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ON THE LOCAL DIFFERENTIAL GEOMETRY OF COMPLETE INTERSECTIONS

Joseph M. LANDSBERG

A common scenario in differential geometry is that one looks for manifolds satisfying a local geometric condition (e.g. curvature pinching) subject to a global constraint (e.g. compactness). There may be many solutions to the local problem, but when one encodes the global condition into the local geometry (usually via integration by parts), the further restrictions on the local geometry severely cut down the space of solutions. Examples of this phenomena occur in a wide range of problems (e.g. isoparametric submanifolds [C], curvature pinching [R1], and spectral geometry [R2]).

This pattern also occurs in algebraic geometry. For example, in my work on varieties with degenerate secant varieties, [L3], there is a local PDE that has a large space of solutions, but when one imposes the global condition that the variety is smooth, there are no solutions in small codimension and in the critical codimension there is rigidity, in fact homogeneity of solutions.

In what follows I will discuss the local differential geometry of varieties that are complete intersections (or perhaps I should say non-complete intersections, since the non-complete intersections are the pathological objects) and some restrictions on solutions to the partial differential equations for non-complete intersections that result from requiring the variety to be smooth and of small codimension. In this sense, the work follows a familiar pattern. However, the bulk of what I will talk about will be the local differential geometry, since previously there was not a good picture of what complete intersections "look like" geometrically in comparison to non-complete intersections.

Although what follows is certainly motivated by differential geometry, all objects can be defined algebraically and much carries over to fields of arbitrary characteristics.

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Outline

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I. *What is a complete intersection?*

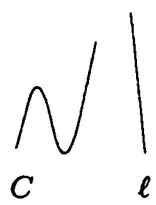
Classical algebraic geometry may roughly be described as the study of varieties X^n in complex projective space $\mathbb{CP}^{n+a}$. Here a *variety* is the zero locus of a collection of homogeneous polynomials satisfying some additional properties. By *geometry*, one means the properties of X that are invariant under linear changes in coordinates on $\mathbb{C}^{n+a+1}$. For example the *degree* of X, i.e. the number of points of intersection of X with a general $\mathbb{P}^a \subset \mathbb{P}^{n+a}$, is a geometric property. An example of an additional property I will require of varieties is that they be *irreducible*, so for example the union of two curves in the plane would not be allowed, one would have to study each curve seperately.

Example. — In $\mathbb{CP}^3$, let Q_1, Q_2 be quadric hypersurfaces. Consider $X = Q_1 \cap Q_2$ (the common zero locus of two degree 2 homogeneous polynomials). X is an algebraic set of dimension one and degree four. Usually, X is a curve of degree four:

but not always. For example, let $(x^1, \ldots, x^4)$ be linear coordinates on $\mathbb{C}^4$, and let

$$Q_1 = x^1x^4 - x^2x^3$$
$$Q_2 = (x^2)^2 - x^1x^3$$

Then X is a curve of degree three plus a line



where $C = \{[s^3, s^2t, st^2, t^3], [s,t] \in \mathbb{P}^1\}, \ell = \{[0,0,u,v], [u,v] \in \mathbb{P}^1\}$.

Since we want our varieties to be irreducible, we need to get rid of one of these components and we may as well get rid of the line by intersecting X with $Q_3 = (x^3)^2 - x^2x^4$ to be left with the cubic curve.

Note that since degree $C = 3$, C cannot be the intersection of two hypersurfaces because degree is multiplicative.

The idea is that the cubic curve is pathology, which motivates the following:

DEFINITION. — *Let $X^n \subset \mathbb{CP}^{n+a} = \mathbb{P}V$ be a variety. X is called a complete intersection if it is describable as the transverse intersection of a hypersurfaces. Equivalently, letting $I_X \subset S^\bullet V^*$ denote the ideal of X, X is a complete intersection if I_X can be generated by a elements.*

The above example shows that the degrees of complete intersections are easier to compute than the degrees of non-complete intersections. In fact, the same is true for a number of topological properties. For example, much of the cohomology of complete intersections is inherited from that of projective space. More precisely, if X is a complete intersection, the restriction map $H^i(\mathbb{P}^{n+a}, \mathbb{Z}) \to H^i(X^n, \mathbb{Z})$ is an isomorphism for $i < n$, and injective for $i = n$ (see e.g. [H]).

In the early 1970's, Barth and Larsen proved theorems to the effect that smooth varieties of small codimension have many of the nice cohomological properties of complete intersections. For example, if X is any smooth variety, the restriction map $H^i(\mathbb{P}^{n+a}, \mathbb{Z}) \to H^i(X^n, \mathbb{Z})$ is an isomorphism for $i \leq n - a$ (see e.g. [H]). Their results motivated:

HARTSHORNE'S CONJECTURE ON COMPLETE INTERSECTIONS ([H], 1974). — *Let $X^n \subset \mathbb{CP}^{n+a}$ be a smooth variety. If $a < n/2$, then X is a complete intersection.*

Hartshorne's conjecture has been a big motivating problem in algebraic geometry for the past twenty years. Although it does not appear to be close to being solved, alot of interesting mathematics has come out of it (e.g. the study of vector bundles on projective space).

Two types of progress have been made on the conjecture to my knowledge. The first type consists of theorems that add some additional hypotheses, e.g. that the degree of X is small with respect to the codimension of X. The best progress along these lines is due to Ran [R] in codimension two.

When Hartshorne made the conjecture on complete intersections, he also conjectured a sort of first approximation to it. The second type of progress towards the conjecture was that this first approximation was proved by Zak ([Z], 2.3). Zak's work is the starting place for mine.

At first glance, it appears that determining whether or not a variety is a complete intersection is not the type of issue one would want to study with a differential-geometric perspective. First of all, the problem is global. For example, by the implicit function theorem, all smooth varieties are local complete intersections (i.e. locally cut out by the right number of equations). Second, even algebraic geometers do not seem to have much of a geometric picture of what a complete intersection "looks like" (although as mentioned above, there is a good cohomological picture).

Before describing the differential geometry of complete intersections, just to frame the talk, here is a result that follows from this perspective:

THEOREM ([L1], 6.24). — *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety with ideal I_X generated by quadrics. Let $b = \dim X_{sing}$. (Set $b = -1$ if X is smooth.) If $a < \frac{n-(b+1)+3}{3}$, then X is a complete intersection.*

II. *Two principles*

In differential geometry one wants to work locally, to take derivatives at a point and extract geometric information. For this project we will need:

- A way to recognize whether or not X is a complete intersection from local differential geometry.

- A way to recognize if X is not too singular from local differential geometry.

- To utilize the fact that the ambient space is projective space and that it has a special topology.

To get an idea of how to determine whether or not X is a complete intersection locally, go back to the pictures

complete intersection   not a complete intersection

These pictures motivate the first principle:

• *If X is not a complete intersection, then X "bends less" than expected.*

Here the expectation will be based on some additional knowledge of the variety, e.g. the degrees of hypersurfaces containing it. In our example, a non-complete intersection cut out by quadrics "bends less" than a complete intersection of quadrics would. (The above picture is only psychologically correct because the type of bending we will be studying only occurs for varieties of dimension greater than one.) I will give a precise version of (•) in part VI.

Next we need a principle to account for smoothness. To obtain this, it is useful to recall the origins of projective space (see e.g. [K]). During the Italian Rennaisannce, the architect Alberti realized that in order to give proper perspective to a painting, parallel lines should meet at infinity. In fact we may almost define projective space as the space where linear spaces always intersect in (at least) the expected dimensions.

This linear intersection property has consequences for the local differential geometry. For example, consider the following two surfaces in affine space

hyperbola cylinder

Note that both the hyperbola and the cylinder are defined by a quadratic equation, and both are ruled by lines. They both can be completed to projective varieties. When one completes the hyperbola, one obtains a smooth surface, but completing the cylinder, one gets a singular cone. The philosophy is that the reason the cylinder becomes singular is that as one travels along a ruling, the embedded tangent space is constant, and this forces the rulings crash into each other at infinity, creating the singularity. Contrast this with the hyperbola where the embedded tangent space rotates as one travels along a ruling with the result that a singularity is avoided at infinity. This picture motivates the second principle:

•• *In order for X to be smooth, it must "bend enough".*

(••) will be made precise shortly, first we need some definitions.

III. *Fundamental forms*

Recall that in Euclidean geometry, to measure how a submanifold is bending, i.e. infinitesimally moving away from its embedded tangent space to first order, one uses the *second fundamental form.* There is a similar object in projective differential geometry. The most näive definition of the second fundamental form in metric geometry is: fix a point $x \in X$, choose local coordinates (x^j, x^μ), $1 \le i,j,k \le n$, $n+1 \le \mu,\nu \le n+a$, centered at x such that $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}$ is an orthonormal basis of T_xX and $\frac{\partial}{\partial x^n},\dots,\frac{\partial}{\partial x^{n+a}}$ is an orthonormal basis of N_xX. Writing X locally as a graph, we get $x^\mu = q^\mu_{ij}x^ix^j + O(|x|^3)$ and the second fundamental form of X at x is

$$II_x = q^\mu_{ij}dx^i \circ dx^j \otimes \frac{\partial}{\partial x^\mu} \in S^2T^*_xX \otimes N_xX.$$

The same definition works in the projective setting, except that the notion of orthonormal doesn't apply and the normal space is a quotient space. As a result, the group action on II_x is larger and II contains less information. Essentially one no longer can measure the magnitude of bending, only its existence. A more geometric definition of II is as follows:

Fixing $x \in X \subset \mathbb{P}V = \mathbb{CP}^{n+a}$ determines a flag of V: $\hat{x} \subset \hat{T}_xX \subset V$ where $\hat{x}$ denotes the line in V corresponding to x, and $\hat{T} = \hat{T}_xX$ is the deprojectivization of $\tilde{T}_xX$, the embedded tangent (projective) space to X at x. In this notation, the (intrinsic) tangent space to X at x is $T_xX = \hat{x}^* \otimes (\hat{T}/\hat{x})$ and the normal space is $N_xX = T_x\mathbb{P}V/T_xX = \hat{x}^* \otimes (V/\hat{T})$.

Let $\mathcal{O}_X(-1)$ denote the line bundle whose fiber at x is $\hat{x}$, let $\mathcal{O}_X(1)$ denote the dual bundle and if E is a vector bundle, let $E(k) = E \otimes \mathcal{O}_X(1)^{\otimes k}$.

There is a natural map associated to the smooth points X_{sm} of a variety $X \subset \mathbb{P}V$, the *Gauss map*

$$\begin{aligned} \gamma : X_{sm} &\to G(n+1, V) \\ x &\mapsto \hat{T}_x \end{aligned}$$

where $G(n+1,V)$ denotes the Grassmannian of $(n+1)$-planes through the origin in V.

Fixing $x \in X$, the derivative of the Gauss map at x is

$$\gamma_* : T_xX \to T_{\hat{T}_x}G(n+1,V) = \hat{T}^* \otimes (V/\hat{T}) = \hat{T}^* \otimes N(-1).$$

γ_* is such that the kernel of the endomorphism $\gamma_*(v) : \hat{T} \to N(-1)$ contains $\hat{x} \subset \hat{T}$ for all $v \in T$. Thus γ_* factors to a map

$$\gamma'_* : T \to (\hat{T}/\hat{x})^* \otimes N(-1) = T^* \otimes N.$$

Furthermore, essentially because the Gauss map is already the derivative of a map and mixed partials commute, γ'_* is symmetric. $\gamma'_* \in S^2T^* \otimes N$ is the projective second fundamental form II_x.

Another definition of II_x used in Riemannian geometry is via covariant derivatives. In the projective setting we don't have a connection on TX or $T\mathbb{P}V$, but we do have an equivalence class of connections. So if we work on a larger space over X that takes into account this ambiguity, e.g. the frame bundle, we can make such a definition "upstairs". A definition of II_x using this method is given in [L2].

I will write II to denote the map $S^2T \to N$ and II^* to denote the adjoint map $N^* \to S^2T^*$.

Geometrically, a line in N^*_xX determines a tangent hyperplane H, and the associated quadratic form Q_H tells how X is moving away from H at x to first order. Considering $\mathbb{P}N^*_xX$ as the space of hyperplanes tangent to X at x, $\mathbb{P}(\ker II^*)$ is the space of hyperplanes tangent at x that X is not moving away from to first order.

The *base locus* of II is defined to be the tangent directions corresponding to vectors v such that $II(v, v) = 0$. Geometrically, the directions in the base locus are those tangent directions for which X is not moving away from its embedded tangent space to first order. The *singular locus* of II is defined to be the tangent directions corresponding to vectors v such that $II(v, w) = 0 \, \forall \, w \in T_xX$. Geometrically, directions in the singular locus are those in the base locus for which the tangent space is not rotating to first order either.

For example, directions of the rulings of the hyperbola are in the base locus, but not the singular locus, while directions of the rulings of the cylinder are in the singular locus.

The higher fundamental forms of X are defined similarly to II. For example, the third fundamental form $III^* : \ker II^* \to S^3T^*$ measures how X is moving away form the hyperplanes in $\mathbb{P}(\ker II^*)$ to second order. To define III using Taylor series, let $q^{\mu}_{ijk}x^ix^jx^k$ denote the third order terms in the expansion described above, then $III = q^{\mu}_{ijk}dx^i \circ dx^j \circ dx^k \otimes (\frac{\partial}{\partial x^{\mu}} \bmod II(S^2T))$. Definitions of the higher fundamental forms via Gauss maps are given in [L1] and definitions via "covariant derivatives" are given in [L2].

Placing closed conditions on II is imposing systems of partial differential equations on X. For example, in Euclidean geometry, trace $II = 0$ is the PDE for minimal submanifolds. In the projective setting, the notion of trace does not make sense, and our systems of PDE will be more subtle. (In order to have geometric meaning, any PDE must be invariant under the group acting on II.)

At this point you may be asking: Why not just fix a metric? After all, there is a natural Kähler metric on the ambient space! The answer is that the properties we are looking at are invariant under a larger group than the isometries of projective space with the Fubini-Study metric. Two varieties that may look very different (locally) from a metric perspective may be projectively equivalent. A more compelling reason is that I will describe systems of PDE that exactly characterize varieties with certain geometric properties. These systems are invariant under the group of projective motions. Were we working in the metric setting, we would have to deal with classes of PDE's instead of a single system. I will describe the PDE's later.

I'll close this section by stating a precise version of (••):

Rank restriction theorem, special case ([L1],4.14). — *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety. Let $b = \dim(X_{sing})$. (Set $b = -1$ if X is smooth.) Let $x \in X$ be a general point.*

1. For any $\alpha \in N_x^*X$,

$$\dim(\text{Singloc } II^*(\alpha)) \leq 2(a-1) + (b+1).$$

2. For generic $\alpha \in N_x^*X$,

$$\dim(\text{Singloc } II^*(\alpha)) \leq a - 1 + (b+1).$$

The rank restriction theorem can be though of as an analogue of "Bochner" type formulae in Riemannian geometry in that global geometric information governs local bending. It plays a role in the proofs as follows: Thinking of geometric conditions on varieties as PDE's and varieties satisfying the conditions as solutions to initial value problems; the effect of the rank restriction theorem is to rule out solutions obtained by starting with pathological initial data.

IV. *A local characterization of complete intersections*

As above let $V = \mathbb{C}^{n+a+1}$ and let $X^n \subset \mathbb{P}V = \mathbb{CP}^{n+a}$ be a variety of dimension n. Let X_{sm} denote the smooth points of X. Let $I_X \subset S^\bullet V^*$ denote the ideal of X and let $I_{X,d} = I_d = S^d V^* \cap I_X$ denote the d-th graded piece of I_X.

In a moment I'll describe a local characterization of complete intersections, but first here is a special case:

Warm up Proposition [L2],1.1. — *Let $X \subset \mathbb{P}V$ be a variety such that $I_X = (I_d)$ (i.e. I_X is generated by I_d) and $I_{d-1} = (0)$. Then the following are equivalent:*

1. X is a complete intersection.

2. Every hypersurface of degree d containing X is smooth at all $x \in X_{sm}$.

3. Let $x \in X_{sm}$. Every hypersurface of degree d containing X is smooth at x.

Note that if X is not a hypersurface, there will always be some $Z \in I_d$ that are singular, the proposition says that the singularities must occur away from the smooth points of X.

The reader may wish to check the proposition with our two examples of curves in $\mathbb{P}^3$, finding the quadric that is singular at each point of the cubic curve and that the singular quadrics in the ideal of a smooth quartic curve are not singlar at any points of the quartic.

I will now describe the general case, first we need a definition:

DEFINITION [L'v]. — *Let $X \subset \mathbb{P}V$ be a variety. Let $P \in I_d$ and let $Z = Z_P \subset \mathbb{P}V$ be the corresponding hypersurface. We will say Z trivially contains X if $P = \ell^1 P_1 + \cdots \ell^m P_m$ with $P_1, \ldots, P_m \in I_{d-1}$ and $\ell^1, \ldots, \ell^m \in V^*$, and otherwise that Z essentially contains X.*

PROPOSITION [L2],1.6. — *Let $X \subset \mathbb{P}V$ be a variety. The following are equivalent:*

1. X is a complete intersection.

2. Every hypersurface essentially containing X is smooth at all $x \in X_{sm}$.

3. Let $x \in X_{sm}$. Every hypersurface essentially containing X is smooth at x.

In addition to working locally, I work one degree at a time:

DEFINITION [L2], 1.8. — *Fix a point $x \in X_{sm}$. We will say X has no excess equations in degree d at x, if every hypersurface of degree d essentially containing X is smooth at x. (Note that X is a complete intersection if and only if X has no excess equations in degree d at x for all d).*

Knowing I_d is not a local property, but we can approximate I_d at a point $x \in X$ using local information. Namely, for each natural number k, we can consider the hypersurfaces of degree d *osculating to order k at x*. Informally, a hypersurface Z osculates to order k at x if when one chooses local coordinates around x and writes Z as a graph and restricts the Taylor series of Z to X, that no terms of order less than $k + 1$ appear. If there is some number k_d such that all the hypersurfaces of degree d osculating to order k_d are smooth at x, then of course X has no excess equations in degree d at x. To find reasonable candidates for the k_d's we need to explore some basics about osculating hypersurfaces.

V. *Osculating hypersurfaces*

Recall that If $x \in X$ is a smooth point, then there is always an $(a-1)$-dimensional space of hyperplanes (degree one hypersurfaces) tangent (osculating to order one) at x. So there is no geometry of osculating hyperplanes until one takes two derivatives, i.e. $k_1 \geq 2$. This fact generalizes to:

PROPOSITION [L2],3.16. — *Let $X^n \subseteq \mathbb{P}V = \mathbb{CP}^{n+a}$ be a variety and let $x \in X_{sm}$. For all $p \leq d$,*

$$dim \left\{ \begin{array}{c} \text{(not necessarily irreducible) hypersurfaces} \\ \text{of degree } d \text{ osculating to order } p \text{ at } x \end{array} \right\}$$
$$= \binom{(n+a+1)+(d-1)}{d} - \left\{ 1 + n + \binom{n+1}{2} + \cdots + \binom{n+p-1}{p} \right\}.$$

So there is no geometry of osculating hypersurfaces of degree d until one takes $d+1$ derivatives, i.e. $k_d \geq d+1$. For $k > d$, the dimension of the space of hypersurfaces of degree $d$osculating to order k depends on the geometry of X. Here a new phenomena occurs due to the presence of hypersurfaces that are singular at x. There are always singular hypersurfaces osculating to order $2d-1$ at x. In fact one has:

PROPOSITION [L2],3.17. — *Let $X^n \subseteq \mathbb{P}V = \mathbb{CP}^{n+a}$ be any variety, and $x \in X$, any smooth point.*

$$dim \left\{ \begin{array}{c} \text{(not necessarily irreducible) hypersurfaces of} \\ \text{degree } d \text{ osculating to order } 2d-1 \text{ at } x \end{array} \right\} \geq \binom{a+d-1}{d} - 1.$$

I should explain where ([L2], 3.16, 3.17) come from. The space of hyperplanes osculating to order k at x is the kernel of the k-th fundamental form at x. To study higher degree hypersurfaces, re-embedd X via the Veronese. Given $X \subset \mathbb{P}V$, consider $v_d(X) \subset \mathbb{P}S^dV$, the d-th Veronese re-embedding of X. ($v_d(X)$ is the restriction of the Veronese mapping $v_d : \mathbb{P}V \to \mathbb{P}S^dV$, $[x] \mapsto [x^d]$ to X.) The space of hypersurfaces osculating to order k at x is the kernel of the k-th fundamental form of $v_d(X)$ at x, and these fundamental forms have a-priori properties as explained in [L2].

It turns out that the fundamental forms of $v_d(X)$ are computable in terms of subtle differential invariants of X, the first of which essentially measures the infinitesmal variation of the second fundamental form. (If we were to reduce to a metric connection, this invariant would reduce to the covariant derivative of the second fundamental form.)

Returning to osculating hyperplanes, the dimension of the space of hyperplanes osculating to order two at x depends on the geometry of X. If codim$X$ $< \binom{n+1}{2}$, then one expects that there will be no hyperplanes osculating to order two at general points. An immediate corollary of the rank restriction theorem is the following:

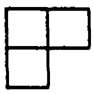
THEOREM. — *Let $X^n \subset \mathbb{CP}^{n+a}$ be a variety with $a < \frac{n-(b+1)}{2} + 1$ (where $b = dimX_{sing}$). Let $x \in X$ be a general point. If a hyperplane H osculates to order two at x, then $X \subset H$.*

In contrast, *every* surface in $\mathbb{P}^6$ has at least one hyperplane osculating to order two at every point. There is also a class of smooth surfaces in $\mathbb{P}^5$, the *Legendrian surfaces*, which have the property that at every point there is at least one hyperplane osculating to order two. (This class includes the ruled surfaces.)

VI. *PDE*

There are differential relations that guarantee a $X \subset \mathbb{P}V$ has no excess equations in degree d at some general $x \in X$. They are rather complicated genericity conditions on the differential invariants of X of order up to d.

I will explain the precise differential relations in the $d = 2$ case. As a $Gl(T)$ module, $S^2T^*\otimes T^*$ decomposes into two irreducible pieces, $S^3T^*\oplus S^{(21)}T^*$ where the second component can be thought of as the kernel of the symmetrization map. It comes from the Young diagram



(hence the notation).

Proposition. — *Let $x \in X$ be a general point, X will have no excess equations in degree two at x if*

$$(II^*(N^*)\otimes T^*) \cap S^3T^* = 0$$
$$\textit{and}\ (II^*(N^*)\otimes T^*) \cap S^{(21)}T^* = 0.$$

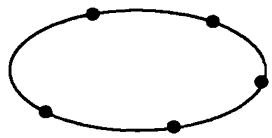
This proposition makes the principle ($\bullet$) precise in the $d = 2$ case and is the key local observation in proving ([L2], 6.24).

The first condition in the proposition is that the prolongation of II is empty. The second condition is that the system of quadrics generated by II has no linear syzygies.

VII. *More PDE*

([L2], 6.24) is not entirely satisfying as it gives no way if telling if I_X is generated by I_2. Fortunately, this can also be determined from the local differential geometry.

Recall that in $\mathbb{CP}^2$, five points determine a conic curve, so if you are handed a curve in $\mathbb{P}^2$ and want to know if it is a conic, it is sufficient to pick six general points on the curve and if they all lie on a conic, then your curve is that conic.



Monge realized that instead of picking six points, one could just as well pick one point and take five derivates, and he derived the *Monge equation*, a fifth order ODE characterizing smooth conic curves in the plane.

When $a < \frac{n-(b+1)}{3} + 1$, I have derived a fifth order PDE system characterizing (necessarily complete) intersections of quadrics which I call the *generalized Monge equation* ([L2], 4.17).

If one assumes certain reasonable genericity conditions on the differential invariants of order up to d, then there are PDE systems of order $2d+1$ that characterize complete intersections whose ideals are generated in degree $\leq d$. ([L2] 3.23).

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