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# Mean Field Limits for Ginzburg-Landau Vortices

Sylvia Serfaty

## Abstract

We review results in the literature on asymptotic limits for the Ginzburg-Landau equations. We then present results where we show, by a modulated energy method, that solutions of the Gross-Pitaevskii equation converge to solutions of the incompressible Euler equation, and solutions to the parabolic Ginzburg-Landau equations converge to solutions of a limiting equation which we identify.

We work in the setting of the whole plane (and possibly the whole three-dimensional space in the Gross-Pitaevskii case), in the asymptotic limit where  $\varepsilon$ , the characteristic lengthscale of the vortices, tends to 0, and in a situation where the number of vortices  $N_\varepsilon$  blows up as  $\varepsilon \rightarrow 0$ . The requirements are that  $N_\varepsilon$  should blow up faster than  $|\log \varepsilon|$  in the Gross-Pitaevskii case, and at most like  $|\log \varepsilon|$  in the parabolic case. Both results assume a well-prepared initial condition and regularity of the limiting initial data, and use the regularity of the solution to the limiting equations.

## 1 The Ginzburg-Landau model and the equations

We are interested in the Ginzburg-Landau equations

$$(1.1) \quad -\Delta u = \frac{u}{\varepsilon^2}(1 - |u|^2) \quad \text{in } \mathbb{R}^2,$$

the Gross-Pitaevskii equation

$$(1.2) \quad i\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \quad \text{in } \mathbb{R}^2$$

and the parabolic Ginzburg-Landau equation

$$(1.3) \quad \partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \quad \text{in } \mathbb{R}^2$$

in the plane, all in the asymptotic limit  $\varepsilon \rightarrow 0$ . We will also consider the three-dimensional version of the Gross-Pitaevskii equation

$$(1.4) \quad i\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \quad \text{in } \mathbb{R}^3.$$

These equations are variational, associated to the energy (say in a domain  $\Omega$ )

$$(1.5) \quad E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}.$$

These are famous equations of mathematical physics, which arise in the modeling of superfluidity, superconductivity, nonlinear optics, etc. The Gross-Pitaevskii equation is an important instance of a nonlinear Schrödinger equation. These equations also come in a version with gauge, more suitable for the modeling of superconductivity, but whose essential mathematical features are similar to these, and which we will discuss briefly below. There is also interest in the “mixed flow” case, sometimes called complex Ginzburg-Landau equation

$$(1.6) \quad (a + ib)\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \quad \text{in } \mathbb{R}^2.$$

For further reference on these models, one can see e.g. [T, TT, AK, SS5].

In these equations, the unknown function  $u$  is complex-valued, and it can exhibit vortices, which are zeroes of  $u$  with non-zero topological degree, and a core size on the order of  $\varepsilon$ . In the plane and when  $\varepsilon \rightarrow 0$ , these vortices are points, whereas in the three-dimensional space they are lines. We are interested in one of the main open problems on Ginzburg-Landau dynamics, which is to understand the dynamics of vortices in the regime in which their number  $N_\varepsilon$  blows up as  $\varepsilon \rightarrow 0$ .

## 2 The setting of a bounded number of vortices

### 2.1 Limits of minimizers and critical points

The asymptotic analysis of vortices in Ginzburg-Landau equations was pioneered by Bethuel-Brezis-Hélein in [BBH]. They studied the case of a bounded domain with the number  $N$  of vortices bounded as  $\varepsilon \rightarrow 0$  (hence it can be assumed to be independent of  $\varepsilon$ ), and imposed via a fixed Dirichlet boundary condition. They showed that vortices for solutions of (1.1), respectively minimizers of  $E_\varepsilon$ , converge to points which are critical points, respectively minimizers, of a so-called “renormalized energy” of the form

$$(2.1) \quad W(a_1, \dots, a_N) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| \\ + \text{other terms related to boundary conditions}$$

where the  $a_i$ ’s are the vortex locations and  $d_i$ ’s their degrees. They also showed that minimizers have only vortices of degree +1 (up to a change of orientation), their number  $N$  being equal to the degree of the Dirichlet boundary condition, and the minimal energy has the expansion

$$(2.2) \quad \min E_\varepsilon = \pi N |\log \varepsilon| + \min W + o(1).$$

It was also proven in [Se0] that stability is preserved in the limit i.e. stable critical points of  $E_\varepsilon$  have vortices which converge to stable critical points of  $W$ . The converse question: i.e. given critical points of  $W$ , find solutions of (1.1) whose asymptotic vortices are the prescribed ones, was answered in details in [PR] via nonlinear local inversion techniques.

## 2.2 The dynamics

For the dynamics of (1.2)–(1.3), again in the case of a fixed number of vortices, it was proven, either in the setting of the whole plane or that of a bounded domain, that, for “well-prepared” initial data, after suitable time rescaling, their limiting positions obey the law

$$(2.3) \quad \frac{da_i}{dt} = (a + \mathbb{J}b)\nabla_i W(a_1, \dots, a_N)$$

where  $\mathbb{J}$  is the rotation by  $\pi/2$  in the plane, and  $W$  is in the setting of the plane the so-called Kirchhoff-Onsager energy

$$(2.4) \quad W(a_1, \dots, a_N) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j|$$

where the  $d_i$ 's are the degrees of the vortices and are assumed to be initially in  $\{1, -1\}$ .

In other words, the vortices move according to the corresponding flow (gradient, Hamiltonian, or mixed) of their limiting interaction energy  $W$ . After some formal results based on matched asymptotics by Pismen-Rubinstein and E in [PR, E1], these results were proven in the setting of a bounded domain by Lin [Li1] and Jerrard-Soner [JS1] in the parabolic case, Colliander-Jerrard [CJ1, CJ2] and Lin-Xin [LX2] with later improvements by Jerrard-Spirn [JSp1] in the Schrödinger case, and Kurzke-Melcher-Moser-Spirn [KMMS] in the mixed flow case. In the setting of the whole plane, the analogous results were obtained by Lin-Xin [LX1] in the parabolic case, Bethuel-Jerrard-Smets [BJS] in the Schrödinger case, and Miot [Mi] in the mixed flow case. A proof based on the idea of relating gradient flows and  $\Gamma$ -convergence was also given in [SS4], it was the initial motivation for the abstract scheme of “ $\Gamma$ -convergence of gradient flows” introduced there. Generalizations to the case with gauge, pinning terms and applied electric field terms were also studied [Sp1, Sp2, KS1, Ti, ST2].

All these results hold for well-prepared data and for as long as the points evolving under the dynamical law (2.3) do not collide. In the parabolic case, Bethuel-Orlandi-Smets showed in the series of papers [BOS1, BOS2, BOS3] how to lift the well-prepared condition and handle the difficult issue of collisions and extend the dynamical law (2.3) beyond them. Results of a similar nature were also obtained in [Se1].

## 2.3 The three-dimensional case

As we mentioned, vortices in three dimensions are lines. These are studied in the framework of geometric measure theory, using currents and varifolds, cf. [Ri, JS1, ABO]. The leading order expansion of the energy replacing (2.2) was then shown to be

$$\min E_\varepsilon = \pi |d| L |\log \varepsilon| + o(|\log \varepsilon|)$$

where  $L$  is the length (or area) of the vortex line (viewed as an integer multiplicity rectifiable current), while minimizers, respectively critical points, have vortex lines which converge

to length minimizing, respectively stationary, currents. This was proved in the works [Ri, LR1, BBO, JS1, ABO, BBM].

Since the leading order to the energy is not trivial (as opposed to the two-dimensional situation) the dynamics of vortex lines is expected to be driven by it, i.e. by their length. Indeed, it was shown in [BOS] (see also results in [LR2]) that the limiting flow as  $\varepsilon \rightarrow 0$  of vortices for (1.3) in  $\mathbb{R}^3$  is the gradient flow of length, i.e. mean curvature motion of a curve in space, to be understood in the sense of Brakke. For (1.2) the expected limiting dynamics of three-dimensional vortex lines is the binormal flow of a curve (studied in [JS]) but in contrast to the two-dimensional case there are only partial results towards establishing this rigorously [J2].

### 3 Large number of vortices

When the number of points  $N_\varepsilon$  blows up as  $\varepsilon \rightarrow 0$ , then the vortices are studied via their density (also called vorticity), in a mean-field limit fashion.

More precisely, for a family of functions  $u_\varepsilon$ , one introduces the supercurrent  $j_\varepsilon$  and the vorticity (or Jacobian)  $\mu_\varepsilon$  of the map  $u_\varepsilon$  which are defined via

$$(3.1) \quad j_\varepsilon := \langle iu_\varepsilon, \nabla u_\varepsilon \rangle \quad \mu_\varepsilon := \text{curl } j_\varepsilon,$$

where  $\langle x, y \rangle$  stands for the scalar product in  $\mathbb{C}$  as identified with  $\mathbb{R}^2$  via  $\langle x, y \rangle = \frac{1}{2}(\bar{x}y + \bar{y}x)$ . The vorticity  $\mu_\varepsilon$  plays the same role as the vorticity in classical fluids, the only difference being that it is essentially quantized at the  $\varepsilon$  level, as can be seen from the asymptotic estimate  $\mu_\varepsilon \simeq 2\pi \sum_i d_i \delta_{a_i}$  as  $\varepsilon \rightarrow 0$ , with  $\{a_i\}$  the vortices of  $u_\varepsilon$  and  $d_i \in \mathbb{Z}$  their degrees (these are the so-called Jacobian estimates, cf [JS2, SS5]).

#### 3.1 The stationary case

The stationary case was studied in [SS2], where it is proven that if  $u_\varepsilon$  is a solution to (1.1) and  $N_\varepsilon \gg 1$  then the normalized vorticity  $\mu_\varepsilon/N_\varepsilon$  converges to a measure  $\mu$ , solution to the formal relation

$$(3.2) \quad \mu \nabla h = 0 \quad h = -\Delta^{-1} \mu.$$

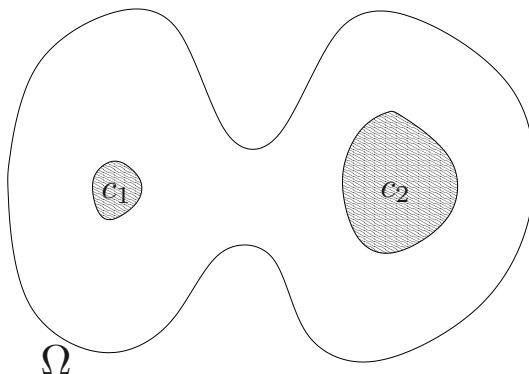
For  $\mu$  a general probability measure, the product  $\mu \nabla h$  does not make sense, and a weak formulation à la Delort [De] must instead be used to give a meaning to (3.2): setting

$$T_\mu := -\nabla h \otimes \nabla h + \frac{1}{2} |\nabla h|^2 \delta_i^j$$

the stress-energy tensor associated to the vorticity, the weak meaning to (3.2) is

$$\text{div } T_\mu = 0$$

(which in certain settings needs to be understood in “finite parts”, see [SS5]). The formal relation (3.2) leads to expecting vortex “patches” (as in 2D Euler) with  $h$  constant on the support of  $\mu$ , and (since  $\mu = -\Delta h$ )  $\mu$  itself of constant density on its support, as illustrated in the figure below:



In other words, the vortices feel a mean field force  $-\nabla h$ , which is the gradient of the logarithmic potential that they generate, and which must vanish on the support of the vorticity. The method of the proof consists in passing to the limit in the “conservative form of the equation” i.e. the vanishing of the stress tensor associated to (1.1), taking advantage of a good control of the size of the set occupied by vortices. This approach seems to fail to extend to the dynamical setting for lack of extension of this good control.

### 3.2 Expected dynamics

In the setting  $N_\varepsilon \gg 1$ , it is expected that the limiting system of ODEs (2.3) should be replaced by its mean-field evolution for the vorticity. In other words, the mean-field evolution for  $\mu = \lim_{\varepsilon \rightarrow 0} \mu_\varepsilon / N_\varepsilon$  can be guessed to be the mean-field limit of (2.3) as  $N \rightarrow \infty$ . Proving this essentially amounts to showing that the limits  $\varepsilon \rightarrow 0$  and  $N \rightarrow \infty$  can be interchanged, which is a delicate question.

In the case of the Gross-Pitaevskii equation (1.2)-(1.4), it is well-known that the Madelung transform formally yields that the limiting evolution equation should be the incompressible Euler equation (for this and related questions, see for instance the survey [CDS]). In the case of the parabolic Ginzburg-Landau equation, it was proposed, based on heuristic considerations by Chapman-Rubinstein-Schatzman [CRS] and E [E2], that the limiting equation should be

$$(3.3) \quad \partial_t \mu - \operatorname{div}(\mu \nabla h) = 0 \quad h = -\Delta^{-1} \mu.$$

where  $\mu$  is the limit of the vortex density, assumed to be nonnegative. Note that this is the time-dependent version of (3.2), a dissipative counterpart to the 2D incompressible Euler equation in vorticity form. In fact, both papers really derived the equation for possibly signed densities, [CRS] did it for the very similar model with gauge in a bounded domain, in which case the coupling  $h = -\Delta^{-1} \mu$  is replaced by  $h = (-\Delta + I)^{-1} \mu$ , and [E2] treated

both situations with and without gauge, also for signed densities, without discussing the domain boundary.

### 3.3 Study of the Chapman-Rubinstein-Schatzman-E equation

After this model was proposed, the equation (3.3) was studied for its own sake. Its properties depend greatly on the regularity of the initial data  $\mu$ . For  $\mu$  a general probability measure, the weak formulation à la Delort [De] must be used; also uniqueness of solutions can fail, although there is always existence of a unique solution which becomes instantaneously  $L^\infty$ .

The works [LZ1, DZ, MZ] showed the existence of weak solutions (à la Delort) by the vortex approximation method, and existence and uniqueness of  $L^\infty$  solutions, which decay in  $1/t$ . Their proofs use pseudo-differential operators.

It also turns out that (3.3) can be interpreted as the gradient flow (as in [O, AGS]) in the space of probability measures equipped with the 2-Wasserstein distance, of the energy functional

$$(3.4) \quad \Phi(\mu) = \int |\nabla h|^2 \quad h = -\Delta^{-1}\mu,$$

which is also the mean field limit of the usual Ginzburg-Landau energy. The equation (3.3) was also studied with that point of view in [AS] in the bounded domain setting (where the possible entrance and exit of mass creates difficulties). This energy point of view allows one to envision a possible (and so far unsuccessful) energetic proof of the convergence of (1.3) based on the scheme of Gamma-convergence of gradient flows, as described in [Se2].

A PDE approach valid in all dimension was proposed in [SV], showing existence via limits in fractional diffusion  $\partial_t \mu + \operatorname{div}(\mu \nabla \Delta^{-s} \mu)$  when  $s \rightarrow 1$ , uniqueness in the class  $L^\infty$ , propagation of regularity, and the existence of the asymptotic self-similar profile

$$\mu(t) = \frac{1}{\pi t} \mathbf{1}_{B_{\sqrt{t}}}.$$

### 3.4 Rigorous convergence of (1.3) and (1.2)

After studying (3.3), the main question remains to prove the convergence of solutions to (1.3) to (3.3) and (1.2) to Euler, as  $\varepsilon \rightarrow 0$  and  $N_\varepsilon \rightarrow \infty$ . The only available results until now were due to Kurzke and Spirn and Jerrard and Spirn, both in the case of very dilute limits: it is proved in [KS2] for (1.3) that convergence to (3.3) holds in the case where  $N_\varepsilon$  grows slower than  $(\log \log |\log \varepsilon|)^{1/4}$ , and in [JS2] for (1.2) that convergence to the Euler equation in vorticity form holds in the case where  $N_\varepsilon$  grows slower than  $\log |\log \varepsilon|^{1/2}$ , assuming in each case some specific well-preparedness conditions on the initial data. To do so, relying on their previous work [KS1, JSp1], they showed that the method of proof for finite number of vortices can be made more quantitative and pushed beyond bounded  $N_\varepsilon$ , controlling the vortex distances and proving that their positions remain close to those of the  $N_\varepsilon$  points solving the ODE system (2.3), and then finally passing to the limit for that

system by applying classical “point vortex methods”, in the manner of, say, Schochet [Sch]. There is however very little hope for extending such an approach to larger values of  $N_\varepsilon$ .

## 4 Main new results

### 4.1 The method

In order to treat a broader regime for  $N_\varepsilon \gg 1$ , we introduce in [Se3] an alternate method, based on a “modulated energy”, which exploits the (assumed) regularity and stability of the solution to the limit equation. The method is robust and works for dissipative as well as conservative equations, as well as for variants with gauge [Se3] or with “pinning” weight [DS]. It avoids the question of understanding the detailed dynamics of the vortices (their distances, etc).

Letting  $\mathbf{v}(t)$  be the expected limiting velocity field (such that  $\frac{1}{N_\varepsilon} \langle \nabla u_\varepsilon, iu_\varepsilon \rangle \rightarrow \mathbf{v}$  and  $\text{curl } \mathbf{v} = 2\pi\mu$ ), then the modulated energy is defined as

$$\mathcal{E}_\varepsilon(u, t) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathbf{v}(t)|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}.$$

It is an energy modelled on the Ginzburg-Landau energy, and the proof relies on showing, via a Gronwall relation, that if it is initially small it remains small. The idea of proving convergence via a Gronwall argument on the modulated energy, while assuming and using the regularity of the limiting solution is similar to the relative entropy method for establishing (the stability of) hydrodynamic limits, first introduced in [Yau] and used for quantum many body problems, mean-field theory and semiclassical limits, one example of the latter arising precisely for the limit of the Gross-Pitaevskii equation in [LZ2]; or Brenier’s modulated entropy method to establish kinetic to fluid limits such as the derivation of the Euler equation from the Boltzmann or Vlasov equations (see for instance [SR] and references therein).

### 4.2 Statements of results

Here, for simplicity we omit a few assumptions dealing with the behavior at infinity, the full statements can be found in [Se3].

**Theorem 1** (cf. [Se3]). *Assume  $u_\varepsilon$  solves (1.2) or (1.4) and let  $N_\varepsilon$  be such that  $|\log \varepsilon| \ll N_\varepsilon \ll 1/\varepsilon$ . Let  $\mathbf{v}$  be a  $L^\infty(\mathbb{R}_+, C^{0,1})$  solution to the incompressible Euler equations*

$$\boxed{\begin{cases} \partial_t \mathbf{v} = 2 \text{div} (\mathbf{v} \otimes \mathbf{v} - \frac{1}{2} |\mathbf{v}|^2 I) + \nabla p & \text{in } \mathbb{R}^n \\ \text{div } \mathbf{v} = 0 & \text{in } \mathbb{R}^n, \end{cases}} \quad (\text{IE})$$

$n = 2, 3$  with  $\text{curl } \mathbf{v} \in L^\infty(L^1)$ .



Let  $\{u_\varepsilon\}_{\varepsilon>0}$  be solutions associated to initial conditions  $u_\varepsilon^0$ , with  $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \leq o(N_\varepsilon^2)$ . Then, for every  $t \geq 0$ , we have  $\mathcal{E}_\varepsilon(u_\varepsilon(t), t) \leq o(N_\varepsilon^2)$ , and in particular

$$\frac{1}{N_\varepsilon} \langle \nabla u_\varepsilon, iu_\varepsilon \rangle \rightarrow v \quad \text{in } L_{loc}^1(\mathbb{R}^n).$$

Note that the result immediately implies the convergence of the vorticity  $\mu_\varepsilon/N_\varepsilon \rightarrow \text{curl } v$ .

**Theorem 2** (cf. [Se3]). Assume  $u_\varepsilon$  solves (1.3) and let  $N_\varepsilon$  be such that  $1 \ll N_\varepsilon \leq O(|\log \varepsilon|)$ . Let  $v$  be a  $L^\infty([0, T], C^{1,\gamma})$  solution to

$$\bullet \text{ if } N_\varepsilon \ll |\log \varepsilon| \quad \boxed{\begin{cases} \partial_t v = -2v \text{curl } v + \nabla p & \text{in } \mathbb{R}^2 \\ \text{div } v = 0 & \text{in } \mathbb{R}^2, \end{cases}} \quad (\text{L1})$$

$$\bullet \text{ if } N_\varepsilon \sim \lambda |\log \varepsilon| \quad \boxed{\partial_t v = \frac{1}{\lambda} \nabla \text{div } v - 2v \text{curl } v \quad \text{in } \mathbb{R}^2.} \quad (\text{L2})$$

Assume  $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \leq \pi N_\varepsilon |\log \varepsilon| + o(N_\varepsilon^2)$  and  $\text{curl } v(0) \geq 0$ . Then  $\forall t \leq T$  we have

$$\frac{1}{N_\varepsilon} \langle \nabla u_\varepsilon, iu_\varepsilon \rangle \rightarrow v \quad \text{in } L_{loc}^1(\mathbb{R}^2).$$

Again, this implies convergence of the vorticity. One observes that taking the curl of the equation yields back (3.3) if  $N_\varepsilon \ll |\log \varepsilon|$ , but *not* if  $N_\varepsilon \propto |\log \varepsilon|$ . Long-time existence for the new type of limiting equation (L2) is proven in [Du].

The assumptions placed on the initial data are well-preparedness assumptions.

One of the difficulties in the proof is that the convergence of  $j_\varepsilon/N_\varepsilon$  to  $v$  is not strong in  $L^2$ , in general, but rather weak in  $L^2$ , due to a concentration of an amount  $\pi |\log \varepsilon|$  of energy at each of the vortex points (this energy concentration can be seen as a defect measure in the convergence of  $j_\varepsilon/N_\varepsilon$  to  $v$ ). In order to take that concentration into account, we need to subtract off of  $\mathcal{E}_\varepsilon$  the constant quantity  $\pi N_\varepsilon |\log \varepsilon|$ . In the regime where  $N_\varepsilon \gg |\log \varepsilon|$ , then  $\pi N_\varepsilon |\log \varepsilon| = o(N_\varepsilon^2)$  and this quantity (or the concentration) happens to become negligible, which is what will make the proof in the Gross-Pitaevskii case much simpler and applicable to the three-dimensional setting as well, but restricted to the regime  $N_\varepsilon \gg |\log \varepsilon|$ . In the parabolic case, the factor of growth of the modulated energy in Gronwall's lemma is bounded by  $C N_\varepsilon / |\log \varepsilon|$  hence the restriction to the regime  $N_\varepsilon \leq O(|\log \varepsilon|)$ . We are not sure whether the formal analogue of (L2), i.e. the equation with  $\lambda = \infty$  (shown to be locally well-posed in [Du]), is the correct limiting equation.

### 4.3 Sketch of the proofs

The proof consists in computing the time derivative of the modulated energy to show that Gronwall's lemma applies. It relies on algebraic identities which reveal only quadratic

terms in this time-derivative. Returning to our definitions of the modulated energy  $\mathcal{E}_\varepsilon$ , the current  $j_\varepsilon$ , and the vorticity  $\mu_\varepsilon$ , we also define the velocity

$$V_\varepsilon = 2\langle i\partial_t u_\varepsilon, \nabla u_\varepsilon \rangle$$

for which we have the identities

$$\partial_t j_\varepsilon = \nabla \langle iu_\varepsilon, \partial_t u_\varepsilon \rangle + V_\varepsilon$$

$$\partial_t \text{curl } j_\varepsilon = \partial_t \mu_\varepsilon = \text{curl } V_\varepsilon.$$

We also define the stress tensor

$$S_\varepsilon := \langle \partial_k u_\varepsilon, \partial_l u_\varepsilon \rangle - \frac{1}{2} \left( |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \right) \delta_{kl}$$

and the “modulated stress-tensor”

$$\tilde{S}_\varepsilon = \langle \partial_k u_\varepsilon - iu_\varepsilon N_\varepsilon v_k, \partial_l u_\varepsilon - iu_\varepsilon N_\varepsilon v_l \rangle - \frac{1}{2} \left( |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon v|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \right) \delta_{kl}.$$

#### 4.3.1 The Gross-Pitaevskii case

We start with the Gross-Pitaevskii case for which the proof is very simple. If  $u_\varepsilon$  solves (1.2) and  $v$  solves (IE), the time-derivative of the modulated energy can be computed to be

$$(4.1) \quad \frac{d\mathcal{E}_\varepsilon(u_\varepsilon(t), t)}{dt} = \int_{\mathbb{R}^2} N_\varepsilon (N_\varepsilon v - j_\varepsilon) \cdot \underbrace{\partial_t v}_{2v^\perp \text{curl } v + \nabla p} - N_\varepsilon V_\varepsilon \cdot v.$$

A linear term in the error  $N_\varepsilon v - j_\varepsilon$  appears. This term is a priori controlled by  $\sqrt{\mathcal{E}}$  which is insufficient to apply Gronwall’s lemma. To remedy this, we use the stress-energy tensor and observe the relation

$$\text{div } \tilde{S}_\varepsilon = -N_\varepsilon (N_\varepsilon v - j_\varepsilon)^\perp \text{curl } v - N_\varepsilon v^\perp \mu_\varepsilon + \frac{1}{2} N_\varepsilon V_\varepsilon.$$

Multiplying it by  $2v$  we are led to

$$\int_{\mathbb{R}^2} 2v \cdot \text{div } \tilde{S}_\varepsilon = \int_{\mathbb{R}^2} -N_\varepsilon (N_\varepsilon v - j_\varepsilon) \cdot 2v^\perp \text{curl } v + N_\varepsilon V_\varepsilon \cdot v.$$

Thus, (4.1) can be rewritten as

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2\tilde{S}_\varepsilon : \nabla v.$$

But  $\tilde{S}_\varepsilon$  is bounded pointwise by  $\mathcal{E}_\varepsilon$  while  $\nabla v$  is bounded by assumption on  $v$ , so we obtain a Gronwall relation and conclude that if  $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \leq o(N_\varepsilon^2)$  it remains true. Here we use crucially that the vortex energy is  $\pi N_\varepsilon |\log \varepsilon| \ll$  hence negligible in the regime  $N_\varepsilon \gg |\log \varepsilon|$ .

### 4.3.2 The parabolic case

If  $u_\varepsilon$  solves (PGL) and  $\mathbf{v}$  solves (L1) or (L2), then we compute

$$(4.2) \quad \frac{d\mathcal{E}_\varepsilon(u_\varepsilon(t), t)}{dt} = - \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log \varepsilon|} |\partial_t u_\varepsilon|^2 + \int_{\mathbb{R}^2} (N_\varepsilon(N_\varepsilon \mathbf{v} - j_\varepsilon) \cdot \partial_t \mathbf{v} - N_\varepsilon V_\varepsilon \cdot \mathbf{v})$$

The same problem as above appears, and again we use an identity of the modulated stress-energy tensor:

$$(4.3) \quad \operatorname{div} \tilde{S}_\varepsilon = \frac{N_\varepsilon}{|\log \varepsilon|} \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathbf{v} \rangle + N_\varepsilon (N_\varepsilon \mathbf{v} - j_\varepsilon)^\perp \operatorname{curl} \mathbf{v} - N_\varepsilon \mathbf{v}^\perp \mu_\varepsilon.$$

where

$$\phi = \mathbf{p} \quad \text{if } N_\varepsilon \ll |\log \varepsilon| \quad \phi = \frac{1}{\lambda} \operatorname{div} \mathbf{v} \quad \text{if not.}$$

Multiplying (4.3) by  $\mathbf{v}^\perp$  and inserting into (4.2), we obtain

$$(4.4) \quad \begin{aligned} \frac{d\mathcal{E}_\varepsilon}{dt} &= \int_{\mathbb{R}^2} 2\tilde{S}_\varepsilon : \nabla \mathbf{v}^\perp - N_\varepsilon V_\varepsilon \cdot \mathbf{v} - 2N_\varepsilon |\mathbf{v}|^2 \mu_\varepsilon \\ &\quad - \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log \varepsilon|} |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 + 2\mathbf{v}^\perp \cdot \frac{N_\varepsilon}{|\log \varepsilon|} \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathbf{v} \rangle. \end{aligned}$$

The vortex energy  $\pi N_\varepsilon |\log \varepsilon|$  is no longer negligible with respect to  $N_\varepsilon^2$ . We now need to prove

$$\frac{d\mathcal{E}_\varepsilon}{dt} \leq C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + o(N_\varepsilon^2)$$

and to conclude from (4.4), we need many of the tools on vortex analysis that have been developed over the years:

- the vortex ball construction [J1, Sa] which allows to bound the energy of the vortices from below in disjoint vortex balls  $B_i$  by  $\pi |d_i| |\log \varepsilon|$  and deduce that the energy outside of  $\cup_i B_i$  is controlled by the excess energy  $\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|$
- the ‘‘product estimate’’ of [SS3] allows to control the velocity:

$$\begin{aligned} \left| \int V_\varepsilon \cdot \mathbf{v} \right| &\leq \frac{2}{|\log \varepsilon|} \left( \int |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 \int |(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathbf{v}) \cdot \mathbf{v}|^2 \right)^{1/2} \\ &\leq \frac{1}{|\log \varepsilon|} \left( \frac{1}{2} \int |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 + 2 \int |(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathbf{v}) \cdot \mathbf{v}|^2 \right) \end{aligned}$$

Thanks to these tools, we may write

$$\begin{aligned} \frac{d\mathcal{E}_\varepsilon}{dt} &= \int_{\mathbb{R}^2} 2 \underbrace{\tilde{S}_\varepsilon}_{\leq C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|)} : \underbrace{\nabla \mathbf{v}^\perp}_{\text{bounded by assumption}} - \underbrace{N_\varepsilon V_\varepsilon \cdot \mathbf{v}}_{\text{controlled by product estimate}} - 2N_\varepsilon |\mathbf{v}|^2 \mu_\varepsilon \\ &\quad - \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log \varepsilon|} |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 + \underbrace{2\mathbf{v}^\perp \cdot \frac{N_\varepsilon}{|\log \varepsilon|} \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathbf{v} \rangle}_{\text{bounded by Cauchy-Schwarz}} \end{aligned}$$

and so

$$\begin{aligned} \frac{d\mathcal{E}_\varepsilon}{dt} &\leq C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log \varepsilon|} \left(\frac{1}{2} + \frac{1}{2} - 1\right) |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 \\ &\quad + \frac{2N_\varepsilon}{|\log \varepsilon|} \int_{\mathbb{R}^2} |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathbf{v}) \cdot \mathbf{v}^\perp|^2 + |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathbf{v}) \cdot \mathbf{v}|^2 - 2N_\varepsilon \int_{\mathbb{R}^2} |\mathbf{v}|^2 \mu_\varepsilon \\ &= C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + \underbrace{\frac{2N_\varepsilon}{|\log \varepsilon|} \int_{\mathbb{R}^2} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathbf{v}|^2 |\mathbf{v}|^2 - 2N_\varepsilon \int_{\mathbb{R}^2} |\mathbf{v}|^2 \mu_\varepsilon}_{\text{bounded by } C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) \text{ by ball construction estimates}}. \end{aligned}$$

Again we may conclude by Gronwall's lemma.

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