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BOUNDARY LAYERS AND INCOMPRESSIBLE NAVIER-STOKES-FOURIER LIMIT OF THE BOLTZMANN EQUATION IN A BOUNDED DOMAIN

NING JIANG AND NADER MASMOUDI

ABSTRACT. In this note, we review the recent work [23] on the boundary layer and incompressible Navier-Stokes-Fourier limit of the Boltzmann equation with a general cut-off collision in a bounded domain. Appropriately scaled families of DiPerna-Lions-(Mischler) renormalized solutions with Maxwell reflection boundary conditions are shown to have fluctuations that converge as the Knudsen number goes to zero. Every limit point is a weak solution to the Navier-Stokes-Fourier system with different types of boundary conditions depending on the ratio between the accommodation coefficient and the Knudsen number.

The main new result is that this convergence is strong in the case of Dirichlet boundary condition. Indeed, we prove that the acoustic waves are damped immediately, namely they are damped in a boundary layer in time. This damping is due to the presence of viscous and kinetic boundary layers in space. As a consequence, we also justify the first correction to the infinitesimal Maxwellian that one obtains from the Chapman-Enskog expansion with Navier-Stokes scaling.

1. Introduction

The hydrodynamic limits from the Boltzmann equation got a lot of interest in the previous two decades. Hydrodynamic regimes are those where the Knudsen number ε is small. The Knudsen number is the ratio of the mean free path and the macroscopic length scales. The incompressible Navier-Stokes-Fourier (NSF) system can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the number density F about an absolute Maxwellian M are scaled to be on the order ε, see [2].

The program that justifies the hydrodynamic limits from the Boltzmann equation in the framework of DiPerna-Lions [11] was initiated by Bardos-Golse-Levermore [2, 3] in late 80's. Since then, there has been lots of contributions to this program [4, 12, 18, 19, 22, 27, 30, 31, 33, 39]. In particular the work of Golse and Saint-Raymond [18] is the first complete rigorous justification of NSF limit from the Boltzmann equation in a class of bounded collision kernels, without making any nonlinear weak compactness hypothesis. They have recently extended their result to the case of hard potentials [19]. With some new nonlinear estimates, Levermore and Masmoudi [27] treated a broader class of collision kernels which includes all hard potential cases and, for the first time in this program, soft potential cases.

All of the above mentioned works were carried out in either the periodic spatial domain or the whole space, except for [33] and [40]. In [33], the linear Stokes-Fourier system was recovered with the same collision kernels assumption as in [12], while in [40], the Navier-Stokes limit was derived with the same kernels assumption as in [19], i.e. hard potential kernels. In [33] and [40], the fluctuations of renormalized solutions to the Boltzmann equation in a bounded domain (see [37]) was proved to pass to the limit and recovered fluid boundary conditions, either Dirichlet, or Navier slip boundary condition, depending on the relative sizes of the accommodation coefficient and the Knudsen number.

The dependance of the boundary conditions of the limiting fluid equations on the relative importance of the accommodation coefficient and the Knudsen number was observed by Sone and his collaborators. Their results, mostly formal, are presented in Chapter 3 and 4 in [44] for several types of kinetic boundary conditions. The work [33] and [40] rigorously justified the incompressible Stokes and Navier-Stokes equations from Boltzmann equation imposed with Maxwell reflection boundary condition.

In his survey paper [45], Ukai proposed the following question: *"As far as the Boltzmann equation in a bounded domain is concerned, some progress has been made recently. In [37] , the convergence of the Boltzmann equation to the (linear) Stokes-Fourier equation was proved together with the convergence of the boundary conditions. It is a big challenging problem to extend the result to the nonlinear case and to strength the convergence so as to make visible the boundary layer."* (In the above citation of Ukai's survey, the reference *[37]* is the Saint-Raymond and Masmoudi's paper [33].)

In this work, we study the incompressible NSF limit in a bounded domain from the Boltzmann equation with the Maxwell reflection boundary condition in which the accommodation might depend on the Knudsen number. We consider a bounded domain $\Omega \subset \mathbb{R}^{\mathrm{D}}$, $\mathrm{D} \geq 2$, with boundary $\partial\Omega \in C^2$. The NSF system governs the fluctuations of mass density, bulk velocity, and temperature $(\rho, \mathrm{u}, \theta)$ about their spatially homogeneous equilibrium values in a Boussinesq regime. Specifically, after a suitable choice of units, these dimensionless fluctuations satisfy the incompressibility and Boussinesq relations

$$\nabla_x \cdot \mathrm{u} = 0\,, \quad \rho + \theta = 0\,, \tag{1.1}$$

while their evolution is determined by the Navier-Stokes and heat equations

$$\begin{aligned} \partial_t \mathrm{u} + \mathrm{u}\cdot\nabla_x \mathrm{u} + \nabla_x p = \nu \Delta_x \mathrm{u}\,, \quad \mathrm{u}|_{t=0} = \mathrm{u}_0\,, \\ \partial_t \theta + \mathrm{u}\cdot\nabla_x \theta = \tfrac{2}{\mathrm{D}+2} \kappa \Delta_x \theta\,, \quad \theta|_{t=0} = \theta_0\,, \end{aligned} \tag{1.2}$$

where $\nu > 0$ is the kinematic viscosity and $\kappa > 0$ is the heat thermal conductivity.

Traditionally, two types of natural physical boundary conditions could be imposed for the incompressible NSF system (1.2). The first is the homogeneous Dirichlet boundary condition, namely,

$$\mathrm{u} = 0\,, \quad \theta = 0 \quad \text{on} \ \ \mathbb{R}^+ \times \partial\Omega\,. \tag{1.3}$$

The other is the so-called Navier slip boundary condition, which was proposed by Navier [38]:

$$\begin{aligned} &[2\nu d(\mathrm{u})\cdot\mathrm{n} + \chi \mathrm{u}]^{\tan} = 0\,, \quad \mathrm{u}\cdot\mathrm{n} = 0 \quad \text{on} \ \ \mathbb{R}^+ \times \partial\Omega\,, \\ &\kappa \partial_{\mathrm{n}} \theta + \chi \tfrac{\mathrm{D}+1}{\mathrm{D}+2} \theta = 0 \quad \text{on} \ \ \mathbb{R}^+ \times \partial\Omega\,, \end{aligned} \tag{1.4}$$

where $d(\mathrm{u}) = \frac{1}{2}(\nabla_x \mathrm{u} + \nabla_x \mathrm{u}^\top)$ denotes the symmetric part of the stress tensor and ∂_{n} denotes the directional derivative along the outer normal vector $\mathrm{n}(x), x \in \partial\Omega$. In the above Navier boundary condition, $\chi > 0$ is the reciprocal of the slip length which depends on the material of the container.

In the current work, for general cut-off collision kernels, namely in the framework of [27], we justify the NSF system. Regarding the *weak* convergence results, our proof is basically the same as in [33] and [40]: the boundary conditions of the limiting NSF system depend on the ratio of the accommodation coefficient and the Kundsen number, namely when $\frac{\alpha_\varepsilon}{\varepsilon} \to \infty$ as $\varepsilon \to 0$, Dirichlet condition is derived, while when $\frac{\alpha_\varepsilon}{\varepsilon} \to \sqrt{2\pi}\chi$, the Navier-slip boundary condition is derived. The main difference is that [40] used the same renormalizations of [19], applicable for hard potentials, while in the current work, we use the renormalization of [27], which works for more general cut-off kernels, including soft potentials.

The main novelty of the current work is the treatment of the Dirichlet boundary condition case. Indeed, we prove that when $\frac{\alpha_\varepsilon}{\varepsilon} \to \infty$, the convergence is *strong*. Furthermore, as a consequence of this strong convergence, the first correction to the infinitesimal Maxwellian, which is a quadratic term obtained from the Chapman-Enskog expansion with the Navier-Stokes scaling, is rigorously justified. We point out that in all the previous works mentioned above, the convergence is in w-L^1, unless the initial data is well-prepared, i.e. is hydrodynamic and satisfies the Boussinesq and incompressibility relations. This weak convergence is caused by the persistence of fast acoustic waves. In the Navier-Stokes regime, the Reynold number Re is order $O(1)$, then the von Kámán relation $\varepsilon = \frac{Ma}{Re}$ implies that in the fluid limit $\varepsilon \to 0$, the Mach number Ma must go to zero. As is well know physically, one expects that as $Ma \to 0$, fast acoustic waves are generated and carry the energy of the potential part of the flow. For the periodic flows, or for some particular boundary conditions such as Navier condition (1.4), these waves subsist forever and their frequency grows with ε. Mathematically, this means that the convergence is only *weak*. This phenomenon happens in many singular limits of fluid equations among which we only mention [28, 29].

One of the ingredients of the convergence proof is the treatment of the acoustic waves which are highly oscillating. A compensated compactness type argument was used by Lions and Masmoudi [30] to prove that these acoustic waves have no contribution on the equation satisfied by the weak limit. This argument was previously used in the compressible incompressible limit [29].

In [10], a striking phenomenon, namely the damping of acoustic waves caused by the Dirichlet boundary condition was found by Desjardins, Grenier, Lions, and Masmoudi in considering the incompressible limit of the isentropic compressible Navier-Stokes equations. In the case of a viscous flow in a bounded domain with Dirichlet boundary condition, and under a generic assumption on the domain (related to the so-called Schiffer's conjecture and the Pompeiu problem [9]), they showed that the acoustic waves are instantaneously (asymptotically) damped, due to the formation of a thin boundary layer in time. This layer is caused by a boundary layer in space and dissipates the energy carried by the acoustic waves. From a mathematical point of view, strong convergence was obtained.

Inspired by the idea of [10], the current paper considers the much more involved kinetic-fluid coupled case. We prove that if the accommodation coefficient is bigger than the Knudsen number, there is no need for the argument in [29] since we can prove that the acoustic waves are damped instantaneously. Our work is based on the construction of viscous and kinetic Knudsen boundary layers of size $\sqrt{\varepsilon}$ and ε. The main idea is to use a family of test functions which solve approximately a scaled stationary linearized Boltzmann equation and can capture the propagation of the fast acoustic waves. These test functions are constructed through considering a family of approximate eigenfunctions of a *dual* operator with a *dual* kinetic boundary condition with respect to the original Boltzmann equation. The approximate eigenvalue is the sum of several terms with different order of ε: the leading term is purely imaginary, which describes the acoustic mode, and the real part of the next order term is strictly *negative* which gives the strict dissipation when applying the test functions to the renormalized Boltzmann equation.

In contrast to [10], the approximate eigenfunctions include interior part and two boundary layers: fluid viscous layer and kinetic Knudsen layer, while in [10], only a fluid boundary layer was necessary. Another important difference is that a generic assumption on the domain had to be made in [10] (in particular there are modes which are not damped in the disc), while in the current work, this assumption is not needed. The reason is that we deal with the full acoustic system, namely including the temperature. The NSF system has also some dissipation in the temperature equation which is ignored in the isentropic model. (in particular this dissipation property holds in the case of the ball). This was also considered in [24] in which we reinforced the result of [10].

When the accommodation coefficient α_ε is asymptotically larger than the Knudsen number ε in the sense that $\alpha_\varepsilon/\varepsilon \to \infty$ as $\varepsilon \to 0$, the fluid limit is the NSF equations with Dirichlet boundary condition. For example, we can assume $\alpha_\epsilon = \chi\varepsilon^\beta$ with $0 \leq \beta < 1$. We found that $\beta = 1/2$ is a threshold in the sense that the kinetic-fluid coupled boundary layers behave differently for $0 \leq \beta < 1/2$ and $1/2 \leq \beta < 1$, but for both cases the kinetic-fluid layers have damping effect. The current paper focuses on the threshold case $\beta = 1/2$ and we leave the other cases for a separate paper due to the more complex construction of the boundary layers.

One of the difficulties of the construction happens in the case the Laplace operator $-\Delta_x$ with Neumann boundary condition has multiple eigenvalues. As a consequence, the dimension of the null space of the the operator $\mathcal{A} - i\lambda_0^k$ is greater than one, where $\mathcal{A}$ denotes the acoustic operator, and $\frac{\mathrm{D}}{\mathrm{D}+2}[\lambda_0^k]^2$ are eigenvalues for $k \in \mathbb{N}$ (for details see Section 5.2). Thus, as each stage of the construction of boundary layers, the terms in the null space of $\mathcal{A} - i\lambda_0^k$ can not be determined uniquely. To completely determine all the terms in the ansaza of boundary layers, we have to add some orthogonality conditions. Surprisingly, all these orthogonality conditions are consistent, at least for the threshold case $\beta = \frac{1}{2}$ treated in the current paper. Similar idea has been used in [24] which can be applied to the compressible-incompressible limit of the full Navier-Stokes-Fourier system in a bounded domain.

A key role is played by the linearized kinetic boundary layer equation in the coupling of viscous and kinetic layers. More specifically, its solvability provides the boundary conditions of the fluid variables in the interior and viscous boundary layers which satisfy the acoustic systems with source terms and second order ordinary differential equations respectively. This linearized kinetic boundary layer equation has been studied extensively (see [1, 8, 15, 14, 46]). Applying the boundary layer equations to construct the two layer eigenfunctions is the main novelty of the current paper. To the best of our knowledge these two layer eigenfunctions are new even in the applied literature.

2. Boltzmann Equation in Bounded Domain

Here we introduce the Boltzmann equation in a bounded domain, only so far as to set our notations, which are essentially those of [3] and [33]. More complete introduction to the Boltzmann equation can be found in [6, 7, 16, 44].

2.1. **Maxwell Boundary Condition.** We consider Ω, a smooth bounded domain of $\mathbb{R}^{\mathrm{D}}$, and $\mathcal{O} = \Omega \times \mathbb{R}^{\mathrm{D}}$, the space-velocity domain. Let $\mathrm{n}(x)$ be the outward unit normal vector at $x \in \partial\Omega$ and let $\mathrm{d}\sigma_x$ be the Lebesgue measure on the boundary $\partial\Omega$. We define the outgoing and incoming sets Σ_+ and Σ_- by

$$\Sigma_\pm = \{(x, v) \in \Sigma : \pm \mathrm{n}(x)\cdot v > 0\} \quad \text{where} \quad \Sigma = \partial\Omega \times \mathbb{R}^{\mathrm{D}} .$$

Denoted by γF the trace of F over Σ, the boundary condition takes the form of a balance between the values of the outgoing and incoming parts of γF, namely $\gamma_\pm F = \mathbf{1}_{\Sigma_\pm}\gamma F$. In order to describe the interaction between particles and the wall, Maxwell [34] proposed in 1879 the following phenomenological law which splits into a local reflection and a diffuse reflection

$$\gamma_- F = (1-\alpha) L\gamma_+ F + \alpha K \gamma_+ F \quad \text{on} \quad \Sigma_- , \tag{2.1}$$

where $\alpha \in [0\,,1]$ is a constant, called the "accommodation coefficient." The local reflection operator L is given by

$$L\phi(x\,,v) = \phi(x\,, R_x v) , \tag{2.2}$$

where $R_x v = v - 2\,[\mathrm{n}(x)\cdot v]\,\mathrm{n}(x)$ is the velocity before the collision with the wall. The diffuse reflection operator K is given by

$$K\phi(x\,,v) = \sqrt{2\pi}\,\widetilde{\phi}(x) M(v) ,$$

where $\widetilde{\phi}$ is the outgoing mass flux

$$\widetilde{\phi}(x) = \int\limits_{v\cdot\mathrm{n}(x)>0} \phi(x\,,v) v\cdot\mathrm{n}(x)\,\mathrm{d}v ,$$

and M is the absolute Maxwellian $M(v) = \frac{1}{(2\pi)^{\mathrm{D}/2}} \exp\left(-\frac{1}{2}|v|^2\right)$.

2.2. **Nondimensionalized Form of the Boltzmann Equation.** We consider a sequence of renormalized solutions $F_\varepsilon(t,x,v)$ to the rescaled Boltzmann equation

$$\begin{aligned} \varepsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon &= \frac{1}{\varepsilon}\mathcal{B}(F_\epsilon\,,F_\epsilon) \quad \text{on} \quad \mathbb{R}^+\times\mathcal{O}\,,\\ F_\epsilon(0\,,x\,,v) &= F_\epsilon^{\text{in}}(x\,,v)\geq 0 \quad \text{on} \quad \mathcal{O}\,,\\ \gamma_- F_\epsilon &= (1-\alpha)L\gamma_+F_\epsilon + \alpha K\gamma_+ F_\epsilon \quad \text{on} \quad \mathbb{R}^+\times\Sigma_-\,. \end{aligned} \tag{2.3}$$

The Boltzmann collision operator $\mathcal{B}$ acts only on the v argument of F and is formally given by

$$\mathcal{B}(F\,,F) = \iint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^{\mathrm{D}}} (F_1'F' - F_1F)b(\omega\,,v_1-v)\,\mathrm{d}\omega\,\mathrm{d}v_1\,.$$

The collision kernel b is a positive, locally integrable function and has the classical form

$$b(\omega\,,v) = |v|\Sigma(|\omega\cdot\hat{v}|\,,|v|)\,,$$

where $\hat{v}=v/|v|$ and Σ is the specific differential cross section. This symmetry implies that the quantity $\int b(\omega\,,v)\,\mathrm{d}\omega$ is a function of $|v|$ only. The DiPerna-Lions theory requires that b satisfies

$$\lim_{|v|\to\infty}\frac{1}{1+|v|^2}\iint\limits_{\mathbb{S}^{\mathrm{D}-1}\times K} b(\omega\,,v_1-v)\,\mathrm{d}\omega\,\mathrm{d}v_1 = 0 \tag{2.4}$$

for any compact set $K\subset\mathbb{R}^{\mathrm{D}}$. There are some additional assumptions on b needed as in [27].

2.3. **Navier-Stokes Scaling.** The incompressible NSF system can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the kinetic densities F_ε about the absolute Maxwellian M are scaled to be of order ε. More precisely, we take

$$F_\varepsilon = MG_\varepsilon = M(1+\varepsilon g_\varepsilon)\,. \tag{2.5}$$

In terms of g_ε the system (2.3) finally reads

$$\begin{aligned} \varepsilon\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon + \tfrac{1}{\varepsilon}\mathcal{L}g_\varepsilon &= \mathcal{Q}(g_\varepsilon\,,g_\varepsilon) \quad \text{on} \quad \mathbb{R}^+\times\mathcal{O}\,,\\ g_\varepsilon(0\,,x\,,v) &= g_\varepsilon^{\text{in}}(x\,,v) \quad \text{on} \quad \mathcal{O}\,,\\ \gamma_- g_\varepsilon &= (1-\alpha)L\gamma_+g_\varepsilon + \alpha\langle\gamma_+g_\varepsilon\rangle_{\partial\Omega} \quad \text{on} \quad \mathbb{R}^+\times\Sigma_-\,. \end{aligned} \tag{2.6}$$

2.4. **DiPerna-Lions-(Mischler) Solutions.** We will work in the setting of renormalized solutions which were initially constructed by DiPerna and Lions [11] over the whole space $\mathbb{R}^{\mathrm{D}}$ for any initial data satisfying natural physical bounds. Recently, their result was extended to the case of a bounded domain by Mischler [35, 36, 37] with general Maxwell boundary conditions (2.1).

The DiPerna-Lions-(Mishler) theory does not yield solutions that are known to solve the Boltzmann equation in the usual weak sense. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial value problems:

$$\begin{aligned} (\varepsilon\partial_t + v\cdot\nabla_x)\Gamma(G_\epsilon) &= \frac{1}{\varepsilon}\Gamma'(G_\epsilon)\mathcal{Q}(G_\epsilon\,,G_\epsilon) \quad \text{on} \quad \mathbb{R}^+\times\mathcal{O}\,,\\ G_\epsilon(0\,,\cdot\,,\cdot) &= G_\epsilon^{\text{in}}\geq 0 \quad \text{on} \quad \mathcal{O}\,. \end{aligned} \tag{2.7}$$

Here the admissible function $\Gamma:[0\,,\infty)\to\mathbb{R}$ is continuously differentiable and for some constant $C_\Gamma<\infty$ its derivative satisfies

$$|\Gamma'(z)|\sqrt{1+z}\leq C_\Gamma\,. \tag{2.8}$$

The weak formulation of the renormalized Boltzmann equation (2.7) is given by

$$
\begin{aligned}
&\varepsilon \int_{\Omega} \langle \Gamma(G_\epsilon(t_2))Y\rangle \,\mathrm{d}x - \varepsilon \int_{\Omega} \langle \Gamma(G_\epsilon(t_1))Y\rangle \,\mathrm{d}x \\
&- \int_{t_1}^{t_2} \int_{\Omega} \langle \Gamma(G_\epsilon) v\cdot\nabla_x Y\rangle \,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \Gamma(\gamma G_\epsilon) Y[\mathrm{n}(x)\cdot v]\rangle \,\mathrm{d}\sigma_x\,\mathrm{d}t \\
&= \frac{1}{\varepsilon}\int_{t_1}^{t_2} \int_{\Omega} \langle \Gamma'(G_\epsilon)\mathcal{Q}(G_\epsilon\,, G_\epsilon) Y\rangle \,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}
\tag{2.9}
$$

for every $Y \in C^1 \cap L^\infty(\bar{\Omega}\times\mathbb{R}^{\mathrm{D}})$ and every $[t_1\,, t_2] \subset [0\,,\infty]$. Moreover, the boundary condition is also understood in the renormalized sense:

$$
\Gamma(\gamma_- G_\epsilon) = \Gamma\left((1-\alpha)L\gamma_+ G_\epsilon + \alpha\widetilde{F_\epsilon}\right) \quad \text{on} \quad \mathbb{R}^+ \times \Sigma_-\,, \tag{2.10}
$$

where the equality holds almost everywhere and in the sense of distribution.

3. Statement of the Main Results

3.1. **Dirichlet Boundary Condition.** The main theorem of this work is the following strong convergence to the NSF system with Dirichlet boundary condition when the accommodation coefficient α_ϵ is much larger than the Knudsen number ε, i.e. $\frac{\alpha_\epsilon}{\varepsilon} \to \infty$ as $\varepsilon \to 0$.

Theorem 3.1. (Dirichlet Boundary Condition) *Let b be a collision kernel that satisfies conditions. Let $g_\varepsilon^{\mathrm{in}}$ be the associated family of fluctuations given by $G_\varepsilon^{\mathrm{in}} = 1 + \varepsilon g_\varepsilon^{\mathrm{in}}$. Assume that the families $G_\varepsilon^{\mathrm{in}}$ and $g_\varepsilon^{\mathrm{in}}$ satisfy*

$$
H(G_\varepsilon^{\mathrm{in}}) \le C^{\mathrm{in}}\varepsilon^2\,, \tag{3.1}
$$

and

$$
\lim_{\varepsilon\to 0}\left(\langle g_\varepsilon^{\mathrm{in}}\rangle\,, \langle v g_\varepsilon^{\mathrm{in}}\rangle\,, \langle(\tfrac{|v|^2}{\mathrm{D}} - 1) g_\varepsilon^{\mathrm{in}}\rangle\right) = (\rho^{\mathrm{in}}, \mathrm{u}^{\mathrm{in}}, \theta^{\mathrm{in}})\,, \tag{3.2}
$$

in the sense of distributions for some $(\rho^{\mathrm{in}}, \mathrm{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(\mathrm{d}x\,;\mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R})$. Let G_ε be any family of DiPerna-Lions renormalized solutions to the Boltzmann equation that have $G_\varepsilon^{\mathrm{in}}$ as initial values, and the accommodation coefficient α_ϵ satisfies

$$
\alpha_\varepsilon = \sqrt{2\pi}\,\chi\,\sqrt{\varepsilon}\,. \tag{3.3}
$$

Then the family of fluctuations g_ε given by (2.5) is relatively compact in $L^1_{loc}(\mathrm{d}t; L^1(\sigma M \mathrm{d}v\mathrm{d}x))$. Every limit point g of g_ε has the infinitesimal Maxwellian form

$$
g = v\cdot\mathrm{u} + \left(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}+2}{2}\right)\theta\,, \tag{3.4}
$$

where $(\mathrm{u},\theta) \in C([0,\infty); L^2(\mathrm{d}x\,;\mathbb{R}^{\mathrm{D}}\times\mathbb{R})) \cap L^2(\mathrm{d}t\,; H^1(\mathrm{d}x\,;\mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$ with mean zero over Ω, and it satisfies the NSF system with Dirichlet boundary condition (1.1), (1.2), *and* (1.3), *where kinematic viscosity ν and thermal conductivity κ are given by*

$$
\nu = \tfrac{1}{(\mathrm{D}-1)(\mathrm{D}+2)}\langle \widehat{\mathrm{A}} : \mathcal{L}\widehat{\mathrm{A}}\rangle\,, \quad \kappa = \tfrac{1}{\mathrm{D}}\langle \widehat{\mathrm{B}}\cdot\mathcal{L}\widehat{\mathrm{B}}\rangle\,. \tag{3.5}
$$

The initial data is given by

$$
\mathrm{u}^0 = \mathbb{P}\mathrm{u}^{\mathrm{in}}\,, \quad \theta^0 = \tfrac{\mathrm{D}}{\mathrm{D}+2}\theta^{\mathrm{in}} - \tfrac{2}{\mathrm{D}+2}\rho^{\mathrm{in}}\,. \tag{3.6}
$$

Here the operator $\mathbb{P}$ is the Leray's projection on the space of divergence free vector fields. Moreover, every subsequence g_{ε_k} of g_ε that converges to g as $\varepsilon_k \to 0$ also satisfies

$$
\begin{aligned}
\langle v g_{\varepsilon_k}\rangle &\to \mathrm{u} \quad \textit{in} \quad L^p_{loc}(\mathrm{d}t; L^1(\mathrm{d}x;\mathbb{R}^{\mathrm{D}}))\,, \\
\langle(\tfrac{1}{\mathrm{D}}|v|^2 - 1) g_{\varepsilon_k}\rangle &\to \theta \quad \textit{in} \quad L^p_{loc}(\mathrm{d}t; L^1(\mathrm{d}x;\mathbb{R})) \quad \textit{for every} \quad 1 \le p < \infty\,.
\end{aligned}
\tag{3.7}
$$

Furthermore, $\frac{1}{\varepsilon}\mathcal{P}^{\perp} g_{\varepsilon}$ *is relatively compact in* $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\sigma M \mathrm{d}v\mathrm{d}x))$. *For every subsequence* ε_k *so that* g_{ε_k} *converges to* g*,*

$$\begin{aligned}\frac{1}{\varepsilon}\mathcal{P}^{\perp} g_{\varepsilon_k} \to & \tfrac{1}{2}\mathrm{A} : \mathrm{u}\otimes\mathrm{u} + \mathrm{B}\cdot\mathrm{u}\theta + \tfrac{1}{2}\mathrm{C}\theta^2 \\ & - \widehat{\mathrm{A}} : \nabla_x \mathrm{u} - \widehat{\mathrm{B}}\cdot\nabla_x\theta \,, \quad \text{in } \ w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\sigma M \mathrm{d}v\mathrm{d}x))\,,\end{aligned} \tag{3.8}$$

as $\varepsilon_k \to 0$.

Remark: In the formal Chapman-Enskog expansion,

$$g_{\varepsilon} = g + \varepsilon\mathcal{P}^{\perp} g_1 + \varepsilon\mathcal{P} g_1 + \varepsilon^2 g_2 + \cdots \,,$$

where g is given by (3.4) and $\mathcal{P}^{\perp} g_1$ is the righthand side term in (3.8). In previous works [18, 19, 27], under the assumptions (3.1) and (3.2), the convergence to (3.4) and (3.7) are only in $w\text{-}L^1$. So the convergence to the quadratic term (3.8), which is the first correction to the infinitesimal Maxwellian that one obtains from the Chapman-Enskog expansion with the Navier-Stokes scaling, could not be obtained. In Theorem 3.1, by showing the acoustic waves are instantaneously damped, we justify not only the strong convergence to the leading order term g, but also weak convergence to the kinetic part of the next order corrector (3.8).

3.2. **Navier Boundary Condition.** The second result is about Navier boundary condition. For this case, although the coupled viscous boundary layer and the Knudsen layer still have dissipative effect, however, the damping happens a longer time scale $O(1)$. Consequently, unlike the Dirichlet boundary condition case, the fast acoustic waves can be damped, but *not instantaneously*. Nevertheless, we can show the weak convergence result, thus justify the NSF limit with slip Navier boundary condition, while the linear Stokes-Fourier limit was justified in [33].

Theorem 3.2. (Navier Boundary Condition) *With the same assumptions with Theorem 3.1, except that the accommodation coefficients satisfy*

$$\frac{\alpha_{\epsilon}}{\sqrt{2\pi}\,\varepsilon} \to \chi\,, \quad \text{as} \quad \varepsilon \to 0\,. \tag{3.9}$$

Then the family g_{ε} *is relatively compact in* $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\sigma M \mathrm{d}v\mathrm{d}x))$. *Every limit point* g *of* g_{ε} *in* $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\sigma M \mathrm{d}v\mathrm{d}x))$ *has the infinitesimal Maxwellian form as* (3.4) *in which* $(\mathrm{u},\theta) \in C([0,\infty); L^2(\mathrm{d}x\,; \mathbb{R}^{\mathrm{D}}\times\mathbb{R})) \cap L^2(\mathrm{d}t; H^1(\mathrm{d}x\,; \mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$ *is a Larey solution of the NSF system with Navier boundary condition* (1.1), (1.2), *and* (1.4), *where kinematic viscosity* ν *and thermal conductivity* κ *are given by* (3.5), *the initial data is given by* (3.6).

Moreover, every subsequence g_{ε_k} *of* g_{ε} *that converges to* g *as* $\varepsilon_k \to 0$ *also satisfies*

$$\begin{aligned}\mathbb{P}\langle v g_{\varepsilon_k}\rangle &\to \mathrm{u} \quad \text{in } \ C([0,\infty)\,; \mathcal{D}'(\Omega\,; \mathbb{R}^{\mathrm{D}}))\,, \\ \langle (\tfrac{1}{\mathrm{D}+2}|v|^2 - 1) g_{\varepsilon_k}\rangle &\to \theta \quad \text{in } \ C([0,\infty)\,; w\text{-}L^1(\Omega\,; \mathbb{R}))\,.\end{aligned} \tag{3.10}$$

Remark: For the Navier-slip boundary condition case, since the convergence is weak, the convergence (3.8), i.e. the justification of the first correction to the infinitesimal Maxwellian in the Chapman-Enskog expansion can not be obtained.

4. Acoustic operator and analysis of the Kinetic Boundary Layer Equation

In this section, we collect results about the acoustic operator and the linear kinetic boundary layer equation which will be used to determine the boundary conditions of the fluid variables.

We first define the acoustic operator $\mathcal{A}$:

$$\mathcal{A}\begin{pmatrix}\rho\\ \mathrm{u}\\ \theta\end{pmatrix} = \begin{pmatrix}\nabla_x\cdot\mathrm{u}\\ \nabla_x(\rho+\theta)\\ \frac{2}{\mathrm{D}}\nabla_x\cdot\mathrm{u}\end{pmatrix}\,, \tag{4.1}$$

over the domain

$$\mathrm{Dom}(\mathcal{A}) = \{U = (\rho\,, \mathrm{u}\,, \theta) \in \mathbb{V} : \mathrm{u}\cdot\mathrm{n} = 0 \text{ on } \partial\Omega\}\ .$$

The null space of $\mathcal{A}$ and its orthogonal with respect to the usual inner product are characterized as

$$\mathrm{Null}(\mathcal{A}) = \{(-\varphi\,, \mathrm{w}\,, \varphi) \in \mathbb{V} : \nabla_x\cdot\mathrm{w} = 0 \quad \text{and} \quad \mathrm{w}\cdot\mathrm{n} = 0 \quad \text{on} \quad \partial\Omega\}\,, \tag{4.2}$$

and

$$\mathrm{Null}(\mathcal{A})^\perp = \{(\rho\,, \mathrm{u}\,, \theta) \in \mathbb{V} : \theta = \tfrac{2}{\mathrm{D}}\rho\,, \mathrm{u} = \nabla_x\phi\,, \quad \text{for some} \quad \phi \in H^1(\Omega)\}\,, \tag{4.3}$$

respectively. Because $\mathrm{Null}(\mathcal{A})$ includes the incompressibility and Boussinesq relations, we call it *incompressible* regime. We will see in the next subsection that $\mathrm{Null}(\mathcal{A})^\perp$ is spanned by the eigenspaces of the acoustic operator $\mathcal{A}$, so we call it *acoustic* regime.

For any $U = (\rho\,, \mathrm{u}\,, \theta) \in \mathbb{H}$, we can define Π and $\Pi^\perp$ the projections to the incompressible regime $\mathrm{Null}(\mathcal{A})$ and acoustic regime $\mathrm{Null}(\mathcal{A})^\perp$ respectively as follows:

$$\Pi U = \left(\tfrac{2}{\mathrm{D}+2}\rho - \tfrac{\mathrm{D}}{\mathrm{D}+2}\theta\,, \mathbb{P}\mathrm{u}\,, \tfrac{\mathrm{D}}{\mathrm{D}+2}\theta - \tfrac{2}{\mathrm{D}+2}\rho\right)\,,$$

$$\Pi^\perp U = \left(\tfrac{\mathrm{D}}{\mathrm{D}+2}(\rho+\theta)\,, \mathbb{Q}\mathrm{u}\,, \tfrac{2}{\mathrm{D}+2}(\rho+\theta)\right)\,.$$

We define the kinetic boundary layer operator $\mathcal{L}^{BL}$, reflection boundary operator $L^{\mathcal{R}}$ and diffusive boundary operator $L^{\mathcal{D}}$ acting on functions $\{g^{\mathrm{bb}}(x, v, \xi) : (x, v, \xi) \in \Omega^\delta \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}_+\}$ as follows:

$$\mathcal{L}^{BL} g^{\mathrm{bb}} := -(v\cdot\nabla_x \mathrm{d})\partial_\xi g^{\mathrm{bb}} + \mathcal{L} g^{\mathrm{bb}}\,, \tag{4.4}$$

where $\mathcal{L}$ is the linearized Boltzmann operator.

$$L^{\mathcal{R}} g^{\mathrm{bb}} := \gamma_+ g^{\mathrm{bb}} - L\gamma_- g^{\mathrm{bb}}\,, \quad \text{and} \quad L^{\mathcal{D}} g^{\mathrm{bb}} := \sqrt{2\pi}\,\chi\left[\langle\gamma_- g^{\mathrm{bb}}\rangle_{\partial\Omega} - L\gamma_- g^{\mathrm{bb}}\right]\,.$$

Lemma 4.1. *Considering the following linear kinetic boundary layer equation of $g^{\mathrm{bb}}(x, v, \xi)$ in half space:*

$$\begin{aligned} \mathcal{L}^{BL} g^{\mathrm{bb}} &= S^{\mathrm{bb}}\,, \quad \text{in} \quad \xi > 0\,, \\ g^{\mathrm{bb}} &\longrightarrow 0\,, \quad \text{as} \quad \xi \to \infty\,, \end{aligned} \tag{4.5}$$

with boundary condition

$$L^{\mathcal{R}} g^{\mathrm{bb}} = H^{\mathrm{bb}}\,, \quad \text{on} \quad \xi = 0\,, \quad v\cdot\mathrm{n} > 0\,. \tag{4.6}$$

In the above equations, the boundary source term H^{bb} is taken of the following form:

$$H^{\mathrm{bb}} = -L^{\mathcal{R}} g + L^{\mathcal{D}} f\,, \tag{4.7}$$

where g and f are of the forms:

$$\begin{aligned} g =& \rho_g + \mathrm{u}_g\cdot v + \theta_g\left(\tfrac{|v|^2}{2} - \tfrac{\mathrm{D}}{2}\right) \\ &-(\partial_\zeta \mathrm{u}^{\mathrm{b}}\otimes\mathrm{n} : \widehat{\mathrm{A}} + \partial_\zeta\theta^{\mathrm{b}}\mathrm{n}\cdot\widehat{\mathrm{B}}) + (\partial_{\pi^\alpha}\widetilde{\mathrm{u}}^{\mathrm{b}}\otimes\nabla_x\pi^\alpha : \widehat{\mathrm{A}} + \partial_{\pi^\alpha}\widetilde{\theta}^{\mathrm{b}}\nabla_x\pi^\alpha\cdot\widehat{\mathrm{B}}) \\ &+(\nabla_x\mathrm{u}^{\mathrm{int}} : \widehat{\mathrm{A}} + \nabla_x\theta^{\mathrm{int}}\cdot\widehat{\mathrm{B}}) + S_g\,, \end{aligned} \tag{4.8}$$

and

$$f = \rho_f + \mathrm{u}_f\cdot v + \theta_f\left(\tfrac{|v|^2}{2} - \tfrac{\mathrm{D}}{2}\right) + S_f\,, \tag{4.9}$$

and where $S_g\,, S_f \in \mathrm{Null}(\mathcal{L})^\perp$ are source terms.

Then there exists a solution $g^{\mathrm{bb}}(x, v, \xi)$ of the equation (4.5) *if and only if the following boundary conditions are satisfied by the fluid variables:*

(*i*) *On the boundary $\partial\Omega$, the normal components of velocity is*

$$\mathrm{u}_g\cdot\mathrm{n} = \int_0^\infty \langle S^{\mathrm{bb}}\rangle\,\mathrm{d}\xi\,. \tag{4.10}$$

(ii) On the boundary $\partial\Omega$, the tangential components of velocities and temperature satisfy

$$\begin{aligned}[\mathrm{u}_f]^{\tan} =& \tfrac{\nu}{\chi}\left[\partial_\zeta \mathrm{u}^{\mathrm{b}}\right]^{\tan} - \tfrac{\nu}{\chi}\left[2d(\mathrm{u}^{\mathrm{int}})\cdot \mathrm{n}\right]^{\tan} - \tfrac{\nu}{\chi}\nabla_\pi[\widetilde{\mathrm{u}}^{\mathrm{b}}\cdot\mathrm{n}] \\ &+ \left[\int_{v\cdot\mathrm{n}>0}(L^{\mathcal{D}}S_f)v(v\cdot\mathrm{n})M\,\mathrm{d}v\right]^{\tan} - \tfrac{1}{\chi}\langle(v\cdot\mathrm{n})vS_g\rangle^{\tan} + \tfrac{1}{\chi}\int_0^\infty\langle S^{\mathrm{bb}}v\rangle^{\tan}\,\mathrm{d}\xi\,,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\theta_f =& \tfrac{\mathrm{D}+2}{\mathrm{D}+1}\tfrac{\kappa}{\chi}\partial_\zeta\theta^{\mathrm{b}} - \tfrac{\mathrm{D}+2}{\mathrm{D}+1}\tfrac{\kappa}{\chi}\partial_{\mathrm{n}}\theta^{\mathrm{int}} + \tfrac{\sqrt{2\pi}}{2(\mathrm{D}+1)}\mathrm{u}_f\cdot\mathrm{n} + \tfrac{\sqrt{2\pi}}{\mathrm{D}+1}\int_{v\cdot\mathrm{n}>0}(L^{\mathcal{D}}S_f)|v|^2(v\cdot\mathrm{n})M\,\mathrm{d}v \\ &- \tfrac{1}{(\mathrm{D}+1)\chi}\langle(v\cdot\mathrm{n})|v|^2S_g\rangle + \tfrac{\mathrm{D}+2}{\mathrm{D}+1}\tfrac{1}{\chi}\int_0^\infty\langle S^{\mathrm{bb}}(\tfrac{|v|^2}{\mathrm{D}+2}-1)\rangle\,\mathrm{d}\xi\,,\end{aligned} \tag{4.12}$$

where kinematic viscosity ν and thermal conductivity κ are given by (3.5)*, $\mathrm{u}^{\tan}$ denotes the tangential components of the vector* u*, and ∇_π denotes the tangential derivative.*

5. Approximate Eigenfunctions-Eigenvalues

5.1. **Motivation.** We define the operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}_\varepsilon^*$ as

$$\mathcal{L}_\varepsilon := \frac{1}{\varepsilon}\mathcal{L} - v\cdot\nabla_x\,, \qquad \mathcal{L}_\varepsilon^* := \frac{1}{\varepsilon}\mathcal{L} + v\cdot\nabla_x\,.$$

Formally, $\mathcal{L}_\varepsilon$ and $\mathcal{L}_\varepsilon^*$ are "dual" in the following sense:

$$\langle\mathcal{L}_\varepsilon^* g^*\,,g\rangle = \langle g^*\,,\mathcal{L}_\varepsilon g\rangle\,, \tag{5.1}$$

provided that g^* satisfies the Maxwell reflection boundary condition

$$\gamma_- g^* = (1-\alpha)L\gamma_+ g^* + \alpha\langle\gamma_+ g^*\rangle_{\partial\Omega} \quad \text{on} \quad \Sigma_-\,, \tag{5.2}$$

and g satisfies the dual boundary condition

$$\gamma_+ g = (1-\alpha)L\gamma_- g + \alpha\langle\gamma_- g\rangle_{\partial\Omega} \quad \text{on} \quad \Sigma_+\,. \tag{5.3}$$

If g_ε is the fluctuation defined in (2.5), then g_ε obeys the scaled Boltzmann equation (2.6) in which $\mathcal{L}_\varepsilon^* g_\varepsilon$ appears and g_ε satisfies the boundary condition (5.2). Then from (5.1), $\mathcal{L}_\varepsilon g_\varepsilon^{BL}$ appears in the weak formulation of the Boltzmann equation if we take g_ε^{BL} as a test function. Thus, it is natural to construct eigenfunctions and eigenvalues of $\mathcal{L}_\varepsilon$ satisfying the dual boundary condition (5.3). Specifically, we consider the kinetic eigenvalue problem:

$$\mathcal{L}_\varepsilon g_\varepsilon^{BL} = -i\lambda_\varepsilon^{BL}g_\varepsilon^{BL}\,, \tag{5.4}$$

with g_ε^{BL} satisfying the dual Maxwell boundary condition (5.3), where the accommodation coefficient α takes the value $\alpha_\epsilon = \sqrt{2\pi}\chi\sqrt{\varepsilon}$. By doing so, formally the equation (2.6) becomes an ordinary differential equation of $b_\varepsilon = \int_\Omega\langle g_\varepsilon\,,g_\varepsilon^{BL}\rangle\,\mathrm{d}x$:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}b_\varepsilon + \tfrac{i\lambda_\varepsilon^{BL}}{\varepsilon}b_\varepsilon = c_\varepsilon\,.$$

To solve the eigenvalue problem (5.4) and (5.3), a key observation is that the solutions must include interior and two boundary layer terms: the *fluid viscous boundary layer* with thickness $\sqrt{\varepsilon}$, and the *kinetic Knudsen layer* with thickness ε. We make the ansatz of g_ε^{BL} and λ_ε^{BL} as

$$g_\varepsilon^{BL} = \sum_{m\geq 0}\left[g_m^{\mathrm{int}}(x,v) + g_m^{\mathrm{b}}(\pi(x),\tfrac{\mathrm{d}(x)}{\sqrt{\varepsilon}},v)\right]\varepsilon^{m/2} + \sum_{m\geq 1}g_m^{\mathrm{bb}}(\pi(x),\tfrac{\mathrm{d}(x)}{\varepsilon},v)\varepsilon^{m/2}\,, \tag{5.5}$$

and

$$\lambda_\varepsilon^{BL} = \sum_{m\geq 0}\lambda_m\varepsilon^{m/2}\,. \tag{5.6}$$

Each g_m^{b} and g_m^{bb} are defined in Ω^δ. After rescaling by $\sqrt{\varepsilon}$ and ε respectively,

$$g_m^{\mathrm{b}}, g_m^{\mathrm{bb}} : (\partial\Omega\times\mathbb{R}^+)\times\mathbb{R}^{\mathrm{D}} \longrightarrow \mathbb{R}\,.$$

Both $g_m^{\rm b}$ and $g_m^{\rm bb}$ will vanish in the outside of Ω^δ. Thus $g_m^{\rm b}$ and $g_m^{\rm bb}$ are required to be rapidly decreasing to 0 in the ζ and ξ respectively, which are defined by $\zeta = \frac{{\rm d}(x)}{\sqrt{\varepsilon}}$ and $\xi = \frac{{\rm d}(x)}{\varepsilon}$.

In the ansatz (5.5), g_ε^{BL} consists three types of terms: the interior terms $g_m^{\rm int}$, the fluid viscous boundary layer terms $g_m^{\rm b}$, and the kinetic Knudsen layer terms $g_m^{\rm bb}$. They are coupled through the boundary condition (5.3).

5.2. **Statement of the Proposition.** Now we state the proposition which can be considered as a kinetic analogue of the Proposition 2 in [10].

Proposition 5.1. *Let Ω be a C^2 bounded domain of $\mathbb{R}^{\rm D}$ and the accommodation coefficient $\alpha_\varepsilon = \sqrt{2\pi}\chi\sqrt{\varepsilon}$. Then, for every acoustic mode $k \geq 1$, non-negative integer N, and each $\tau \in \{+\,,-\}$, there exists approximate eigenfunctions $g_{\varepsilon,N}^{\tau,k}$ and eigenvalues $-i\lambda_{\varepsilon,N}^{\tau,k}$ of $\mathcal{L}_\varepsilon$, and error terms $R_{\varepsilon,N}^{\tau,k}$ and $r_{\varepsilon,N}^{\tau,k}$ respectively, such that*

$$\mathcal{L}_\varepsilon g_{\varepsilon,N}^{\tau,k} = -i\lambda_{\varepsilon,N}^{\tau,k} g_{\varepsilon,N}^{\tau,k} + R_{\varepsilon,N}^{\tau,k}\,, \tag{5.7}$$

and $g_{\varepsilon,N}^{\tau,k}$ satisfy the approximate dual Maxwell boundary condition:

$$L^{\mathcal{R}} g_{\varepsilon,N}^{\tau,k} = \sqrt{\varepsilon} L^{\mathcal{D}} g_{\varepsilon,N}^{\tau,k} + r_{\varepsilon,N}^{\tau,k} \quad on \quad \Sigma_+\,. \tag{5.8}$$

Moreover, there exits complex numbers $\lambda_1^{\tau,k}$, such that $i\lambda_{\varepsilon,N}^{\tau,k}$ has the following expansions:

$$i\lambda_{\varepsilon,N}^{\tau,k} = i\lambda_0^{\tau,k} + i\lambda_1^{\tau,k}\sqrt{\varepsilon} + O(\varepsilon)\,, \quad with \quad \mathrm{Re}(i\lambda_1^{\tau,k}) < 0\,. \tag{5.9}$$

Furthermore, for all $1 < r, p \leq \infty$, we have error estimates:

$$\|R_{\varepsilon,N}^{\tau,k}\|_{L^r({\rm d}x, L^p(a^{1-p}M{\rm d}v))} = O(\sqrt{\varepsilon}^{N-1})\,, \tag{5.10}$$

and

$$\|g_{\varepsilon,N}^{\tau,k} - g_0^{\tau,k,{\rm int}}\|_{L^r({\rm d}x, L^p(a^{1-p}M{\rm d}v))} = O(\varepsilon^{1/2r})\,. \tag{5.11}$$

where $g_0^{\tau,k,{\rm int}}$ is defined as

$$g^{\tau,k} = \sqrt{\tfrac{{\rm D}+2}{2{\rm D}}}\left\{\tfrac{\rm D}{{\rm D}+2}\Psi^k + v\cdot\tfrac{\nabla_x\Psi^k}{i\lambda^{\tau,k}} + \tfrac{2}{{\rm D}+2}\Psi^k(\tfrac{|v|^2}{2} - \tfrac{\rm D}{2})\right\}. \tag{5.12}$$

Here Ψ^k is the eigenfunction of the Laplace operator. We also have the boundary error estimates:

$$\|r_{\varepsilon,N}^{\tau,k}\|_{L^r({\rm d}\sigma_x, L^p(a^{1-p}M{\rm d}v))} = O\left(\sqrt{\varepsilon}^{N+1}\right). \tag{5.13}$$

6. Proof of the Strong Convergence in Theorem 3.1

We choose the renormalization:

$$\Gamma(Z) = \frac{Z-1}{1+(Z-1)^2}\,, \tag{6.1}$$

and define $\widetilde{g}_\epsilon = \frac{1}{\varepsilon}\Gamma(G_\epsilon)$, and its associated fluid moments:

$$\widetilde{U}_\varepsilon = (\widetilde{\rho}_\varepsilon\,, \widetilde{\rm u}_\varepsilon\,, \widetilde{\theta}_\varepsilon) = (\langle\widetilde{g}_\epsilon\rangle\,, \langle v\widetilde{g}_\epsilon\rangle\,, \langle(\tfrac{|v|^2}{\rm D} - 1)\widetilde{g}_\epsilon\rangle)\,.$$

$\widetilde{U}_\varepsilon$ can be orthogonally decomposed as parts in the null space of the acoustic operator $\mathcal{A}$ and its orthogonal:

$$\begin{aligned}\widetilde{U}_\varepsilon &= \Pi\widetilde{U}_\varepsilon + \Pi^\perp\widetilde{U}_\varepsilon\\ &= \left(\langle(1-\tfrac{|v|^2}{{\rm D}+2})\widetilde{g}_\epsilon\rangle\,, \mathbb{P}\langle v\widetilde{g}_\epsilon\rangle\,, \langle(\tfrac{|v|^2}{{\rm D}+2}-1)\widetilde{g}_\epsilon\rangle\right)\\ &\quad+ \left(\langle\tfrac{|v|^2}{{\rm D}+2}\widetilde{g}_\epsilon\rangle\,, \mathbb{Q}\langle v\widetilde{g}_\epsilon\rangle\,, \langle\tfrac{2|v|^2}{{\rm D}({\rm D}+2)}\widetilde{g}_\epsilon\rangle\right),\end{aligned} \tag{6.2}$$

in which we call $\Pi\widetilde{U}_\varepsilon$ and $\Pi^\perp\widetilde{U}_\varepsilon$ the incompressible and acoustic parts of $\widetilde{U}_\varepsilon$ respectively.

Using the same method as in [33] and [40], we can prove that the incompressible part of the fluid moments $\widetilde{U}_\varepsilon$, i.e. $\Pi\widetilde{U}_\varepsilon$ converges only *weakly* to solutions of the incompressible NSF equations. This weak convergence is caused by the persistence of fast acoustic part $\Pi^\perp\widetilde{U}_\varepsilon$, as in the periodic domain [27]. If $\Pi^\perp\widetilde{U}_\varepsilon$ vanishes in some strong sense as ε goes to zero, we can improve the convergence of $\Pi\widetilde{U}_\varepsilon$ from weak to strong. The main novelty of this paper is to prove that in the bounded domain Ω, when $\alpha_\varepsilon = O(\sqrt{\varepsilon})$, the acoustic part will be damped *instantaneously.* This damping effect comes from the kinetic-fluid coupled boundary layers. More precisely, we have the following proposition:

Proposition 6.1. *Let $\Pi^\perp\widetilde{U}_\varepsilon$ be defined as (6.2). If $\alpha_\varepsilon = O(\sqrt{\varepsilon})$, then*

$$\Pi^\perp\widetilde{U}_\varepsilon \to 0 \quad in \quad L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x))\,,$$

as $\varepsilon \to 0$.

Then we can apply Proposition 6.1 to prove the Main Theorem 3.1.

6.1. **Strong Convergence in L^1: Proof of Theorem 3.1.** We first show that we can improve the relative compactness of the family of fluctuations g_ε from weak to strong in $L^1_{loc}(\mathrm{d}t; L^1(\sigma M\mathrm{d}v\mathrm{d}x))$. Indeed, g_ε can be decomposed as

$$\begin{aligned} g_\varepsilon =& \mathcal{P}\widetilde{g}_\epsilon + \mathcal{P}^\perp\widetilde{g}_\epsilon + \frac{\varepsilon^2 g_\varepsilon^3}{N_\varepsilon} \\ =& v\cdot\mathbb{P}\widetilde{\mathrm{u}}_\varepsilon + \left(\tfrac{\mathrm{D}}{\mathrm{D}+2}\widetilde{\theta}_\varepsilon - \tfrac{2}{\mathrm{D}+2}\widetilde{\rho}_\varepsilon\right)\left(\tfrac{|v|^2}{2} - \tfrac{\mathrm{D}+2}{2}\right) + v\cdot\mathbb{Q}\widetilde{\mathrm{u}}_\varepsilon + \tfrac{|v|^2}{\mathrm{D}+2}\left(\widetilde{\rho}_\varepsilon + \widetilde{\theta}_\varepsilon\right) \\ &+ \mathcal{P}^\perp\widetilde{g}_\epsilon + \frac{\varepsilon^2 g_\varepsilon}{\sqrt{N_\varepsilon}}\frac{g_\varepsilon^2}{\sqrt{N_\varepsilon}}\,, \end{aligned}$$

where $\mathcal{P}$ is the projection to Null($\mathcal{L}$), $\mathbb{P}$ is the Leray projection, and $\mathbb{Q} = \mathbb{I} - \mathbb{P}$.

It has been proved in [27] that $\mathcal{P}^\perp\widetilde{g}_\epsilon \to 0$ in $L^2_{loc}(\mathrm{d}t; L^2(aM\mathrm{d}v\mathrm{d}x))$, (see (6.41) in [27]). We can also show that

$$\mathbb{P}\widetilde{\mathrm{u}}_\varepsilon \to \mathrm{u}\,, \quad \tfrac{\mathrm{D}}{\mathrm{D}+2}\widetilde{\theta}_\varepsilon - \tfrac{2}{\mathrm{D}+2}\widetilde{\rho}_\varepsilon \to \theta\,, \quad \text{in} \quad L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x))\,. \tag{6.3}$$

Indeed, this convergence is justified in Lemma 5.6 in [18]. Although the renormalization and decomposition of g_ε are different in [18] and the current paper, the proof of the convergence (6.3) can follow the argument in the proof of Lemma 5.6 in [18]. Furthermore, the Proposition 6.1 yields that

$$v\cdot\mathbb{P}^\perp\widetilde{\mathrm{u}}_\varepsilon + \tfrac{|v|^2}{\mathrm{D}+2}\left(\widetilde{\rho}_\varepsilon + \widetilde{\theta}_\varepsilon\right) \to 0 \quad \text{in} \quad L^2_{loc}(\mathrm{d}t; L^2(M\mathrm{d}v\mathrm{d}x))\,.$$

Thus $\mathcal{P}\widetilde{g}_\epsilon \to g = v\cdot\mathrm{u} + \left(\frac{1}{2}|v|^2 - \frac{\mathrm{D}+2}{2}\right)\theta$ in $L^2_{loc}(\mathrm{d}t; L^2(M\mathrm{d}v\mathrm{d}x))$, as $\varepsilon \to 0$. The key nonlinear estimate in [3] claims that

$$\sigma\frac{g_\varepsilon^2}{\sqrt{N_\varepsilon}} = O(|\log\varepsilon|) \quad \text{in} \quad L^\infty(\mathrm{d}t; L^1(aM\mathrm{d}v\mathrm{d}x))\,.$$

It is easy to see that $\frac{\varepsilon\widetilde{g}_\epsilon}{\sqrt{N_\varepsilon}}$ is bounded, hence

$$\frac{\varepsilon^2 g_\varepsilon^3}{N_\varepsilon} \to 0 \quad \text{in} \quad L^1_{loc}(\mathrm{d}t; L^1(\sigma M\mathrm{d}v\mathrm{d}x))\,. \tag{6.4}$$

We deduce that g_ε is relatively compact in $L^1_{loc}(\mathrm{d}t; L^1(\sigma M\mathrm{d}v\mathrm{d}x))$ and that every limit g has the form (3.4), combining the above estimates.

Next, we can also improve the convergence of the moments of g_ε. In [27], it was proved that the incompressible part $(\mathbb{P}\langle vg_\varepsilon\rangle\,, \langle(\frac{1}{\mathrm{D}+2}|v|^2 - 1)g_\varepsilon\rangle)$ converge to $(\mathrm{u}\,,\theta)$ in $C([0,\infty); w\text{-}L^1(\mathrm{d}x))$. We also have $(\mathbb{P}\langle vg_\varepsilon\rangle\,, \langle(\frac{1}{\mathrm{D}+2}|v|^2 - 1)g_\varepsilon\rangle)$ converge to $(\mathrm{u}\,,\theta)$ in $L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x))$. Now, from

Proposition 6.1, we know that the acoustic part $\mathbb{Q}\langle v\widetilde{g}_\epsilon\rangle$ and $\langle(\frac{1}{\mathrm{D}+2}|v|^2\widetilde{g}_\epsilon\rangle$ converge strongly to 0 in $L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x))$. So combining this with (6.4), we get

$$\begin{aligned}\langle vg_\varepsilon\rangle \to \mathrm{u} \quad &\text{in} \quad L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x;\mathbb{R}^{\mathrm{D}}))\cap C([0,\infty); w\text{-}L^1(\mathrm{d}x;\mathbb{R}^{\mathrm{D}}))\,,\\ \langle(\tfrac{1}{\mathrm{D}}|v|^2-1)g_\varepsilon\rangle \to \theta \quad &\text{in} \quad L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x;\mathbb{R}))\cap C([0,\infty); w\text{-}L^1(\mathrm{d}x;\mathbb{R}))\,.\end{aligned}$$

Furthermore, since now we have $\widetilde{\mathrm{u}}_\varepsilon \to \mathrm{u}$ and $\widetilde{\theta}\to\theta$ in $L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x))$, we can improve the Quadratic Limit Theorem 13.1 in [27] to

$$\widetilde{\mathrm{u}}_\varepsilon\otimes\widetilde{\mathrm{u}}_\varepsilon \to \mathrm{u}\otimes\mathrm{u}\,,\quad \widetilde{\theta}_\varepsilon\widetilde{\mathrm{u}}_\varepsilon\to\mathrm{u}\theta\,,\quad \widetilde{\theta}^2_\varepsilon\to\theta^2 \quad\text{in}\quad L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x))\,, \tag{6.5}$$

as $\varepsilon\to 0$.

Let $p = 2+\frac{1}{s-1}$, so that $p=2$ when $s=\infty$. Let $\hat\xi\in L^p(aM\mathrm{d}v)$ be such that $\mathcal{P}\hat\xi = 0$ and set $\xi = \mathcal{L}\hat\xi$, hence,

$$\frac{1}{\varepsilon}\langle\xi\widetilde{g}_\epsilon\rangle = \frac{1}{\varepsilon}\langle\xi\mathcal{P}^\perp\widetilde{g}_\epsilon\rangle = \langle\hat\xi\mathcal{Q}(\widetilde{g}_\epsilon,\widetilde{g}_\epsilon)\rangle - \langle\!\langle\hat\xi\widetilde{q}_\varepsilon\rangle\!\rangle + \langle\!\langle\hat\xi T_\varepsilon\rangle\!\rangle\,.$$

We know from in [27] that

$$\langle\!\langle\hat\xi T_\varepsilon\rangle\!\rangle \to 0 \quad\text{in}\quad L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x))\,, \tag{6.6}$$

and

$$\langle\!\langle\hat\xi\widetilde{q}_\varepsilon\rangle\!\rangle \to \langle\xi\widehat{\mathrm{A}}\rangle : \nabla_x u + \langle\xi\widehat{\mathrm{B}}\rangle\cdot\nabla_x\theta \quad\text{in}\quad w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x))\,. \tag{6.7}$$

Note that

$$\begin{aligned}\langle\hat\xi\mathcal{Q}(\widetilde{g}_\epsilon,\widetilde{g}_\epsilon)\rangle = \langle\hat\xi\mathcal{Q}(\mathcal{P}\widetilde{g}_\epsilon,\mathcal{P}\widetilde{g}_\epsilon)\rangle &+ 2\langle\hat\xi\mathcal{Q}(\mathcal{P}\widetilde{g}_\epsilon,\mathcal{P}^\perp\widetilde{g}_\epsilon)\rangle\\ &+\langle\hat\xi\mathcal{Q}(\mathcal{P}^\perp\widetilde{g}_\epsilon,\mathcal{P}\widetilde{g}_\epsilon)\rangle\,.\end{aligned}$$

It is easy to show that the last two terms above vanish as $\varepsilon\to 0$. For the first term,

$$\begin{aligned}\langle\hat\xi\mathcal{Q}(\mathcal{P}\widetilde{g}_\epsilon,\mathcal{P}\widetilde{g}_\epsilon)\rangle &= \tfrac12\langle\xi\mathcal{P}^\perp(\mathcal{P}\widetilde{g}_\epsilon)^2\rangle\\ =&\tfrac12\langle\xi\mathrm{A}\rangle : (\widetilde{\mathrm{u}}_\varepsilon\otimes\widetilde{\mathrm{u}}_\varepsilon) + \langle\xi\mathrm{B}\rangle\cdot\widetilde{\mathrm{u}}_\varepsilon\widetilde{\theta}_\varepsilon + \tfrac12\langle\xi\mathrm{C}\rangle\widetilde{\theta}^2_\varepsilon\,.\end{aligned} \tag{6.8}$$

Applying the quadratic limit (6.5), (6.8) can be taken limit in $L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x))$ strongly. Combining with convergence (6.6) and (6.7), we get

$$\frac{1}{\varepsilon}\langle\xi\mathcal{P}^\perp\widetilde{g}_\epsilon\rangle \to \left\langle\xi\left(\tfrac12\mathrm{A}:\mathrm{u}\otimes\mathrm{u} + \mathrm{B}\cdot\mathrm{u}\theta + \tfrac12\mathrm{C}\theta^2 - \widehat{\mathrm{A}}:\nabla_x\mathrm{u} - \widehat{\mathrm{B}}\cdot\nabla_x\theta\right)\right\rangle$$

in $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))$. Since $g_\varepsilon - \widetilde{g}_\epsilon \to 0$ in $L^\infty(\mathrm{d}t; L^1(\sigma M\mathrm{d}v\mathrm{d}x))$, the convergence above implies (3.8). Thus we finish the proof of the Main Theorem 3.1.

6.2. **Proof of Proposition 6.1.** We can reduce the proof of the Proposition 6.1 to show that the projection of $\widetilde{U}_\varepsilon$ on each *fixed* acoustic mode goes to zero in $L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x))$. Furthermore, the relation

$$\langle\widetilde{U}_\varepsilon, U^{\tau,k}\rangle_{\mathbb{H}} = \int_\Omega \left\langle\widetilde{g}_\epsilon\,, g_0^{\tau,k,\mathrm{int}}\right\rangle \mathrm{d}x$$

implies that the proof of Proposition 6.1 is reduced to showing that :

Proposition 6.2. *Assume that $\alpha_\varepsilon = O(\sqrt{\varepsilon})$ and let $\widetilde{g}_\epsilon$ be the renormalized fluctuation, satisfying the scaled Boltzmann equation, and $g_0^{\tau,k,int}$ (τ is + or -) be the infinitesimal Maxwellian of acoustic mode $k\geq 1$:*

$$g_0^{\tau,k,int} = \tfrac{\mathrm{D}}{\mathrm{D}+2}\Psi^k + \tfrac{\nabla_x\Psi^k}{\tau i\lambda^k}\cdot v + \tfrac{2}{\mathrm{D}+2}\Psi^k(\tfrac{|v|^2}{2}-\tfrac{\mathrm{D}}{2})\,.$$

Then, for any fixed mode k,

$$\int_\Omega\left\langle\widetilde{g}_\epsilon\,, g_0^{\tau,k,\mathrm{int}}\right\rangle\mathrm{d}x \to 0 \quad \textit{in}\quad L^2(0,T)\,,\quad \textit{as}\quad \varepsilon\to 0\,.$$

Proof. We start from the weak formulation of the rescaled Boltzmann equation (2.9) with the renormalization Γ defined in (6.1) and the test function Y taken to be the approximate eigenfunctions of $\mathcal{L}_\varepsilon$ constructed in Proposition 5.1 to the order $N = 4$, namely $Y = g^{\tau,k}_{\varepsilon,4}$:

$$\begin{aligned}
&\int_\Omega \langle \widetilde{g}_\epsilon(t_2) g^{\tau,k}_{\varepsilon,4}\rangle \,\mathrm{d}x - \int_\Omega \langle \widetilde{g}_\epsilon(t_1) g^{\tau,k}_{\varepsilon,4}\rangle \,\mathrm{d}x \\
&+ \frac{1}{\varepsilon}\int_{t_1}^{t_2}\int_\Omega \langle \widetilde{g}_\epsilon \mathcal{L}_\varepsilon g^{\tau,k}_{\varepsilon,4}\rangle \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_{t_1}^{t_2}\int_{\partial\Omega} \langle \gamma\widetilde{g}_\epsilon \gamma g^{\tau,k}_{\varepsilon,4}(v\cdot \mathrm{n})\rangle \,\mathrm{d}\sigma_x\,\mathrm{d}t \\
&= \frac{1}{\varepsilon}\int_{t_1}^{t_2}\int_\Omega \langle\!\langle R_\varepsilon g^{\tau,k}_{\varepsilon,4}\rangle\!\rangle \,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned} \tag{6.9}$$

where

$$R_\varepsilon = \Gamma'(G_\varepsilon)q_\varepsilon + \frac{1}{\varepsilon}\left(\frac{g_{\varepsilon 1}}{N_{\varepsilon 1}} + \frac{g_\varepsilon}{N_\varepsilon} - \frac{g'_{\varepsilon 1}}{N'_{\varepsilon 1}} - \frac{g'_\varepsilon}{N'_\varepsilon}\right).$$

Define

$$\widetilde{b}^{\tau,k}_\varepsilon(t) = \int_\Omega \langle \widetilde{g}_\epsilon(t) g^{\tau,k}_{\varepsilon,4}\rangle \,\mathrm{d}x\,.$$

Then from (6.9) $\widetilde{b}^{\tau,k}_\varepsilon(t)$ satisfies

$$\widetilde{b}^{\tau,k}_\varepsilon(t_2) - \widetilde{b}^{\tau,k}_\varepsilon(t_1) - \frac{1}{\varepsilon}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}\int_{t_1}^{t_2}\widetilde{b}^{\tau,k}_\varepsilon(t)\,\mathrm{d}t = \int_{t_1}^{t_2} c^{\tau,k}_\varepsilon(t)\,\mathrm{d}t\,, \tag{6.10}$$

where $c^{\tau,k}_\varepsilon(t)$ is:

$$\begin{aligned}
c^{\tau,k}_\varepsilon(t) = &-\frac{1}{\varepsilon}\int_\Omega \langle \widetilde{g}_\epsilon(t) R^{\tau,k}_{\varepsilon,4}\rangle \,\mathrm{d}x - \frac{1}{\varepsilon}\int_{\partial\Omega}\langle \gamma\widetilde{g}_\epsilon \gamma g^{\tau,k}_{\varepsilon,4}(v\cdot \mathrm{n})\rangle \,\mathrm{d}\sigma_x \\
&+ \frac{1}{\varepsilon}\int_\Omega \langle\!\langle R_\varepsilon g^{\tau,k}_{\varepsilon,4}\rangle\!\rangle \,\mathrm{d}x\,.
\end{aligned} \tag{6.11}$$

We claim that the boundary contribution in (6.11) is zero as $\varepsilon \to 0$, i.e.

Lemma 6.1. *Let $g^{\tau,k}_{\varepsilon,4}$ be the approximate eigenfunction of $\mathcal{L}_\varepsilon$ constructed in Proposition 5.1. Then,*

$$\frac{1}{\varepsilon}\int_{\partial\Omega}\langle \gamma\widetilde{g}_\epsilon \gamma g^{\tau,k}_{\varepsilon,4}(v\cdot \mathrm{n})\rangle \,\mathrm{d}\sigma_x = \Gamma^{\tau,k}_1 + \Gamma^{\tau,k}_2\,, \tag{6.12}$$

where $\Gamma^{\tau,k}_1$ is bounded in $L^p_{loc}(\mathrm{d}t)$ for $p > 1$, and $\Gamma^{\tau,k}_2$ vanishes in $L^1_{loc}(\mathrm{d}t)$ as $\varepsilon \to 0$.

6.3. **Estimates of $\widetilde{b}^{\tau,k}_\varepsilon$.** From (6.10), $\widetilde{b}^{\tau,k}_\varepsilon$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{b}^{\tau,k}_\varepsilon - \frac{1}{\varepsilon}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}\widetilde{b}^{\tau,k}_\varepsilon = c^{\tau,k}_{1,\varepsilon}(t) + c^{\tau,k}_{2,\varepsilon}(t)\,. \tag{6.13}$$

The solution to (6.13) is given by

$$\widetilde{b}^{\tau,k}_\varepsilon(t) = \widetilde{b}^{\tau,k}_\varepsilon(0)e^{\frac{1}{\varepsilon}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}t} + \int_0^t [c^{\tau,k}_{1,\varepsilon}(s) + c^{\tau,k}_{2,\varepsilon}(s)]e^{-\frac{1}{\varepsilon}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}(s-t)}\,\mathrm{d}s\,. \tag{6.14}$$

From the Proposition 5.1, $i\lambda^{\tau,k}_{\varepsilon,4} = \tau i\lambda^k + i\lambda^{\tau,k}_1\sqrt{\varepsilon} + i\widetilde{\lambda}^{\tau,k}_1\varepsilon$, where $\widetilde{\lambda}^{\tau,k}_1 = O(1)$.

$$\begin{aligned}
\frac{1}{\varepsilon}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}t = &\frac{1}{\sqrt{\varepsilon}}\left[\mathrm{Re}(i\lambda^{\tau,k}_1) + \sqrt{\varepsilon}\,\mathrm{Re}(i\widetilde{\lambda}^{\tau,k}_1)\right]t \\
&- i\left[\tau\frac{1}{\varepsilon}\lambda^k + \frac{1}{\sqrt{\varepsilon}}\mathrm{Im}(i\lambda^{\tau,k}_1) + \mathrm{Im}(i\widetilde{\lambda}^{\tau,k}_1)\right]t\,.
\end{aligned} \tag{6.15}$$

Using (6.15), the first term in (6.14) is estimated as follows:

$$\begin{aligned}&\|\widetilde{b}^{\tau,k}_\varepsilon(0)e^{\frac{1}{\varepsilon}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}t}\|_{L^2(0,T)}\\ =&|\widetilde{b}^{\tau,k}_\varepsilon(0)|\left[-2\left(\mathrm{Re}(i\lambda^{\tau,k}_1)+\sqrt{\varepsilon}\mathrm{Re}(i\widetilde{\lambda}^{\tau,k}_1)\right)\right]^{-1/2}\left(1-e^{\frac{1}{\sqrt{\varepsilon}}[\mathrm{Re}(i\lambda^{\tau,k}_1)+\sqrt{\varepsilon}\mathrm{Re}(i\widetilde{\lambda}^{\tau,k}_1)]T}\right)^{1/2}\varepsilon^{1/4}\,.\end{aligned}$$

To estimate $|\widetilde{b}^{\tau,k}_\varepsilon(0)|$, from

$$\widetilde{b}^{\tau,k}_\varepsilon(0)=\int_\Omega\langle\widetilde{g}^{\mathrm{in}}_\epsilon\,,g^{k,\mathrm{int}}_0\rangle\,\mathrm{d}x+\int_\Omega\langle\widetilde{g}^{\mathrm{in}}_\epsilon\,,g^{\tau,k}_{\varepsilon,4}-g^{k,\mathrm{int}}_0\rangle\,\mathrm{d}x\,,$$

noticing that $g^{\tau,k,\mathrm{int}}_0\in$Null($\mathcal{L}$) and $\|\langle\zeta(v)\widetilde{g}^{\mathrm{in}}_\epsilon\rangle\|_{L^2(\mathrm{d}x)}$ is bounded for every $\zeta(v)\in$Null($\mathcal{L}$), and the error estimate for $g^{\tau,k}_{\varepsilon,4}-g^{\tau,k,\mathrm{int}}_0$ in (5.11), we deduce that $|\widetilde{b}^{\tau,k}_\varepsilon(0)|$ is bounded. Using the key fact that $\mathrm{Re}(i\lambda^{\tau,k}_1)<0$, we deduce that for any $0<T<\infty$, sufficiently small ε:

$$\|\widetilde{b}^{\tau,k}_\varepsilon(0)e^{-\frac{1}{\varepsilon^2}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}t}\|_{L^2(0,T)}\leq C\varepsilon^{1/4}\,.$$

In order to estimate the remaining term in (6.14), we observe that for any $a\in L^p(0,t)$ and $1\leq p,r\leq\infty$, such that $p^{-1}+r^{-1}=1$, we have

$$\Big|\int_0^t a(s)e^{-\frac{1}{\varepsilon}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}(s-t)}\,\mathrm{d}s\Big|\leq C\int_0^t e^{-\frac{1}{\sqrt{\varepsilon}}\mathrm{Re}(i\lambda^{\tau,k}_1)(s-t)}|a(s)|\,\mathrm{d}s\,.$$

Direct calculations show that

$$\left\|e^{-\frac{1}{\sqrt{\epsilon}}\mathrm{Re}(i\lambda^{\tau,k}_1)(t-s)}\right\|_{L^r(0,t)}=\varepsilon^{1/2r}\left[\frac{1}{-r\mathrm{Re}(i\lambda^{\tau,k}_1)}\left(e^{-\frac{r}{\sqrt{\epsilon}}\mathrm{Re}(i\lambda^{\tau,k}_1)t}-1\right)\right]^{1/r}e^{-\frac{1}{\sqrt{\varepsilon}}\mathrm{Re}(i\lambda^{\tau,k}_1)t}\,.$$

Using the fact $\mathrm{Re}(i\lambda^{\tau,k}_1)<0$ again, we have

$$\Big|\int_0^t a(s)e^{-\frac{1}{\varepsilon}\overline{i\lambda^{\tau,k}_{\varepsilon,4}}(s-t)}\,\mathrm{d}s\Big|\leq C\|a\|_{L^p(0,t)}\varepsilon^{1/2r}\,. \tag{6.16}$$

Now applying $a(t)$ in (6.16) to $c^{\tau,k}_{1,\varepsilon}$ and $c^{\tau,k}_{2,\varepsilon}$, finally we get:

$$\widetilde{b}^{\tau,k}_\varepsilon\to 0\,,\quad\text{strongly in}\quad L^2_{loc}(\mathrm{d}t)\,.$$

To finish the proof of the Proposition, we notice that

$$\int_\Omega\left\langle\widetilde{g}_\epsilon\,,g^{\tau,k,\mathrm{int}}_0\right\rangle\,\mathrm{d}x=\widetilde{b}^{\tau,k}_\varepsilon+\int_\Omega\langle\widetilde{g}_\epsilon\,,g^{\tau,k,\mathrm{int}}_0-g^{\tau,k}_{\varepsilon,4}\rangle\,\mathrm{d}x\,.$$

Applying the error estimate (5.11) in Proposition 5.1, we finish the proof of the Proposition 6.2. □

Consequently, we prove the Proposition 6.1.

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