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ON MULTIPLY CONNECTED MESOSCOPIC SUPERCONDUCTING STRUCTURES

Jacob RUBINSTEIN and Michelle SCHATZMAN

Abstract

An introduction to models of superconductivity is given. The Ginzburg-Landau theory is applied to multiconnected mesoscopic superconductors, i.e. thin structures whose thickness is small with respect to the coherence length. From the mathematical point of view, this amounts to minimizing the Ginzburg-Landau functional over an open set $\mathcal{O}(\varepsilon)$ which is a fattening of an imbedded graph M in $\mathbf{R}^2$: away from the vertices of the graph, M is fattened so as to be of thickness 2ε; close to the junctions, in order to avoid all difficulties related to angular domains, the fattening is $O(\varepsilon)$. The unknowns in the full Ginzburg-Landau functional are the two-dimensional vector potential A and the complex order parameter u.

The model in the limit as the thickness tends to zero is a variational problem on M whose unknowns are a scalar function and n integers, n being the number of independent cycles of the graph, or alternately the dimension of the cohomology group of degree 1 of $\mathcal{O}(\varepsilon)$. We prove that a subsequence of minimizers of the approximating problem tends to a solution of the limit problem if we define an appropriate notion of convergence.

1. Introduction

In this article we describe how the study of a set of thin superconductiong network of strips and rings leads to a simple variational problem on an imbedded graph.

It turns out that this variational problem on a singular manifold is obtained as the limit of a more complicated problem on a thin manifold with boundary, which is an

Classification math. : 35Q40, 82D55, 35B25, 35J50.

appropriate fattening of the graph.

A similar problem has been treated in [45] for the case of a ring. From the physical point of view, this case is related to the Little–Parks experiment [36], [37] (which is one of the well known demonstrations of the physical effect of a vector potential in a region where its curl, the magnetic field, vanishes.)

The article [45] gave a justification of the equation used in [12] and [13] to predict previously unobserved behavior in the Little–Parks setup.

Parks conjectured in [44] that for a multiply connected mesoscopic structure, the order parameter could vanish at some points (in the one dimensional model) stopping the current from flowing in the corresponding branch. Bruydoncx *et al.* recently performed a set of experiments on multiconnected mesoscopic superconducting strips [14], and they found oscillations which differ qualitatively from those described by Little and Parks. The theory of these oscillations has not yet been formulated, but it is clear that there is much interest in mesoscopic superconducting structures.

The reason for the presence of this article in this volume is that the limiting process that we perform is reminiscent of works of Yves Colin de Verdière [25], Bruno Colbois and Colette Anné [4], [5], [6], [7]. The second author of this article realized in March 1997 that studying the limit of a Laplace operator on an almost singular manifold, for instance a smooth manifold with thin handles glued to it, or a smooth manifold with small holes, had been very active subjects, with few connections to physical applications.

On the other hand, we were aware of an extensive literature of applied mathematics around Ciarlet, Le Dret, Destuynder and other authors [19], [20], [18], [21], [27], [34] who systematically derived mechanical models of lower dimensional structures by passing to the limit on the thickness of structures of higher dimension, possibly with junctions between structures of different dimensionality. At the beginning of this systematic effort the thin structures themselves were flat. The theory of shells is the typical scientific subject where thin structures and geometry meet; much has been done there, and much remains to be done: see for instance [22], [23], [24] and their extensive bibliography, and also, with time or pseudo-time dependence, [8], [26].

It is a pleasure to thank here the organizers of the seminar, and in particular H. Pesce, to whose memory this volume is dedicated, for giving the opportunity to the second author to talk to an audience of geometers about problems of physical origin. The exercise has been quite unusual, and quite educative, at least for the speaker, and hopefully also for the audience of the seminar.

We are also grateful to Giles Richardson for discussions pertaining to this article, and a careful reading of an early version.

2. About models of superconductivity

For this section, we have intensively used [28] and [51] which are among the basic books of the field; moreover, the recent book [35] has proved very useful.

Superconductivity has been discovered in 1911 by Kamerlingh Onnes in Leiden, who observed that "the electrical resistance of various metals such as mercury, lead and tin, disappeared completely in a small temperature range at a critical temperature T_c" [33]. Another feature of superconductivity discovered by Meissner and Ochsenfeld [42] is that a magnetic field applied to a superconductor is expelled from it as it is cooled below T_c: this is the Meissner effect. Thus, the superconductor is a diamagnetic material: the induction B vanishes inside the sample.

The existence of such a reversible Meissner effect implies that the superconductivity will be destroyed by a critical magnetic field H_c. This thermodynamic critical field H_c has been found empirically to be quite well approximated by a parabolic law

$$H_c(T) \approx H_c(0)[1 - (T/T_c)^2].$$

A more detailed study shows that the field penetrates on a very short distance inside the sample. The characteristic length of the penetration of the magnetic field is denoted λ and is called the penetration length. Empirically, λ is found to be approximately described by

$$\lambda(T) \approx \lambda(0)[1 - (T/T_c)^4]^{-1/2}.$$

When observing the behavior of superconductors subjected to an increasing or a decreasing magnetic field at fixed temperature, one finds two different types of behaviors: certain superconductors remain diamagnetic when the applied field is increased up to $H_c(T)$, and become normal beyond H_c: they are called type I superconductors; on the other hand, when starting from a normal initial state and decreasing the magnetic field, this type of material becomes superconducting again at the nucleation field $H_{c_2}(T) < H_c(T)$, because it is difficult for type I superconductors to nucleate superconducting regions inside the sample.

The corresponding phase diagram is of the form given in fig. 1, at least for type I superconductors. This diagram applies to a long cylinder, with magnetic field applied parallel to the axis. In reality, the nucleation at superconductivity depends on the geometry, and in a different setup geometry dependent effects appear: in particular, for fields between $(1 - \eta)H_c(T)$ and $H_c(T)$, normal and superconducting regions coexist; η depends on the geometry of the sample.

For some other superconductors, the perfect diamagnetism holds only for $H \leqslant H_{c_1}(T) < H_c(T)$; beyond H_{c_1}, the magnetic field progressively penetrates in flux tubes, called vortices: magnetization progressively decreasing until the nucleation field H_{c_2} is reached, where superconductivity disappears. In such materials, called type II superconductors, it is very easy to nucleate superconducting regions, and for this reason, H_{c_2} can be much larger than the thermodynamic critical field H_c. This is a useful property in applications (e.g. superconducting magnets). Between H_{c_1} and H_{c_2}, the material can be considered as subdivided into superconducting regions, with vanishing resistance, and a network of flux tubes which can be considered as normal regions: this state is the

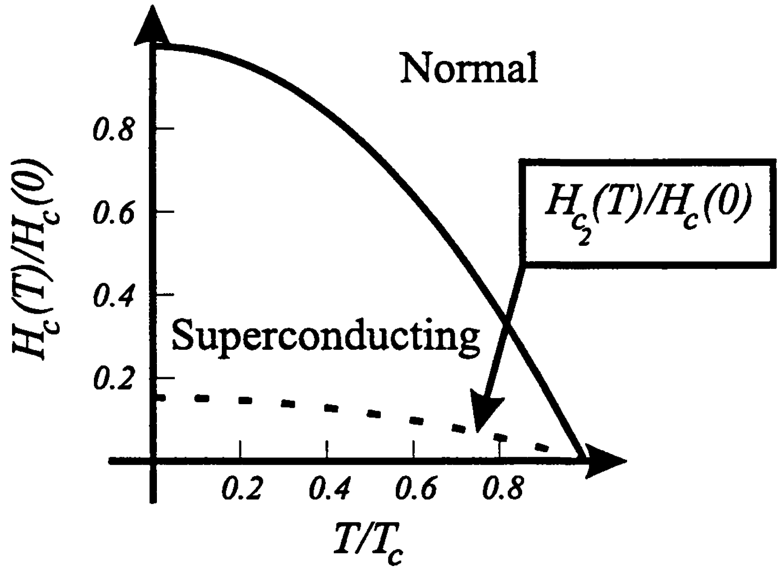

Figure 1: Phase transition diagram for type I superconductors.

mixed state, also known as the Shubnikov phase [48], [49]. Finally, at the boundary of the sample with an insulating material, Saint-James and de Gennes [47] have shown that it is possible to observe a "surface" supraconductivity up to $H_{c_3} \approx 1.695 H_{c_2}$. Hence the complicated phase diagram for a cylindrical type II superconductor, with axial applied field, in figure 2. At the H_c transition curve (dashed line), no transition is observed.

Once again, this phase diagram does not tell all the story: it depends on the geometry; between H_{c_2} and H_{c_3}, there are some very weak vortices in the bulk; between H_{c_1} and H_{c_2}, the vortices in the bulk may look like an Abrikosov lattice, ([1] and see also below); as the field nears H_{c_1}, the vortices are expected to be more and more distant. Crossing the H_{c_1} line either by increasing or by decreasing the magnetic field may involve superheating or supercooling phenomena.

For a mathematical study of this phenomenon, see for instance [17] and [10].

In both cases, the phase transitions are altered when the material is "dirty", i.e. contains impurities or defects; in particular, for type II superconducting materials, the vortices can be pinned at the defects, which causes some very important hysteresis phenomena.

The first theory of superconductivity is the London theory [38], [39], [40], [41] that accounted for the expulsion of the magnetic fields and the quantization of the magnetic fluxoids.

The Ginzburg-Landau (GL) theory of superconductivity was proposed in 1950 [29]; Ginzburg and Landau introduced a complex pseudowave function u as an order

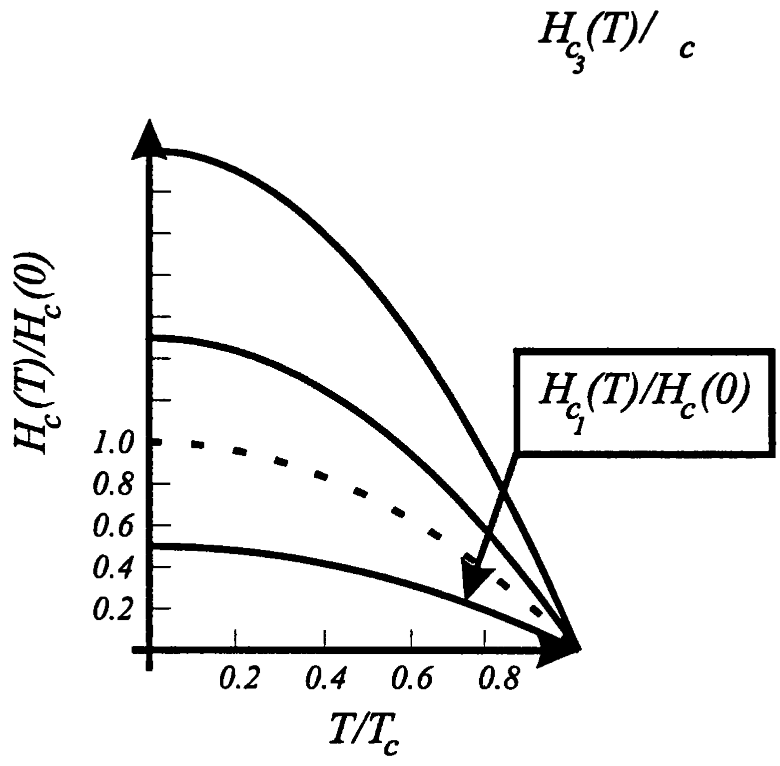

Figure 2: Phase transition diagram for type II superconductors.

parameter; u describes the superconducting electrons, and is such that at x the density of superconducting electrons is $|u(x)|^2$.

Ginzburg and Landau wrote down an empirical functional, the GL energy, which they postulated took the form

$$F_{n_0} + \int_{\mathbf{R}^3} \frac{|\nabla \times A - H_0|^2}{8\pi}\, dx + \int_\Omega \left(\frac{\hbar}{4m} \left| \left(\nabla - \frac{2ie}{\hbar c} A \right) u \right|^2 + a|u|^2 + \frac{b}{2}|u|^4 \right) dx. \quad (2.1)$$

Here F_{n_0} is the free energy of the normal state, Ω is the domain occupied by the superconductor, H_0 is the applied magnetic field, A is the magnetic vector potential, e and m are respectively the electron's charge and mass, c is the velocity of light, $\hbar = h/2\pi$ is Planck's constant; $a < 0$ and $b > 0$ are two constants depending on the material and temperature. It is assumed that a is proportional to $T - T_c$. Finally, the applied field H_0 must be given.

The coherence and penetration lengths are given by

$$\xi = \frac{\hbar}{2\sqrt{m|a|}} \text{ and } \lambda = \left(\frac{bmc^2}{8\pi|a|e^2} \right)^{1/2} \quad (2.2)$$

and the scales of spatial variation of u and A are respectively ξ and λ. It should be observed that λ and ξ diverge as $(T_c - T)^{-1/2}$ for T close to T_c. It turns out that the GL parameter $\kappa = \lambda/\xi$ is approximately independent of temperature.

In 1957, a microscopic theory based on first principles was proposed by Bardeen, Cooper and Schrieffer [9]; the main idea of this theory is that superconducting electrons are bound in pairs; the electron pairs follow statistics similar to the Bose-Einstein statistics instead of the Fermi statistics, that single electrons follow. At very low temperatures and magnetic fields, the electron pair states are energetically favored over the single electrons states.

Gorkov showed in 1959 [30], [31], [32] that the BCS theory provides a microscopic foundation for the GL theory, for T sufficiently close to T_c. This interpretation related the coherence length ξ to the distance between the members of a pair of coupled BCS electrons.

However, it is important to see that the BCS theory is not usable except over microscopic states and there remains a wide field of application of the GL theory (which is valid over a surprisingly large temperature range).

If κ is large instead of small, Abrikosov predicted in 1957 [1], [2], [3] that the energy associated to a domain wall between the normal and the superconducting states is negative; the break point between the two regimes is $\kappa = 1/\sqrt{2}$. For materials with $\kappa > 1/\sqrt{2}$, he found that the superconductor sample would be subdivided into domains of microscopic scale ξ; hence there would be a mixed state where the flux penetrates in an array of flux tubes; within each cell of the array there is a vortex of supercurrent concentrating the flux toward the vortex center. The core of the vortex center is almost normal and $|u|$ vanishes at the center of the tube.

This short history cannot be concluded without mentioning two other facts: the first one is that the discovery of the first high temperature superconductor with a critical temperature of 35 K [11] has challenged theoreticians and experimentators for the last eleven years. Since that time, the largest T_c attained is about 130 K for mixed oxydes of thallium, baryum, calcium and copper; the second is that there does not yet exist a good theory of time dependent superconductivity on which physicists definitely agree.

3. A GL model on multiconnected thin strips

The graph M is an imbedded finite planar graph; it has a set of edges $\mathcal{E}$ identified with curves in $\mathbf{R}^2$, and a set of vertices $\mathcal{V}$. Each edge is numbered by $j \in \{1, \dots, |\mathcal{E}|\}$ and is parameterized by a mapping ψ_j of class C^2 and of rank one from an interval $[a_j, b_j]$ to $\mathbf{R}^2$; without loss of generality, the parameter is the arc length. Loops are not forbidden. We ask that the arcs intersect one another only at the vertices of the imbedded graph.

For each vertex $v \in \mathcal{V}$, define

$$J(v) = \{(j, a_j, +1) : \psi_j(a_j) = v\} \cup \{(j, b_j, -1) : \psi_j(b_j) = v\}. \tag{3.3}$$

If ζ belongs to $J(v)$, its components are denoted $(\zeta[1], \zeta[2], \zeta[3])$.

The last condition we impose on M is a transversality condition: for every $v \in \mathcal{V}$, and for every distinct elements ζ and η of $J(v)$,

$$\zeta[3]\psi'_{\zeta[1]}(\zeta[2]) \neq \eta[3]\psi'_{\eta[1]}(\eta[2]). \tag{3.4}$$

Thus, two edges cannot leave a vertex along the same tangent and in the same direction.

Let ε be a strictly positive number, and let I be a subinterval of $[a_j, b_j]$ and let ρ be the rotation by an angle of $\pi/2$; we define the following set

$$S_j(\varepsilon, I) = \{\psi_j(\theta) + t\rho\psi'_j(\theta) : \theta \in I, |t| \leqslant \varepsilon\}.$$

If I equals (a_j, b_j), we use the simpler notation $S_j(\varepsilon) = S_j(\varepsilon, (a_j, b_j))$ and we say that $S_j(\varepsilon)$ is the strip of width 2ε centered in the j-th arc of M; if I is one of the intervals $(a_j, (a_j + b_j)/2)$ or $((a_j + b_j)/2, b_j))$, we say that $S_j(\varepsilon, I)$ is a half-strip.

We denote $\hat{\mathcal{O}}(\varepsilon)$ the union of the strips $S_j(\varepsilon)$.

For all $\eta > 0$, let $V(\eta)$ the union of the balls of radius η around the vertices of M. Our assumptions, and in particular (3.4), imply that there exist strictly positive numbers ε_0 and R such that for all $\varepsilon \in (0, \varepsilon_0)$, any two distinct half strips intersect only inside $V(R\varepsilon)$.

The "lace" $\mathcal{O}(\varepsilon)$ is defined for $\varepsilon \in (0, \varepsilon_0)$: it contains $\hat{\mathcal{O}}(\varepsilon)$ and is contained in $\hat{\mathcal{O}}(\varepsilon) \cup V(R\varepsilon)$; the boundary of the intersection $\mathcal{O}(\varepsilon) \cup V(R\varepsilon)$ is smoothed in such a fashion that $\mathcal{O}(\varepsilon)$ is an open set with boundary of class C^2. The details of the smoothing do not affect the asymptotic limit.

We will consider branches of $\mathcal{O}(\varepsilon)$, i.e. the sets $S(\varepsilon, [a_j + \sqrt{\varepsilon}, b_j - \sqrt{\varepsilon}])$; the connected components of the complement of these branches in $\mathcal{O}(\varepsilon)$ are called junctions, and there is one around each vertex $v \in \mathcal{V}$.

We rescale the Ginzburg-Landau functional defined in (2.1; we scale the magnetic flux with respect to the quantum fluxoid, and we scale the distances in terms of the scale of the domain, and we obtain:

$$\begin{aligned} G_\varepsilon(u, A) &= \int_{\mathcal{O}(\varepsilon)} \left(\mu[-|u|^2 + |u|^4/2] + |(i\nabla + A)u|^2\right) dx \\ &\quad + \kappa^2 \int_{\mathbf{R}^2} |\nabla \times A - H_e|^2 \, dx. \end{aligned} \tag{3.5}$$

Notice that the parameter μ is temperature dependent. The applied magnetic field H_e is a given vector field in the direction orthogonal to the plane, and μ and κ are positive parameters. The set $\mathbf{R}^2 \setminus \mathcal{O}(\varepsilon)$ without its unbounded component is denoted $\mathcal{F}(\varepsilon)$. We observe that $\mathcal{F}(\varepsilon)$ consists of finitely many components $F_l(\varepsilon)$. As ε tends to 0, the boundary of $F_l(\varepsilon)$ tends to a cycle of the graph M; this set of cycles is denoted $\mathcal{L}$, the number of components of $\mathcal{F}$ is equal to $|\mathcal{L}|$, which is also the number of independent cycles of M. The flux of H_e through F_l is denoted by $\Phi_{l,\varepsilon}$.

Since the network is thin, the induced current has a negligible effect on the magnetic field. More precisely, the following estimate holds

THEOREM 3.1. — *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of G_ε; write $A_\varepsilon = A_i + A_e$, where $\nabla \times A_e = H_e$. Then*

1. *$\nabla \times A_i$ vanishes in the unbounded component of $\mathbf{R}^2 \setminus \mathcal{O}(\varepsilon)$ and on every component $F_l(\varepsilon)$ of $\mathcal{F}(\varepsilon)$ it is equal to a constant of size $O(\varepsilon)$.*
2. *$|\nabla \times A_i|_{L^2(\mathcal{O}(\varepsilon))} = O(\varepsilon^{3/2})$.*
3. *$|A_i|_{L^\infty(\mathcal{O}(\varepsilon))} \leqslant O(\kappa^{-2})\varepsilon\sqrt{\ln(1/\varepsilon)}$.*

The proof is essentially the same as in Lemmas 4 and 5 of [45].

To study the limit of the minimizers $(u_\varepsilon, A_\varepsilon)$, we have to study the limit of the bilinear energy form on $H^1(\mathcal{O}(\varepsilon)) \times H^1(\mathcal{O}(\varepsilon))$ given by

$$E_\varepsilon(u, \hat{u}) = \int_{\mathcal{O}(\varepsilon)} \nabla u \cdot \nabla \hat{u} \, dx. \tag{3.6}$$

The (positive) Laplacian on $\mathcal{O}(\varepsilon)$ is $\mathrm{Lap}_\varepsilon = -\Delta$ with Neumann boundary conditions.

Similarly, if f is a continuous function from M to $\mathbf{R}$, such that $f \circ \psi_j$ is in $H^1(a_j, b_j)$, we say that f belongs to $H^1(M)$, and we denote $f \circ \psi_j = f_j$. The energy bilinear form on $H^1(M) \times H^1(M)$ is given in local coordinates by

$$E(f,\hat{f}) = \sum_{1 \leqslant j \leqslant |\mathcal{E}|} \int_{a_j}^{b_j} f'_j \hat{f}'_j \, d\theta, \tag{3.7}$$

while the L^2 scalar product is

$$(f,\hat{f}) = \sum_{1 \leqslant j \leqslant |\mathcal{E}|} \int_{a_j}^{b_j} f_j \hat{f}_j \, d\theta. \tag{3.8}$$

Thus, Lap_M is the operator defined by

$$(\mathrm{Lap}_M f, \hat{f}) = E(f,\hat{f}), \quad \forall \hat{f} \in H^1(M). \tag{3.9}$$

An element f of the domain of Lap_M verifies $f_j \in H^2(a_j, b_j)$ for all j; moreover, integration by parts yields the following Kirchhoff-like transmission condition at all the vertices $v \in \mathcal{V}$:

$$\sum_{\zeta \in J(v)} \zeta[3] \, f'_{\zeta[1]}(\zeta[2]) = 0. \tag{3.10}$$

We relate the energy on $\mathcal{O}(\varepsilon)$ and the energy on M via the following construction: there exists a mapping P_1 from $H^1(M)$ to $H^1(\mathcal{O}(\varepsilon))$, which acts almost as an extension and a mappping P_2 from $H^1(\mathcal{O}(\varepsilon))$ to $H^1(M)$, which acts almost as a projection.

Roughly speaking, P_1 extends functions on each edge as constants in the direction normal to the edge, while P_2 replaces functions in $H^1(\mathcal{O}(\varepsilon))$ by their normal averages. The construction is appropriately modified in the junctions. It is similar in principle to the construction of [50], but the structure of P_i and the estimates near the singular set $\mathcal{V}$ are very delicate [46]. The same techniques work for a fattened graph imbedded in $\mathbf{R}^N$. A similar result for integer lattice graphs was independently proved in [16].

Related results can be found in [25], which obtains estimates close to ours in order to construct metrics on a Riemannian surface for which the multiplicity of the first eigenvalue of the Laplace-Beltrami operator can be arbitrarily large.

We are able to prove estimates which substantiate this intuitive description:

LEMMA 3.2. — *There exists a constant C such that for all sufficiently small ε, for all f in $H^1(M)$ and all $u \in H^1(\mathcal{O}(\varepsilon))$ the following inequalities hold:*

$$\begin{aligned} 2\varepsilon(1+C\varepsilon)|f|^2_{L^2(M)} + C\varepsilon E(f,f) &\geqslant |P_1 f|^2_{L^2(\mathcal{O}(\varepsilon))} \\ &\geqslant 2\varepsilon(1-C\varepsilon)|f|^2_{L^2(M)}, \end{aligned} \tag{3.11}$$

$$2\varepsilon(1+C\varepsilon))E(f,f) \geqslant E_\varepsilon(P_1 f, P_1 f), \tag{3.12}$$

$$E(P_2u, P_2u) \leqslant \frac{1 + C\sqrt{\varepsilon}}{2\varepsilon} E_\varepsilon(u, u), \tag{3.13}$$

$$E(P_2u, P_2u) \geqslant \frac{1}{2\varepsilon}\left((1 - C\sqrt{\varepsilon}) \int_{\mathcal{O}(\varepsilon)} |u|^2\, dy - C\sqrt{\varepsilon} \int_{\mathcal{O}(\varepsilon)} |\operatorname{grad} u|^2\, dy\right), \tag{3.14}$$

A consequence of Lemma 3.2 is that the p-th eigenvalue (counted with its multiplicity) of Lap_ε converges to the p-th eigenvalue of Lap_M (counted with its multiplicity) as ε tends to 0.

We introduce now the functional

$$G_0(f) = \sum_{1 \leqslant j \leqslant |\mathcal{E}|} \int_{a_j}^{b_j} \left(|f_j' - iA_j f_j|^2 + \mu[-|f_j|^2 + |f_j|^4/2]\right) d\theta \tag{3.15}$$

where A_j is the tangential component of $A_e \circ \psi_j$.

Using Theorem 3.1 and estimates analogous to those of Lemma 3.2 we get

THEOREM 3.3. — *Let $(u_\varepsilon, A_\varepsilon)$ be the sequence of minimizers of $G_\varepsilon(\cdot,\cdot)$. Then every subsequence of $w_\varepsilon = P_2(u_\varepsilon)$ converges strongly in $H^1(M)$ to a minimizer f of G_0.*

The local current density for the minimizer of G_ε is given by

$$I_\varepsilon = \Im(\overline{u}_\varepsilon(\nabla - iA_\varepsilon)u_\varepsilon). \tag{3.16}$$

Consider the j-th branch of $\mathcal{O}(\varepsilon)$. Let Γ be a curve in this branch connecting one boundary of $\mathcal{O}(\varepsilon)$ to another. The current flux through Γ is defined as $\mathscr{C}_{j,\varepsilon} = \int_\Gamma I_\varepsilon \cdot \nu$, where ν is the normal to Γ. It is constant along the branch.

The Euler-Lagrange equation for G_0 is

$$-\left(\frac{d}{d\theta} - iA_j\right)^2 f_j + \mu(|f_j|^2 - 1)f_j = 0, \quad \theta \in (a_j, b_j), \quad j \in \{1, \dots, |\mathcal{E}|\}, \tag{3.17}$$

together with the Kirchhoff conditions for all $v \in \mathcal{V}$

$$\sum_{\zeta \in J(v)} \zeta[3]\left(f_{\zeta[1]}'(\zeta[2]) - iA_{\zeta[1]}(\zeta[2])\, f_{\zeta[1]}(\zeta[2])\right) = 0. \tag{3.18}$$

The current I_j in the j-th edge is defined by the constant

$$I_j = \Im(\overline{f}_j(f_j' - iA_j f_j)). \tag{3.19}$$

Since the currents are the main physical observable quantity, the following theorem is of much practical importance.

THEOREM 3.4. — *The following holds*

$$\lim_{\varepsilon\to 0} \mathscr{C}_{j,\varepsilon}/2\varepsilon = I_j, \quad \forall\, j \in \{1,\dots,|\mathscr{E}|\}. \tag{3.20}$$

The minimization of G_0 concerns a complex valued function f on M and the data are given by the auxiliary function A_e. It is therefore remarkable that it can be reduced to a minimization problem involving a *scalar* function y on M and a finite number of integers; the data are now a finite set of real numbers. For this purpose we introduce the vector k with components

$$k_l = 2\pi N_l - \Phi_l, \quad l = 1,\dots,|\mathscr{L}| \tag{3.21}$$

where the N_l's are integers, and $\Phi_l = \lim_{\varepsilon\to 0}\Phi_{l,\varepsilon}$, and the matrix Λ is defined by

$$\Lambda = \operatorname{diag}(\Lambda_1,\dots,\Lambda_{|\mathscr{L}|}), \quad \Lambda_l = \sum_{j\in\partial F_l(0)} \int_{a_j}^{b_j} |f_j|^{-2}. \tag{3.22}$$

The edge-node incidence matrix is an $|\mathscr{E}| \times |\mathscr{V}|$ matrix denoted $\mathscr{A}$. The component $\mathscr{A}_{ev}$ is equal to 1 if v is the origin of the edge e, but not its end, to -1 if it is its end, but not its origin, and 0 otherwise. The edge-loop incidence matrix $\mathscr{B}$ is an $|\mathscr{E}| \times |\mathscr{L}|$ matrix whose entry $\mathscr{B}_{el}$ is equal to 1 if the edge e belongs to the loop l and has the same orientation as l, -1 if e belongs to l and has the opposite orientation to l, and 0 otherwise [15].

With these definitions, the integration of the phase equation derived from (3.17) and (3.18) leads to the following linear system for $I = (I_1,\dots,I_{|\mathscr{E}|})$:

$$\mathscr{B}^T \Lambda I = k, \quad \mathscr{A}^T I = 0 \tag{3.23}$$

Generalizing classical graph theory arguments, we can prove

THEOREM 3.5. — *The system (3.23) possesses a unique solution $I(k)$ that satisfies the identity $I(k)^T \Lambda I(k) = k^T(\mathscr{B}^T\Lambda\mathscr{B})^{-1}k$.*

Moreover, minimizing the energy G_0 is equivalent to minimizing

$$H_0(y,k) = \sum_{1\leqslant j\leqslant|\mathscr{E}|} \int_{a_j}^{b_j} (y_j'(\theta)^2 + \mu[-|y_j(\theta)|^2 + |y_j(\theta)|^4/2])\,d\theta + k^T(\mathscr{B}^T\Lambda\mathscr{B})^{-1}k. \tag{3.24}$$

Here $y = |f|$ and k is a vector taking discrete values (3.21). The number $|\mathscr{L}|$ of unknown integers is the dimension of the cohomology group of degree 1 of $\mathscr{O}(\varepsilon)$; it is also the number of independent cycles of M.

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