Séminaire Laurent Schwartz

EDP et applications

Année 2011-2012

Eduard Feireisl

Domain sensitivity in singular limits of compressible viscous fluids

Séminaire Laurent Schwartz — EDP et applications (2011-2012), Exposé n° XII, 16 p.

<http://slsedp.cedram.org/item?id=SLSEDP_2011-2012____A12_0>

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2011-2012
Exposé n° XII, 1-16

Domain sensitivity in singular limits of compressible viscous fluids

Eduard Feireisl*

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

1 Introduction

Besides their natural applications in computational science and engineering, multiscale problems pose challenging theoretical questions in the framework of mathematical fluid dynamics. The large range of different scales produces a very large family of unknowns that can be treated only at the expense of higher computational costs. The observable macroscopic behavior of a system described by the methods of *continuum* mechanics may depend sensitively on purely microscopic phenomena as interactions of molecules and impurities of the physical boundaries occurring on much smaller space and time scales. The method of model reduction provides *effective equations* for a particular choice of scale(s) of interest. Rigorous derivation of the effective equations typically involves performing a singular limit process in a more general *primitive system* where some of the characteristic numbers are small or become infinite. Typical examples in fluid mechanics are the high Reynolds number limit, where the fluid becomes inviscid, the low Mach number regime driving the fluid flow to incompressibility, or the highly rotating fluids (low Rossby number) occuring in geophysical problems, among others.

In this note, we discuss several recently developed methods for studying stability of a singular limit process with respect to the shape of the underlying physical space. As a model example, we consider a compressible viscous barotropic fluid occupying a spatial domain $\Omega \subset R^3$. In what follows, we describe two rather different problems: (i) the choice of effective boundary conditions; (ii) the fluid flow in the low Mach number regime. In

*The work supported by Grant 201/09/0917 of GA ČR as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

the remaining part of the paper, we analyze these two issues simultaneously comparing the impact of different scales on the form of the resulting effective equations as well as the boundary conditions. Such a "synthesis" of several mathematical techniques may be useful in analyzing much broader class of multiscale problems.

1.1 Effective boundary conditions for viscous fluids

Recently, a new discussion has been initiated concerning the effective boundary conditions satisfied by a velocity field describing the motion of a *viscous* fluid, see Priezjev and Troian [31]. The commonly well-accepted hypothesis asserts *no-slip*, meaning, in particular, the velocity vanishes on a solid wall provided the latter is at rest. On the other hand, some gases as well as fluids are known to obey a kind of *slip behavior* described by Navier's condition

$$\mathbf{u} \cdot \mathbf{n} = 0, \ \beta [\mathbf{u}]_{\rm tan} + [\mathbb{S}\mathbf{n}]_{\rm tan} = 0, \ \beta \geq 0 \tag{1.1}$$

where $\mathbf{u}$ is the fluid velocity, $\mathbb{S}$ denotes the viscous stress, and $\mathbf{n}$ is the normal vector to the boundary, see Bulíček et al. [5], Coron [8]. Moreover, the slip boundary conditions have been identified as the *effective* boundary conditions in domains with rough boundaries, see Jaeger and Mikelič [19], Mohammadi, Pironneau, and Valentin [29].

Another possibility how to interpret the no-slip condition was proposed by Richardson [32], and later developed by Amirat et al. [1], Casado-Diaz, Fernandez-Cara, and Simon [6]. It is shown that the no-slip condition

$$\mathbf{u}|_{\partial\Omega} = 0 \tag{1.2}$$

emerges as an inevitable consequence of fluid trapping by boundary asperities. A general form of the boundary conditions that can be obtained in this way was identified in [3], for more specific results in this direction see Casado-Diaz, Luna-Laynez, and Suarez-Grau [7].

In order to illustrate the previous discussion, consider a family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset R^3$ satisfying uniform $\delta-$cone condition, in particular, we may assume that

$$\Omega_\varepsilon \to \Omega \text{ in the sense of Hausdorff complementary topology,}$$

see Henrot and Pierre [18, Chapter 2]. In addition, we suppose that the boundaries $\partial\Omega_\varepsilon$ "oscillate" in the following sense:

$$\lim_{r\to 0} \left(\liminf_{\varepsilon \to 0} \frac{1}{|\partial\Omega_\varepsilon \cap B_r(y)|} \int_{\partial\Omega_\varepsilon \cap B_r(y)} |\mathbf{n} \cdot \mathbf{w}| \ {\rm d}S_x \right) > 0 \tag{1.3}$$

for q.a. $y \in \partial\Omega$, and all $|\mathbf{w}| = 1, \mathbf{w} \cdot \mathbf{n}(y) = 0$, where $\mathbf{n}(y)$ denotes the outer normal vector at $y \in \partial\Omega$. Under these circumstances, let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be a sequence of vector fields such that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } W^{1,2}(R^3;R^3),\ \ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0.$$

Applying the general result obtained in [4, Theorem 4.1] we may infer that the limit velocity $\mathbf{u}$ satisfies the no-slip condition (1.2).

1.2 Incompressible limits

The so-called *incompressible limits* provide a rigorous justification of the incompressible Navier-Stokes systems as a singular limit of a more complex barotropic model of a compressible viscous fluid in the low Mach number regime, see the surveys by Danchin [10], Gallagher [17], Masmoudi [28], and Schochet [33], among others. On condition that the characteristic speed of a compressible fluid is largely dominated by the speed of sound, the time evolution of the fluid density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ is governed by the scaled Navier-Stokes system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.4}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.5}$$

where the (small) parameter ε is the Mach number, $p = p(\varrho)$ is the barotropic pressure, and the $\mathbb{S}(\nabla_x \mathbf{u})$ is the viscous stress, here given by Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I},\ \ \mu > 0,\ \ \eta \geq 0. \tag{1.6}$$

Furthermore, we suppose the fluid is confined to an unbounded (exterior) domain $\Omega \subset R^3$, with smooth boundary, and impose the slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \ [\mathbb{S}(\nabla_x \mathbf{u})\mathbf{n}]_{\tan}|_{\partial\Omega} = 0, \tag{1.7}$$

along with conditions at "infinity":

$$\mathbf{u} \to 0,\ \ \varrho \to \overline{\varrho} > 0 \text{ as } |x| \to \infty. \tag{1.8}$$

Under these conditions, it is natural to assume that solutions of the evolutionary problem (1.4 - 1.8) satisfy the total energy balance:

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho) - \partial_\varrho H(\overline{\varrho})(\varrho - \overline{\varrho}) - H(\overline{\varrho}) \right) \right)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \tag{1.9}$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \left(H(\varrho_{0,\varepsilon}) - \partial_\varrho H(\overline{\varrho})(\varrho_{0,\varepsilon} - \overline{\varrho}) - H(\overline{\varrho}) \right) \right) \ \mathrm{d}x$$

where we have introduced the initial data

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon}, \ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}$$

characterizing the original state of the system. The symbol H stands for the potential energy determined in terms of the pressure, specifically,

$$H''(\varrho) = \frac{1}{\varrho} p'(\varrho);$$

in particular, the function H is strictly convex, provided the pressure p is an increasing function of the fluid density.

In what follows, we shall assume that $p = p(\varrho) \in C[0,\infty) \cap C^2(0,\infty)$ satisfies

$$p(0) = 0, \ p'(\varrho) > 0 \text{ for all } \varrho > 0, \ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = p_\infty > 0 \text{ for a certain } \gamma > 3/2. \tag{1.10}$$

The technical restriction $\gamma > 3/2$ appears also as a kind of critical exponent in the *existence* theory related to system (1.4 - 1.6), see [16], Lions [26].

The energy inequality (1.9) yields suitable uniform bounds independent of the scaling parameter $\varepsilon \to 0$ provided we are able to control the quantity on the right-hand side given in terms of the initial data. Accordingly, we assume the data are *prepared*, specifically,

$$\varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon r_{0,\varepsilon}, \ \text{with } \{r_{0,\varepsilon}\}_{\varepsilon > 0} \text{ bounded in } L^2 \cap L^\infty(\Omega), \tag{1.11}$$

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon > 0} \text{ is bounded in } L^2(\Omega; R^3). \tag{1.12}$$

Consequently, using convexity of the function H we obtain

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^q(\Omega)} \leq c, \ 1 \leq q \leq \min\{\gamma, 2\},$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon\|_{L^2(\Omega;R^3)} \leq c,$$

and, by virtue of Korn's inequality,

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega;R^3)} \leq c$$

for any family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of (weak) solutions to problem (1.4 - 1.8) emanating from the initial data satisfying (1.11), (1.12), cf. Lions and Masmoudi [27]. In particular,

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^2 + L^q(\Omega), \tag{1.13}$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)), \tag{1.14}$$

where

$$\mathrm{div}_x \mathbf{U} = 0.$$

Strong convergence with respect to *time* in (1.14) is a more delicate issue and will be discussed below. As soon as established, it is easy to check that the limit function $\mathbf{U}$ represents a (weak) solution to the incompressible Navier-Stokes system

$$\overline{\varrho}\,(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U})) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}).$$

1.3 Synthesis: Incompressible limits on families of domains

Our goal is to apply the methods presented in Sections 1.1, 1.2 to problems of *domain sensitivity* of incompressible limits. To this end, we fix the Mach number to be ε and introduce a family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset R^3$ enjoying the following properties:

- $\{\Omega_\varepsilon\}_{\varepsilon>0}$ satisfy uniform $\delta-$cone condition, in particular,

$$\Omega_\varepsilon \to \Omega \text{ in the sense of Hausdorff complementary topology.};$$

- $\{\Omega_\varepsilon\}_{\varepsilon>0}$ are exterior domains with a compact and regular boundaries satisfying "oscillation" hypothesis (1.3);

- for each $x \in \partial\Omega_\varepsilon$ there are two open balls B_i, B_e of radius $r \geq c_b \varepsilon^\beta$ such that

$$x \in \overline{B}_i \cap \overline{B}_e,\ B_i \subset \Omega_\varepsilon,\ B_e \subset R^3 \setminus \Omega_\varepsilon,$$

  with $0 < \beta < 1/4$.

The last hypothesis states that oscillations of the boundaries are relatively slow with respect to the Mach number. As we will see below, the low Mach number limit passage is stable with respect to the class of domains satisfying the above stated hypotheses uniformly with respect to $\varepsilon \to 0$. As the crucial issue is to establish strong (a.a. pointwise) convergence of the velocities $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, the main problem is to show that the gradient component of the velocity fields, associated to propagation of *acoustic waves*, converges strongly (locally a.a.) to zero in Ω.

2 Helmholtz decomposition, acoustic waves, and related problems

The momentum fields $\varrho_\varepsilon \mathbf{u}_\varepsilon$ can be written in the form

$$\mathbf{u}_\varepsilon = \mathbf{H}_\varepsilon[\varrho_\varepsilon \mathbf{u}_\varepsilon] + \nabla_x \Phi_\varepsilon$$

where the function Φ_ε is termed *acoustic potential* determined as the (unique) solution of the elliptic problem

$$\Delta \Phi_\varepsilon = \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon),\ (\nabla_x \Phi_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon)\cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ \Phi_\varepsilon \to 0 \text{ as } |x| \to \infty.$$

Of course, the operator $\mathbf{H}_\varepsilon$ is nothing other than the standard *Helmholtz* projector onto the space of solenoidal functions in Ω_ε, with vanishing normal trace. Under the hypotheses on the family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ introduced in Section 1.3, the result of Farwig, Kozono, and Sohr [13] yields the bound

$$\|\mathbf{H}_\varepsilon[\mathbf{v}]\|_{L^p\cap L^2(\Omega_\varepsilon;R^3)} \le \varepsilon^{-\beta\left(\frac{3}{2}-\frac{3}{p}\right)} c(p) \|\mathbf{v}\|_{L^p\cap L^2(\Omega_\varepsilon;R^3)},\ 2 \le p < \infty,$$

with $c(p)$ independent of ε.

Clearly, the solenoidal part $\mathbf{H}_\varepsilon[\varrho_\varepsilon \mathbf{u}_\varepsilon]$ satisfying the projected equation (1.5) is apparently more regular than its gradient counterpart, in particular, the Lions-Aubin argument can be used to deduce that

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \overline{\varrho}\mathbf{U} \text{ pointwise a.a. in } (0,T)\times K,$$

and

$$(\mathbf{H} - \mathbf{H}_\varepsilon)[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to 0 \text{ pointwise a.a. in } (0,T)\times K$$

for any compact $K \subset \Omega$, where $\mathbf{H}$ denotes the Hemholtz projection in the limit domain Ω. Note that all functions can be extended outside Ω_ε as the latter satisfies the uniform $\delta-$condition.

Writing

$$\varrho_\varepsilon \mathbf{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] + (\mathbf{H}_\varepsilon - \mathbf{H})[\varrho_\varepsilon \mathbf{u}_\varepsilon] + \nabla_x \Phi_\varepsilon$$

we therefore focus on the gradient part,

$$\nabla_x \Phi_\varepsilon \to 0 \text{ weakly-(*) in } L^\infty(0,T; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega, R^3)),$$

where "weakly" concerns the time variable as spatial compactness is ensured by (1.13), (1.14). The time evolution of the potential Φ_ε is governed by *acoustic equation* discussed in the next section.

2.1 Acoustic equation

Lighthill [23], [24] proposed to rewrite the compressible Navier-Stokes system in the form:

$$\varepsilon \partial_t r_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0,$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + p'(\overline{\varrho}) \nabla_x r_\varepsilon = \varepsilon \mathrm{div}_x \mathbb{L}_\varepsilon,$$

where

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon,$$

and the symbol $\mathbb{L}_\varepsilon$ stands for the so-called Lighthill's tensor

$$\mathbb{L}_\varepsilon = \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \frac{1}{\varepsilon^2}\Big(p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho}) \Big) \mathbb{I}.$$

Furthermore, Lighthill's system may be written in terms of the acoustic potential Φ_ε as a *wave equation*

$$\varepsilon \partial_t r_\varepsilon + \Delta_{N,\varepsilon} \Phi_\varepsilon = 0, \tag{2.1}$$

$$\varepsilon \partial_t \Phi_\varepsilon + p'(\overline{\varrho}) r_\varepsilon = \varepsilon \Delta^{-1}_{N,\varepsilon} \mathrm{div}_x \mathrm{div}_x \mathbb{L}, \tag{2.2}$$

supplemented with the boundary condition

$$\nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0, \tag{2.3}$$

where the symbol $\Delta_{N,\varepsilon}$ denotes the standard Neumann Laplacian in Ω_ε. We point out that the slip boundary condition (1.7) is *absolutely necessary* in order to justify this step.

2.2 Abstract formulation

For the sake of simplicity, we will assume that $p'(\overline{\varrho}) = 1$. System of equations (2.1), (2.2) can be written in the abstract form:

$$\varepsilon \partial_t r_\varepsilon + \Delta_{N,\varepsilon} \Phi_\varepsilon = 0, \tag{2.4}$$

$$\varepsilon \partial_t \Phi_\varepsilon + r_\varepsilon = \varepsilon F(-\Delta_{N,\varepsilon})[g_\varepsilon], \tag{2.5}$$

where $-\Delta_{N,\varepsilon}$ is the self-adjoint extension of the Neumann Laplacian on the Hilbert space $L^2(\Omega_\varepsilon)$, $g_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon))$, and $F = F(y)$ is a suitably chosen function defined on the half-line $(0,\infty)$ that may become singular for $y \to 0$, $y \to \infty$.

In order to identify the function F, we use the elliptic estimates that may be deduced by a scaling argument:

$$\|\nabla_x^2 v\|_{L^p(\Omega_\varepsilon)} \leq c(p)\Big(\|\Delta_{N,\varepsilon} v\|_{L^p(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}}\|v\|_{L^p(\Omega_\varepsilon)}\Big) \text{ for } 1 < p < \infty, \tag{2.6}$$

where the constant $c(p)$ is independent of $\varepsilon \to 0$.

Now, we have

$$\left|\int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x^2 \Delta_{N,\varepsilon}^{-1}[\varphi] \ \mathrm{d}x\right| \leq \|\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \|\nabla_x^2 \Delta_{N,\varepsilon}^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)},$$

where, by virtue of (2.6),

$$\|\nabla_x^2 \Delta_{N,\varepsilon}^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)} \leq c\left(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-2\beta}\|\Delta_{N,\varepsilon}^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)}\right).$$

Thus we conclude that

$$\Delta_{N,\varepsilon}^{-1}\mathrm{div}_x\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) = F(-\Delta_{N,\varepsilon})g_\varepsilon,\ g_\varepsilon = \varepsilon^{-2\beta}h_\varepsilon,\ \{h_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega_\varepsilon)),$$

with

$$F(y) = 1 + \frac{1}{y}.$$

As the remaining terms on the right-hand side of (2.2) can be handled in a similar way, we may write (2.4), (2.5) in the form

$$\varepsilon\partial_t r_\varepsilon + \Delta_{N,\varepsilon}\Phi_\varepsilon = 0, \tag{2.7}$$

$$\varepsilon\partial_t\Phi_\varepsilon + r_\varepsilon = \varepsilon^{1-2\beta}F(-\Delta_{N,\varepsilon})[h_\varepsilon],\ \{h_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega_\varepsilon)), \tag{2.8}$$

see [15] for details.

2.3 Dispersive estimates

Solutions of system (2.7), (2.8) can be written by means of *variation-of-constants formula*:

$$\Phi_\varepsilon(t) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}}\frac{t}{\varepsilon}\right)\left[\Phi_{0,\varepsilon} + \mathrm{i}\frac{1}{\sqrt{-\Delta_{N,\varepsilon}}}[r_{0,\varepsilon}]\right] \tag{2.9}$$

$$+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}}\frac{t}{\varepsilon}\right)\left[\Phi_{0,\varepsilon} - \mathrm{i}\frac{1}{\sqrt{-\Delta_{N,\varepsilon}}}[r_{0,\varepsilon}]\right]$$

$$+\frac{\varepsilon^{-2\beta}}{2}\int_0^t \left[\exp\left(\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}}\frac{t-s}{\varepsilon}\right)+\exp\left(-\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}}\frac{t-s}{\varepsilon}\right)\right] F(-\Delta_{N,\varepsilon})[h_\varepsilon]\ \mathrm{d}s.$$

Apparently, the last term on the right-hand side of (2.9) becomes unbounded for $\varepsilon \to 0$. The idea is to use the *dispersive estimates* for the wave propagator

$$\exp\left(\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}}\frac{t}{\varepsilon}\right)$$

to compensate for the singular term $\varepsilon^{-2\beta}$. The dispersive estimates are *local* in the physical space and apply to a compact "frequency" range represented by a cut-off function $G(-\Delta_{N,\varepsilon})$, $G \in C_c^\infty(0,\infty)$. In particular, the dispersive estimates are not compatible with the presence of *trapped modes* representing proper eigenvectors of the Neumann Laplacian in Ω_ε. Accordingly, after a short inspection of (2.9), we focus on integrals in the form

$$\int_{\Omega_\varepsilon} \exp\left(\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}}\frac{t}{\varepsilon}\right) G(-\Delta_{N,\varepsilon})[\psi]\varphi\ \mathrm{d}x$$

that can be conveniently expressed in terms of the spectral measure $\mu_{\varphi,\varepsilon}$ associated to the function $\varphi \in C_c^\infty(\Omega)$. Specifically, we get

$$\int_{\Omega_\varepsilon} \exp\left(\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}}\frac{t}{\varepsilon}\right) G(-\Delta_{N,\varepsilon})[\psi]\varphi\ \mathrm{d}x = \int_0^\infty \exp\left(\mathrm{i}\sqrt{y}\frac{t}{\varepsilon}\right) G(y)\tilde{\psi}(y)\ \mathrm{d}\mu_{\varphi,\varepsilon} \tag{2.10}$$

for a certain function $\tilde{\psi} \in L^2_{\mu_{\varphi,\varepsilon}}[0,\infty)$,

$$\|\tilde{\psi}\|_{L^2_{\mu_{\varphi,\varepsilon}}[0,\infty)} \leq \|\psi\|_{L^2(\Omega_\varepsilon)}.$$

The main advantage of the new formula (2.10) is that the $\varepsilon-$dependence is concentrated in the spectral measure $\mu_{\varphi,\varepsilon}$. Following Last [21] we write

$$\int_0^T \left|\int_{\Omega_\varepsilon} \exp\left(\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}}\frac{t}{\varepsilon}\right) G(-\Delta_{N,\varepsilon})[\psi]\varphi\ \mathrm{d}x\right|^2\ \mathrm{d}t \tag{2.11}$$

$$= \int_0^T\int_0^\infty\int_0^\infty \exp\left(\mathrm{i}\left(\sqrt{y}-\sqrt{x}\right)\frac{t}{\varepsilon}\right) G(y)G(x)\tilde{\psi}(y)\overline{\tilde{\psi}(x)}\ \mathrm{d}\mu_{\varphi,\varepsilon}(x)\ \mathrm{d}\mu_{\varphi,\varepsilon}(y)\ \ \mathrm{d}t$$

$$\leq e\int_0^\infty\int_0^\infty \left(\int_{-\infty}^\infty \exp\left(-\left(\frac{t}{T}\right)^2\right)\exp\left(\mathrm{i}\left(\sqrt{y}-\sqrt{x}\right)\frac{t}{\varepsilon}\right)\ \mathrm{d}t\right)\times$$

$$\times G(y)G(x)\tilde{\psi}(y)\overline{\tilde{\psi}(x)}\ \mathrm{d}\mu_{\varphi,\varepsilon}(x)\ \mathrm{d}\mu_{\varphi,\varepsilon}(y)$$

$$\leq eT\sqrt{\pi}\int_0^\infty\int_0^\infty \exp\left(-\frac{|\sqrt{y}-\sqrt{x}|^2}{\varepsilon^2}\frac{T^2}{4}\right) G(y)G(x)\tilde{\psi}(y)\overline{\tilde{\psi}(x)}\ \mathrm{d}\mu_{\varphi,\varepsilon}(x)\ \mathrm{d}\mu_{\varphi,\varepsilon}(y),$$

where

$$\int_0^\infty \exp\left(- \frac{|\sqrt{y} - \sqrt{x}|^2}{\varepsilon^2} \frac{T^2}{4} \right) \mathrm{d}\mu_{\varphi,\varepsilon}(y) \tag{2.12}$$

$$= \sum_{n=0}^\infty \int_{\varepsilon n \leq |\sqrt{y}-\sqrt{x}| \leq \varepsilon(n+1)} \exp\left(- \frac{|\sqrt{y} - \sqrt{x}|^2}{\varepsilon^2} \frac{T^2}{4} \right) \ \mathrm{d}\mu_{\varphi,\varepsilon}(y)$$

$$\leq \sup_{n\geq 0} \int_{\varepsilon n \leq |\sqrt{y}-\sqrt{x}| \leq \varepsilon(n+1)} 1 \ \mathrm{d}\mu_{\varphi,\varepsilon}(y) \sum_{n=0}^\infty \exp\left(- \frac{n^2 T^2}{4} \right) \ \text{for } x \in \mathrm{supp}[G].$$

Consequently, in order to obtain uniform bounds with respect to ε, we have to control the values of the spectral measure $\mu_{\varphi,\varepsilon}$ on any compact subinterval of $(0,\infty)$ containing $\mathrm{supp}[G]$. In particular, our goal will be to show that $\mu_{\varphi,\varepsilon}$ is (locally) *Lipschitz continuous* with respect to the standard Lebesgue measure, specifically,

$$\mu_{\varphi,\varepsilon}[I] \leq c(\delta)|I| \text{ for any interval } I \subset (\delta, 1/\delta),\ \delta > 0. \tag{2.13}$$

Note that Last [21] considered more general Hölder type estimates

$$\mu_{\varphi,\varepsilon}[I] \leq c(\delta)|I|^\alpha \text{ for suitable } 0 < \alpha \leq 1.$$

If (2.13) holds, it is easy to deduce from (2.11), (2.12) that

$$\int_0^T \left| \int_{\Omega_\varepsilon} \exp\left(\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}} \frac{t}{\varepsilon} \right) G(-\Delta_{N,\varepsilon})[\psi]\varphi \ \mathrm{d}x \right|^2 \ \mathrm{d}t \leq \varepsilon c(G,\varphi) \|\psi\|^2_{L^2(\Omega_\varepsilon)}. \tag{2.14}$$

It is crucial to make sure that the constant $c(\delta)$ in (2.13) is independent of ε. If this is the case, relation (2.13) provides the desired *stability* of the low Mach number limit process with respect to the underlying spatial domain.

It is remarkable that relation (2.14) provides also a piece of information concerning the *rate* of decay, here of order $\sqrt{\varepsilon}$ that seems optimal. Such a result is in the spirit of the abstract theory developed by Kato [20] and intimately related to the Limiting Absorption Principle for the Neumann Laplacian, see Eidus [12]. Weaker results of the type

$$\int_0^T \left| \int_{\Omega_\varepsilon} \exp\left(\mathrm{i}\sqrt{-\Delta_{N,\varepsilon}} \frac{t}{\varepsilon} \right) G(-\Delta_{N,\varepsilon})[\psi]\varphi \ \mathrm{d}x \right|^2 \ \mathrm{d}t \leq \omega(\varepsilon, G, \varphi) \|\psi\|^2_{L^2(\Omega_\varepsilon)}, \tag{2.15}$$

with

$$\omega(\varepsilon, G, \varphi) \to 0 \text{ as } \varepsilon \to 0,$$

can be deduced by means of the celebrated RAGE theorem, see [14]. Although (2.15) can be verified for a substantially larger class of domains, its form is not convenient for our purposes here as it may be very unstable with respect to the underlying geometries.

2.4 Lipschitz continuity of the spectral measures

Our goal is to show relation (2.13), with a constant $c(\delta)$ independent of the scaling parameter ε. Revoking the standard *Stone's formula* we get

$$\mu_{\varphi,\varepsilon}(a,b) = \lim_{\delta\to 0+} \lim_{\eta\to 0+} \int_{a+\delta}^{b-\delta} \int_{\Omega_\varepsilon} \varphi \left(\frac{1}{-\Delta_{N,\varepsilon} - \lambda - \mathrm{i}\eta} - \frac{1}{-\Delta_{N,\varepsilon} - \lambda + \mathrm{i}\eta} \right) [\varphi] \, \mathrm{d}x \, \mathrm{d}\lambda. \quad (2.16)$$

Since Ω_ε are exterior domains with regular boundaries, the Neumann Laplacian $\Delta_{N,\varepsilon}$ satisfies the Limiting Absorption Principle (see Leis [22], Vainberg [34]), specifically, operators

$$\mathcal{V} \circ (-\Delta_{N,\varepsilon} - \lambda \pm \mathrm{i}\eta)^{-1} \circ \mathcal{V},$$

with

$$\mathcal{V}[v] = (1 + |x|^2)^{-s/2} v, \ s > 1,$$

are bounded uniformly for λ belonging to compact subsets of $(0,\infty)$. In particular, as $\varphi \in C_c^\infty(\Omega_\varepsilon)$, we can perform the limits in (2.16) to obtain

$$\mu_{\varphi,\varepsilon}(a,b) = \int_a^b \int_{\Omega_\varepsilon} \left(w_{\lambda,\varepsilon}^- - w_{\lambda,\varepsilon}^+ \right) \varphi \, \mathrm{d}x \, \mathrm{d}\lambda, \ 0 < a < b < \infty,$$

where $w_{\lambda,\varepsilon}^\pm$ is the solution of the elliptic problem

$$\Delta w_{\lambda,\varepsilon}^\pm + \lambda w_{\lambda,\varepsilon}^\pm = \varphi \text{ in } \Omega_\varepsilon, \ \nabla_x w_{\lambda,\varepsilon}^\pm \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \quad (2.17)$$

determined uniquely by *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r \left(\partial_r \pm \mathrm{i}\sqrt{\lambda} \right) w_{\lambda,\varepsilon}^\pm = 0, \ r = |x|. \quad (2.18)$$

Problem (2.17), (2.18) can be "localized" in space as φ has compact support, say,

$$\mathrm{supp}[\varphi] \subset \{|x| \le R\};$$

whence solutions satisfying Sommerfeld radiation condition are uniquely determined *outside* the ball B_R by its value on the sphere $\{|x| = R\}$. Specifically, if

$$w_{\lambda,\varepsilon}^\pm = \sum_{l=0}^\infty \sum_{m=-l}^{l} a_l^m Y_l^m(\theta,\phi) \text{ for } |x| = R,$$

where (r,θ,ϕ) are polar coordinates, and Y_l^m spherical harmonics of order l, then

$$w_{\lambda,\varepsilon}^\pm = \sum_{l=0}^\infty \sum_{m=-l}^{l} a_l^m Y_l^m(\theta,\phi) \frac{h_l^{(1)}(\pm\sqrt{\lambda} r)}{h_l^{(1)}(\pm\sqrt{\lambda} R)} \text{ for all } |x| \ge R,$$

where $h_l^{(1)}$ are spherical Bessel functions, see Nédélec [30].

The localized problem can be treated easily by means of the domain perturbation methods used in spectral analysis, see Arrieta and Krejčiřík [2] and the references cited therein. In particular, the hypotheses imposed on the family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ guarantee validity of (2.13), and, consequently, the decay estimate (2.14) uniformly for $\varepsilon \to 0$. Note that the rate of convergence of order $\sqrt{\varepsilon}$ claimed in (2.14) is necessary to eliminate the singularity $\varepsilon^{-2\beta}$ in (2.9), that means, we need

$$0 < \beta < 1/4,$$

in accordance with the list of hypotheses introduced in Section 1.3.

Thus we have established convergence

$$\left\{ t \mapsto \int_{\Omega_\varepsilon} \varphi G(-\Delta_{N,\varepsilon})[\Phi_\varepsilon] \ \mathrm{d}x \right\} \to 0 \text{ in } L^2(0,T) \tag{2.19}$$

for any $\varphi \in C_c^\infty(\Omega)$, $G \in C_c^\infty(0,\infty)$. Although (2.19) may look like a relatively weak result, it can be combined with the spatial compactness of $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ to obtain the desired conclusion

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ (strongly) in } L^2((0,T)\times K; R^3) \text{ for any compact } K \subset \Omega, \tag{2.20}$$

see [15] for details.

3 Convergence via RAGE theorem

If Ω is a fixed domain, we need not to establish uniformity of the decay of acoustic waves, and, accordingly, formula (2.15) provides the desired conclusion. Inspecting relation (2.11) we recall that

$$\int_0^T \left| \int_\Omega \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right) G(-\Delta_N)[\psi]\varphi \ \mathrm{d}x \right|^2 \ \mathrm{d}t$$
$$\leq c(|G|) \int_0^\infty \int_0^\infty \exp\left(- \frac{|\sqrt{y}-\sqrt{x}|^2}{\varepsilon^2}\frac{T^2}{4}\right) \tilde{\psi}(y)\overline{\tilde{\psi}(x)} \ \mathrm{d}\mu_\varphi(x) \ \mathrm{d}\mu_\varphi(y);$$

whence a direct application of Cauchy-Schwartz inequality gives rise to

$$\int_0^T \left| \int_\Omega \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right) G(-\Delta_N)[\psi]\varphi \ \mathrm{d}x \right|^2 \ \mathrm{d}t \leq \omega^2(\varepsilon,\varphi) c(|G|) \|\psi\|^2_{L^2(\Omega)},$$

where

$$\omega(\varepsilon,\varphi) = \left(\int_0^\infty \int_0^\infty \exp\left(- \frac{|\sqrt{y}-\sqrt{x}|^2}{\varepsilon^2}\frac{T^2}{4}\right) \ \mathrm{d}\mu_\varphi(x) \ \mathrm{d}\mu_\varphi(y) \right)^{1/4}.$$

It is easy to see that

$$\omega(\varepsilon, \varphi) \to 0 \text{ as } \varepsilon \to 0 \text{ for each fixed } \varphi \in C_c^\infty(\Omega)$$

only if the spectral measure μ_φ does not charge points, meaning, only if the point spectrum of the Neumann Laplacian in Ω is empty. The argument is exactly the same as in the proof of celebrated RAGE theorem (see Cycon et al. [9, Theorem 5.8]):

Theorem 3.1 *Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto H_c, where*

$$H = H_c \oplus \mathrm{cl}_H \Big\{ \mathrm{span}\{ w \in H \mid w \textit{ an eigenvector of } A \} \Big\}.$$

Then

$$\left\| \frac{1}{\tau} \int_0^\tau \exp(-\mathrm{i}tA) C P_c \exp(\mathrm{i}tA) \ \mathrm{d}t \right\|_{\mathcal{L}(H)} \to 0 \textit{ for } \tau \to \infty.$$

The presence or absence of eigenvalues and the associated eigenfuctions (trapped modes) of the Neumann Laplacian in *unbounded* spatial domains is a delicate issue and may depend sensitively on the geometry of the problem, see D'Ancona and Racke [11], Linton and McIver [25], and the references cited therein.

References

[1] A. A. Amirat, D. Bresch, J. Lemoine, and J. Simon. Effect of rugosity on a flow governed by stationary Navier-Stokes equations. *Quart. Appl. Math.*, **59**:768–785, 2001.

[2] J. M. Arrieta and D. Krejčiřík. Geometric versus spectral convergence for the Neumann Laplacian under exterior perturbations of the domain. In *Integral methods in science and engineering. Vol. 1*, pages 9–19. Birkhäuser Boston Inc., Boston, MA, 2010.

[3] D. Bucur, E. Feireisl, and Nečasová. Boundary behavior of viscous fluids: Influence of wall roughness and friction-driven boundary conditions. *Arch. Rational Mech. Anal.*, **197**:117–138, 2010.

[4] D. Bucur, E. Feireisl, Š. Nečasová, and J. Wolf. On the asymptotic limit of the Navier–Stokes system on domains with rough boundaries. *J. Differential Equations*, **244**:2890–2908, 2008.

[5] M. Bulíček, J. Málek, and K.R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear- rate dependent viscosity. *Indiana Univ. Math. J.*, **56**:51–86, 2007.

[6] J. Casado-Díaz, E. Fernández-Cara, and J. Simon. Why viscous fluids adhere to rugose walls: A mathematical explanation. *J. Differential Equations*, **189**:526–537, 2003.

[7] Luna-Laynez M. Suárez-Grau F.J. Casado-Díaz, J. Asymptotic behavior of a viscous fluid with slip boundary conditions on a slightly rough wall. *Math. Models Meth. Appl. Sci.*, **20**:121–156, 2010.

[8] F. Coron. Derivation of slip boundary conditions for the Navier-Stokes system from the Boltzmann equation. *J. Statistical Phys.*, **54**:829–857, 1989.

[9] H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators: with applications to quantum mechanics and global geometry.* Texts and monographs in physics, Springer-Verlag, Berlin,Heidelberg, 1987.

[10] R. Danchin. Zero Mach number limit for compressible flows with periodic boundary conditions. *Amer. J. Math.*, **124**:1153–1219, 2002.

[11] P. D'Ancona and R. Racke. Evolution equations in non-flat waveguides. 2010. arXiv:1010.0817.

[12] D. M. Eidus. Limiting amplitude principle (in Russian). *Usp. Mat. Nauk*, **24**(3):91–156, 1969.

[13] R. Farwig, H. Kozono, and H. Sohr. An L^q-approach to Stokes and Navier-Stokes equations in general domains. *Acta Math.*, **195**:21–53, 2005.

[14] E. Feireisl. Local decay of acoustic waves in the low mach number limits on general unbounded domains under slip boundary conditions. *Commun. Partial Differential Equations*, 2010. Submitted.

[15] E. Feireisl, T. Karper, O. Kreml, and J. Stebel. Stability with respect to domain of the low Mach number limit of compressible viscous fluids. 2011. Preprint.

[16] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids. *J. Math. Fluid Mech.*, **3**:358–392, 2001.

[17] I. Gallagher. Résultats récents sur la limite incompressible. *Astérisque*, (299):Exp. No. 926, vii, 29–57, 2005. Séminaire Bourbaki. Vol. 2003/2004.

[18] A. Henrot and M. Pierre. *Variation et optimisation de formes*, volume 48 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin, 2005. Une analyse géométrique. [A geometric analysis].

[19] W. Jaeger and A. Mikelić. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations*, **170**:96–122, 2001.

[20] T. Kato. Remarks on the zero viscosity limit for nonstationary Navier–Stokes flows with boundary. *In Seminar on PDE's, S.S. Chern (ed.), Springer, New York*, 1984.

[21] Y. Last. Quantum dynamics and decomposition of singular continuous spectra. *J. Funct. Anal.*, **142**:406–445, 1996.

[22] R. Leis. *Initial-boundary value problems in mathematical physics*. B.G. Teubner, Stuttgart, 1986.

[23] J. Lighthill. On sound generated aerodynamically I. General theory. *Proc. of the Royal Society of London*, **A 211**:564–587, 1952.

[24] J. Lighthill. On sound generated aerodynamically II. General theory. *Proc. of the Royal Society of London*, **A 222**:1–32, 1954.

[25] C.M. Linton, P.-L., and P. McIver. Embedded trapped modes in water waves and acoustics. *Wave Motion*, **45**:16–29, 2007.

[26] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models*. Oxford Science Publication, Oxford, 1998.

[27] P.-L. Lions and N. Masmoudi. Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl.*, **77**:585–627, 1998.

[28] N. Masmoudi. Examples of singular limits in hydrodynamics. *In Handbook of Differential Equations, III, C. Dafermos, E. Feireisl Eds., Elsevier, Amsterdam*, 2006.

[29] B. Mohammadi, O. Pironneau, and F. Valentin. Rough boundaries and wall laws. *Int. J. Numer. Meth. Fluids*, **27**:169–177, 1998.

[30] J.-C. Nédélec. *Acoustic and electromagnetic equations*. Springer-Verlag, Heidelberg, 2001.

[31] N. V. Priezjev and S.M. Troian. Influence of periodic wall roughness on the slip behaviour at liquid/solid interfaces: molecular versus continuum predictions. *J. Fluid Mech.*, **554**:25–46, 2006.

[32] S. Richardson. On the no-slip boundary condition. *J. Fluid Mech.*, **59**:707–719, 1973.

[33] S. Schochet. The mathematical theory of low Mach number flows. *M2ANMath. Model Numer. anal.*, **39**:441–458, 2005.

[34] B. R. Vaĭnberg. *Asimptoticheskie metody v uravneniyakh matematicheskoi fiziki*. Moskov. Gos. Univ., Moscow, 1982.