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CLOSED GEODESICS AND FLAT TORI IN SPECTRAL THEORY ON SYMMETRIC SPACES [1]

David GURARIE

Closed geodesics are known to play an important role in spectral theory of Laplacians on Riemannian manifolds $\mathcal{M}$. They also contribute to spectral theory of Schrödinger operators $\Delta + V$, typically in the form of higher order correction to the principal (Laplace) eigenvalue. We give a brief survey of the classical "Shape of metric" and "Shape of potential" problems of spectral theory, and explore the role of "*length spectrum*" (the length of all closed path/geodesics), and the related "*Radon transform of V*". Then we outline some recent progress in special cases: the n-sphere theory, and Schrödinger operators on higher rank symmetric spaces. The latter case brings in new players: the flat tori. They naturally appear in higher rank compact symmetric spaces and play the role of closed geodesics here. We conclude by a list of open problems.

1. **Classical spectral invariants; length spectrum and invariant Lagrangians.** In spectral theory of Laplacian $H = -\Delta$ or Schrödinger operators $H = -\Delta + V$ on a compact Riemannian manifold $\mathcal{M}$ one is interested in the connection between spectrum of H on one hand and the geometry of $\mathcal{M}, V$ as well as the dynamics of the underlaying classical hamiltonian flow of H in the phase-space, identified with the cotangent bundle $T^*(\mathcal{M})$. Mark Kac [Ka] asked his celebrated question "Can one hear the shape of the drum?" and gave some examples. Similar "shape problems" can be posed for the metric on $\mathcal{M}$, potential V, etc. The well known examples of audible geometry include

• The classical *heat/Weyl* invariants, like $b_0 = vol(\mathcal{M})$; $b_1 = \text{area}(\partial\mathcal{M})$; integral of curvature $\int_{\mathcal{M}} K\,dv$, etc. ([We]; [MP]; [McS]; [Gi]). The name reflects their derivation via small-time asymptotic expansion of the heat-kernel,

$$e^{tH} \sim t^{-\frac{n}{2}}\{b_0 + b_1 t^{\frac{1}{2}} + b_2 t + \dots\},\ n = \dim(\mathcal{M}). \tag{1}$$

• The next important example goes under the name of *length spectrum*, it refers to all closed trajectories of the underlying classical hamiltonian flow. In the case of Laplacian it becomes the geodesic flow, so one talks about the length of all closed geodesics on $\mathcal{M}$. The general principle states: *spectrum of Δ_M* determines the *length spectrum* $\{\ell(\gamma) = |\gamma|\}$, plus some additional data, like Poincare numbers (that measure the "twist of the flow" along γ), and Morse indices $\sigma(\gamma)$. Precisely, ([BB]; [BT]; [Gut];

[1] An expanded version of the lecture delivered at the Math. Physics seminar of ETH and seminar on Geometry and Spectral theory of the University of Grenoble.

[Ch]; [DG]; [Co3]): *for any closed stable path* $\ell = \ell(\gamma)$ *there exists an almost arithmetic sequence of eigenvalues in spec*$\sqrt{\Delta}$:

$$\lambda_k \simeq \tfrac{2\pi}{\ell}k + \sum m_j\theta_j + \nu(\gamma). \tag{2}$$

Here $\{\theta_j\}$ are Poincare angles and ν - Morse index along γ. There are several ways to establish (2). One is based on the quasi-mode construction [Co3], it effectively produces an embedding of space $L^2(\gamma)\to L^2(M)$, that "almost intertwines" operators $\frac{d^2}{ds^2}|_\gamma$ with the Laplacian Δ_M. Another way is based on the study of the *wave-kernel* $U_t = e^{it\sqrt{H}}$ - the fundamental solution of the wave equation:

$$u_{tt} + H[u] = \delta(x-y) \text{ on } \mathcal{M}\times\mathbb{R}.$$

It turns out that

$$\chi(t) = trU_t = \sum e^{it\sqrt{\lambda_k}},$$

understood as a distribution[2] on $\mathbb{R}$, has singularities located inside the length spectrum $\{\ell(\gamma)\}$ of $\mathcal{M}$, including the "big singularity" at t = 0 (the latter would correspond to all trivial one-point path). Precisely, χ is made (expanded into the sum) of contributions of different closed path $\{\gamma\}$, $\chi = \chi_0 + \sum \chi_\gamma$. The Fourier transform $\hat{\chi}_0$ of the main singularity has an asymptotic expansion

$$\hat{\chi}_0(\mu) \sim c_0\mu^{n-1} + c_1\mu^{n-2} + \dots, \text{ as } \mu\to\infty,$$

whose coefficients $\{c_0; c_1; \dots\}$ are simply related to the Weyl (heat) invariants (0)-(2). The sum of nonzeros terms yield

$$(\chi-\chi_0)(t) \sim \frac{1}{2\pi}\sum_\gamma\sum_{m=1}^{\infty} \frac{\ell(\gamma)e^{i\frac{\pi}{2}\sigma(\gamma)}}{(t-\ell(\gamma^m))\sqrt{det(I-P(\gamma)^m)}}. \tag{3}$$

Here γ denotes a primitive closed classical path (geodesics), γ^m - its m^{th} iterate, $\sigma(\gamma)$ - the Morse index of γ, and $P(\gamma)$ - the Poincare map along γ. We remark that singular terms of the type $[t-\ell(\gamma^m)]^{-1}$ in (3) correspond to isolated closed path γ. In degenerate cases (i.e. when orbits of a given period form a d-parameter family) more singular distributions of order $\frac{d+1}{2}$ appear in place of $(t-\ell)^{-1}$.

Formula (3) extends the classical *Poisson summation* for the wave-kernel $e^{it\sqrt{\Delta}}$ on the n-torus $\mathbb{T}^n$, and has a noncommutative analog, the celebrated *Selberg-trace formula* on hyperbolic spaces $M = \mathbb{H}/\Gamma$, Poincare half-plane modulo a discrete

[2]Unitary operators U_t have no "trace" in the usual sense, but integrating U_t against smooth compactly supported test-functions f on $\mathbb{R}$, produce trace class operators $U_f = \int f(t)U_t dt$ on $L^2(M)$. So distribution χ can be defined via pairing $\langle\chi; f\rangle = trU_f$.

(Fuschian) subgroup Γ of $SL(2;\mathbb{R})$, acting on $\mathbb{H}$ by fractional linear transformations [Sel]; [HST]; [Mc]. It is worth to note, however, that in classical cases (Poisson. Selberg), asymptotic relation (3) becomes exact.

• Our next example exhibits higher-D geometric structures that have spectral content. Those are invariant Lagrangians (tori) in the phase-space $T^*(\mathcal{M})$, preserved by the hamiltonian flow of H. One assumes that $T^*(\mathcal{M})$ is foliated into the n-parameter family of such Lagrangian $\Lambda(c) = \Lambda(c_1;...c_n)$, those could be for instance, the joint level sets of n Poisson commuting integrals $\{I_i(x;p) = c_i\}$.

To quantize family $\{\Lambda(c)\}$ one picks a system of fundamental cycles $\{\gamma_j(c): 1 \leq j \leq n-1\}$ in each $\Lambda(c)$ and writes the generalized Born-Sommerfeld (EBK)-quantization rules ([Ke]; [MF]; [Gut]):

$$\begin{cases} \dots \\ T_j(c) = \oint_{\gamma_j} p \cdot dx = \pi(2m_j + \frac{1}{2}\nu(\gamma_j)) \\ \dots \end{cases} \tag{4}$$

Here numbers $\{m_j\}$ vary over the integer lattice $\mathbb{Z}^n_+$ and $\nu(...)$ denotes the Maslov index of Λ, evaluated along the cycle. Solving system (4) we get a quantized set of Lagrangians $\{\Lambda_m = \Lambda(c(m)): m = (m_1;...m_{n-1})\}$, hence a quantized sequence (labeled by lattice points $m \in \mathbb{Z}^n_+$) of eigenvalues of the hamiltonian $H = h(x;\partial)$,

$$\lambda_m \approx h \mid_{\Lambda_m}. \tag{5}$$

EKB-quantization based on invariant Lagrangian provides more rich spectral structure compared to closed path, but is limited in scope to essentially *integrable* classical hamiltonian.

2. Perturbation problems. The best studied class of perturbation problems are one-dimensional regular Sturm-Liouville (S-L) operators $H = -\partial^2 + V(\mathrm{x})$ on $[0;1]$, with any kind of boundary conditions: two-point; periodic; Floquet, etc. Such operators have simple (multiplicity free) eigenvalues $\{\lambda_k\}$, that admit an asymptotics expansion [Bo],

$$\lambda_k = (\pi k)^2 + b_0 + b_1 k^{-2} + ..., \text{ as } k \to \infty. \tag{6}$$

Here $b_0 = \int V dx$, $b_1 = \int (V - b_0)^2 dx$, and higher $\{b_j\}$ involve certain polynomial expressions in V and its derivatives. The correspondence $V \to \mathrm{spec}(H_V)$ is highly nonunique, there are typically large (∞-D) isospectral classes both in the periodic/Floquet and the "2-point boundary" case ([IMT]; [PT]). So inverse spectral problem requires an infinite auxiliary set of data, like the higher KdV-parameters or norming constants.

The multidimensional Schrödinger problems are believed to be spectrally rigid in the sense that their isospectral classes, Iso(V), consist of small (finite-dimensional) families obtained by natural (geometric) symmetries of the Laplacian, i.e. isometries of $\mathcal{M}$, rather than hidden KdV-type symmetries. This *spectral rigidity hypothesis* was confirmed in a number of cases, the foremost is the case of negatively curved manifolds. It was shown (under some additional technical conditions [GK]) that potential V on a hyperbolic manifold $\mathcal{M}$ is in fact uniquely determined by spec(H_V), negatively curved manifolds have typically no continuous internal symmetries.

Another example is the flat torus $\mathbb{T}^n$, whose symmetries are made of all translations and reflections. It was shown that generic potentials[3] V on $\mathbb{T}^n$ are also spectrally rigid ([ERT]; [MN]). The case of positively curved (and highly symmetric) n-sphere proved to be more difficult. The general rigidity hypothesis remains open, but there is a number partial results ([Wei]; [Gui]; [Co]; [Ur]; [Wi]; [Gur4-5]), that we shall now discuss.

To proceed we make a general comment on the role of closed path for perturbation problems: they typically enter spectral asymptotics in the form of integrals of V along γ, $\int_\gamma V ds$ (Radon transform), or certain functionals of V. Most known results starting from the 1-D Borg formula (6) to the multi-D flat ($\mathbb{T}^n$), hyperbolic or spherical cases involve such "Radon transforms" in that or other form. The latter case will be illustrated in the next section.

3. The n-sphere theory. The n-sphere Laplacian has regular distributed and highly degenerate spectrum:

$$\{\lambda_k = k(k+n-1) = (k+\rho)^2 - \rho^2 : k=0;1;...\},\ \rho = \tfrac{n-1}{2},$$

the multiplicity $d_k = d(\lambda_k)$ increasing with k as $O(k^{n-1})$. The degeneracy results from the underlying rotational symmetry $SO(n+1)$. Each eigensubspace $\mathcal{H}_k \subset L^2(S)$ is invariant under $SO(n+1)$ and defines an irreducible representation π^k. Potential V breaks the rotational symmetry of the problem, so spec(H_V) splits into clusters of simple (less degenerate) eigenvalues $\Lambda_k = \{\lambda_{km} = \lambda_k + \mu_{km} : m = 1;...d_k\}$. The clusters

[3] Of course, special potentials, like $V = V_1(x_1) + ... + V_n(x_n)$ may possess "large" (infinite) isospectral classes, obtained by the KdV-flows applied to each 1-D component $\{V_j(x_j)\}$.

are asymptotically well localized. Namely, the cluster size,

$$|\Lambda_k| = \begin{cases} \mathcal{O}(1), \text{ for even/generic } V \\ \mathcal{O}(k^{-2}), \text{ for odd } V \end{cases},$$

while the distance between neighboring Λ_k increases in proportion to k [Gui2]. So one looks for spectral invariants, associated with distribution of shifts within clusters.

Such distribution is naturally described by *cluster-measures*,

$$\mathrm{d}\nu_k = \frac{1}{d_k}\sum_m \delta(x - \mu_{km}).$$

Weinstein [Wei] studied asymptotics of measures $\{\mathrm{d}\nu_k\}$ and proved that sequence $\{\mathrm{d}\nu_k\}$ converges to a continuous measure $\beta_0(\lambda)\mathrm{d}\lambda$ on $\mathbb{R}$. The limiting density (not surprising) turned out to be the distribution function of the *Radon transform* $\tilde{V}$. So for any test function f on $\mathbb{R}$ one has

$$\langle f \mid \beta_0 \rangle = \int_{\mathcal{O}} f \circ \tilde{V} \mathrm{d}S, \tag{7}$$

integration over the space $\mathcal{O}$ of closed geodesics (great circles)[4] on S^n (fig.1). Moreover, sequence $\{\mathrm{d}\nu_k\}$ can be expanded in powers of k^{-1}, similar to Borg's expansion (7),

$$\mathrm{d}\nu_k \sim \beta_0 + k^{-1}\beta_1 + \ldots \tag{8}$$

Coefficients $\{\beta_0;\beta_1\ldots\}$ are certain distributions on $\mathbb{R}$ depending on V. Weinstein named them *band-invariants*, and calculated β_0. The higher band-invariants involve complicated expressions of potential and its derivatives, only two of them $\{\beta_1;\beta_2\}$ have been computed explicitly [Ur1-2], [Gur4].

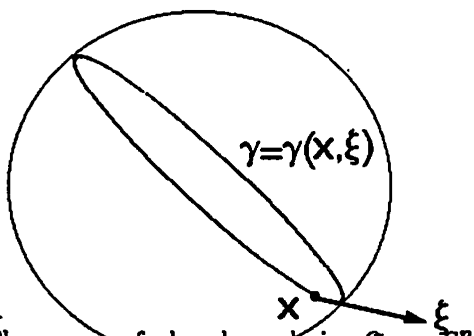

Fig.1: Closed geodesics on S^n are great circles $\gamma = \gamma(x;\xi)$ through point x in the direction ξ.

[4]The space of closed geodesics $\mathcal{O}$ on S^n differs from the hyperbolic or flat torus cases. The latter are either a discrete sequence of isolated path $\{\gamma_m\}$ (hyperbolic), or a discrete union of cells $\mathcal{O}_\ell = \{\gamma : |\gamma| = \ell\} \simeq \mathbb{T}^{n-1}$, of different length $\ell = (\sum m_j^2)^{1/2}$ (torus), while the spherical space $\mathcal{O}$ consists of a single "fat" cell $\mathcal{O} \simeq S^*(S^n)/\mathbb{T}$, of dim $= 2(n-1)$. It also possesses some other nice structures, e.g. a homogeneous space of $SO(n+1)$, complex projective variety [Gui]; [Ur]. Clearly, the entire geodesic flow on S^n is periodic of period 2π.

Turning to the inverse problem the band invariants provide some useful information, in particular they allow to prove spectral rigidity for special classes of potentials (low degree spherical harmonics) on S^2 [Gui]. Their use however is limited by the fact that $\{\beta_j\}$ describe *distributions of values* of the Radon (and related) transforms. rather than the transform itself. The latter could have been easily inverted to recover potential V, but in general one cannot recover a function (even single-variable) from the distribution of its values.

In recent papers [Gur2-4] we studied the class of *zonal (axisymmetric) potentials* on S^n, i.e. functions $\{V\}$ invariant under the subgroup $SO(n)$ of rotations about the "north pole". Such V clearly depend on a single variable angle θ between point on S^n and the vertical axis (see fig.2). The corresponding Schrödinger operators possess an auxiliary $SO(n)$-symmetry. On S^2 it is generated by the z-component of the angular momentum operator $J = i\partial_\theta$, while on S^n the role of J is played either by the Lie algebra $so(n)$, or after a suitable "symmetry reduction" by a single operator $J = \sqrt{-\Delta_{n-1}}$ - the lower-dimensional Laplacian on S^{n-1}. Such hamiltonians are integrable in the Liouville sense (algebra $so(n)$ generates $n-1$ integrals Poisson-commuting with classical hamiltonian of H. Those could be naturally quantized to produce a commutative family of operators $(J_1;...J_{n-1})$ commuting with H. So one can study the *joint spectral* decomposition of H and $\{J_i\}$, or H and J. Thus spectral shifts $\{\mu_{km}\}$ of the k-th cluster acquire an additional (bigraded) structure, index m labeling the angular momentum of the k-th eigenfunction ψ_{km},

$$\begin{aligned} H[\psi_{km}] &= (\lambda_k+\mu_{km})\psi_{km} \\ J[\psi_{km}] &= m\psi_{km} \end{aligned} \qquad (9)$$

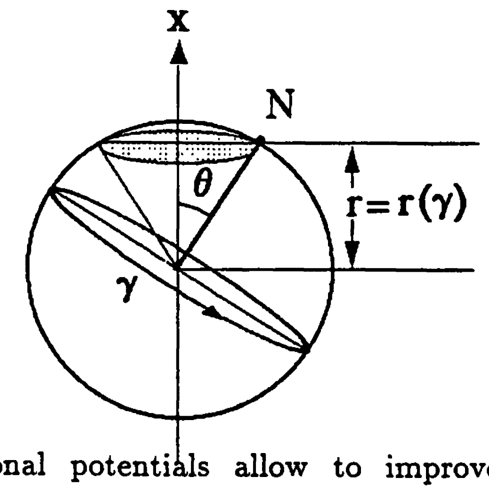

Fig.2: Zonal axisymmetric potentials depend only on angle θ between the point on the sphere and the vertical axis. Their Radon transform is also a single variable function of the height r of the north pole N of a great circle γ.

Zonal potentials allow to improve the Weinstein's result (7) by replacing *asymptotics of cluster-measures* $\{d\nu_k\}$ by asymptotics of *individual spectral shifts* $\{\mu_{km}\}$. The main result [Gur4] states,

Theorem 1: *Spectral shifts* $\{\mu_{km}\}$ *of the joint* H,J*-eigenvalue problem* (9) *admit an asymptotic expansion,*

$$\mu_{km} \sim a(\tfrac{m}{k}) + k^{-1}b(\tfrac{m}{k}) + k^{-2}c(\tfrac{m}{k}) + \dots, \tag{10}$$

The coefficients $a(x); b(x); c(x)$ *are computed explicitly in terms of certain transforms of* V *(its even and odd parts* $V = V_{ev} + V_{od}$*) on* $[0;1]$. *Namely, the reduced Radon transform*[5] $\Re$,

$$\Re: f(x) \to \tilde{f}(r) = \frac{2}{\pi}\int_0^{\sqrt{1-r^2}} f(x)\frac{dx}{\sqrt{1-r^2-x^2}};$$

the Gegenbauer-Legendre operator (reduced Laplacian)

$$\mathfrak{L} = \frac{d}{dx}(1-x^2)\frac{d}{dx} + (\frac{\alpha x}{1-x^2})^2 - \alpha, \text{ with } \alpha = \frac{n-2}{2},$$

and operator $\mathfrak{E} = x\frac{d}{dx}$, *Precisely,*

$$a(r) = \Re(V_{ev}) = \tilde{V}_{ev}(r);$$

$$b(r) = -\tfrac{1}{4}\Re\mathfrak{L}(V_{ev});$$

$$c(r) = \tfrac{1}{32}\Re\{\mathfrak{L}^2 + 2(n+1)\mathfrak{L}\}(V_{ev}) +$$
$$+\tfrac{1}{2}\{\Re(V_{ev}^2 + V_{od}^2) - (\Re V_{ev})^2\} - \frac{r^2}{2(1-r^2)}\Re\{x\tfrac{d}{dx}(V_{ev}^2 + V_{od}^2)\}.$$

In special cases of even [Gur2] and odd [Gur3] zonal potentials V on S^2, we get

$$\mu_{km} = \begin{cases} \tilde{V}\left(\frac{m}{k}\right) + O(k^{-1}), & \text{even } V, \\ k^{-2}U\left(\frac{m}{k}\right) + O(k^{-3}), & \text{odd } V, \end{cases}$$

Function $U(x)$ is obtained by transform $\frac{1}{4}(\Re - \frac{r^2}{1-r^2}\Re\mathfrak{E})$ applied to V_{od}^2.

Theorem 1 provides a unique and explicit solution of the inverse problem for the joint (H,J)-spectrum. It also yields *local spectral rigidity* for generic zonal potential on S^n, via rigidity of almost arithmetic sequences. The details are given in [Gur2-4].

Remark: Theorem 1 can be interpreted in terms of semiclassical quantization (4) of an integrable hamiltonian H. It gives the effect of perturbation V on spectral shifts $\{\mu_{km}\}$ of H_V in terms of two commuting integrals: $A = \sqrt{-\Delta}$ and $J = \sqrt{-\Delta_{n-1}}$ (on

[5] We remark that zonal functions $\{f(x)\}$, as well as their Radon transforms $\{\tilde{f}(r)\}$, depend on a single variable $x \in [-1;1]$ of f that runs along the symmetry axis, while variable r of $\tilde{f}$ measures the x-coordinate of the "north pole" of the great circle γ, i.e. $\tilde{f}(\gamma) = \tilde{f}(r(\gamma))$.

S^2, $J = i\partial_\theta$), in the form

$$V_{eff} \approx a(J/A) + A^{-1}b(J/A) + A^{-2}c(J/A) + \ldots, \tag{11}$$

where $a = \mathrm{Radon}(V)$, $b = \ldots$ etc. When both operators A and J are quantized by the EBK-rules (4) to their classical levels: $A_{cl} = k$, $J_{cl} = m$, on a family of 2-D (reduced) invariant tori then (10) follows from (11).

4. Averaging method. We shall briefly outline some basic methods employed in the n-sphere theory. They involve a suitable *averaging procedure* [Wei], [Gui], [Ur], *Symbolic calculi* on S^n [Ur], and (in the zonal case) *Zonal reduction* [Gur2-4]. One usually works with operators $\sqrt{-\Delta}$ and $\sqrt{H}$, those are more convenient from the viewpoint of symbolic calculus[6] than Δ and H. Precisely, we take operator $A = \sqrt{-\Delta + (\frac{n-1}{2})^2} - (\frac{n-1}{2})$, and write $H_V = (A+B)^2$, so $B \approx \frac{1}{2}V/\sqrt{-\Delta} + \ldots$ One of the main obstacles in diagonalizing operators like H_V is that Laplacian Δ and perturbation V do not commute. Weinstein's remedy was to *average* B with a unitary group generated by A. One conjugates B

$$B(t) = e^{-itA}Be^{itA},$$

and defines the average operator

$$\bar{B} = \frac{1}{2\pi}\int_0^{2\pi} B(t)dt. \tag{12}$$

Clearly, $\bar{B}$ commutes with A, and one can show that $A+B$ and $A+\bar{B}$ are "almost unitarily equivalent",

$$U^{-1}(A+B)U = A + \bar{B} + R,$$

the remainder R being small relative to A, its modulus estimated by

$$|R| = (RR^*)^{\frac{1}{2}} \le \mathrm{Const}A^{-3}.$$

As a consequence, spectral shifts $\{\mu_{km}\}$ of $A+B$ are approximated by the averaged shifts $\{\bar{\mu}_{km}\}$ of $A+\bar{B}$, equal to eigenvalues of $\bar{B}$,

$$|\mu_{km} - \bar{\mu}_{km}| \le \mathrm{Const}\ k^{-3}.$$

Asymptotic distribution of the latter is then computed using the symbolic calculus on S^n (so called Szegö limit Theorem [Gui]). To get β_0 one takes a test function f on $\mathbb{R}$ paired with cluster-measures $\{d\nu_k\}$, and passes to the limit $k \to \infty$,

[6] Taking square-roots of Laplacians, Schrödinger operators is a convenient technical device that allows to reduce many symbolic (semiclassical) manipulations with functions $\{f(\Delta)\}$, from the entire unbounded phase-space $T^*(\mathcal{M})$ to its compact part the cosphere bundle $S^*(\mathcal{M})$.

$$\langle f; d\nu_k\rangle = \frac{1}{d_k}\sum_m f(\mu_{km}) \approx \frac{tr\,[P_k f(\bar{B})P_k]}{rkP_k} \to \int_{S^*(S^n)} (f\circ \mathrm{symbol}\bar{B})\ dS.$$

Symbol of the conjugate operator (ψdo) $B(t)$ is obtained by composing $\sigma_B = V(x)/2|\xi|$ with the hamiltonian (geodesic) flow, $\exp(t\Xi)$, of "symbol$A$" $=|\xi|$. Here Ξ denotes the hamiltonian vector field of "symbol A". This yields symbol of the average operator $\bar{B}$,

$$\sigma_{\bar{B}} = \frac{1}{2\pi}\int_0^{2\pi} \sigma_B\circ\exp(t\Xi)dt,$$

and the latter is nothing but the Radon transform $\widetilde{V}(\gamma)$, evaluated on a closed path (great circle) $\gamma = \gamma_{x\xi}$, passing through $x \in S^n$ in the direction $\xi \in T_x{}^*$. Thus we get the first band-invariant β_0 in terms of the Radon transform $\widetilde{V}$.

The above argument proves Weinstein's Theorem. It requires further elaboration and considerable effort to prove Theorem 1 (see [Gur2-4]). These results demonstrate some essential features of the n-sphere Schrödinger theory, that relies heavily on *spherical symmetries* of the problem and periodicity, both on the classical level (geodesic flow), and the quantum level (operator $A = \sqrt{-\Delta}$ with periodic spectrum). Both features, symmetry and periodicity persist, in any rank-one symmetric space, so all the above results can be extended to such $\mathcal{M}$, as well as more general *Zoll manifolds* [Wei], [Ku]. However, the higher rank spaces pose a new and very different situation. They clearly lack periodicity, both the classical and quantum alike: the eigenvalues of $\sqrt{\Delta + \mathrm{Const}}$ are no more integers, and a typical geodesics does not close, but rather wrap up densely around a flat tori. Although such space could still have a fair number of closed path the dynamics would be dominated by aperiodic γ. The lack of periodicity requires a modification of the basic averaging techniques: the Weinstein averages (12) should be replaced by the *ergodic averages*,

$$\lim\left\{\bar{B}_T = \frac{1}{T}\int_0^T B(t)dt\right\},\ \text{as } T\to\infty.$$

In [Gur5] we developed such multiparameter averaging procedure in the context of multi-D anharmonic oscillators on $\mathbb{R}^n$. These methods proved to be applicable also to Schrödinger operators on symmetric spaces.

5. Symmetric spaces. Symmetric spaces $\mathcal{M}$ are natural generalizations of three basic models in geometry: the flat $\mathbb{R}^n$, spherical S^n and hyperbolic $\mathbb{H}^n$. All three spaces possess large symmetry groups G of isometries. Those are the Euclidian motion group

$\mathbb{E}_n = \mathbb{R}^n \rhd SO(n)$ for $\mathbb{R}^n$, the orthogonal group $G = SO(n+1)$ for S^n, and pseudo-orthogonal (Lorentz) $G = SO(1;n)$ for $\mathbb{H}_n$. Symmetries act transitively on each of three spaces, so $\mathcal{M} = K\backslash G$ (G modulo stabilizer K of some $x_0 \in \mathcal{M}$). Furthermore, any pair of points $\{x, y\}$ is conjugated to any other equidistant pair $\{x'; y'\}$ by an element of G. Hence the only relevant parameter of any G-invariant operator on such space is the distance $\mathrm{d}(x;y)$. Another characterization of all 3 spaces $K\backslash G$ is that stabilizer K acts transitively on the geodesic sphere $S_r(x_0) = \{y: d(x_0; y) = r\}$, centered at x_0.

Three basic geometries $\mathbb{R}^n$; S^n; $\mathbb{H}^n$ are examples of rank-one symmetric spaces. In general, symmetric spaces $\mathcal{M}$ is defined as a quotient $K\backslash G$ of a semisimple Lie group G, modulo its maximal compact subgroup K. Here we are mostly interested in *compact symmetric spaces*, i.e. quotients of compact Lie groups G. It is known that such group G (and its Lie algebra $\mathfrak{G}$) has an involutive automorphism θ ($\theta^2 = 1$), that stabilizes subgroup $K = \{u \in G: \theta(u) = u\}$. Such θ splits Lie algebra $\mathfrak{G}$ into the direct (orthogonal) sum $\mathfrak{K} \oplus \mathfrak{P}$, the subalgebra $\mathfrak{K}$ of K and a subspace $\mathfrak{P}$. The latter could be identified with the tangent space of $\mathcal{M}$ at $x_0 = \{K\}$. Furthermore, $\theta\ \mathfrak{K} \simeq I$, $\theta\ \mathfrak{P} \simeq -I$, so the resulting Lie brackets between the $\mathfrak{K}$ and $\mathfrak{P}$ -components of $\mathfrak{G}$ become

$$[\mathfrak{K};\mathfrak{K}] \subset \mathfrak{K};\ [\mathfrak{K};\mathfrak{P}] \subset \mathfrak{P};\ [\mathfrak{P};\mathfrak{P}] \subset \mathfrak{K}.$$

Space $\mathfrak{P}$ contains a maximal abelian (*Cartan*) subalgebras $\mathfrak{A} \simeq \mathbb{R}^r$, whose dimension $r = \dim\mathfrak{A}$ determines the rank of $\mathcal{M}$. The image of $\mathfrak{A}$ under the exponential map $\mathfrak{G} \to G$, forms a maximal geodesically flat torus $\gamma = \exp\mathfrak{A} \simeq \mathbb{T}^r$ in $\mathcal{M}$. Geometrically one could view space $\mathcal{M}$ in the following way: as above group G acts transitively on $\mathcal{M}$, so each point x is carried into any other point, but this is no more the case with pairs of equidistant points, or the geodesic sphere at x_0. The latter is not covered by stabilizer K, but rather is foliated into the r-parameter family of K-orbits. These r-radii ($\mathrm{r} = \mathrm{rank}\mathcal{M}$) play the role of a single geodesic distance d(x;y) in the rank-one case.

Let us remark that space $\mathcal{O}_r$ of all flat r-tori $\{\gamma\}$ of a compact symmetric space is itself a smooth manifold, whose dimension depends on $\dim G$ and rank. Group G acts on $\mathcal{O}_r$, turning it into a homogeneous space $K_0\backslash G$, K_0 - stabilizer of $\mathfrak{A}$ in G.

Symmetric spaces (both compact and non-compact) were completely classified by Cartan (see for instance [He]; [Gur6]). Here we bring the basic examples of *irreducible globally symmetric compact spaces*:

(i) $SU(n)/SO(n)$ $\quad\theta: X \to \bar{X}$ (complex conjugation) $\quad r = n-1$

(*ii*) $SU(2n)/Sp(n)$	$\theta(X) = J_n \bar{X} J_n^{-1}$, where J_n is symplectic matrix $\begin{pmatrix} & I \\ -I & \end{pmatrix}$ in $\mathbb{C}^{2n}$	$r = n-1$
(iii) $SU(p+q)/S(U(p)\times U(q))$	$\theta(X) = I_{pq}\bar{X}I_{pq}$, where I_{pq} is the matrix of the indefinite $(p;q)$-form: $\sum_1^p x_j y_j - \sum_{p+1}^{p+q} x_j y_j$, in $\mathbb{C}^{p+q}$;	$r = min(p;q)$
(iv) $SO(p+q)/SO(p)\times SO(q)$	$\theta(X) = I_{pq} X I_{pq}$, same I_{pq} as above in $\mathbb{R}^{p+q}$;	$r = min(p;q)$
(v) $SO(2n)/U(n)$	$\theta(X) = J_n X J_n^{-1}$	$r = [\frac{n}{2}]$
(vi) $Sp(n)/U(n)$	$\theta(X) = \bar{X} = J_n X J_n^{-1}$	$r = n.$

Those along with a few exceptional cases exhaust the list. More general (reducible) spaces can be decomposed into products of irreducibles $\mathcal{M}_1 \times \mathcal{M}_2 \times \ldots$ The simplest example of the sort is quotient $SO(4)/SO(2)\times SO(2) \simeq (S^2\times S^2)/\mathbb{Z}_2$ - the product of two spheres. This example is exceptional among orthogonal types (iv), as all others are irreducible.

Spectra of Laplacians on symmetric spaces are well known:

$$\lambda_\alpha = (\alpha+\rho)^2 - \rho^2,$$

where α varies over the so called *highest weight lattice* $\Lambda = \{\alpha = \alpha_0 + \sum m_i \alpha_i; m_i \in \mathbb{Z}_+\}$ in $\mathfrak{A}$. The latter is spanned by a few basic weights $\{\alpha_0\}$ and all linear integral combinations of positive roots $\{\alpha_i\}$ of $\mathfrak{G}$ (weights of the adjoint action $\mathrm{ad}_X(Y) = [X;Y]$) that lie in $\mathfrak{A}$. Weight $\rho = \frac{1}{2}\sum \alpha_i$ represents the half sum of all positive roots taken with their multiplicities. Let us also mention that all highest weights lie in the *Weyl chamber* a sector in $\mathfrak{A}$ made of the intersection of half-spaces $\Gamma_+(\alpha_j) = \{X: \langle \alpha_j \mid X\rangle > 0\}$, which depends on the choice of the positive root system in $\mathfrak{A}$.

We shall illustrate the foregoing with 2 examples:

1) *The n-sphere:* $S^n \simeq SO(n+1)/SO(n)$. Here subalgebra $\mathfrak{K} \simeq so(n)$, subspace $\mathfrak{P}$ consists of matrices: $X = X_b = \begin{pmatrix} 0 & b \\ -{}^T b & 0 \end{pmatrix}$ with columnar vector $b \in \mathbb{R}^n$; isotropy subgroup $K = SO(n)$ acts on $\mathfrak{P}$ by rotations: $u^{-1} X_b u = X_{u(b)}$. Maximal abelian subalgebras $\mathfrak{A} \subset \mathfrak{P}$ are 1-dimensional (rank 1), and the exponential image of $\mathfrak{A}$ becomes a closed geodesics (great circle γ) in S^n. Lattice $\Lambda \subset \Gamma_+$ (half-line) consists of integers k, and the basic positive root $\{1\}$ has multiplicity $n-1$, hence ρ $\frac{n-1}{2}$.

The k-th eigen of the laplacian Δ_{S^n}, $\lambda_k = k(k+n-1) = (k+\rho)^2 - \rho^2$, and the eigensubspace $\mathcal{H}_k$ (spherical harmonics of degree k) coincides with a π^k-irreducible component of the regular representation R of $G = SO(n+1)$ on S^n of weight $k \simeq (k;0;0;...)$.

2) *Space* $SU(n)/SO(n)$. Here the Cartan $\mathfrak{P}$-component is realized by real symmetric matrices,

$$Z = X + iY,\ X \in so(n) = \mathfrak{K},\ Y \in \mathrm{Sym}_n = \mathfrak{P}.$$

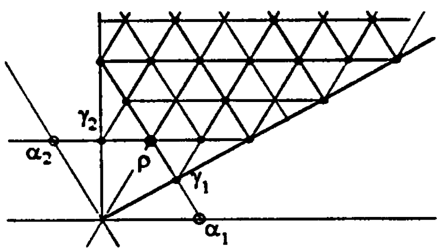

Fig.3. $SU(3)$ weight lattice (marked points) is obtained from 2 basic weights:
$\gamma_1 = (1; -\frac{1}{2}; -\frac{1}{2})$ and $\gamma_2 = (\frac{1}{2};\frac{1}{2}; -1)$
by adding all integral combinations of two basic roots,
$\alpha_1 = (1; -1;0)$ and $\alpha_2 = (0;1; -1)$.
So any α in Λ is equal to
$\alpha = \gamma_{1,2} + m_1\alpha_1 + m_2\alpha_2$.
The half-sum of positive roots ρ coincides with highest positive root
$\alpha_3 = \alpha_1 + \alpha_2$.

Subgroup $K = SO(n)$ acts on $\mathfrak{P}$ by conjugation: $X \to u^{-1}Xu$; Cartan subalgebra $\mathfrak{A} \subset \mathfrak{P}$ consists of all real diagonal matrices, $\mathfrak{A} \simeq \mathbb{R}^{n-1}$ (hence rank $= n-1$). The image of any $\mathfrak{A}$ under the exponential map $\exp:\mathfrak{P} \to$ [illegible] becomes a geodesically flat torus in $\mathcal{M}$: $\gamma = \exp(\mathfrak{A}) \simeq \mathbb{T}^{n-1}$. Positive roots in $\mathfrak{A} = \mathfrak{H} \cap \mathfrak{P} \simeq \mathbb{R}^{n-1}$ are of the form

$$H_{ij} = \mathrm{diag}(0;...1; ... -1;...);\ \text{on the } i\text{-th and } j\text{-th place},$$

and the basis of positive roots con[illegible] $H_j = \mathrm{diag}(...1; -1;...)$. The Weyl chamber is a sector of angle $\frac{\pi}{3}$ on the right of the vertical axis (fig.3). Highest weights are obtained from two basic weights $(1; -\frac{1}{2}; -\frac{1}{2})$ and $(\frac{1}{2};\frac{1}{2}; -1)$ by adding all integer combinations of two basic roots (see fig.3):

$$\alpha = \sum_j k_j H_j = \mathrm{diag}(k_1; k_2 - k_1; ...; -k_{n-1}).$$

The half sum of positive roots: $\frac{1}{2}\rho = \frac{1}{2}(n-1; n-3; ...; -n+1)$, while the inner product (Killing form),

$$\langle H; H' \rangle = tr(ad_H ad_{H'}) = \sum_{i<j} (h_i - h_j)(h_i' - h_j').$$

So the eigenvalues of Δ,

$$\lambda_k = \sum_{i<j} (k_{ij} + \rho_{ij})^2 - \rho_{ij}^2;$$

where $k_{ij}=(k_i-k_{i-1})-(k_j-k_{j-1})$; $\rho_{ij}=\rho_i-\rho_j=2(j-i)$.

6. Schrödinger operators on symmetric spaces. Turning to Schrödinger operators on symmetric spaces, our first goal is to find the proper analog of the band-invariant β_0. As in the n-sphere theory (§3) spectrum of $H=-\Delta+V$ consists of clusters $\{\lambda_k+\mu_{km}:1\le m\le d(\alpha)\}$, resulting from splitting degenerate eigenvalues of Δ. One would like label α to vary over the lattice points $\alpha\in\Gamma_+$ in the Weyl chamber of $\mathfrak{A}$. However, clusters $\{\Lambda_\alpha\}$ of different weights $\{\alpha\}$ may overlap, if eigenvalue $\lambda_\alpha=\lambda_\beta$ are equal. So we need to combine clusters $\{\Lambda_\alpha\}$ corresponding to equal eigenvalues $\{\alpha:\lambda_\alpha=k^2\}$ into larger super-clusters Λ_k. Here label $k=k_\alpha=\sqrt{|\alpha+\rho|^2-\rho^2}$ is a real number. Another apparent difficulty is that the distances between eigenvalues of Δ need not grow as $\alpha\to\infty$, so cluster (or super-clusters) may overlap even for large distinct α. This would limit our results to small (in norm) perturbations V.

We define a sequence of cluster-measures labeled by reals $k=k_\alpha$ to be

$$d\nu_k=\frac{1}{d(k)}\sum\delta(\lambda-\mu_{km}), \tag{13}$$

where d(k) sums multiplicities $\{d_\alpha\}$ of all α in a super-cluster k (i.e. $\lambda_\alpha=k^2$).

One are interested in the asymptotic distribution of measures $\{d\nu_k\}$, when k$\to\infty$. Guillemin [Gui3] studied a similar problem for asymptotic distribution of "normalized" weight diagrams: $\Sigma_\alpha=\{-\alpha\le\beta\le\alpha\}$ of irreducible representations[7] π^α, associated to weights α in Γ_+. He introduced a sequence of discrete distribution-measures for normalized weight-diagrams $\Sigma'=\{\frac{\beta}{|\alpha|}:\beta\in\Sigma_\alpha\}$,

$$d\nu_\alpha=\frac{1}{d(\alpha)}\sum_{\beta\in\Sigma}\delta(\lambda-\frac{\beta}{|\alpha|}),\ \text{on } \mathfrak{A},\ \alpha\in æ<_+ \tag{14}$$

and proved the following

Szegö-type Theorem (Guillemin): *Consider a sequence of normalized weights $\alpha_k/|\alpha_k|\to\alpha_0$ in the Cartan subalgebra $\mathfrak{H}\subset\mathfrak{G}$, the corresponding representation spaces $\mathcal{V}_k=\mathcal{V}(\alpha_k)\subset L^2(G)$ with highest weights $\{\alpha_k\}$, and orthogonal projections $P_k:L^2\to\mathcal{V}_k$. Then*

[7]We recall that any irreducible π^α is determined by a finite set of its weights $\{\beta\}$ (linear functionals on $\mathfrak{H}$), obtained by restricting representation-operators $\{\pi^\alpha{}_H\}$ on all Cartan elements $\{H\in\mathfrak{H}\}$, $\pi^\alpha|\mathfrak{H}\simeq\oplus\langle\beta|H\rangle$. One of them α (highest) uniquely determines π^α, all other $\{\beta\}$ in the weight diagram $\Lambda_\alpha{}^\beta$ are squeezed between -α and α, and transformed one into the other by Weyl elements, a finite group generated by all (reflectional) symmetries of the root system of the pair in $\mathfrak{H}\subset\mathfrak{G}$.

(i) *Sequence of measures* $d\nu_k$ *(14) converges to a continuous distribution* β_0, *supported on the weight-diagram of* α.

(ii) *Distribution* β_0 *coincides with a projection of the natural G-invariant measure* $d_{\mathcal{O}}$ *on the co-adjoint orbit* $\mathcal{O}_0 = \mathcal{O}(\alpha_0) = \{\mathrm{ad}_g(\alpha_0): g \in G\}$, *through* $\alpha_0 \in \mathfrak{H}$,

$$\mathrm{d}\nu_{\alpha_k} \to \mathrm{d}\beta(\lambda) = \mathrm{proj}_{\mathfrak{H}}[\mathrm{d}_{\mathcal{O}}(\ldots)], \text{ as } k\to\infty, \tag{15}$$

where $\mathrm{proj}_{\mathfrak{H}}$ *stands for the natural projection:* $\mathfrak{G}\to\mathfrak{H}$, *determined by the Killing inner product on Lie algebra* $\mathfrak{G}$.

(iii) *For any 0-th order pseudo-differential operator B on G, sequence*

$$\frac{tr(P_k B P_k)}{\mathrm{d}(\alpha)} \to \int_{\mathcal{O}(\alpha)} \mathrm{symb}(B) d_{\mathcal{O}}(x;\xi).$$

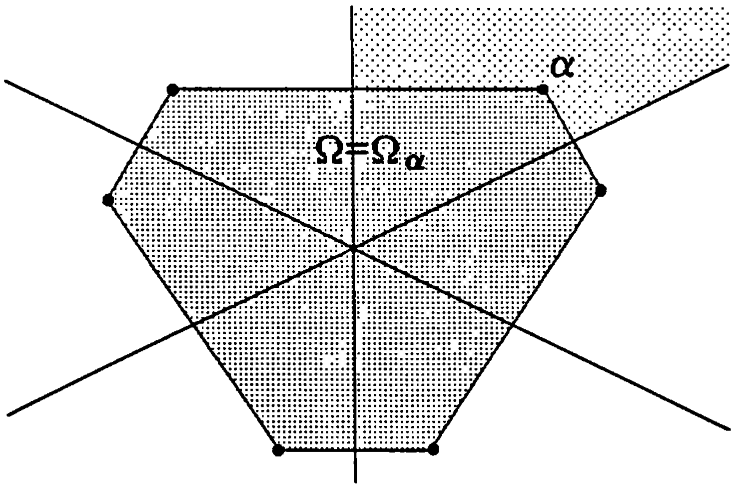

Fig.4 shows a weight diagram of any positive weight α in the Weyl chamber (lightly shaded sector) of . Lie algebra $\mathfrak{G} = su(3)$. The same figure describes restricted weights of symmetric space $SU(3)/SO(3)$. The G-orbit of element α in $su(3)$ projects down onto its weight-diagram (the dark shaded region $\Omega = \Omega_\alpha$), the convex hull of α, reflected by all elements of the Weyl group $\mathbb{W}$ of $su(3)$.

Let us notice that a G-orbit $\mathcal{O} \subset \mathfrak{G}$, passing through α, projects onto the region in $\mathfrak{H}$, bounded by the weight-diagram Λ_α (shaded region Ω in fig.4), and gives certain density on Ω. A similar result holds for symmetric spaces $\mathcal{M} = G/K$, but this time one takes restricted weights: $\alpha \in \mathfrak{A} \subset \mathfrak{P}\cap\mathfrak{H}$, and a G-orbit $\mathcal{O}$ in the cotangent bundle $T^*(M)$, naturally embedded in $\mathfrak{G}$.

As a simple example illustration of Guillemin's Theorem we take the 2-sphere $S^2 \simeq SO(3)/SO(2)$. Here $\alpha = k$ - integer, its weight-diagram $\{\beta = m\}$ coincides with interval $\{-k \le m \le k\}$, the diagram measures,

$$\mathrm{d}\nu_k(x) = \frac{1}{2k+1}\sum_{-k}^{k} \delta(x - \tfrac{m}{k}),$$

clearly converges to the uniform (Lebesgue) distribution $\mathrm{d}x$ on $[-1;1]$, which is easily verified to coincide with the projection of the invariant measure $\{\mathrm{d}S\,|_{\mathcal{O}}\}$ on orbit

$\mathcal{O} \simeq S^2 \subset S^*(S^2)$.

Our main objective are spectral clusters of Schrödinger operators on $\mathcal{M}$ and the corresponding cluster-measures $\{d\nu\}$. We shall establish the analogue of (15), where the role of invariant measure $d_{\mathcal{O}}$ will be played by a generalized *Radon transform* of V. The latter is defined via integration of V over flat tori in $\mathcal{M}$,

$$\widetilde{V}(\gamma) = \int_{\gamma} V \, d^r S.$$

Precisely, we take any sequence of cluster labels $\{k = k_j \to \infty\}$, and consider the limiting set of all normalized sequences of weights $\{\alpha_j\}$, that belong to $\{k_j\}$ (in the sense that $\lambda_{\alpha_j} = k_j^2$),

$$\Sigma = \Sigma(\{k_j\}) = \{\alpha_0 = \lim \alpha_j / |\alpha_j| : \alpha_j \in k_j\}.$$

Now we take a G-invariant set $\mathcal{O}(\Sigma)$, made up of all orbits $\{\mathcal{O}(\alpha_0) : \alpha_0 \in \Sigma\}$.

Theorem 2: *Given an increasing sequence $\{k_j\}$ of cluster-labels, the sequence of the corresponding cluster-measures $d\nu_{k_j}$* converges to a continuous limit,

$$d\nu_{k_j} \to \beta(\lambda) d\lambda; \text{ as } k \to \infty.$$

The resulting density $\beta(\lambda)$ is equal to the distribution function of the generalized Radon transform $\widetilde{V}(\gamma)$ restricted on the G-invariant set [8] $\mathcal{O}(\Sigma) \subset S^*(\mathcal{M})$.

Theorem 2 gives an improved and corrected version of the main result of [Gur7] (Theorem 2). The argument outlined in [Gur7] remains valid, so we refer to the paper for further details. Here we just mention that the method of [Gur7] exploits ergodic averaging of [Gur5] (a modification of the Weinstein's averaging for non-periodic hamiltonians), as well as Guillemin's Szego-type Theorem.

Concluding remarks.

• Theorem 2 represents a first step in a long range program of extending the n-sphere Schrödinger theory to higher rank spaces. It shows that the role of closed geodesics and the related Radon transform would be played by the flat tori. Next steps should include derivation of higher band-invariants $\{\beta_j\}$ (cf. [Ur]), and the *zonal theory* along the lines of [Gur2-4]. Let us remark that symmetric spaces possess the natural analog of zonal potentials, the K-invariant functions on $\mathcal{M} = K \backslash G$. These are known to

[8] Let us remark an important difference between rank-one and higher ranks: the former have a single G-orbit to cover the entire co-sphere bundle $S^*(\mathcal{M})$, while the latter have continuous families of such orbits.

play an important role in the representation theory of semisimple groups.

• There is a number of related problems for Laplacians on locally-symmetric hyperbolic spaces $\mathcal{M} = \mathbb{H}/\Gamma$. Here $\mathbb{H} = K\backslash G$ denotes a quotient of a noncompact semisimple Lie group, modulo the maximal compact subgroup, and Γ - a uniform lattice in G, so both quotients $\Gamma\backslash G$ and $\Gamma\backslash\mathbb{H}$ are compact. The classical examples is the Poincare plane $\mathbb{H} = SL_2/SO(2)$, modulo a Fuschian subgroup Γ. The connection between spectrum of the Laplacian and the closed geodesics is well known for Riemann surfaces, as a consequence of the Selberg-trace formula (see [Sel];[Mc]). We conjecture, that flat tori would play the role of closed geodesics and would contribute "asymptotic lattices" of eigenvalues to spec($\Delta_{\mathcal{M}}$). Such trace-formula should be connected in some to the known higher-rank versions of "Selberg trace formula" (cf. [DKV]; [HST]; [Sel]; [Va]). It would be interesting to recast such results into the spectral form (cf. [Mc]) and to deduce a proper analog of the Poincare map and Maslov-Morse indices.

• In connection with the last remark let us mention another class of tori known in spectral theory. These are the phase-space invariant tori of intermediate dimensions between 1-D (closed path) and n-D (invariant Lagrangians) studied by Voros [Vo]. He showed the connection between "path-quantization" and "EBK-quantization", and derived a suitable form of EBK-rules in this context. These results are interesting, but hardly applicable in our geometric setup. Firstly, it is not clear whether flat tori could be lifted from $\mathcal{M}$ to isotropic tori in the phase-space $T^*(\mathcal{M})$. Secondly, even if this were possible, one is not likely to get a "nice foliations" of $T^*(\mathcal{M})$ by lifted tori. In this regard we raise yet more general problem. Given a manifold $\mathcal{M}$ with a stable (in an apropriate sense) flat torus γ is there a quasimode construction, based on such γ? In other words could one construct an embedding of the flat Laplacian Δ_γ into $\Delta_{\mathcal{M}}$, the same way one does for closed path γ (cf. [Co3]). If this were possible one could produce the trace-formula based on tori, as well as asymptotic lattices of eigenvalues, rather than arithmetic sequences.

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