n)^n$, we finally obtain

$$|\operatorname{Jac} G| \leqslant \left(\frac{\delta}{n-1}\right)^n (\operatorname{tr} A)^{n/2}.$$

By (3.3), G induces a mapping from $X = \Omega_0/\Gamma$ to B^n/Γ_0, which we denote by the same symbol G. It is clear that G induces an isomorphism between the fundamental groups of X and B^n/Γ; in particular, G is a homotopy equivalence. Since $\deg G = \pm 1$,

$$\begin{aligned}\int_{B^n/\Gamma_0} v_h &= \left|\int_X G^* v_h\right| \\ &= \left|\int_X \operatorname{Jac} G\, v_g\right| \\ &\leqslant \left(\frac{\delta}{n-1}\right)^n \int_X \left(\operatorname{tr}_g A\right)^{n/2} v_g.\end{aligned}$$

Using (2.1), we obtain the following assertion on the total scalar curvature:

THEOREM 3.1. — *Let Γ_0, Γ and Ω_0 be as above, g the canonical metric restricted to $X = \Omega_0/\Gamma$, and assume $n \geqslant 3$. Then we have*

$$\int_X \left(-S_g\right)^{n/2} v_g \geqslant \left(\frac{n-1}{\delta}\right)^n \left(\frac{2\delta - n + 2}{n}\right)^{n/2} \int_{B^n/\Gamma_0} (-S_h)^{n/2} v_h.$$

The constant on the right-hand side is equal to 1 when $\delta = n-1$, and it is monotone decreasing from 1 to $(\frac{n-1}{n})^n(\frac{n+2}{n})^{n/2}$ in the range $n-1 \leqslant \delta \leqslant n$. On the other hand, it is known [2] that the hyperbolic metric h is a local minimizer of the functional

$$g \mapsto \int_{B^n/\Gamma_0} \left|S_g\right|^{n/2} v_g$$

defined on the space of all Riemannian metrics on B^n/Γ_0.

4. Cohomology and L^2-Harmonic Forms of Hyperbolic Manifolds

A complete orientable hyperbolic manifold can be written in the form H^m/Γ, where H^m is hyperbolic m-space and Γ is a torsion-free discrete subgroup of $\mathrm{M\ddot{o}b}_0(m-1)$, the group of orientation-preserving isometries of H^m. Throughout this section, we assume that Γ is convex-cocompact, so that the manifold-with-boundary $Y = [B^m \cup \Omega(\Gamma)]/\Gamma$ is compact. Via a standard embedding $\mathrm{M\ddot{o}b}_0(m-1) \subset \mathrm{M\ddot{o}b}_0(m+k)$, unique up to conjugation, we may regard Γ as a group of conformal automorphisms of S^{m+k}. The conformal compactification of the product manifold $H^m/\Gamma \times S^k$ is then identified with $\mathscr{X} = \Omega^{m+k}(\Gamma)/\Gamma$, where $\Omega^{m+k}(\Gamma)$ is the domain of discontinuity of Γ in S^{m+k}. By elementary algebraic topology, one can show

$$H^p(\mathscr{X}; \mathbb{R}) \cong H^p(Y; \mathbb{R}) \oplus H^{p-k}(Y, \partial Y; \mathbb{R}).$$

Note that the second cohomology group in the right-hand side is isomorphic to the compactly-supported de Rham cohomology group $H_c^{p-k}(H^m/\Gamma; \mathbb{R})$. On the other hand, as a consequence of the vanishing theorem proved in [18], we have:

(i) If $\delta + 1 < p < (m+k) - \delta - 1$, then $H^p(\mathscr{X}; \mathbb{R}) = 0$;

(ii) If δ is an integer, $\delta \leqslant \frac{(m+k)-2}{2}$ and $H^{\delta+1}(\mathscr{X}; \mathbb{R}) \neq 0$, then $\Lambda(\Gamma)$ is a round δ-sphere.

We now suppose that $p > \delta + 1$. Choosing $k = \max(2p - m, 0)$ so that the assumption of (i) is satisfied, we obtain $H^p(\mathscr{X}; \mathbb{R}) = 0$. This in turn implies $H^p(Y; \mathbb{R}) = H^{p-k}(Y, \partial Y; \mathbb{R}) = 0$ as well as $H^{m-p}(Y, \partial Y; \mathbb{R}) = H^{m-p+k}(Y; \mathbb{R}) = 0$ by the Poincaré duality. On the other hand, it follows from (ii) that if δ is an integer, $p = \delta + 1$ and either $H^p(Y; \mathbb{R})$ or $H^{p-k}(Y, \partial Y; \mathbb{R})$ (with k as above) do not vanish, then $\Lambda(\Gamma)$ is a round δ-sphere. We have proved

THEOREM 4.1. — *Let H^m/Γ be a complete orientable hyperbolic manifold with Γ convex-cocompact. If $p > \delta + 1$, then $H^p(Y; \mathbb{R}) = H^{m-p}(Y, \partial Y; \mathbb{R}) = 0$. If $\delta + 1 < p < m/2$, we also have that $H^{m-p}(Y; \mathbb{R}) = H^p(Y, \partial Y; \mathbb{R}) = 0$. If δ is an interger and $H^{\delta+1}(H^m/\Gamma; \mathbb{R}) \neq 0$ (or $H^{m-\delta-1}(H^m/\Gamma; \mathbb{R}) \neq 0$ if $\delta < m/2 - 1$), then $\Lambda(\Gamma)$ is a round δ-sphere.*

Let $\mathscr{H}^q(H^m/\Gamma; \mathbb{R})$ be the space of L^2-harmonic q-forms on H^m/Γ. A result of R. Mazzeo and R. Phillips [15] states that if Γ is convex-cocompact and $q < m/2$,

$$\mathscr{H}^q(H^m/\Gamma; \mathbb{R}) \cong H^q(Y, \partial Y; \mathbb{R}).$$

On the other hand, the star operator induces an isomorphism

$$\mathscr{H}^q(H^m/\Gamma; \mathbb{R}) \cong \mathscr{H}^{m-q}(H^m/\Gamma; \mathbb{R}),$$

which corresponds to the Poincaré duality

$$H^q(Y, \partial Y; \mathbb{R}) \cong H^{m-q}(Y; \mathbb{R}).$$

As an immediante consequence of Theorem 4.1, we obtain

COROLLARY 4.2. — *Let H^m/Γ be as in Theorem 4.1. If $p > \delta+1$ and $p \neq m/2$, $\mathscr{H}^p(H^m/\Gamma; \mathbb{R}) = \mathscr{H}^{m-p}(H^m/\Gamma; \mathbb{R})=0$. If δ is an interger, $\delta \neq m/2-1$ and $\mathscr{H}^{\delta+1}(H^m/\Gamma; \mathbb{R})\neq 0$, then $\Lambda(\Gamma)$ is a round δ-sphere.*

5. Cohomology of Flat Bundles and Its Application

In this section, we assume that Γ is not only torsion-free but orientation preserving; namely, Γ is a torsion-free discrete subgroup of the identity component $\mathrm{M\ddot{o}b}_0(n)$ of $\mathrm{M\ddot{o}b}(n)$. However, after an appropriate modification, most of what we will show in this section is valid without this assumption.

Let cd G (resp. hd G) be the cohomological (resp. homological) dimension of a group G, namely,

$$\mathrm{cd}\, G = \max\{k \mid H^k(G; \mathscr{S}) \neq 0 \text{ for some } \mathbb{Z}G\text{–module } \mathscr{S}\},$$
$$\mathrm{hd}\, G = \max\{k \mid H_k(G; \mathscr{S}) \neq 0 \text{ for some } \mathbb{Z}G\text{–module } \mathscr{S}\},$$

where $H^k(G;\mathscr{S})$ (resp. $H_k(G;\mathscr{S})$) is the k-th cohomology (resp. homology) group of G with coefficients in $\mathscr{S}$. If we take a Kleinian group $\Gamma \subset \mathrm{M\ddot{o}b}(n)$ as G, $H^*(\Gamma;\mathscr{S})$ (resp. $H_*(\Gamma;\mathscr{S})$) is isomorphic to $H^*(Y,\mathscr{S})$ (resp. $H_*(Y;\mathscr{S})$), where $Y = [B^{n+1} \cup \Omega(\Gamma)]/\Gamma$ and $\mathscr{S}$ is regarded as a local system on Y. If Γ is convex-cocompact, Γ is of so-called type FL, and $\mathrm{cd}\,\Gamma = \mathrm{hd}\,\Gamma$ holds (see [9, p. 204]). As was shown in [11, Proposition 4.13], the inequality $\mathrm{cd}\,\Gamma - 1 \leqslant \delta(\Gamma)$ holds for any convex-cocompact Kleinian group Γ. We will examine the equality case of this inequality. A naive conjecture is that convex-cocompact Kleinian groups satisfying the equality have round spheres as their limit sets. At least this is true for Γ isomorphic (as a group, not as a Kleinian group) to cocompact lattice in $\mathrm{M\ddot{o}b}(m-1)$ as we have mentioned in §3, and m turns out to be $\mathrm{cd}\,\Gamma$ in this case. Another way to state this is:

THEOREM 5.1. — *([8], [25], [6], [4], [11]) Let $\Gamma_0 \subset \mathrm{M\ddot{o}b}(m-1)$ be a cocompact lattice. Suppose that $\rho : \Gamma_0 \longrightarrow \mathrm{M\ddot{o}b}(n)$, $n \geqslant m-1$, is a faithful discrete representation and that $\Gamma = \rho(\Gamma_0)$ is convex-cocompact. If $\delta(\Gamma) + 1 = m (= \mathrm{cd}\,\Gamma = \mathrm{cd}\,\Gamma_0)$, then $\Lambda(\Gamma)$ is a round $\delta(\Gamma)$-sphere.*

It should be mentioned that in [25], [6], and [4], $\mathrm{M\ddot{o}b}(n)$ in the theorem above has been replaced by the isometry group of more general negatively curved manifolds.

Using Theorem 5.4 below, which is a generalization of [17, Theorem 5.2] to an arbitrary flat Hilbert space bundle, we can prove the following extension of Theorem 5.1 and [11, Theorem 5.2]

THEOREM 5.2. — *Let $\Gamma_0 \subset \mathrm{M\ddot{o}b}(m)$ be convex-cocompact with $\mathrm{cd}\,\Gamma_0 = m$. Suppose that $\rho : \Gamma_0 \longrightarrow \mathrm{M\ddot{o}b}(n)$, $n \geqslant m-1$, is a faithful discrete representation and that $\Gamma = \rho(\Gamma_0)$ is convex-cocompact. If $\delta(\Gamma) + 1 = m$, then $\Lambda(\Gamma)$ is a round $\delta(\Gamma)$-sphere.*

Remark. Since a cocompact lattice Γ_0 is a subgroup of $\mathrm{M\ddot{o}b}(\mathrm{cd}\,\Gamma_0 - 1)$ and $\mathrm{M\ddot{o}b}(\mathrm{cd}\,\Gamma_0 - 1) \subset \mathrm{M\ddot{o}b}(\mathrm{cd}\,\Gamma_0)$, Γ in Theorem 5.1 satisfies the assumption of Theorem 5.2. If Γ satisfies the assumption of [11, Theorem 5.2], then we can take Γ itself as Γ_0 in Theorem 5.2. Therefore Theorem 5.2 includes both Theorem 5.1 and [11, Theorem 5.2].

The first thing we have to recall is the relation between the cohomology of Γ and that of $X = \Omega(\Gamma)/\Gamma$. Note that, through the homomorphism $\pi_1(X) \longrightarrow \pi_1(Y)$ induced from the inclusion $X \longrightarrow Y$, the restriction of a local system on Y to X gives rise to a local system on X. Thus $\mathbb{Z}\Gamma$-module defines a local system not only on Y but on X.

LEMMA 5.3. — *Let $\Gamma \subset \mathrm{M\ddot{o}b}(n)$ be a convex-cocompact Kleinian group and $\mathscr{S}$ a $\mathbb{Z}\Gamma$-module.*

(a) *For $p < n - \mathrm{cd}\,\Gamma$, $H^p(X;\mathscr{S})$ is isomorphic to $H^p(\Gamma;\mathscr{S})$.*

(b) *For $p < n - \mathrm{cd}\,\Gamma$, $H^{n-p}(X;\mathscr{S})$ is isomorphic to $H_p(\Gamma;\mathscr{S})$.*

Proof.

(a) Since Y is a compact manifold-with-boundary, by the Poincaré-Lefschetz duality, we have $H^p(Y, X; \mathscr{S}) \cong H_{n+1-p}(Y; \mathscr{S}) \cong H_{n+1-p}(\Gamma; \mathscr{S})$. Since $\mathrm{hd}\,\Gamma = \mathrm{cd}\,\Gamma$, $H_q(\Gamma; \mathscr{S}) = 0$ for $q > \mathrm{cd}\,\Gamma$, hence $H^p(Y, X; \mathscr{S}) = 0$ for $p < n + 1 - \mathrm{cd}\,\Gamma$. Applying the cohomology exact sequence for the pair (Y, X), we obtain the desired result.

(b) Since $H^{n-p}(Y; \mathscr{S}) \cong H^{n-p}(\Gamma; \mathscr{S}) = 0$ for $n - p > \mathrm{cd}\,\Gamma$, by cohomology exact sequence for the pair (Y, X), $H^{n-p}(Y; \mathscr{S}) \cong H^{n-p+1}(Y, X; \mathscr{S})$ for $n - p > \mathrm{cd}\,\Gamma$. By the Poincaré-Lefschetz duality, $H^{n-p+1}(Y, X; \mathscr{S}) \cong H_p(Y; \mathscr{S}) \cong H_p(\Gamma, \mathscr{S})$. This proves (b). □

Take a Kleinian group $\Gamma \subset \mathrm{Möb}(n)$ and a unitary representation ρ of Γ. We denote by $\mathscr{H}_\rho$ the Hilbert space with Γ action via ρ. The group Γ acts on $\Omega(\Gamma) \times \mathscr{H}_\rho$ diagonally, and we have a Hilbert space bundle E_ρ over $X = \Omega(\Gamma)/\Gamma$, the quotient of $\Omega(\Gamma) \times \mathscr{H}_\rho$ by the diagonal action. There is a natural metric on the bundle E, and we also have a flat connection compatible with this metric. We call such a bundle a *flat Hilbert space bundle*. Since the connection is flat, the covariant differentiation D defined for E-valued p-forms satisfies $D^2 = 0$, and hence we have a cochain complex consists of E-valued p-forms with coboundary operator D. We denote by $H^*(X; E)$ the cohomology of this cochain complex.

On the other hand, since $\mathscr{H}_\rho$ is a $\mathbb{Z}\Gamma$-module, $\mathscr{H}_\rho$ defines a local system on X as we have explained above. We denote by the same symbol $\mathscr{H}_\rho$ the local system on X defined from $\mathscr{H}_\rho$. Since X is a manifold, the cohomology of this local system $\mathscr{H}_\rho$ agrees with Čech cohomology of the locally constant sheaf naturally defined by the local system $\mathscr{H}_\rho$. Moreover, by a standard argument (for example, imitate the argument in [5, §8]), we can prove that this Čech cohomology is isomorphic to $H^*(X; E)$. Thus, in particular, $H^*(X; \mathscr{H}_\rho)$ is isomorphic to $H^*(X; E)$.

As a generalization of the vanishing theorem in [17], we obtain the following vanishing result for $H^*(X; E)$. We will use the second part of this theorem to prove Theorem 5.2.

THEOREM 5.4. — *Let $\Gamma \subset \mathrm{Möb}(n)$, $n \geqslant 3$, be a Kleinian group such that $\Omega(\Gamma)/\Gamma$ is compact, and E a flat Hilbert space bundle over $X = \Omega(\Gamma)/\Gamma$. Denote by δ the critical exponent of Γ.*

(a) *Suppose $\delta < (n-2)/n$. Then, for integers p satisfying $\delta + 1 < p < n - \delta - 1$, $H^p(X, E) = 0$.*

(b) *Suppose δ is an integer, $\delta \leqslant (n-2)/2$, and either $H^{\delta+1}(X; E) \neq 0$ or $H^{n-\delta-1}(X; E) \neq 0$ holds. Then $\Lambda(\Gamma)$ is a round δ-sphere.*

Proof. The first thing we have to do is to construct a new cochain complex whose cohomology is isomorphic to $H^*(X; E)$.

Let $A^p(E)$ and D be the linear space consisting of smooth p-forms with values in E and covariant differentiation respectively. By definition, $H^*(X; E)$ is the cohomology of

the cochain complex $\{(A^*(E), D)\}$. The standard L^2-inner product on $A^p(E)$ is given by

$$(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle v_g, \quad \alpha, \beta \in A^p(E), (0 \leqslant p \leqslant n),$$

where $\langle \cdot, \cdot \rangle$ is the inner product on each fiber coming from the canonical metric g on $\Omega(\Gamma)/\Gamma$ and the inner product on $\mathscr{H}_\rho$, and v_g is the volume form of g. Let us consider another inner product given by

$$((\alpha, \beta)) = (\alpha, \beta) + (D\alpha, D\beta).$$

Denote by $W^p(E)$ the completion of $A^p(E)$ with respect to the norm $\| \cdot \|_W$ defined by means of this inner product. It is obvious that $D : A^p(E) \to A^{p+1}(E)$ is bounded with respect to the norm $\| \cdot \|_W$, and hence D can be uniquely extended to the bounded operator $\bar{D} : W^p(E) \to W^{p+1}(E)$. Clearly $\bar{D}^2 = 0$, and we obtain a cochain complex $\{(W^*(E), \bar{D})\}$. Let us denote the cohomology of this new cochain complex by $H^*(X; W)$. We are going to sketch a proof of $H^*(X; E) \cong H^*(X; W)$. Let U be an open subset of X diffeomorphic to $\mathbb{R}^n$. Then it is not so difficult to see that the Poincaré Lemma holds for $W^p(U)$, where $W^p(U)$ is the completion of the space of smooth p-forms on U with values in E which are bounded with respect to the norm $\| \cdot \|_W$. In other words, the following sequence is exact:

$$0 \longrightarrow \mathscr{H}_\rho \xrightarrow{\iota} W^0(U) \xrightarrow{\bar{D}} W^1(U) \xrightarrow{\bar{D}} W^2(U) \xrightarrow{\bar{D}} \cdots,$$

where $\mathscr{H}_\rho$ is regarded as the set of constant $\mathscr{H}_\rho$-valued functions on U and ι is the natural inclusion map. Then, by a slight modification of the proof of $H^*(X; E) \cong H^*(X; \mathscr{H}_\rho)$, one can prove $H^*(X; W) \cong H^*(X; \mathscr{H}_\rho)$. (To prove this one needs to assume X is compact.) This implies $H^*(X; E) \cong H^*(X; W)$.

Let us turn to the proof of (a). Note that $W^p(E)$ can be considered as the domain of the closure of D regarded as a densely defined operator $L_2^p(E) \longrightarrow L_2^{p+1}(E)$, where $L_2^p(E)$ is the completion of $A^p(E)$ with respect to the ordinary L^2-norm $\| \cdot \|_L = (\cdot, \cdot)^{1/2}$. Thus we may denote the closure of D by $\bar{D}$. Let us denote the adjoint of $D : L_2^p(E) \longrightarrow L_2^{p+1}(E)$ by D^*, and the (positive) Laplacian $DD^* + D^*D$ by Δ. Since E is flat, writing down the Weitzenböck formula

$$(\Delta\alpha, \alpha) = (\nabla\alpha, \nabla\alpha) - \int_X \mathscr{R}(\alpha, \alpha) v_g,$$

we see that the curvature term $\mathscr{R}$, which is a bilinear form on each fiber, is completely determined by the curvature of cotangent bundle and essentially the same as that in the Weitzenböck formula calculated in [17, §5]. Thus, under the assumption of (a), $\mathscr{R}(\alpha, \alpha) \leqslant -c\langle \alpha, \alpha \rangle$ at each point for some positive constant c as in the proof of [17, Theorem 5.2] Since

$$(\Delta\alpha, \alpha) = (\bar{D}\alpha, \bar{D}\alpha) + (D^*\alpha, D^*\alpha),$$

on the kernel of $\bar{D}$, we have

$$(D^*\alpha, D^*\alpha) \geqslant c \operatorname{Vol}(X, g)(\alpha, \alpha).$$

Therefore $D^* : \operatorname{Ker} \bar{D} \longrightarrow I$ has the bounded inverse, where $I = D^*(\operatorname{Ker} \bar{D})$ and it must be closed. Thus so does $\bar{D} = D^{**} : I \longrightarrow \operatorname{Ker} \bar{D}$. In other words, for any $\alpha \in \operatorname{Ker} \bar{D}$, there is $\beta \in I$ such that $\bar{D}(\beta) = \alpha$. Thus $H^p(X; E) = H^p(X; W) = 0$.

On the other hand, under the assumption of (b), we have $\mathscr{R}(\alpha, \alpha) \leqslant 0$ as in the proof of [17, Theorem 5.2]. Now assume $H^{\delta+1}(X; E) \neq 0$. If $\mathscr{R}(\cdot, \cdot)$ for $(\delta+1)$-forms is negative definite everywhere, then there must be a positive constant c such that $(D^*\alpha, D^*\alpha) \geqslant c \operatorname{Vol}(X, g)(\alpha, \alpha)$ holds for any $\alpha \in \operatorname{Ker} \bar{D}$. This implies $H^{\delta+1}(X; E) = H^{\delta+1}(X; W) = 0$ as we have seen above. A contradiction. Thus $\mathscr{R}(\cdot, \cdot)$ must have zero eigenvalue (as an endomorphism on each fiber) at some point. By the equation (20) of [17], this implies the tensor A defined in [17, §3] (see also §2 of this paper) must have $(n-\delta-1)$-dimensional zero eigenspace. The same is true if we assume $H^{n-\delta-1}(X; E) \neq 0$. By [17, Lemma 3.2 (b)], $\Lambda(\Gamma)$ must be contained in a round $(\delta+1)$-sphere S. Since $\delta + 1 < n$, Γ is convex-cocompact by [11, Lemma 2.2 (2)]. Therefore $\operatorname{cd} \Gamma \leqslant \delta + 1$ and we have $n-\delta-1 \leqslant n - \operatorname{cd} \Gamma$. Suppose $\operatorname{cd} \Gamma < \delta+1$. Then we have $\delta+1 \leqslant n-\delta-1 < n-\operatorname{cd} \Gamma$ (we are assuming $\delta \leqslant (n-2)/2$). Together with our assumption, Lemma 5.3 (a) implies either $H^{\delta+1}(\Gamma; \mathscr{H}_p) \neq 0$ or $H^{n-\delta-1}(\Gamma; \mathscr{H}_p) \neq 0$. On the other hand, we have $\operatorname{cd} \Gamma < \delta+1 \leqslant n-\delta-1$. A contradiction. Therefore $\operatorname{cd} \Gamma \geqslant \delta+1$, and hence $\operatorname{cd} \Gamma = \delta+1$.

Suppose the round $(\delta+1)$-sphere S above is the minimal round sphere containing $\Lambda(\Gamma)$. Then Γ leaves this S invariant. By [11, Theorem 5.2], $\Lambda(\Gamma)$ is a round δ-sphere. If S is not the minimal round sphere containing $\Lambda(\Gamma)$, then the minimal one S' has the dimension strictly smaller than $\delta+1$. On the other hand, since $\delta = d(\Gamma)$ by convex-cocompactness, the dimension of S' cannot be strictly smaller than δ. Hence the dimension of S' is δ. Since Γ leaves S' invariant, Γ leaves the hyperbolic $(\delta+1)$-plane whose boundary is S' invariant. Since $\operatorname{cd} \Gamma = \delta+1$, Γ must act cocompactly on this hyperbolic $(\delta+1)$-plane. Therefore $\Lambda(\Gamma)$ coincides with S'. This completes the proof.

Remark 1. Since our bundles in Theorem 5.4 may be infinite dimensional, we cannot expect that each cohomology class is represented by a harmonic form. Therefore the ordinary Bochner technique breaks down in our situation. However, as we have seen, Weitzenböck formula still tells us the vanishing of cohomology. The first author learned this method from P. Pansu's talk [20].

Remark 2. The important point in Theorem 5.4 is that the second part is valid for any flat Hilbert space bundle. In [17], the proof of the corresponding part of Theorem 5.2 was carried out by a rather standard method using the de Rham decomposition and the classification of product conformally flat manifolds. Though this method cannot be applied to our present situation, we have been able to prove the second part for any flat Hilbert space bundle using some remarkable properties of our canonical metric.

Let

$$\operatorname{cd}_{\mathscr{H}} \Gamma = \max\{k \mid H^k(\Gamma; \mathscr{H}_\rho) \neq 0 \text{ for some } \mathscr{H}_\rho\},$$
$$\operatorname{hd}_{\mathscr{H}} \Gamma = \max\{k \mid H_k(\Gamma; \mathscr{H}_\rho) \neq 0 \text{ for some } \mathscr{H}_\rho\}.$$

Clearly $\operatorname{cd} \Gamma \geqslant \operatorname{cd}_{\mathscr{H}} \Gamma$ and $\operatorname{hd} \Gamma \geqslant \operatorname{hd}_{\mathscr{H}} \Gamma$ hold, though it is not clear whether the equality

holds or not. For a convex-cocompact Kleinian group Γ, we also have $\mathrm{cd}\,\Gamma = \mathrm{hd}\,\Gamma \geqslant \mathrm{hd}_{\mathscr{H}}\,\Gamma$.

LEMMA 5.5. — *Let Γ be a convex-cocompact Kleinian group. If either $\delta(\Gamma)+1 = \mathrm{cd}_{\mathscr{H}}\,\Gamma$ or $\delta(\Gamma)+1 = \mathrm{hd}_{\mathscr{H}}\,\Gamma$ holds, then $\Lambda(\Gamma)$ is a round $\delta(\Gamma)$-sphere.*

Proof. Put $\delta = \delta(\Gamma)$. By extending the action of Γ to $S^{n'}$ for sufficiently large n' through the inclusion $\mathrm{Möb}(n) \longrightarrow \mathrm{Möb}(n')$ and replacing n with n' if necessarry, we may assume both $\delta+1 < n-\mathrm{cd}\,\Gamma$ and $\delta \leqslant (n-2)/2$. If $\mathrm{cd}_{\mathscr{H}}\,\Gamma = \delta+1$, then there is $\mathscr{H}_\rho$ such that $H^{\delta+1}(\Gamma;\mathscr{H}_\rho) \neq 0$. By Lemma 5.3 (a), $H^{\delta+1}(\Gamma;\mathscr{H}_\rho) \cong H^{\delta+1}(X;\mathscr{H}_\rho) \cong H^{\delta+1}(X;E) \neq 0$, where E is the flat Hilbert space bundle defined from $\mathscr{H}_\rho$. Similarly, if $\mathrm{hd}_{\mathscr{H}}\,\Gamma = \delta + 1$, by Lemma 5.3 (b), there is $\mathscr{H}'_\rho$ such that $H_{\delta+1}(\Gamma;\mathscr{H}'_\rho) \cong H^{n-\delta-1}(X;\mathscr{H}'_\rho) \neq 0$. Thus there is a flat Hilbert space bundle E' such that $H^{n-\delta-1}(X;E') \neq 0$. Now apply Theorem 5.4 (b). □

LEMMA 5.6. — *Let $\Gamma_0 \subset \mathrm{Möb}(m)$ be convex-cocompact with $\mathrm{cd}\,\Gamma_0 = m$. Then there exists a Hilbert space $\mathscr{H}_\rho$ with unitary action of Γ_0 such that $H_m(\Gamma_0;\mathscr{H}_\rho) \neq 0$. In particular, $\mathrm{hd}_{\mathscr{H}}\,\Gamma_0 = \mathrm{cd}\,\Gamma_0$.*

Proof. By [11, Proposition 4.6], $H^m(\Gamma_0, \mathbb{Z}\Gamma_0)$ is isomorphic to the 0th reduced homology $\tilde{H}_0(\Omega(\Gamma_0),\mathbb{Z})$ of $\Omega(\Gamma_0)$. Note that $H_0(\Omega(\Gamma_0),\mathbb{Z})$ is a free abelian group and its basis consists of connected components of $\Omega(\Gamma_0)$. Thus there is a natural Γ_0-action on $H_0(\Omega(\Gamma_0),\mathbb{Z})$ which permutes the basis. Therefore $H_0(\Omega(\Gamma_0),\mathbb{Z}) \otimes \mathbb{C}$ (or $H_0(\Omega(\Gamma_0),\mathbb{Z}) \otimes \mathbb{R}$) admits a Γ_0-invariant inner product. Let $\mathscr{H}$ be the completion of $H_0(\Omega(\Gamma_0),\mathbb{Z}) \otimes \mathbb{C}$ with respect to the norm given by this inner product. We denote by $\mathscr{H}_\rho$ this Hilbert space with unitary Γ_0-action coming from the action of Γ_0 on $H_0(\Omega(\Gamma_0),\mathbb{Z})$. Since $\tilde{H}_0(\Gamma_0,\mathbb{Z})$ is the kernel of the augmentation map, it is a Γ_0-invariant subgroup of $H_0(\Gamma_0,\mathbb{Z})$. Thus the natural inclusion turns out to be a Γ-equivariant monomorphism $\tilde{H}_0(\Gamma_0,\mathbb{Z}) \longrightarrow \mathscr{H}_\rho$. By the universal coefficient theorem (see [9, p. 204]), $H_m(\Gamma_0,\mathscr{H}_\rho) \neq 0$. □

Proof of Theorem 5.2. The assumption of Theorem 5.2 and Lemma 5.6 implies $\mathrm{hd}_{\mathscr{H}}\,\Gamma = \mathrm{hd}_{\mathscr{H}}\,\Gamma_0 = \delta + 1$. Then apply Lemma 5.5 to Γ. □

6. Metrics on Quasi-Balls

Let Ω be a domain in $\mathbb{R}^n$ with compact smooth boundary Σ. Following the idea which we used to construct the canonical metric, we define a Riemannian metric g_Ω on Ω by

$$g_\Omega = \left(\frac{1}{\omega_{n-1}}\int_\Sigma \varphi(\cdot,y)^{1-n}\,d\sigma(y)\right)^{\frac{2}{n-1}} g_0, \tag{6.1}$$

where ω_{n-1} is the volume of the unit $(n-1)$-sphere, g_0 denotes the Euclidean metric and σ is the volume element of Σ defined with respect to the induced metric. The constant

$1/\omega_{n-1}$ is put so that the metric coincide with the Poincaré metric when Ω is the unit ball. Let $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a Möbius transformation. Let $\Omega^* = \gamma^{-1}(\Omega)$, and let σ^* be the volume element of $\Sigma^* = \gamma^{-1}(\Sigma)$. By the same computation as was used to show the Γ-invariance of the canonical metric (see §2), we obtain

$$\gamma^* g_\Omega = g_{\Omega^*}. \tag{6.2}$$

In particular, g_Ω is invariant under the Möbius automorphism group of Ω, i.e. the group of Möbius transformations preserving Ω. When γ is allowed to map the point at infinity into Σ and thus Σ^* is noncompact, we may still define g_{Ω^*} by (6.1), and (6.2) continues to hold in this case.

We now consider special examples of quasi-balls, namely, intersections or unions of two intersecting round balls. Let B_1 and B_2 be such balls, and S_1 and S_2 their boundaries, respectively. We denote by α the angle, viewed from the interior of $B_1 \cap B_2$ (or $B_1 \cup B_2$), at which S_1 and S_2 meet. We pick two points p_1, p_2 from $S_1 \cap S_2$ and choose a Möbius transformation of $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ such that $\gamma(0) = p_1$ and $\gamma(\infty) = p_2$. Then $\Omega = \gamma^{-1}(B_1 \cap B_2)$ (or $\gamma^{-1}(B_1 \cup B_2)$) is a sector bounded by the union Σ of two half-hyperplanes Σ_1 and Σ_2 which meet with the angle α. We arrange them so that

$$\Sigma_1 = \{(x_1, \ldots, x_n) \mid x_{n-1} \geqslant 0, x_n = 0\}$$

and

$$\Sigma_2 = \{(x_1, \ldots, x_n) \mid x_{n-1} = r\cos\alpha, x_n = r\sin\alpha, r \geqslant 0\}.$$

Similarities $x \mapsto rx$, $r > 0$, and translations $x \mapsto x + v$, $v \in \mathbb{R}^{n-2} = \{x_{n-1} = x_n = 0\}$, form the identity component of the Möbius automorphism group of Ω, and its orbits are half-hyperplanes bounded by $\mathbb{R}^{n-2}$. Thus the set $\{x_\theta = (0, \ldots, 0, \cos\theta, \sin\theta) \mid 0 < \theta < \alpha\}$ represents the orbit space.

Now let

$$u(x) = \frac{1}{\omega_{n-1}} \int_\Sigma \varphi(x,y)^{1-n} d\sigma(y)$$

and

$$u_i(x) = \frac{1}{\omega_{n-1}} \int_{\Sigma_i} \varphi(x,y)^{1-n} d\sigma(y), \quad i = 1, 2,$$

and compute

$$\begin{aligned} u_1(x_\theta) &= \frac{2^{n-1}}{\omega_{n-1}} \int_{\Sigma_1} |x_\theta - y|^{2(1-n)} d\sigma(y) \\ &= \frac{2^{n-1}\omega_{n-3}}{\omega_{n-1}} \int_0^\infty \left(\int_0^\infty \left(r^2 + (t - \cos\theta)^2 + \sin^2\theta\right)^{1-n} r^{n-3} dr \right) dt. \end{aligned}$$

Since

$$\int_0^\infty \left(r^2 + a\right)^{1-n} r^{n-3} dr = a^{-n/2} \int_0^\infty \left(r^2 + 1\right)^{1-n} r^{n-3} dr (=: a^{-n/2} c_n),$$

we get

$$u_1(x_\theta) = \frac{2^{n-1}\omega_{n-3}c_n}{\omega_{n-1}} \int_0^\infty \left((t-\cos\theta)^2 + \sin^2\theta\right)^{-n/2} dt.$$

By change of variables, this is simplified into

$$\frac{2^{n-1}\omega_{n-3}c_n}{\omega_{n-1}} \sin^{1-n}\theta \int_\theta^\pi \sin^{n-2}\xi\, d\xi. \tag{6.3}$$

This expression makes sense even at $\theta = \pi$, since

$$\sin^{1-n}\theta \int_\theta^\pi \sin^{n-2}\xi\, d\xi$$

has a well-defined limit $1/(n-1)$ as $\theta \to \pi$. We may obtain the formula for $u_2(x_\theta)$ just by replacing θ by $\alpha - \theta$ in (6.3):

$$u_2(x_\theta) = \frac{2^{n-1}\omega_{n-3}c_n}{\omega_{n-1}} \sin^{1-n}(\alpha-\theta) \int_{\alpha-\theta}^\pi \sin^{n-2}\xi\, d\xi.$$

Hence

$$\begin{aligned} u(x_\theta) &= u_1(x_\theta) + u_2(x_\theta) \\ &= \frac{2^{n-1}\omega_{n-3}c_n}{\omega_{n-1}} \left(\sin^{1-n}\theta \int_\theta^\pi \sin^{n-2}\xi\, d\xi + \sin^{1-n}(\alpha-\theta) \int_{\alpha-\theta}^\pi \sin^{n-2}\xi\, d\xi\right). \end{aligned}$$

When $\alpha = \pi$, this becomes

$$u(x_\theta) = \frac{2^{n-1}\omega_{n-3}c_n}{\omega_{n-1}} \sin^{1-n}\theta \int_0^\pi \sin^{n-2}\xi\, d\xi,$$

and thus

$$g_\Omega = \left(\frac{2^{n-1}\omega_{n-3}c_n}{\omega_{n-1}} \int_0^\pi \sin^{n-2}\xi\, d\xi\right)^{\frac{2}{n-1}} \sin^{-2}\theta\, g_0$$

at x_θ. Since this metric coincides with the Poincaré metric $(1/{x_n}^2)\, g_0$, we must have

$$\frac{2^{n-1}\omega_{n-3}c_n}{\omega_{n-1}} = \left(\int_0^\pi \sin^{n-2}\xi\, d\xi\right)^{-1}.$$

Therefore

$$\begin{aligned} u(x_\theta) = &\left(\int_0^\pi \sin^{n-2}\xi\, d\xi\right)^{-1} \left(\sin^{1-n}\theta \int_\theta^\pi \sin^{n-2}\xi\, d\xi \right. \\ &\left. + \sin^{1-n}(\alpha-\theta) \int_{\alpha-\theta}^\pi \sin^{n-2}\xi\, d\xi\right). \end{aligned}$$

Since g_Ω is invariant under the similarities/translations as above, u satisfies $u(rx) = r^{1-n}u(x)$, $r > 0$, and $u(x+v) = u(x)$, $v \in \mathbb{R}^{n-2}$. Hence

$$u(x) = \left(\int_0^\pi \sin^{n-2}\xi\, d\xi\right)^{-1} \left((r\sin\theta)^{1-n}\int_\theta^\pi \sin^{n-2}\xi\, d\xi + (r\sin(\alpha-\theta))^{1-n}\int_{\alpha-\theta}^\pi \sin^{n-2}\xi\, d\xi\right),$$

where r and θ are such that $x_{n-1} = r\cos\theta$ and $x_n = r\sin\theta$.

When $n = 2$, this becomes

$$u(x) = \frac{1}{r\sin\theta}\left(1 - \frac{\theta}{\pi}\right) + \frac{1}{r\sin(\alpha-\theta)}\left(1 - \frac{\alpha-\theta}{\pi}\right) \quad \left(=: \frac{1}{r}\varphi(\theta)\right).$$

The Gaussian curvature of $g_\Omega = u^2 g_0$ is a function of θ alone, and it is given by the formula

$$K = \frac{\varphi'^2 - \varphi\varphi''}{\varphi^4}.$$

K is symmetric with respect to $\theta = \alpha/2$, negative everywhere and asymptotic to -1 as $\theta \to 0$. Computation using *Mathematica* indicates: $K \geqslant -1$ everywhere; if $\alpha < \pi$, K is monotone increasing in $(0, \alpha/2)$, and $K(\alpha/2) \nearrow c\ (\leqslant -1/2)$ as $\alpha \searrow 0$; if $\alpha > \pi$, there exists $\theta(\alpha) \in (0, \alpha/2)$ such that K is monotone increasing in $(0, \theta(\alpha))$ and monotone decreasing in $(\theta(\alpha), \alpha/2)$, and $\theta(\alpha) \searrow 0$, $K(\theta(\alpha)) \nearrow c\ (\leqslant 0)$ and $K(\alpha/2) \nearrow c'\ (\leqslant -4/5)$ as $\alpha \nearrow 2\pi$.

It is interesting to compare our metric (6.1) with the metric (of class $C^{1,1}$) constructed by W. Thurston (unpublished), B. Apanasov [1] and R. Kulkarni-U. Pinkall [13], which is also Möbius-invariant. For example, for the quasi-ball above with $n = 2$ and $\alpha > \pi$, the latter is given by $h_\Omega = v^2 g_0$ with

$$v(x) = \begin{cases} 1/(r\sin\theta), & 0 \leqslant \theta \leqslant \frac{\pi}{2}, \\ 1/r, & \frac{\pi}{2} \leqslant \theta \leqslant \alpha - \frac{\pi}{2}, \\ 1/(r\sin(\alpha-\theta)), & \alpha - \frac{\pi}{2} \leqslant \theta \leqslant \alpha. \end{cases}$$

The ratio u/v attains its minimum at $\theta = \alpha/2$, and hence

$$\frac{u}{v} \geqslant \frac{1}{\sin(\alpha/2)}\left(2 - \frac{\alpha}{\pi}\right)$$
$$\frac{2}{\pi} = 0.63\ldots \quad \text{as} \quad \alpha \nearrow 2\pi.$$

On the other hand, computer experiment indicates that $\sup u/v \nearrow c\ (\leqslant 2)$ as $\alpha \nearrow 2\pi$.

It will be also interesting to compare our metric with that constructed (for $n \geqslant 3$) by H. Leutwiler [14].

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Hiroyasu IZEKI
Mathematical Institute
Tôhoku University
SENDAI 980 - 8578, (Japan)
e-mail: izeki@math.tohoku.ac.jp

Shin NAYATANI
Graduate School of Mathematics
Nagoya University
NAGOYA 464 - 8602 (Japan)
e-mail: nayatani@math.nagoya-u.ac.jp