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**On the spectral instability of parallel shear flows**

# On the spectral instability of parallel shear flows

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### Abstract

This short note is to announce our recent results [2, 3] which provide a complete mathematical proof of the viscous destabilization phenomenon, pointed out by Heisenberg (1924), C.C. Lin and Tollmien (1940s), among other prominent physicists. Precisely, we construct growing modes of the linearized Navier-Stokes equations about general stationary shear flows in a bounded channel (channel flows) or on a half-space (boundary layers), for sufficiently large Reynolds number $R \to \infty$. Such an instability is linked to the emergence of Tollmien-Schlichting waves in describing the early stage of the transition from laminar to turbulent flows.

## 1   Introduction

Study of hydrodynamics stability and the inviscid limit of viscous fluids is one of the most classical subjects in fluid dynamics, going back to the most prominent physicists including Lord Rayleigh, Orr, Sommerfeld, Heisenberg, among many others. It is documented in the physical literature (see, for instance, [6, 1]) that laminar viscous fluids are unstable, or become turbulent, in a small viscosity or high Reynolds number limit. In particular, generic stationary shear flows are linearly unstable for sufficiently large Reynolds numbers.

Specifically, let $u_0 = (U(z), 0)^{tr}$ be a stationary shear flow. We are interested in the linearization of the incompressible Navier-Stokes equations about the shear profile:

$$v_t + u_0 \cdot \nabla v + v \cdot \nabla u_0 + \nabla p = \frac{1}{R} \Delta v \tag{1.1a}$$

$$\nabla \cdot v = 0 \tag{1.1b}$$

posed on $\Omega = \mathbb{R} \times [0, 2]$ or $\Omega = \mathbb{R} \times \mathbb{R}_+$, together with the classical no-slip boundary conditions on the walls:

$$v_{|\partial\Omega} = 0. \tag{1.2}$$

Here $v$ denotes the usual velocity perturbation of the fluid, and $p$ denotes the corresponding pressure. Of interest is the Reynolds number $R$ sufficiently large, and whether the

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linearized problem is spectrally unstable: the existence of unstable modes of the form $(v, p) = (e^{\lambda t} \tilde{v}(y, z), e^{\lambda t} \tilde{p}(y, z))$ for some $\lambda$ with $\Re \lambda > 0$.

The spectral problem is a very classical issue in fluid mechanics. A huge literature is devoted to its detailed study. We in particular refer to [1, 9] for the major works of Heisenberg, C.C. Lin, Tollmien, and Schlichting. The studies began around 1930, motivated by the study of the boundary layer around wings. In airplanes design, it is crucial to study the boundary layer around the wing, and more precisely the transition between the laminar and turbulent regimes, and even more crucial to predict the point where boundary layer splits from the boundary. A large number of papers has been devoted to the estimation of the critical Reynolds number of classical shear flows (plane Poiseuille flow, Blasius profile, exponential suction/blowing profile, among others).

It were Sommerfeld and Orr [10, 7], who initiated the study of the spectral problem via the Fourier normal mode theory. Precisely, they search for the unstable solutions of the form $e^{i\alpha(y-ct)}(\hat{v}(z), \hat{p}(z))$, defined through the stream function $\psi$,

$$v = \nabla^{\perp} \psi = (\partial_z, -\partial_y)\psi, \qquad \psi(t, y, z) := \phi(z)e^{i\alpha(y-ct)}. \tag{1.3}$$

It follows that $\phi(z)$ solves the well-known Orr-Sommerfeld equations

$$\epsilon(\partial_z^2 - \alpha^2)^2 \phi = (U - c)(\partial_z^2 - \alpha^2)\phi - U''\phi, \tag{1.4}$$

with $\epsilon = 1/(i\alpha R)$, where $\phi(z)$ denotes the corresponding stream function, with $\phi$ and $\partial_z \phi$ vanishing at the boundary $z = 0$. When $\epsilon = 0$, (1.4) reduces to the classical Rayleigh equation, which corresponds to inviscid flows. The singular perturbation theory was developed to construct Orr-Sommerfeld solutions from those of Rayleigh solutions.

**Inviscid unstable profiles.** If the profile is unstable for the Rayleigh equation, then there exist a spatial frequency $\alpha_{\infty}$, an eigenvalue $c_{\infty}$ with $\Im c_{\infty} > 0$, and a corresponding eigenvalue $\phi_{\infty}$ that solve (1.4) with $\epsilon = 0$ or $R = \infty$. We can then make a perturbative analysis to construct an unstable eigenmode $\phi_R$ of the Orr-Sommerfeld equation with an eigenvalue $\Im c_R > 0$ for any large enough $R$. This can be done by adding a boundary sublayer to the inviscid mode $\phi_{\infty}$ to correct the boundary conditions for the viscous problem.

**Inviscid stable profiles.** There are various criteria to check whether a shear profile is stable to the Rayleigh equation. The most classical one was due to Rayleigh [8]: *A necessary condition for instability is that $U(z)$ must have an inflection point*, or its refined version by Fjortoft [1]: *A necessary condition for instability is that $U''(U - U(z_0)) < 0$ somewhere in the flow, where $z_0$ is a point at which $U''(z_0) = 0$.* For instance, the classical Blasius boundary layer profile is linearly stable to the Rayleigh equation.

For such a stable profile, all the spectrum of the Rayleigh equation is imbedded on the imaginary axis: Re $(-i\alpha c_{\infty}) = \alpha \Im c_{\infty} = 0$, and thus it is not clear whether a perturbative argument to construct solutions $(c_R, \phi_R)$ to (1.4) would yield stability ($\Im c_R < 0$) or instability ($\Im c_R > 0$). It is documented in the physical literature that *generic shear profiles (including those which are inviscid stable) are linearly unstable for large Reynolds numbers.* Heisenberg [4, 5], then Tollmien and C. C. Lin [6] were among the first physicists to use asymptotic expansions to study the instability; see also Drazin and Reid [1] for a complete
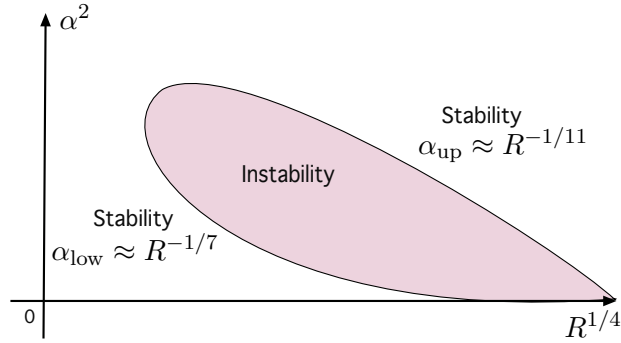
Figure 1: *Illustrated are the stability curves for channel shear flows ([2, Figure 2]).*

account of the physical literature on the subject. Roughly speaking, there are lower and upper marginal stability branches $\alpha_{\text{low}}(R), \alpha_{\text{up}}(R)$ so that whenever $\alpha \in [\alpha_{\text{low}}(R), \alpha_{\text{up}}(R)]$, there exist an unstable eigenvalue $c_R$ and an eigenfunction $\phi_R(z)$ to the Orr-Sommerfeld problem. The asymptotic behavior of these branches $\alpha_{\text{low}}$ and $\alpha_{\text{up}}$ depends on the shear profile:

- for channel flows (including the plane Poiseuille flow):

$$\alpha_{\text{low}}(R) = A_{1c}R^{-1/7} \qquad \text{and} \qquad \alpha_{\text{up}}(R) = A_{2c}R^{-1/11} \qquad (1.5)$$

- for generic boundary layers:

$$\alpha_{\text{low}}(R) = A_{1c}R^{-1/4} \qquad \text{and} \qquad \alpha_{\text{up}}(R) = A_{2c}R^{-1/6} \qquad (1.6)$$

- for Blasius boundary layer:

$$\alpha_{\text{low}}(R) = A_{1c}R^{-1/4} \qquad \text{and} \qquad \alpha_{\text{up}}(R) = A_{2c}R^{-1/10}. \qquad (1.7)$$

Their formal analysis has been compared with modern numerical computations and also with experiments, showing a very good agreement; see Figure 1 or [1, Figure 5.5] for a sketch of the marginal stability curves.

In his works [11, 12, 13], Wasow developed the turning point theory to rigorously validate the formal asymptotic expansions used by the physicists in a full neighborhood of the turning points (or the critical layers in our present paper). Wasow wrote ([11, pp. 868–870]): *"It also turns out that the formal theory alone does not give sufficient information about the actual asymptotic behavior. We are not going to apply our theory to the stability problem proper, but we shall mention two points which are left somewhat obscure in previous investigations..."*. In his book ([13, Chapter 1]), Wasow pointed out again the need of a complete mathematical justification of the linear stability theory. Even though Drazin and Reid ([1]) indeed provide many delicate asymptotic analysis in different regimes with different matching conditions near the critical layers, it is mathematically unclear how to combine their "local" analysis into a single convergent "global expansion" to produce an exact growing mode for the Orr-Sommerfeld equation. To our knowledge, remarkably, after all these efforts, a complete rigorous construction of an unstable growing mode is still elusive for such a fundamental problem.

## 1.1 Main results

Our main results from [2, 3] are as follows, completing the mathematical justification of the linear stability theory of shear flows.

**Theorem 1.1** ([2]; instability of channel flows)**.** *Let $\Omega = \mathbb{R} \times [0, 2]$ and $U(z)$ be an arbitrary shear profile that is analytic and symmetric about $z = 1$ with $U(0) = 0$, $U'(0) > 0$ and $U'(1) = 0$. Let $\alpha_{\mathrm{low}}(R)$ and $\alpha_{\mathrm{up}}(R)$ be defined as in (1.5).*

*Then, there is a critical Reynolds number $R_c$ so that for all $R \geq R_c$ and all $\alpha \in (\alpha_{\mathrm{low}}(R), \alpha_{\mathrm{up}}(R))$, there exist a triple $c(R), \hat{v}(z; R), \hat{p}(z; R)$, with $\mathrm{Im}\, c(R) > 0$, such that $v_R := e^{i\alpha(y-ct)}\hat{v}(z; R)$ and $p_R := e^{i\alpha(y-ct)}\hat{p}(z; R)$ solve the problem (1.1a)-(1.1b) with the no-slip boundary conditions. In the case of instability, there holds the following estimate for the growth rate of the unstable solutions:*

$$\alpha \Im c(R) \quad \approx \quad (\alpha R)^{-1/2},$$

*as $R \to \infty$. In addition, the horizontal component of the unstable velocity $v_R$ is odd in $z$, whereas the vertical component is even in $z$.*

**Theorem 1.2** ([3]; instability of boundary layers)**.** *Let $\Omega = \mathbb{R} \times \mathbb{R}_+$ and $U(z)$ be an arbitrary analytic shear profile with $U(0) = 0$ and $U'(0) > 0$ and satisfy*

$$\sup_{z \geq 0} |\partial_z^k (U(z) - U_+)e^{\eta_0 z}| < +\infty, \qquad k = 0, \dots, 4,$$

*for some constants $U_+$ and $\eta_0 > 0$. Let $\alpha_{\mathrm{low}}(R)$ and $\alpha_{\mathrm{up}}(R)$ be defined as in (1.6) for general boundary layer profiles, or defined as in (1.7) for the Blasius profiles: those with additional assumptions: $U''(0) = U'''(0) = 0$.*

*Then, there is a critical Reynolds number $R_c$ so that for all $R \geq R_c$ and all $\alpha \in (\alpha_{\mathrm{low}}(R), \alpha_{\mathrm{up}}(R))$, there exist a nontrivial triple $c(R), \hat{v}(z; R), \hat{p}(z; R)$, with $\mathrm{Im}\, c(R) > 0$, such that $v_R := e^{i\alpha(y-ct)}\hat{v}(z; R)$ and $p_R := e^{i\alpha(y-ct)}\hat{p}(z; R)$ solve the problem (1.1a)-(1.1b) with the no-slip boundary conditions. In the case of instability, there holds the following estimate for the growth rate of the unstable solutions:*

$$\alpha \Im c(R) \quad \approx \quad R^{-1/2},$$

*as $R \to \infty$.*

The instability of generic shear flows presented in the theorems is linked directly to the emergence of Tollmien-Schlichting instability waves which are commonly used in the literature to describe the early stage of the transition from laminar to turbulent flows; see [1, 9]. Indeed, it was first pointed out by Reynolds back in 1883 in his seminal experiments that flows at a high Reynolds number experience turbulence. In other words, well-organized flows can become chaotic under infinitesimal disturbances when the Reynolds number exceeds a critical number. The transition from laminar to turbulent flows is striking, but not fully understood, with the formation of complicated patterns. To physicists, a typical boundary layer is of the Blasius type; that is, it is steady, self-similar, and in particular has no inflection point. The latter implies that a typical boundary layer is spectrally stable to the inviscid or Euler flows by a view of the classical Rayleigh's stability condition. The linear

instability, or the formation of Tollmien-Schlichting waves, found near the boundary is thus due to the presence of small viscosity, as pointed out by Heisenberg, Lin, and Tollmien, among others. Our theorems analytically confirm the emergence of the instability waves.

**Small parameters.** Throughout the paper, there are three small independent parameters $(\alpha, c, \epsilon)$:

$$(\alpha, c, \epsilon) \quad \approx \quad (0, 0, 0), \tag{1.8}$$

in which $\alpha$ is the spatial frequency, $\epsilon = 1/i\alpha R$, and $c$ is the complex number. Two other small parameters are the critical layer $z_c$, defined through the relation $U(z_c) = c$, and the critical layer thickness $\delta = (\epsilon/U'(z_c))^{1/3}$, defined as in (1.10). Once all the solutions to the Orr-Sommerfeld equations are constructed, the existence of a small complex parameter $c = c(\alpha, \epsilon)$, for each small numbers $(\alpha, \epsilon)$, will be proved through the dispersion relation. Motivated by the physical literature, we then restrict to the range of $(\alpha, \epsilon)$ so that $\alpha^{10} \lesssim \epsilon \lesssim \alpha^6$ (channel flows) or $\alpha^5 \lesssim \epsilon \lesssim \alpha^3$ (boundary layers), with which we establish the instability theorem. The relation of the spatial frequency $\alpha$ and the Reynolds number $R$ (as stated in the main theorems) then follows from the definition $\epsilon = 1/i\alpha R$.

Delicacy in the construction is primarily due to the formation of critical layers. To see this, let $(c, \phi_0)$ be a solution to the Rayleigh equation (equation (1.4) with $\epsilon = 0$). Let $z_c$ be the point at which

$$U(z_c) = c. \tag{1.9}$$

Since the coefficient of the highest-order derivative in the Rayleigh equation vanishes at $z = z_c$, the Rayleigh solution $\phi_0(z)$ has a singularity of the form: $1 + (z - z_c) \log(z - z_c)$. A perturbation analysis to construct an Orr-Sommerfeld solution $\phi_\epsilon$ out of $\phi_0$ will face a singular source $\epsilon(\partial_z^2 - \alpha^2)^2 \phi_0$ at $z = z_c$. To deal with the singularity, we need to introduce the critical layer $\phi_{\mathrm{cr}}$ that solves

$$\epsilon \partial_z^4 \phi_{\mathrm{cr}} = (U - c) \partial_z^2 \phi_{\mathrm{cr}}$$

When $z$ is near $z_c$, $U - c$ is approximately $z - z_c$, and the above equation for the critical layer becomes the classical Airy equation for $\partial_z^2 \phi_{\mathrm{cr}}$. This shows that the critical layer mainly depends on the fast variable: $\phi_{\mathrm{cr}} = \phi_{\mathrm{cr}}(Y)$ with $Y = (z - z_c)/\delta$, in which the critical layer thickness is defined by

$$\delta = \left(\frac{\epsilon}{U'(z_c)}\right)^{1/3} = e^{-i\pi/6}\left(\frac{|\epsilon|}{U'(z_c)}\right)^{1/3} \tag{1.10}$$

in which we recall that $\epsilon = 1/i\alpha R$, and we have taken $i^{1/3} = e^{i\pi/6}$.

In the literature, the point $z_c$ is occasionally referred to as a turning point, since the eigenvalues of the associated first-order ODE system cross at $z = z_c$ (or more precisely, at those which satisfy $U(z_c) = c$), and therefore it is delicate to construct asymptotic solutions that are analytic across different regions near the turning point. In his work, Wasow fixed the turning point to be zero, and were able to construct asymptotic solutions in a full neighborhood of the turning point. Our iterative approach avoids dealing with inner and outer asymptotic expansions, but instead constructs the Green's function, and therefore the inverse, of the corresponding Rayleigh and Airy operators. The Green's function of the critical layer (Airy) equation is complicated by the fact that we have to deal with the second primitive Airy functions, not to mention that the argument $Y$ is complex.

## 1.2 Asymptotic behavior as $z \to +\infty$

We focus on the case of boundary layers. In order to construct the independent solutions of (1.4), let us study their possible behavior at infinity. One observes that as $z \to +\infty$, solutions of (1.4) must behave like solutions of constant-coefficient limiting equation:

$$\varepsilon \partial_z^4 \phi = (U_+ - c + 2\varepsilon\alpha^2)\partial_z^2 \phi - \alpha^2(\varepsilon\alpha^2 + U_+ - c)\phi, \tag{1.11}$$

with $U_+ = U(+\infty)$. Solutions to (1.11) are of the form $Ce^{\lambda z}$ with $\lambda = \pm\lambda_s$ or $\lambda = \pm\lambda_f$, where

$$\lambda_s = \pm\alpha + \mathcal{O}(\alpha^2\sqrt{\varepsilon}), \qquad \lambda_f = \pm\frac{1}{\sqrt{\varepsilon}}(U_+ - c)^{1/2} + \mathcal{O}(\alpha).$$

Therefore, we can find two solutions $\phi_1, \phi_2$ with a "slow behavior" $\lambda \approx \pm\alpha$ (one decaying and the other growing) and two solutions $\phi_3, \phi_4$ with a fast behavior where $\lambda$ is of order $\pm 1/\sqrt{\varepsilon}$ (one decaying and the other growing). It follows that the first two slow-behavior solutions $\phi_1$ and $\phi_2$ are perturbations of eigenfunctions of the Rayleigh equation. The other two, $\phi_3$ and $\phi_4$, are specific to the Orr Sommerfeld equation and linked to the solutions of the critical layers. A solution to the Orr-Sommerfeld problem (1.4) with the zero boundary conditions is defined as a linear combination of the two decaying solutions $\phi_1$ and $\phi_3$.

## 1.3 The onset of instability

Let us formally point out how the lower and upper stability branches, defined as in (1.5)-(1.7), arise. Again, we focus on the case of boundary layers. As discussed, bounded Orr-Sommerfeld solutions are constructed as a linear combination of slow and fast decaying modes:

$$A\phi_1 + B\phi_3$$

for arbitrary constants $A, B$. To determine the correct constants $A, B$ solving the Orr-Sommerfeld problem, we use the boundary conditions at $z = 0$. The solvability of $A, B$ is equivalent to the existence of parameters $(\alpha, \epsilon, c)$ so that the following so-called dispersion relation holds

$$\frac{\phi_1}{\phi_1'}\Big|_{z=0} = \frac{\phi_3}{\phi_3'}\Big|_{z=0}.$$

Here, we recall that $\phi_1 \approx \phi_{\text{Ray}}$ and $\phi_3 \approx Ai(2, \delta^{-1}\eta(z))$, the second primitive Airy function that decays as $z \to \infty$ (see Sections 4.1 and 4.2, respectively). Here, $\eta(z)$ denotes the Langer's variable, defined as in (3.1), so that the critical layer equation becomes the classical Airy equation for $\partial_z^2 \phi$; see Lemma 3.1. One observes that $\phi = U - c$ is an exact solution to the Rayleigh equation with $\alpha = 0$. That is, the Rayleigh solution $\phi_{\text{Ray}} \approx U - c + \mathcal{O}(\alpha)$, as $\alpha \to 0$; see Lemma 4.1. Roughly speaking, the dispersion relation yields

$$\frac{U(0) - c}{U'(0)} + \mathcal{O}(\alpha) \approx \delta\frac{Ai(2, \delta^{-1}\eta(0))}{Ai(1, \delta^{-1}\eta(0))} \tag{1.12}$$

with the Langer's variable $\eta(z) \approx z - z_c$ as $z$ is near the critical layer $z_c$. In particular, $\eta(0) \approx -z_c$. By recalling that $U'(0) > 0$, it suffices to study the imaginary part of the

right-hand side in the dispersion relation. For this, we let $Y = \delta^{-1}\eta(0)$, and observe that to leading order, the right hand side is simply the classical Tietjens function

$$T(Y) = \delta \frac{Ai(2, Y)}{Ai(1, Y)}$$

whose imaginary part changes sign from positive at $Y = 0$ to negative and remains so for $Y$ is sufficiently large; see Lemma 4.2 for the precise statement. *This change of sign is the onset of instability.*

Let us now point out how the ranges of $\alpha$, as predicted in (1.6), arise. First, since $c = U(z_c)$ and $0 = U(0)$, taking the real part of the dispersion relation yields that $z_c \approx \alpha + |\delta|$. Next, using the asymptotic description of Airy functions (see Section 3.1), we may rewrite the dispersion relation (1.12) as

$$-\frac{\Im c}{U'(0)} + \mathcal{O}(\alpha^2 \log \alpha) \approx \delta(1 + |z_c/\delta|)^{-1/2} \tag{1.13}$$

for sufficiently large $|z_c/\delta|$ (and so, $Y = \eta(0)/\delta$ is sufficiently large). Here, it is crucial to point out that the term of order $\mathcal{O}(\alpha)$ does not appear in the imaginary part of the dispersion relation, for the reason that the singular solution of the Rayleigh problem only enters in the expansion at order one in $\alpha$ (see Lemma 4.1).

### 1.3.1 The lower stability branch: $\alpha_{\text{low}} \approx R^{-1/4}$

Let $|z_c/\delta|$ be sufficiently large, but remain bounded, so that the instability arises (due to the change of sign of the Tietjens function). This is the case when $\delta \approx z_c$. In addition, we have $z_c \approx \alpha$. By view of the definition of the critical layer thickness $\delta \approx (\alpha R)^{-1/3}$, the numerical computation of the lower stability branch follows from the approximation $\delta \approx \alpha$.

### 1.3.2 The upper stability branch: $\alpha_{\text{up}} \approx R^{-1/6}$

The instability remains as long as $|z_c/\delta|$ is sufficiently large and the $\mathcal{O}(\alpha^2 \log \alpha)$ term appearing on the left hand side of the dispersion relation (1.13) remains neglected. We note that in all cases, $z_c \approx \alpha$. The computation of the upper stability branch thus follows from the approximation that

$$\delta(1 + |\alpha/\delta|)^{-1/2} \approx \alpha^2$$

which yields $\delta \approx \alpha^{5/3}$ or equivalently, $\alpha \approx R^{-1/6}$. Beyond this range of $\alpha$, the effect of the critical layers is neglected and the slow dynamics of Rayleigh modes becomes dominant. One expects to recover the stability of Orr-Sommerfeld equations from that of the Rayleigh problem.

### 1.3.3 Blasius boundary layer: $\alpha_{\text{up}} \approx R^{-1/10}$

In the case of the classical Blasius boundary layer, we have additional information: $U''(0) = U'''(0) = 0$. The singular solution to the Rayleigh problem is of the form

$$\phi_{2,0} = -\frac{1}{U'_c} + \mathcal{O}(z_c^2)(z - z_c)\log(z - z_c) + holomorphic$$

near $z = z_c$. That is, the singularity $(z - z_c) \log(z - z_c)$ appears at order $\mathcal{O}(z_c^2)$, instead of order $\mathcal{O}(1)$, as in the general case. This improves the dispersion relation (1.13), yielding

$$-\frac{\Im c}{U'(0)} + \mathcal{O}(\alpha^4 \log \alpha) \approx \delta(1 + |\alpha/\delta|)^{-1/2} \tag{1.14}$$

recalling that $z_c \approx \alpha$. A simple calculation shows that the right hand side, which has a negative imaginary part, remains to dominate $\mathcal{O}(\alpha^4 \log \alpha)$ as long as $\alpha \ll \alpha_{\mathrm{up}} \approx R^{-1/10}$.

## 2 An iterative scheme

We now outline our iterative construction of solutions to the Orr-Sommerfeld equations (1.4). For convenience, we rewrite (1.4) as

$$\mathrm{Orr}(\phi) := \mathrm{Ray}_\alpha(\phi) - \epsilon \Delta_\alpha^2 \phi, \tag{2.1}$$

in which $\Delta_\alpha = \partial_z^2 - \alpha^2$, and $\mathrm{Ray}_\alpha := (U - c)\Delta_\alpha - U''$ denotes the corresponding Rayleigh operator. For sake of presentation, we start our construction from the Rayleigh solution $\phi_{\mathrm{Ray}}$ so that

$$\mathrm{Ray}_\alpha(\phi_{\mathrm{Ray}}) = f$$

for some given source $f$. By definition, we have

$$\mathrm{Orr}(\phi_{\mathrm{Ray}}) = f - \epsilon \Delta_\alpha^2 \phi_{\mathrm{Ray}}. \tag{2.2}$$

Clearly, if the operator $\mathrm{Iter} := \epsilon \Delta_\alpha^2 \circ \mathrm{Ray}_\alpha^{-1}$ were well-defined and contractive in some function spaces, a solution to the problem $\mathrm{Orr}(\phi_{\mathrm{Orr}}) = f$ could be constructed via the usual (regular) iterative scheme. That is,

$$\phi_{\mathrm{Orr}} := \phi_{\mathrm{Ray}} + \mathrm{Ray}_\alpha^{-1} \circ \sum_{k \geq 1} \mathrm{Iter}^k(f). \tag{2.3}$$

However, we observe that the Rayleigh solution $Ray_\alpha^{-1}(f)$ must have a singularity of order $(z - z_c) \log(z - z_c)$ near the critical layer $z = z_c$. Indeed, as $\alpha \to 0$, one solution to the Rayleigh equation is $\phi_{1,0} = U - c$, which implies that the other solution is of the form

$$\phi_{2,0}(z) = (U - c) \int_0^z \frac{1}{(U - c)^2} dy$$

which is of the form $(z - z_c) \log(z - z_c)$. The singularity remains in the inverse $\mathrm{Ray}_\alpha^{-1}$, as $\alpha$ is sufficiently small. As a consequence, $\epsilon \Delta_\alpha^2 \circ \mathrm{Ray}_\alpha^{-1}$ consists of singularities of orders $\log(z - z_c)$ and $(z - z_c)^{-k}$, for $k = 1, 2, 3$. To deal with the singularity, we need to examine the leading operator in Orr-Sommerfeld equations near the singular point $z = z_c$, which is the Airy operator defined by

$$\mathrm{Airy}(\phi) := \epsilon \partial_z^4 \phi - (U - c + 2\epsilon \alpha^2) \partial_z^2 \phi. \tag{2.4}$$

We then study the following modified Iter operator

$$\mathrm{Iter} := \underbrace{\mathrm{Reg}}_{\text{regular part}} \circ \underbrace{\mathrm{Airy}^{-1}}_{\text{critical layer}} \circ \underbrace{\epsilon \Delta_\alpha^2}_{\text{error}} \circ \underbrace{\mathrm{Ray}_\alpha^{-1}}_{\text{inviscid}}.$$

in which the regular part is defined by Reg := Orr + Airy (that is, the zero order term in Orr(·)). It suffices to show that the iterative operator is indeed contractive in some suitable function spaces. This operator is indeed contractive in the case of bounded channel flows in the function space $X_p$, $p \geq 0$, consisting of measurable functions $f = f(z)$ such that the norm

$$\|f\|_{X_p} := \sup_{z \in [0,1]} \sum_{k=0}^{p} |(z - z_c)^k \partial_z^k f(z)|$$

is finite. See [2, Lemma 6.2] for the contraction of Iter operator.

However, in the case of boundary layers, the inverse of the Airy(·) operator introduces some linear growth in the spatial variable, since it is defined up to any polynomial of order one in $z$. As a consequence, the operator $\mathrm{Airy}^{-1} \circ \epsilon \Delta_\alpha^2 \circ \mathrm{Ray}_\alpha^{-1}$ fails to be contractive in usual function spaces. To treat the loss of decay at infinity, we again modify the iteration operator as follows:

$$\mathrm{Iter} := \mathrm{Reg} \circ \Big[ \mathrm{Airy}^{-1} \circ \chi \epsilon \Delta_\alpha^2 + \partial_z^{-2} \mathcal{A}_a^{-1} \circ (1 - \chi) \epsilon \Delta_\alpha^2 \Big] \circ \mathrm{Ray}_\alpha^{-1} \tag{2.5}$$

in which $\mathcal{A}_a(\partial_z^2 \phi) = \mathrm{Airy}(\phi)$, $\partial_z^{-1} = -\int_z^\infty$, and $\chi(z)$ is a smooth cut-off function such that $\chi = 1$ on $[0,1]$ (near the critical layer) and zero on $[2, \infty)$ (near the infinity). That is, near the infinity, we have replaced $\mathrm{Airy}^{-1}$ by $\partial_z^{-2} \mathcal{A}_a^{-1}$. It follows that $\mathcal{A}_a = \varepsilon \partial_z^2 - (U - c)$ plays a role as the classical Airy operator, and hence, its Green function is exponentially localized (see Section 3.1). By a view of $\partial_z^{-1}$, this additional correction will preserve the same decay property at infinity as that of $\mathrm{Ray}_\alpha^{-1}$.

In the case of boundary layers, we modify the function space $X_p$ to be $X_{p,\eta}$, for $p \geq 0$ and $\eta > 0$, consisting of measurable functions $f = f(z)$ such that the norm

$$\|f\|_{X_{p,\eta}} := \sup_{|z - z_c| \leq 1} \sum_{k=0}^{p} |(z - z_c)^k \partial_z^k f(z)| + \sup_{|z - z_c| \geq 1} \sum_{k=0}^{p} |e^{\eta z} \partial_z^k f(z)|$$

is finite. It follows that Iter is contractive in $X_{2,\eta}$, for some $\eta > 0$; see [3, Lemma 6.5].

## 3   Airy operator

We now study the Airy operator, defined as in (2.4). Note that $\partial_z^2 \phi$ does not exactly solve the classical Airy equation: $\partial_z^2 u - z u = 0$. We make a change of variables and unknowns in order to go back to the classical Airy equation. This change is very classical in physical literature, and called the Langer's transformation: $(z, \phi) \mapsto (\eta, \Phi)$, with $\eta = \eta(z)$ defined by

$$\eta(z) = \Big[ \frac{3}{2} \int_{z_c}^z \Big( \frac{U - c}{U_c'} \Big)^{1/2} dz \Big]^{2/3} \tag{3.1}$$

and $\Phi = \Phi(\eta)$ defined by the relation

$$\partial_z^2 \phi(z) = \dot{z}^{1/2} \Phi(\eta), \tag{3.2}$$

in which $\dot{z} = dz(\eta)/d\eta$ and $z = z(\eta)$ is the inverse of the map $\eta = \eta(z)$. By a view of the definition (3.1), we note that $(U - c)\dot{z}^2 = U_c' \eta$, with $U_c' = U'(z_c)$. The following lemma links the Airy operator (2.4) with the classical Airy equation.

**Lemma 3.1.** *Let $(z,\phi) \mapsto (\eta, \Phi)$ be the Langer's transformation defined as above. The function $\Phi(\eta)$ solves the classical Airy equation:*

$$\epsilon\partial^2_\eta \Phi - U'_c \eta\Phi = f(\eta) \qquad (3.3)$$

*if and only if the function $\phi = \phi(z)$ solves*

$$\text{Airy}(\phi) = \dot{z}^{-3/2} f(\eta(z)) + \epsilon[\partial^2_z \dot{z}^{1/2} \dot{z}^{-1/2} - 2\alpha^2]\partial^2_z \phi(z). \qquad (3.4)$$

*Proof.* The lemma follows from direct calculations. □

## 3.1 The classical Airy

Thanks to the Langer's transformation, we first solve the classical Airy equation (3.3) for $\Phi$. Let us denote

$$\delta = \left(\frac{\epsilon}{U'_c}\right)^{1/3} = e^{-i\pi/6}(\alpha R U'_c)^{-1/3}$$

to be the critical layer size, and introduce the notation $Z = \delta^{-1}\eta$. Then $\Psi(Z) = \Phi(\eta)$ solves the truly classical Airy: $\Psi'' - Z\Psi = U'_c \delta f(\delta Z)$. We use the following classical lemma:

**Lemma 3.2.** *The classical Airy equation $\Psi'' - z\Psi = 0$ has two independent solutions $Ai(z)$ and $Ci(z)$ so that the Wronskian determinant of $Ai$ and $Ci$ equals to one. In addition, $Ai(e^{i\pi/6}x)$ and $Ci(e^{i\pi/6}x)$ converge to $0$ as $x \to \pm\infty$ ($x$ being real), respectively. Furthermore, there hold asymptotic bounds:*

$$\left|Ai(k, e^{i\pi/6}x)\right| \le C|x|^{-k/2-1/4} e^{-\sqrt{2|x|}x/3}, \qquad k \in \mathbb{Z}, \quad x \in \mathbb{R},$$

*and*

$$\left|Ci(k, e^{i\pi/6}x)\right| \le C|x|^{-k/2-1/4} e^{\sqrt{2|x|}x/3}, \qquad k \in \mathbb{Z}, \quad x \in \mathbb{R},$$

*in which $Ai(0,z) = Ai(z)$, $Ai(k,z) = \partial_z^{-k} Ai(z)$ for $k \le 0$, and $Ai(k,z)$ is the $k^{th}$ primitives of $Ai(z)$ for $k \ge 0$. The Airy functions $Ci(k,z)$ for $k \ne 0$ are defined similarly.*

Hence, the Green kernel of the classical Airy equation (3.3) can be defined as follows:

$$G_a(X,Z) = \delta\epsilon^{-1} \begin{cases} Ai(X)Ci(Z), & \text{if} \quad \xi > \eta, \\ Ai(Z)Ci(X), & \text{if} \quad \xi < \eta, \end{cases}$$

in which $X = \delta^{-1}\xi, Z = \delta^{-1}\eta$.

## 3.2 Green kernel for Airy operator

Let us take $\xi = \eta(x)$ and $\eta = \eta(z)$ where $\eta(\cdot)$ is the Langer's transformation and denote $\dot{x} = 1/\eta'(x)$ and $\dot{z} = 1/\eta'(z)$. By a view of (3.2), we define the function $G(x,z)$ so that

$$\partial^2_z G(x,z) = \dot{x}^{3/2} \dot{z}^{1/2} G_a(\delta^{-1}\eta(x), \delta^{-1}\eta(z)), \qquad (3.5)$$

in which the factor $\dot{x}^{3/2}$ was added simply to normalize the jump of $G(x,z)$. It then follows from Lemma 3.1 together with $\delta_{\eta(x)}(\eta(z)) = \delta_x(z)$ that

$$\text{Airy}(G(x,z)) = \delta_x(z) + \epsilon[\partial^2_z \dot{z}^{1/2} \dot{z}^{-1/2} - 2\alpha^2]\partial^2_z G(x,z). \qquad (3.6)$$

That is, $G(x, z)$ is indeed an approximate Green function of the Airy operator, defined as in (2.4), up to a small error term of order $\epsilon \partial_z^2 G = \mathcal{O}(\delta)$. It remains to solve (3.5) for $G(x, z)$, retaining the jump conditions on $G(x, z)$ across $x = z$. In view of primitive Airy functions, let us denote

$$\widetilde{Ci}(1, z) = \delta^{-1} \int_0^z \dot{y}^{1/2} Ci(\delta^{-1} \eta(y)) \, dy, \qquad \widetilde{Ci}(2, z) = \delta^{-1} \int_0^z \widetilde{Ci}(1, y) \, dy$$

and

$$\widetilde{Ai}(1, z) = \delta^{-1} \int_\infty^z \dot{y}^{1/2} Ai(\delta^{-1} \eta(y)) \, dy, \qquad \widetilde{Ai}(2, z) = \delta^{-1} \int_\infty^z \widetilde{Ai}(1, y) \, dy.$$

Thus, together with our convention that the Green function $G(x, z)$ should vanish as $z$ goes to $+\infty$ for each fixed $x$, we are led to introduce

$$G(x, z) = i\delta^3 \pi \epsilon^{-1} \dot{x}^{3/2} \begin{cases} \left[ Ai(\delta^{-1}\eta(x))\widetilde{Ci}(2, z) + \delta^{-1}a_1(x)(z - x) + a_2(x) \right], & \text{if } x > z, \\ Ci(\delta^{-1}\eta(x))\widetilde{Ai}(2, z), & \text{if } x < z, \end{cases}$$

in which $a_1(x), a_2(x)$ are chosen so that the jump conditions (see below) hold. Clearly, by definition, $G(x, z)$ solves (3.5), and hence (3.6). Here the jump conditions on the Green function read:

$$[G(x, z)]_{|x=z} = [\partial_z G(x, z)]_{|x=z} = [\partial_z^2 G(x, z)]_{|x=z} = 0 \tag{3.7}$$

and

$$[\epsilon \partial_z^3 G(x, z)]_{|x=z} = 1. \tag{3.8}$$

We note that from (3.5) and the jump conditions on $G_a(X, Z)$ across $X = Z$, the above jump conditions of $\partial_z^2 G$ and $\partial_z^3 G$ follow easily. In order for the jump conditions on $G(x, z)$ and $\partial_z G(x, z)$, we take

$$\begin{aligned} a_1(x) &= Ci(\delta^{-1}\eta(x))\widetilde{Ai}(1, x) - Ai(\delta^{-1}\eta(x))\widetilde{Ci}(1, x), \\ a_2(x) &= Ci(\delta^{-1}\eta(x))\widetilde{Ai}(2, x) - Ai(\delta^{-1}\eta(x))\widetilde{Ci}(2, x). \end{aligned} \tag{3.9}$$

This defines an approximate Green function for the Airy operator. Up to an error of order $\epsilon$, we introduce

$$\text{Airy}^{-1}(f) := G \star f.$$

## 3.3   Singularities and contraction of Iter operator

In this section, we study the smoothing effect of the modified Airy function. Precisely, let us consider the Airy equation with a singular source:

$$\text{Airy}(\phi) = \epsilon \partial_z^4 f(z) \tag{3.10}$$

in which $f \in Y_{4,\eta}$, that is, $f(z)$ and its derivatives decay exponentially at infinity and behaves as $(z - z_c) \log(z - z_c)$ near the critical layer $z = z_c$. The singular source $\epsilon \partial_z^4 f$ arises as an error of the inviscid solution when solving the full viscous problem. The key for the contraction of the iteration operator lies in the following lemma:

**Lemma 3.3.** *Assume that $\delta \lesssim z_c$. Let $\mathrm{Airy}^{-1}(\cdot)$ be the inverse of the $\mathrm{Airy}(\cdot)$ operator, and let $f \in Y_{4,\eta}$. There holds the estimate:*

$$\left\| \mathrm{Airy}^{-1}(\epsilon \partial_x^4 f) \right\|_{X_{2,\eta'}} \leq C_\eta \|f\|_{Y_{4,\eta}} \delta (1 + |\log \delta|)(1 + |z_c/\delta|) \tag{3.11}$$

*for arbitrary $\eta' < \eta$.*

*Proof.* The rough idea is that the convolution can be computed as

$$G \star \epsilon \partial_z^4 f = -\epsilon \partial_z^3 G \star \partial_z f,$$

in which $\epsilon \partial_z^3 G$ is bounded and is localized near the critical layer of the size of order $\delta$. This indicates the bound by $\delta \log \delta$ as stated in the estimate (3.11). The factor $1 + |z_c/\delta|$ is precisely due to the linear growth in $z$ in the Green kernel $G(x, z)$. We refer to the paper, [3, Section 5], for details of the proof. □

# 4 Orr-Sommerfeld solutions

## 4.1 Slow modes

In this paragraph we explicitly compute the boundary contribution of the first terms in the expansion of the slow Orr-Sommerfeld modes, which are obtained from the Rayleigh solutions:

$$\phi_1(z; c) = \phi_{\mathrm{Ray}}(z; c) + \mathrm{Airy}^{-1}(\epsilon \Delta_\alpha^2 \phi_{\mathrm{Ray}})(z; c) + \cdots \tag{4.1}$$

in which the second term is obtained by the interation via the Iter operator, plus higher order terms. We recall that the Rayleigh solution, again obtained via a perturbative analysis, is of the form:

$$\phi_{\mathrm{Ray}}(z; c) = e^{-\alpha z}(U - c + \mathcal{O}(\alpha)).$$

It is crucial to note that the possible $(z - z_c)\log(z - z_c)$ singularity in the Rayleigh solution arises only at the order of $\alpha$. That is, we apply the Airy smoothing operator, Lemma 3.3, precisely to the $\mathcal{O}(\alpha)$ term, yielding

$$\| \mathrm{Airy}^{-1}(\epsilon \Delta_\alpha^2 \phi_{\mathrm{Ray}}) \|_\eta \leq C\epsilon + C\alpha \delta (1 + |\log \delta|)(1 + |z_c/\delta|).$$

This yields at once the following lemma:

**Lemma 4.1.** *Let $\phi_1$ be the slow mode constructed above. For small $z_c, \alpha, \delta$, such that $\delta \lesssim z_c$, there hold*

$$\frac{\phi_1(0; c)}{\partial_z \phi_1(0; c)} = \frac{1}{U_0'} \left[ U_0 - c + \alpha \frac{(U_+ - U_0)^2}{U_0'} + \mathcal{O}(\alpha^2 \log \alpha) \right]. \tag{4.2}$$

## 4.2 Fast modes

Similarly, the fast modes are constructed as a perturbation from the second primitive Airy solutions:

$$\phi_{f,0}(z) := \gamma_0 Ai(2, \delta^{-1}\eta(z)), \qquad \gamma_0 := Ai(2, \delta^{-1}\eta(0))^{-1}.$$

Here, $\gamma_0$ is to normalize the possible blow-up value of $Ai(2, \cdot)$ on the boundary $z = 0$, since $\delta^{-1}\eta(0) \approx e^{i7\pi/6}|z_c/\delta|$ could be arbitrarily large. By construction, there holds the following expansion of the fast mode $\phi_3$ on the boundary $z = 0$:

$$\phi_3(0) = \phi_{f,0}(0) + \mathcal{O}(\delta), \qquad \phi_3'(0) = \phi_{f,0}'(0) + \mathcal{O}(1).$$

By definition, we have $\phi_{f,0}(0) = 1$ and

$$\phi_{f,0}'(0) = \delta^{-1}\frac{Ai(1, \delta^{-1}\eta(0))}{Ai(2, \delta^{-1}\eta(0))}.$$

In the study of the linear dispersion relation, we are interested in the ratio $\phi_3/\phi_3'$. From the above estimates on $\phi_3(0)$ and $\phi_3'(0)$, it follows at once that

$$\frac{\phi_3(0)}{\phi_3'(0)} = \frac{\delta C_{Ai}(\delta^{-1}\eta(0))}{1 + \mathcal{O}(\delta)C_{Ai}(\delta^{-1}\eta(0))}(1 + \mathcal{O}(\delta)), \qquad C_{Ai}(Y) := \frac{Ai(2, Y)}{Ai(1, Y)}.$$

As will be calculated below, $\delta C_{Ai}(\delta^{-1}\eta(0)) \approx \delta(1 + |\eta(0)/\delta|)^{-1/2} \ll 1$. Hence, the above ratio is estimated by

$$\frac{\phi_3(0)}{\phi_3'(0)} = \delta C_{Ai}(\delta^{-1}\eta(0))(1 + \mathcal{O}(\delta)). \tag{4.3}$$

Here, we recall that $\delta = e^{-i\pi/6}(\alpha R U_c')^{-1/3}$, and $\eta(0) = -z_c + \mathcal{O}(z_c^2)$. Therefore, we are interested in the ratio $C_{Ai}(Y)$ for complex $Y = -e^{i\pi/6}y$, for $y$ being in a small neighborhood of $\mathbb{R}^+$. Without loss of generality, in what follows, we consider $y \in \mathbb{R}^+$. Directly from the asymptotic behavior of the Airy functions, we obtain the following lemma:

**Lemma 4.2.** *Let* $C_{Ai}(\cdot)$ *be defined as above. Then,* $C_{Ai}(\cdot)$ *is uniformly bounded on the ray* $Y = e^{7i\pi/6}y$ *for* $y \in \mathbb{R}^+$. *In addition, there holds*

$$C_{Ai}(-e^{i\pi/6}y) = -e^{5i\pi/12}y^{-1/2}(1 + \mathcal{O}(y^{-3/2}))$$

*for all large* $y \in \mathbb{R}^+$. *At* $y = 0$, *we have* $C_{Ai}(0) = -3^{1/3}\Gamma(4/3)$.

This yields at once the following estimate on the ratio (4.3):

**Lemma 4.3.** *As long as* $z_c/\delta$ *is sufficiently large, there holds*

$$\frac{\phi_3(0)}{\phi_3'(0)} = -e^{\pi i/4}|\delta||z_c/\delta|^{-1/2}(1 + \mathcal{O}(|z_c/\delta|^{-3/2})) \tag{4.4}$$

*In particular, the imaginary part of* $\phi_3/\phi_3'$ *becomes negative when* $z_c/\delta$ *is large (the ratio has a positive imaginary part when* $z_c/\delta$ *is small).*

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