Séminaire de Théorie spectrale et géométrie

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Séminaire de Théorie spectrale et géométrie, tome 13 (1994-1995), p. 123-133
<http://www.numdam.org/item?id=TSG_1994-1995__13__123_0>

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GRENOBLE
1994–1995 (123–133)

GENERIC RESULT FOR THE EXISTENCE OF A FREE SEMI-GROUP

Pierre-Alain CHERIX

Abstract

The main result of this note is the following: for a finitely presented group $\Gamma = \langle X : R \rangle$, the semi-group generated by X is generically free (in the sense of Gromov). And so we get the generic value of the spectral radius of h_X, the transition operator associated with the simple random walk on the directed Cayley graph of Γ : $r(h_X) = \frac{1}{\sqrt{\#X}}$.

1. Introduction

Let Γ be a finitely generated group. Fix a finite, not necessarily symmetric generating subset X, and let $S = X \cup X^{-1}$ be the symmetrization of X. With X and S are classically associated the usual Cayley graph $G(\Gamma, S)$, but also the Cayley digraph (or directed graph) $G(\Gamma, X)$; in the latter the set of vertices is Γ and, for any $\gamma \in \Gamma$ and $s \in X$, an oriented edge is drawn from γ to γs.

We consider the normalized adjacency operators , or transition operators, h_X and h_S; these are operators of norm at most 1 on $l^2(\Gamma)$, defined by:

$$\begin{aligned}(h_X\xi)(x) &= \frac{1}{\#X}\sum_{s\in X}\xi(xs)\\ (h_S\xi)(x) &= \frac{1}{\#S}\sum_{s\in S}\xi(xs) \qquad (\xi \in l^2(\Gamma), x \in \Gamma).\end{aligned}$$

We denote by $\#E$ the number of elements in the set E. The motivation for this paper came from the following result due to de la Harpe, Robertson and Valette [8] which says that

The author was supported by grant 20-40.405.94 of the Swiss National Fund for Scientific Research.

THEOREM 1.1. — *Assume* $\#X \geq 2$. *Set* $\sigma(X) = \limsup_{k\to\infty} \| h_X^k \|_2^{1/k}$, *where* h_X *is now viewed as the normalized characteristic function of* X *and* h_X^k *denotes the* k^{th} *convolution power of* h_X. *Then*

$$\frac{1}{\sqrt{\#X}} \leq \sigma(X) \leq r(h_X)$$

with $\frac{1}{\sqrt{\#X}} = \sigma(X)$ *if and only if* X *generates a free semi-group, and* $\sigma(X) = r(h_X)$ *if either* X *is symmetric or* Γ *is hyperbolic in the sense of Gromov (but not in general).*

In a joint paper with A. Valette [4], we looked at some consequences of such kind of results (relating group theory and harmonic analysis) for one-relator groups. In particular, we got the following statistical result. For presentations $\Gamma = \langle X : r\rangle$ with a fixed number of generators $\#X$ and one relation r, the ratio

$$\frac{\#\{\text{presentation } r \text{ with } r(h_X) = (\#X)^{-1/2} \text{ and } |r| = N\}}{\#\{\text{presentation } r \text{ with } |r| = N\}}$$

tends (exponentially fast) to 1 when N tends to $+\infty$. This means that "most" presentations $\Gamma = \langle X : r\rangle$ give $r(h_X) = \frac{1}{\sqrt{\#X}}$ (which implies in particular that the semi-group generated by X in Γ is free). This is exactly the sense of genericity introduced by Gromov ([6], 0.2(A)), and studied further by Champetier [2].

The main tool in the proof of the preceding result is small cancellation theory, which is frequent with one-relator groups. Unfortunately, small cancellation is not frequent in the general case of finitely presented group.

The main result of this note is :

THEOREM 1.2. — *For finite presentations,* $\langle X, R\rangle$, *the property* $\rho(h_X) = \frac{1}{\sqrt{\#X}}$ *is generic in the sense of Gromov.*

I thank C. Champetier and A. Valette for many useful discutions and for proof reading the article.

2. Some definitions and notations

For r a word in $\mathbb{F}_X$ (the free group generated by X), we will denote by $|r|$ its word length. It is always possible to write r as an alternating product of words with positive exponants (i.e. $r = \omega_1^{\pm1}\omega_2^{\mp1}\cdots\omega_n^{\pm1}$, where the ω_i's are positive words in X). We denote by $n_+(r)$ (resp. $n_-(r)$) the number of generators appearing in r with a positive exponent $+1$. (resp. with a negative exponent -1).

If r is beginning by a positive word ($r = \omega_1^{+1}\omega_2^{-1}\cdots\omega_n^{\pm1}$), then we get

- $n_+(r) = \sum_i |\omega_{2i-1}|$
- $n_-(r) = \sum_i |\omega_{2i}|$
- $n_+(r) + n_-(r) = |r|$

When r begins by a negative word, we juste interchange the odd and even summations in the preceding formulas.

DEFINITION 2.1. — *For a fixed $\epsilon > 0$, a word $r \in \mathbb{F}_X$ is ϵ-balanced if the decomposition of r in an alternating product of positive words ($r = \omega_1^{\pm 1}\omega_2^{\mp 1}\cdots\omega_n^{\pm n}$) has the following property: for all i, ω_i is such that $|\omega_i| < \epsilon|r|$.*

This implies in particular, that the number of changes of sign is greater or equal to $1/\epsilon$.

We say that a presentation $\langle X, R\rangle$ is ϵ-balanced if every r in R^* is ϵ-balanced (where R^* is the set of all cyclic permutations of r or r^{-1} for all relations $r \in R$).

DEFINITION 2.2. — *A word $r \in \mathbb{F}_X$ has the property E_δ for $\delta > 0$, if for all subwords u of r of length $|u| \geq |r|/4$ we have,*

either $1 \leq \frac{n_+(u)}{n_-(u)} \leq 1 + \delta$

or $1 \leq \frac{n_-(u)}{n_+(u)} \leq 1 + \delta$.

DEFINITION 2.3. — *If P is a property of words in $\mathbb{F}_X$, we say that P is generic if,*

$$lim_{n\to\infty} \frac{\#\{r \in \mathbb{F}_X \mid r \text{ cyclically reduced}, |r| = n, r \text{ with } P\}}{\#\{r \in \mathbb{F}_X \mid \quad r \text{ cyclically reduced}, |r| = n\}} = 1.$$

Set $\#X = k$ and $\#R = n$, and denote by $Pr(k, m_1, \cdots, m_n)$ the set defined by

$$\{\langle X, R\rangle \mid \#X = k, R = \{r_1, \cdots, r_n\}, |r_i| = m_i, r_i \text{cyclically reduced}\}.$$

A property P of finitely presented groups is generic if

$$\lim_{min_{\{m_i\}}\to\infty} \frac{\#\{\langle X,R\rangle \in Pr(k,m_1,\cdots,m_n) \mid \langle X,R\rangle \text{ with } P\}}{\#Pr(k,m_1,\cdots,m_n)} = 1.$$

For a word $\omega \in \mathbb{F}_X$ representing the identity in $\Gamma = \langle X, R\rangle$, we recall that Δ is a Van Kampen diagram of ω, if Δ is a 2-complex for which the 1-skeleton is a planar graph, each edge of that graph being labelled by a element of X or X^{-1} such that when we read the labelling of every 2-cell of the complex, we get a word in R^* and such that the labelling of the border of the complex Δ is the word ω. For more details about Van Kampen diagram, see the appendix on small cancellation of [5] or [3].

We denote by $I(\Delta)$ (resp. $E(\Delta)$ and $\#(\Delta)$) the number of internal edges of Δ (resp. the number of external edges of Δ and the total number of edges of Δ).

DEFINITION 2.4. — *The combinatorial area of a diagram Δ is the number of 2-cells and we say that Δ is a reduced diagram of ω if it has the minimal combinatorial area among all diagrams representing ω.*

For every $\omega \in \mathbb{F}_X$ representing the identity in $\Gamma = \langle X, R\rangle$, the existence of such a reduced diagram of ω is proved in [3].

DEFINITION 2.5. — *A finite presentation $\langle X, R\rangle$ satisfies a θ-condition, if for a fixed $0 < \theta < 1$ and for all reduced diagrams Δ, we get $I(\Delta) < \theta(\#\Delta)$.*

In [10], Ol'shanskii proved that for every fixed $\theta > 0$, the property of satisfying a θ-condition is generic.

3. The proof of theorem 1.2

We begin with some lemmas.

LEMMA 3.1. — *For a fixed m_0 in $\mathbb{N}$, $m_0 \geq 3$, set*

$$\alpha(n) = \frac{1}{2^{nm_0}} \sum_{i=0}^{n} \binom{nm_0}{i}, \text{ and } \beta(n) = \frac{1}{2^n} \sum_{i=0}^{\lfloor n/m_0 \rfloor} \binom{n}{i}$$

(where $\lfloor x \rfloor$ is the integral part of the real number x). There exist constants A, $C > 0$, $C < 1$ depending on m_0 such that $\alpha(n) \leq A\, C^{m_0 n}$ for all n in $\mathbb{N}$ and C becomes smaller when m_0 decreases. Furthermore, if $n_0 \equiv 0 (mod\ \ m_0)$, then $\alpha(n_0/m_0) = \beta(n_0)$ and for all $i = 0, \cdots, m_0 - 2$:

$$\beta(n_0 + i) > \beta(n_0 + i + 1).$$

PROOF OF 3.1 We want to estimate $\alpha(n+1) - \alpha(n)$:

$$
\begin{aligned}
&\alpha(n+1)-\alpha(n)\\
&= \sum_{i=0}^{n+1}\frac{1}{2^{(n+1)m_0}}\binom{(n+1)m_0}{i}-\sum_{i=0}^{n}\frac{1}{2^{m_0 n}}\binom{m_0 n}{i}\\
&= \frac{1}{2^{n(m_0+1)}}\left[\binom{m_0 n}{n+1}-\sum_{l=0}^{m_0-2}\binom{m_0 n}{n-l}\left[\sum_{j=l+2}^{m_0}\binom{m_0}{j}\right]\right]\\
&= \frac{(m_0 n)!}{2^{n(m_0+1)}n!((m_0-1)n)!}\left\{\prod_{\mu=0}^{m_0-2}((m_0-1)n+\mu)-\right.\\
&\left.\sum_{l=0}^{m_0-2}\left(\left[\sum_{j=l+2}^{m_0}\binom{m_0}{j}\right]\prod_{\xi_l=0}^{l}(n-\xi_l+1)\prod_{\nu_l=l+1}^{m_0-2}((m_0-1)n+\nu_l)\right)\right\}/\\
&\left\{(n+1)\prod_{\beta=1}^{m_0-2}[(m_0-1)n+\beta]\right\}
\end{aligned}
$$

The dominating terms of the fraction are of the same degree equal to $m_0 - 1$. So that fraction tends to a negative constant when $n \to \infty$.

By Stirling's formula, we see that there exists a positive constant $\tilde{A}$ such that

$$|\alpha(n+1)-\alpha(n)| \le \tilde{A}C^{m_0 n}, \text{ where } C = \frac{m_0}{2(m_0-1)^{(m_0-1)/m_0}} < 1.$$

By the central-limit theorem, there exists a constant $A > 0$ such that $|\alpha(n)| \le AC^{m_0 n}$.

It is easy to see that C is decreasing when m_0 is increasing.

To finish the proof, we just need to see by direct computation that for all $n_0 \equiv 0(\mathit{mod}\, m_0)$ and all i between 0 and $m_0 - 2$, $\beta(n_0 + i) > \beta(n_0 + i + 1)$. □

LEMMA 3.2. — *Let $|X| \ge 2$ and $\delta \ge 8$ be fixed, the property E_δ is generic.*

PROOF OF 3.2 We denote $B(n) = \#\{r \in \mathbb{F}_X \mid |r| = n, r \text{ cyclically reduced}\}$, $A(n) = \#\{r \in B(n) \mid |r| = n,\ r \text{ with } E_\delta\}$ and $C(n) = B(n) - A(n)$. $C(n)$ can be described as

$$C(n) = \#\{r \in B(n) \mid \exists u \text{ subword of } r, \text{ with } |u| \ge |r|/4 \text{ and either } \frac{n_+(u)}{n_-(u)} > 1+\delta, \text{ or } \frac{n_-(u)}{n_+(u)} > 1+\delta\}$$

(1) We want to estimate the number of u of length l such that $\frac{n_+(u)}{n_-(u)} > 1 + \delta$ Denote $h = n_+(u)$, we have $n_-(u) = l - h$. $\frac{h}{l-h} > 1 + \delta$ is equivalent to $h > \frac{1+\delta}{2+\delta}l$. So

we can make exactly $\begin{pmatrix} l \\ h \end{pmatrix} k^h k^{l-h}$ words of length less or equal to l out of the alphabet $X \cup X^{-1}$ having exactly h letters with an exponent $+1$. Thus

$$\#\{u \in \mathbb{F}_X \mid |u| < l,\ u \text{ reduced}, \frac{n_+(u)}{n_-(u)} > 1+\delta\} \leq \sum_{j=\gamma(l)}^{l} \begin{pmatrix} l \\ i \end{pmatrix} k^l$$

where $\gamma(l) = \begin{cases} \frac{l(1+\delta)}{2+\delta} + 1 & \text{if } \frac{l(1+\delta)}{2+\delta} \in \mathbb{N} \\ \lfloor \frac{l(1+\delta)}{2+\delta} \rfloor & \text{if not} \end{cases}$

By the same way, we estimate the number of words u of length l such that $\frac{n_-(u)}{n_+(u)} > 1+\delta$. We denote

$$\beta(l) = \#\{u \in \mathbb{F}_X \mid u \text{ reduced}, |u| = l, \frac{n_+(u)}{n_-(u)} > 1+\delta \text{ or } \frac{n_-(u)}{n_+(u)} > 1+\delta\},$$

so we have

$$\begin{aligned} \beta(l) &\leq 2\sum_{j=\gamma(l)}^{l} \begin{pmatrix} l \\ j \end{pmatrix} k^l \\ &= 2\sum_{j=0}^{l-\gamma(l)} \begin{pmatrix} l \\ j \end{pmatrix} k^l \end{aligned}$$

(2) We want to estimate the number of words r of length n in $B(n)$ such that r contains a subword of length $l \geq n/4$ which does not satisfy $\frac{n_+(u)}{n_-(u)} \leq 1+\delta$ or $\frac{n_-(u)}{n_+(u)} \leq 1+\delta$. There are $(n-l+1)$ places in r where the subword u can begin. Thus we can write r as $r = r_1 u r_2$ and as r is reduced, r_1 and r_2 are reduced too.We have also $|r_1| + |r_2| = n-l$. That implies $\#\{r_i\} \leq 2k(2k-1)^{|r_i|-1}$. So we can say

$$\begin{aligned} C(n) &\leq \sum_{l=\lfloor n/4 \rfloor}^{n} \beta(l)(n-l+1)(2k)^2(2k-1)^{n-l-2} \\ &\leq \sum_{l=\lfloor n/4 \rfloor}^{n} (k-1/2)^{n-l-2}k^2 2^{n-l}(n-l+1)2\sum_{j=0}^{l-\gamma(l)} \begin{pmatrix} l \\ j \end{pmatrix} k^l \\ &\leq \sum_{l=\lfloor n/4 \rfloor}^{n} (k-1/2)^{n-l-2}k^{2+l}2^{n-l}(n-l+1)2\sum_{j=0}^{l-\gamma(l)} \begin{pmatrix} l \\ j \end{pmatrix} \end{aligned}$$

We can estimate $C(n)/B(n)$,

$$\frac{C(n)}{B(n)} \le \frac{\sum_{l=\lfloor n/4\rfloor}^{n} (k-1/2)^{n-l-2}k^{2+l}2^{n-l}(n-l+1)2\sum_{j=0}^{l-\gamma(l)}\binom{l}{j}}{2^n k(k-1/2)^{n-2}(k-1)}$$
$$= \frac{k}{k-1}\sum_{l=\lfloor n/4\rfloor}^{n}\left(\frac{k}{k-1/2}\right)^{l}(n-l+1)2\sum_{j=0}^{l-\gamma(l)}\binom{l}{j}\frac{1}{2^l}.$$

As $\gamma(l)$ is almost equal to $\lfloor \frac{l(1+\delta)}{2+\delta} \rfloor$, we have $l-\gamma(l) \cong \lfloor \frac{l}{2+\delta} \rfloor$. By lemma 3.1 with $m_0 = 2+\delta$, we have

$$\sum_{j=0}^{l-\gamma(l)}\binom{l}{j}\frac{1}{2^l} \le \tilde{A}C^{\lfloor l/m_0\rfloor m_0}$$

where $C = \left(\frac{m_0}{2(m_0-1)^{(m_0-1)/m_0}}\right)$.

We deduce

$$\frac{C(n)}{B(n)} \le A\sum_{l=\lfloor n/4\rfloor}^{n}\left(\frac{Ck}{k-1/2}\right)^{\lfloor l/m_0\rfloor m_0}\sum_{i=0}^{m_0-1}(n-\lfloor l/m_0\rfloor m_0+1+i)$$
$$\le A\left(\frac{Ck}{k-1/2}\right)^{\lfloor n/4m_0\rfloor m_0}$$
$$\sum_{l=0}^{n-\lfloor n/4m_0\rfloor m_0}\left(\frac{Ck}{k-1/2}\right)^{\lfloor l/m_0\rfloor m_0}\sum_{i=0}^{m_0-1}(n-\lfloor l/m_0\rfloor m_0+1+i)$$

So as the sumation $\sum_{l=0}^{n-\lfloor l/4m_0\rfloor m_0}\left(\frac{Ck}{k-1/2}\right)^{\lfloor l/m_0\rfloor m_0}\sum_{i=0}^{m_0-1}(n-\lfloor l/m_0\rfloor m_0+1+i)$ increases polynomially with n and $\left(\frac{Ck}{k-1/2}\right)^{\lfloor n/4m_0\rfloor m_0}$ decreases exponentially, $\frac{C(n)}{B(n)}$ goes to 0 when n goes to $+\infty$, if we have $\frac{Ck}{k-1/2} < 1$. For $k \ge 2$, to get $\frac{Ck}{k-1/2} < 1$, we have to take $C < 3/4$and we have to choose m_0 such that

$$\frac{m_0}{2(m_0-1)^{(m_0-1)/m_0}} < 0,75.$$

By a direct computation, we see that, as $m_0 = \delta+2$, for $\delta = 8$, $\lfloor \frac{l}{2+\delta} \rfloor \cong \frac{l}{10}$ and that $\frac{10}{2(9)^{9/10}} \cong 0.69$. □

LEMMA 3.3. — *For all fixed $\epsilon > 0$, the property of being ϵ-balanced is generic.*

PROOF OF 3.3 Let $\#X = k$. Denote $C(N)$ the number of cyclically reduced words in $\mathbb{F}_X$. First we see that $C(N)$ is greater or equal to the number of words of length N in $\mathbb{F}(X)$ with the last letter is not the inverse of the first, i.e.

$$C(N) \geq 2k(2k-1)^{N-2}(2k-2). \tag{1}$$

We can now estimate $B(N)$ the number of "bad" presentations, i.e the number of presentations $\langle X : r \rangle$ such there exists $r' \in R^*$, i.e. r' a cyclic conjugate of r, begining with a positive word which has a length bigger than ϵN. As there is not more than $2N$ elements in R^*, we have

$$B(N) \leq 2N \sum_{l=[\epsilon N]+1}^{N} C(N,l)$$

where $C(N,l)$ is the number of cyclically reduced word of length N begining by a positive word of length l exactly. So we have :

$$B(N) \leq 2N \sum_{l=\lfloor \epsilon N \rfloor+1}^{N} k^l(2k-1)^{N-l}. \tag{2}$$

Dividing (2) by (1), We estimate the ration of non ϵ-balanced presentations over the number of presentations :

$$\begin{aligned} \frac{B(N)}{C(N)} &\leq \frac{N(2k-1)^2}{2k(k-1)} \sum_{l=\lfloor \epsilon N \rfloor+1}^{N} k^l(2k-1)^{-l} \\ &= \frac{N(2k-1)^2}{2k(k-1)} \frac{k^{\lfloor \epsilon N \rfloor+1}(2k-1)^{-\lfloor \epsilon N \rfloor-1} - k^{N+1}(2k-1)^{-N-1}}{1-k(2k-1)^{-1}} \end{aligned}$$

As $k \geq 2$, this expression goes exponentially to 0 when $N \to +\infty$. □

This proof appears in [4] for $\epsilon = 1/4$.

LEMMA 3.4. — *Let $\langle X, R \rangle$ be a finite presentation satisfying a θ-condition (with $\theta \leq 1/199$) then for all reduced diagrams Δ, there exists at least one r_i in R^* which is a border of a cell of Δ and which has at least $\frac{99}{100}$ of its elements on the border of the diagram $\partial\Delta$.*

It follows that for all non trivial word ω of $\mathbb{F}_X$ which maps on the identity in $\Gamma = \langle X, R \rangle$, there exists at least one r in R^* which has at least $\frac{99}{100}$ of its elements in ω.

PROOF OF 3.4 The θ-condition tells that for every reduced diagram Δ, $I(\Delta) \leq \theta\#\Delta$ and by definition $\#\Delta = E(\Delta) + I(\Delta)$. We deduce $I(\Delta) \leq \frac{\theta}{1-\theta}E(\Delta)$. It is enough to look at diagrams with a connected interior. In fact, if the reduced diagram Δ does not have a connected interior, each of its parts with a connected interior define a other reduced diagram (relatively to an other word), so the inequality holds for every part of Δ with a connected interior and we conclude by saying that increasing the number of external edges does not change the inequality.

We define the following notation : for a cell f_i of the diagram, we denote $Int(f_i)$ (resp. $Ext(f_i)$) the number of edges of f_i which are internal to the diagram (resp. which are on the border of the diagram). We denote also $\#(f_i)$ the total number of edges of the cell f_i.

To obtain a contradiction, we suppose that all the cells of one diagram Δ have more than 1% of their edges inside the diagram (i.e. for all f_i, we have $100Int(f_i) > \#(f_i)$). It is clear that $E(\Delta) = \sum_i Ext(f_i)$ and that $I(\Delta) = \frac{1}{2}\sum_i Int(f_i)$, because every internal edge belongs exactly to two cells of the diagram and every external edge belongs exactly to one cell of the diagram . So we get :

$$\#(\Delta) = \frac{1}{2}\sum_i Int(f_i) + \sum_i Ext(f_i) = \sum_i \#(f_i) - \frac{1}{2}\sum_i Int(f_i).$$

If for all f_i, we have

$$\begin{aligned} 100Int(f_i) &> \#(f_i) \\ \text{then } 100\sum_i Int(f_i) &> \sum_i \#(f_i) = \#(\Delta) + \frac{1}{2}\sum_i Int(f_i) \\ \frac{199}{2}\sum_i Int(f_i) &> \#(\Delta) \\ 199I(\Delta) &> \#(\Delta). \end{aligned}$$

For this diagram, $I(\Delta) > \frac{1}{199}\#(\Delta)$. This contradicts the θ-condition for $\theta = 1/199$. □

LEMMA 3.5. — *For $\epsilon > 0$ small enough, if r is ϵ-balanced and has property E_δ with $\delta = 8$, if $r = s_{i_1}\cdots s_{i_{|r|}}$ with $s_{i_j} \in S = X \cup X^{-1}$, then every ordered subsequence $(y_1, \cdots, y_l)$ of the ordered sequence $(s_{i_1}, \cdots, s_{i_{|r|}})$ such that $l \geq \frac{99}{100}|r|$ has at least 3 changes of sign.*

PROOF OF 3.5 Set $|r| = n$, $n_+(r) = l$, thus $n_-(r) = n - l$ and $l \geq n - l$, we have $l \geq n/2$. As r has property E_δ, we have

$$\frac{n}{2} \leq l \leq \frac{1+\delta}{2+\delta}n.$$

So there are at least $\frac{1}{2+\delta}n$ negative terms in r.

Let r be a product of 3 words $r = r_1r_2r_3$ with $|r_i| > |r|/4$. As r has property E_δ, every subword u of length bigger than $|r|/4$ is such that either $1 \leq \frac{n_-(u)}{n_+(u)} \leq 1+\delta$, either $1 \leq \frac{n_+(u)}{n_-(u)} \leq 1+\delta$.

So we can suppose that for $i = 1, 2, 3$, we have either $1 \leq \frac{n_+(r_i)}{n_-(r_i)} \leq 1+\delta$,either $1 \leq \frac{n_-(r_i)}{n_+(r_i)} \leq 1+\delta$.

As $\delta = 8$, we can assume that r_1 is such that

$$\begin{array}{ccccc} \frac{n}{2} & \leq & n_+(r_1) & \leq & \frac{9n}{10} \\ \frac{n}{10} & \leq & n_-(r_1) & \leq & \frac{n}{2} \end{array}$$

So we can say that $n_+(r_1) \geq \frac{1}{10}$ and $n_-(r_1) \geq \frac{1}{10}$. By analogous arguments, we have $n_+(r_i) \geq \frac{1}{10}$ and $n_-(r_i) \geq \frac{1}{10}$ for $i = 2, 3$.

Denote by $(y_1, \cdots, y_{m_1})$ the subsequence of $(y_1, \cdots, y_l)$ corresponding to the elements of r_1, by $(y_{m_1+1}, \cdots, y_{m_2})$ the subsequence of $(y_1, \cdots, y_l)$ corresponding to the elements of r_2 and by $(y_{m_2+1}, \cdots, y_l)$ the subsequence of $(y_1, \cdots, y_l)$ corresponding to the elements of r_3. As at worse 1% of all elements of r disappear in $(y_1, \cdots, y_l)$, the sequence $(y_1, \cdots, y_{m_1})$ contains at worse 4% less than r_1 (similary for $(y_{m_1+1}, \cdots, y_{m_2})$, $(y_{m_2+1}, \cdots, y_l)$ with respect r_2, r_3). And as each r_i contain at least 10% of terms of both sign, we get $n_-((y_1, \cdots, y_{m_1})) > 0$ and $n_+((y_1, \cdots, y_{m_1})) > 0$. By the same arguments $(y_{m_1+1}, \cdots, y_{m_2})$ and $(y_{m_2+1}, \cdots, y_l)$ contain terms of both signs. We conclude that the three ordered subsequences $(y_1, \cdots, y_{m_1})$, $(y_{m_1+1}, \cdots, y_{m_2})$ and $(y_{m_2+1}, \cdots, y_l)$ of $(y_1, \cdots, y_l)$ each contain at least one change of sign.

Thus $(y_1, \cdots, y_l)$ at least contains three. □

With these lemmas we can prove the following proposition

PROPOSITION 3.6. — *Let $\Gamma \cong \langle X, R\rangle$ be a finite presentation such that Γ has a θ-condition, with $\theta < 1/199$, and such that every $r \in R$ is ϵ-balanced and has the property E_δ (with ϵ relatively small compared to the minimal length of the relations and $\delta \geq 8$), then X generates a free semi-group in Γ.*

PROOF OF 3.6 We denote by N the normal subgroup generated by R in $\mathbb{F}_X$ and let ω be a non trivial element of N. Choose Δ a reduced diagram for ω (i.e. $\partial\Delta = \omega$). As the presentation $\langle X, R\rangle$ satisfies a θ-condition with θ less than 199, by lemma 3.4, the diagram Δ contains a cell for which the border is a $r \in R$ and such that r has 99% of its generators on the border $\partial\Delta$ of Δ. As r is ϵ-balanced and has the property E_δ, by lemma 3.5, the ordered sequence $(y_1, \cdots, y_l)$ defined by $r \cap \omega$ contains at least 3 changes of sign. So ω contains at least 3 too. For two positive words ω_1, ω_2 in $\mathbb{F}_X$, $\omega_1\omega_2^{-1}$ is a word with only one change of sign, so it does not belong to N, which implies that that the image of $\omega_1\omega_2^{-1}$ in Γ is not trivial, and so ω_1 is different of ω_2 in Γ. We conclude that the semi-group generated by X in Γ is free. □

PROOF OF THEOREM 1.2 We just need to remark that the intersection of a finite number of generic properties is always generic and to appeal to lemmas 3.2, 3.3 and Ol'shanskii's result which asserts that for every fixed $\theta > 0$, the θ-condition is generic (see [10]). We conclude with the proposition 3.6 and the theorem 1.1, hyperbolicity being generic because it follows from a θ-condition (it was independentely proved by Ol'shanskii [10] and Champetier [2]). □

So we have proved that for finitely presented groups $\langle X, R\rangle$, the existence of free semi-group generated by X is very frequent, but it could be interesting to see if it easy to decide whether a particular presentation $\langle X, R\rangle$ has such a property or not, just by looking at the set of relations R. In that direction, it could be interesting to be able to read the θ-condition on R. That would unable us to get more than asymptotic results.

References

[1] C. CHAMPETIER. Cocroissance des groupes à petite simplification. *Bull. London Math. Soc.*, 25:438–444, 1993.

[2] C. CHAMPETIER. Propriétés statistiques des groupes de présentation finie. *to appear in Adv. in Maths.*

[3] C. CHAMPETIER. Introduction à la petite simplification.

[4] P.-A. CHERIX and A. VALETTE. On spectra of simple random walks on one-relator groups (with an appendix of p. jolissaint). *to appear in Pacific J. of math.*

[5] E. GHYS and P. de la HARPE eds. *Sur les groupes hyperboliques d'après M. Gromov.* Number 83 in Progress in Maths. Birkhaüser, 1990.

[6] M. GROMOV. Hyperbolic groups. *"Essays in Group Theory", ed. S.M. Gersten, M.S.R.I. Publ.*, 8:75–263, 1987.

[7] P. de la HARPE, A.G. ROBERTSON, and A. VALETTE. On the spectrum of the sum of generators for a finitely generated group. *Israel J. of Maths.*, 81:65–96, 1993.

[8] P. de la HARPE, A.G. ROBERTSON, and A. VALETTE. On the spectrum of the sum of generators for a finitely generated group ii. *Colloquium Math.*, 65:87–102, 1993.

[9] R.C. LYNDON and P.E. SCHUPP. *Combinatorial group theory.* Number 89 in Ergebnisse der Math. Springer, 1977.

[10] A. OL'SHANSKII. Alomost every group is hyperbolic. *International j. of Algebra and Computation*, 2:1–17, 1992.

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