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Sharp polynomial energy decay for locally undamped waves

Matthieu Léautaud* and Nicolas Lerner†

Abstract

In this note, we present the results of the article [LL14], and provide a complete proof in a simple case. We study the decay rate for the energy of solutions of a damped wave equation in a situation where the *Geometric Control Condition* is violated. We assume that the set of undamped trajectories is a flat torus of positive codimension and that the metric is locally flat around this set. We further assume that the damping function enjoys locally a prescribed homogeneity near the undamped set in traversal directions. We prove a sharp decay estimate at a polynomial rate that depends on the homogeneity of the damping function.

1 Introduction and main results

We consider a smooth connected compact Riemannian manifold (M, g) of dimension n, and denote by Δ_g the associated negative Laplace-Beltrami operator. Given $b \in L^\infty(M)$, we study the decay rates for the damped wave equation on M:

$$\begin{cases} \partial_t^2 u - \Delta_g u + b(x)\partial_t u = 0 & \text{in } \mathbb{R}^+ \times M, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } M. \end{cases} \tag{1.1}$$

The energy of a solution is defined by

$$E(u(t)) = \frac{1}{2}(\|\nabla_g u(t)\|^2_{L^2(M)} + \|\partial_t u(t)\|^2_{L^2(M)}), \tag{1.2}$$

and evolves as

$$\frac{d}{dt}E(u(t)) = -\int_M b|\partial_t u|^2 dx.$$

The energy is thus actually damped when $b \geq 0$ a.e. on M, what we assume from now on. We define the subset of M on which the damping is effective as

$$\omega_b := \bigcup \{U \subset M, U \text{ open}, \operatorname{essinf}_U(b) > 0\}. \tag{1.3}$$

Notice that ω_b is an open set included in the interior of $\operatorname{supp} b$ and thus $\overline{\omega}_b \subset \operatorname{supp} b$.[1] In the usual case where b is continuous, we have $\omega_b = \{b > 0\}$ and $\overline{\omega}_b = \operatorname{supp} b$. As soon as $\omega_b \neq \varnothing$ one has $E(u(t)) \to 0$ as $t \to +\infty$ (see for instance [Leb96]). Moreover, a criterion for uniform (and hence exponential) decay is due to Rauch-Taylor [RT74] (see also [BLR88]): there exist $C > 0, \gamma > 0$ such that for all data,

$$E(u(t)) \leq Ce^{-\gamma t}E(u(0)),$$

if the Geometric Control Condition (GCC) holds: every geodesic starting from S^*M and traveling with unit speed enters the set ω_b in finite time. Reciprocally, if there is a geodesic that never meets $\operatorname{supp}(b)$, then uniform decay does not hold. In the case $b \in \mathscr{C}^0(M)$, the situation is simpler since

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[1] Remark that the converse may fail, taking for instance $b = \mathbb{1}_K$ where K is a compact Cantor set with positive measure satisfying $\mathring{K} = \varnothing$, in which case $\operatorname{supp}(b) = K$ and $\omega_b = \varnothing$.

uniform decay is *equivalent* to the fact that $\omega_b(=\{b>0\})$ satisfies (GCC), as remarked by Burq and Gérard [BG97][2]. As a consequence, when (GCC) is not satisfied, we cannot expect a decay of the energy which is uniform with respect to all data in $H^1(M)\times L^2(M)$. However, Lebeau [Leb96] proved that there is always a uniform decay rate of the energy, with respect to smoother data, say in $H^2(M)\times H^1(M)$. This motivates the following definition.

Definition 1.1. Given $a\in\mathbb{R}$ and a decreasing function $f:[a,+\infty)\to\mathbb{R}_+^*$ such that $f(t)\to 0$ as $t\to+\infty$, we say that the solutions of (1.1) decay at rate $f(t)$ if there exists $C>0$ such that for all $(u_0,u_1)\in H^2(M)\times H^1(M)$, for all $t\geq a$, we have

$$E(u(t))^{1/2}\leq Cf(t)\left(\|u_0\|_{H^2(M)}+\|u_1\|_{H^1(M)}\right).$$

Note that decay at a rate $f(t)$ depends only on (M,g) and on the damping function b. Note also that $f(t)^2$ characterizes the decay of the energy and $f(t)$ that of the associated norm. Lebeau [Leb96] proved that decay at rate $1/\log t$ always holds, independently of (M,g) and b as soon as $\omega_b\neq\varnothing$.

As noticed for instance in [BD08], decay at a rate $f(t)$ implies faster decay for "smoother" data: taking for example $b\in\mathscr{C}^\infty(M)$, decay at rate $f(t)$ implies that for all $s>0$, there exists $C_s>0$ such that for all $(u_0,u_1)\in H^{s+1}(M)\times H^s(M)$, we have

$$E(u(t))^{1/2}\leq C_s f(t/s)^s(\|u_0\|_{H^{s+1}(M)}+\|u_1\|_{H^s(M)}).$$

In view of the Rauch-Taylor theorem mentioned above, it is convenient to introduce the subset of phase-space consisting in points-directions that are never brought into the damping region ω_b by the geodesic flow. Namely, the *undamped set* is defined by

$$S=\{\rho\in S^*M,\ \text{for all } t\in\mathbb{R},\ \varphi_t(\rho)\cap T^*\omega_b=\varnothing\},$$

where φ_t is the geodesic flow. With this definition, (GCC) is equivalent to $S=\varnothing$. In this article, we are concerned with the damped wave equation in a geometric situation where the undamped set S is the cotangent space to a *flat subtorus* of M (of dimension $1\leq n''\leq n-1$) under two main additional assumptions: the metric is locally flat around this subtorus; the damping function b only depends on variables transverse to this torus and enjoys locally a prescribed homogeneity. Such situations may for instance occur on the torus $M=\mathbb{T}^n=(\mathbb{R}/2\pi\mathbb{Z})^n$ endowed with the flat metric. One of our motivations is to understand the best decay rate in the following model problem.

Example 1.2. Let $M=\mathbb{T}^2=(\mathbb{R}/2\pi\mathbb{Z})^2\equiv[-\pi,\pi]^2$, endowed with the flat metric, let $\gamma>0$, and let $b(x_1,x_2)=x_1^{2\gamma}$ near $x_1=0$, positive elsewhere, depending only on x_1. The undamped set consists in two undamped trajectories:

$$S=\{0\}_{x_1}\times\mathbb{T}^1_{x_2}\times\{0\}_{\xi_1}\times\{\pm1\}_{\xi_2}=S^*(\{0\}\times\mathbb{T}^1).$$

For the case where $b=\sin^2 x_1$, Wen Deng communicated to us a direct study in [Den].

Decay rates for the damped wave equation on a flat metric with a lack of (GCC) have already been studied in [LR05, BH07, Phu07, AL14]. In [AL14] it is proved that, on $M=\mathbb{T}^n$, decay at a rate $t^{-1/2}$ always occurs if $\omega_b\neq\varnothing$; on the other hand, the decay cannot be better than t^{-1} as soon as (GCC) is *strongly violated*, i.e. as soon as there exists a neighbourhood $\mathcal{N}$ of a geodesic such that $\mathcal{N}\cap\operatorname{supp}(b)=\varnothing$. In this paper, we are studying the opposite situation, i.e. the case of a *weak lack of damping* on $M=\mathbb{T}^n$: only a positive codimension invariant torus is undamped. In the situation of Example 1.2, for instance, we may expect (and we shall prove) a decay at a stronger polynomial rate than t^{-1}.

According to [Leb96, BD08, BT10, AL14], proving a decay rate for solutions of (1.1) reduces to proving a high-energy estimate for the operators

$$P_\lambda=-\Delta_g-\lambda^2+i\lambda b,\quad \lambda\in\mathbb{R}^*,\quad D(P_\lambda)=H^2(M). \tag{1.4}$$

[2] This is no longer the case in general if b is not continuous, as proved in [Leb92].

The latter are for instance obtained by performing a Fourier transform in the time variable of the damped wave operator $\partial_t^2 - \Delta_g + b(x)\partial_t$, λ being the frequency variable dual to the time t. More precisely, concerning polynomial decay, the optimal result was proved by [BT10] and can be stated as follows (see [AL14, Proposition 2.4]).

Proposition 1.3. *Given $\alpha > 0$, the solutions of (1.1) decay at rate $t^{-\frac{1}{\alpha}}$ if and only if there exist C, λ_0 positive, such that for all $u \in H^2(M)$, for all $\lambda \geq \lambda_0$, we have*

$$C\|P_\lambda u\|_{L^2(M)} \geq \lambda^{1-\alpha}\|u\|_{L^2(M)}. \tag{1.5}$$

Recall that uniform decay is equivalent to the estimate (1.5) with $\alpha = 0$.

Let us now state our main results, the first of which is of negative nature.

Theorem 1.4. *Assume that there exists $1 \leq n'' \leq n-1$, $\varepsilon_0 > 0$, and $C_1 > 0$ such that with $n' = n - n''$, we have*

- $\left(B_{\mathbb{R}^{n'}}(0,\varepsilon_0) \times \mathbb{T}^{n''}, |dx_1'|^2 + \cdots + |dx_{n'}'|^2 + |dx_1''|^2 + \cdots + |dx_{n''}''|^2\right) \subset (M, g),$ (1.6)
- $\nabla_{x''} b = 0$ *in* $\mathcal{N} = B_{\mathbb{R}^{n'}}(0,\varepsilon_0) \times \mathbb{T}^{n''}$, (1.7)
- $0 \leq b(x') \leq C_1 |x'|^{2\gamma}$ *in* $\mathcal{N}$. (1.8)

Then, there exist $C_0 > 0$ and $(u_k)_{k\in\mathbb{N}} \in H^2(M)^{\mathbb{N}}$ with $\|u_k\|_{L^2(M)} = 1$ such that

$$\|P_k u_k\|_{L^2(M)} \leq C_0 k^{1/(\gamma+1)}, \quad \text{for } k \in \mathbb{N}^*.$$

As a consequence, the best estimate we could expect is

$$C\|P_\lambda u\|_{L^2(M)} \geq \lambda^{1/(\gamma+1)}\|u\|_{L^2(M)}, \tag{1.9}$$

i.e. (1.5) with $\alpha = 1 - 1/(\gamma+1)$. Moreover, (see also [BD08, Proposition 3]), our Theorem 1.4 prevents decay at a rate $o\left(t^{-(1+1/\gamma)}\right)$: the best expected decay rate is $t^{-(1+1/\gamma)}$. Let us now state our partial converse of Theorem 1.4: under some additional assumptions on M and b, decay at rate $t^{-(1+1/\gamma)}$ indeed holds.

Theorem 1.5. *Take $1 \leq n'' \leq n-1$ and assume that $(M, g) = (M' \times \mathbb{T}^{n''}, g' + |dx_1''|^2 + \cdots + |dx_{n''}''|^2)$ where (M', g') is a smooth compact Riemannian manifold of dimension $n' = n - n''$ and $(x_1'', \ldots, x_{n''}'')$ denote variables in $\mathbb{T}^{n''}$. Assume that there exist $y' \in M'$, $C_1 \geq 1$ and a neighbourhood $\mathcal{N}'$ of y' such that*

- $g' = |dx_1'|^2 + \cdots + |dx_{n'}'|^2$ *is flat in* $\mathcal{N}'$, (1.10)
- $b \in L^\infty(M)$, $\quad \nabla_{x''} b \in L^\infty(M)$, $\quad$ *and* $\nabla_{x''} b = 0$ *in* $\mathcal{N}' \times \mathbb{T}^{n''}$ (1.11)
- $C_1^{-1}|x' - y'|^{2\gamma} \leq b(x') \leq C_1|x' - y'|^{2\gamma}$ *for* $x' \in \mathcal{N}'$, (1.12)
- *any geodesic starting from* $S^*M \setminus S^*(\{y'\} \times \mathbb{T}^{n''})$ *intersects* ω_b *in finite time.* (1.13)

Then, we have the property (1.5) with $\alpha = 1 - 1/(\gamma+1)$, i.e. decay at rate $t^{-(1+1/\gamma)}$.

Since the work of Lebeau [Leb96] (see also the introduction of [AL14] and the references therein), it is quite well established that the main parameters governing the decay rates when (GCC) fails are the global and local dynamics of the geodesic flow. Our results confirm the idea, raised in [BH07, AL14], that once the geometry (and hence the dynamics) is fixed, the next relevant feature when regarding the best decay rate is the rate at which the damping coefficient b vanishes.

In this note, we shall give a complete proof of this result in a particular case containing the main ideas of the general case. We add two assumptions: b is globally invariant in the direction of $\mathbb{T}^{n''}$, and locally exactly homogeneous. These assumptions are only added to focus on the key points of the proof. The present note hence provides a detailed proof of the following result (which, in particular, tackles the motivating Example 1.2).

Theorem 1.6. *Assume that* (M, g) *is as in Theorem 1.5. Assume that there exist* $y' \in M'$, $C > 0$ *and a neighbourhood* $\mathcal{N}'$ *of* y' *such that*

- $g' = |dx'_1|^2 + \cdots + |dx'_{n'}|^2$ *is flat in* $\mathcal{N}'$, (1.14)
- $b = b \otimes 1$ *does not depend on the variable in* $\mathbb{T}^{n''}$, *and* $b \in L^\infty(M')$, (1.15)
- $b(x')$ *is homogeneous in* $x' - y'$ *of order* 2γ *for* $x' \in \mathcal{N}'$, (1.16)
- $b \geq C$ *a.e. on* $M' \setminus \mathcal{N}'$ *and* $b > 0$ *in* $\mathcal{N}' \setminus \{y'\}$. (1.17)

Then, Property (1.5) *holds with* $\alpha = 1 - 1/(\gamma + 1)$, *i.e. decay occurs at rate* $t^{-(1+1/\gamma)}$.

The main ingredients of the proof of both Theorems 1.5 and 1.6 are the following:

- A usual geometric control estimate outside the undamped region $\{y'\} \times \mathbb{T}^{n''}$;
- A Fourier transform argument near the undamped region to reduce the problem (with one parameter λ) to an estimate for an operator on M' only (depending on two parameters, λ and the spectral parameter coming from the Fourier transform);
- A scaling argument, taking advantage of the homogeneity of b (and the local flatness of the metric of M'), to reduce this two-parameter problem "near a point" to a one parameter problem on the whole $\mathbb{R}^{n'}$;
- A resolvent estimate on the real line for the rescaled operator acting on $\mathbb{R}^{n'}$.

The operator arising after all reductions and scaling arguments takes the form $-\Delta + iW(x)$ on $L^2(\mathbb{R}^{n'})$, where the real positive potential W behaves like $|x|^{2\gamma}$ at infinity (2γ is the "homogeneity" of the damping function b near $\{y'\} \times \mathbb{T}^{n''}$). The main part of the proof is then to obtain an optimal resolvent estimate for this operator on the real line, which is of independent interest. In turn, this estimate provides a bound on the size of the pseudospectrum for this operator, generalizing results of E. B. Davies [Dav99] and K. Pravda-Starov [PS06] in the case of the 1D complex harmonic oscillator, $-\frac{d^2}{dx^2} + e^{i\theta}x^2$.

The problem naturally arises with different large parameters (frequencies in the directions of M' and of $\mathbb{T}^{n''}$, local thickness around the undamped set) and semiclassical régimes, coming with a precise scaling. Therefore, although our proofs are very elementary, they contain implicitely several steps of microlocalizations, i.e. of cutting the phase space into pieces. As a consequence, all estimates proved here could be reread (and the results generalized) in the light of the so-called second microlocalization. This notion was first developed in the analytic category in M. Kashiwara & T. Kawai's article [KK80], followed by G. Lebeau's paper [Leb85]. J.-M. Bony's article [Bon86] and J.-M. Delort's book [Del92] displayed striking applications to propagation of weak singularities for non-linear equations, the J.-M. Bony & N. Lerner's paper [BL89] provided a metrics point of view. More recently, N. Anantharaman & M. Léautaud's work on the damped wave equation [AL14] showed, using techniques of N. Anantharaman & F. Macià [Mac10, AM14], that the second micolocalization could be useful to tackle estimates related to some non-selfadjoint operators. The key tools in the last three papers are the 2-microlocal measures, introduced by L. Miller [Mil97], C. Fermanian-Kammerer and P. Gérard [FK00, FKG02, FK05], which allow to perform (at the level of defect measures) a second microlocalization for bounded sequences in L^2.

Let us now turn to the proof of Theorem 1.6.

2 A sharp estimate for a non-selfadjoint operator on $\mathbb{R}^d$

After a Fourier transformation in the periodic direction and a scaling argument (see the following sections), our main result is reduced to the following theorem. We define on $L^2(\mathbb{R}^d)$ (below, we shall take $d = n'$) the unbounded operator

$$Q_0 = -\Delta + iW(x), \tag{2.1}$$

where W is a real-valued measurable function and $D(Q_0) = \{u \in H^2(\mathbb{R}^d), Wu \in L^2(\mathbb{R}^d)\}$.

Theorem 2.1. *Suppose that W is a real-valued measurable function on $\mathbb{R}^d$ and that there exist $C_1 \geq 1$ and $\gamma > 0$ such that we have*

$$C_1^{-1}|x|^{2\gamma} \leq W(x) \leq C_1\langle x\rangle^{2\gamma} = C_1(1+|x|^2)^{\gamma}. \tag{2.2}$$

Then, there exists $C_0 > 0$ such that for all $\mu \in \mathbb{R}$ and $u \in \mathscr{C}_c^2(\mathbb{R}^d)$, we have

$$C_0\|(Q_0-\mu)u\|_{L^2(\mathbb{R}^d)} \geq \left(\mu^{\gamma/(2\gamma+1)}\mathbf{1}(\mu \geq 1) + |\mu|\mathbf{1}(\mu \leq -1) + 1\right)\|u\|_{L^2(\mathbb{R}^d)}. \tag{2.3}$$

The power $\gamma/(2\gamma+1)$ is optimal in this estimate (see [LL14, Lemma C.1]). To prove Theorem 2.1, we need the following preliminary lemma.

Lemma 2.2. *Suppose that W satisfy Assumption (2.2) and let a be a smooth function on $\mathbb{R}^{2d}$, bounded as well as all its derivatives. Then, there exists $C > 0$ such that for all $u \in \mathscr{C}_c^0(\mathbb{R}^d)$, we have*

$$\|Va^w u\|_{L^2(\mathbb{R}^d)} \leq C\left(\|u\|_{L^2(\mathbb{R}^d)} + \|Vu\|_{L^2(\mathbb{R}^d)}\right),$$

where $V = W^{1/2}$ and a^w stands for the Weyl quantization of the symbol a.

Proof of Lemma 2.2. Using the upper bound in Assumption (2.2) yields

$$\|V(x)a^w u\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} V^2(x)|a^w u|^2 dx \leq C_1 \int_{\mathbb{R}^d} \langle x\rangle^{2\gamma}|a^w u|^2 dx = C_1\|\langle x\rangle^{\gamma} a^w u\|^2_{L^2(\mathbb{R}^d)}.$$

Then, we notice that $\langle x\rangle^{\gamma}$ and $\langle x\rangle^{-\gamma}$ are admissible weight functions for the metric $|dx|^2 + |d\xi|^2$ in the sense of [Ler10, Definition 2.2.15]. As a consequence of symbolic calculus, we have

$$\langle x\rangle^{\gamma} \sharp a \sharp \langle x\rangle^{-\gamma} \in S(1, |dx|^2 + |d\xi|^2),$$

where $S(1, |dx|^2 + |d\xi|^2)$ is the space of smooth functions on $\mathbb{R}^{2d}$ which are bounded as well as all their derivatives. Calderón-Vaillancourt theorem (see e.g. [Ler10, Theorem 1.1.4]) yields

$$\langle x\rangle^{\gamma} a(x,\xi)^w \langle x\rangle^{-\gamma} \in \mathcal{L}(L^2(\mathbb{R}^d)),$$

which implies $\|\langle x\rangle^{\gamma} a^w u\|_{L^2(\mathbb{R}^d)} \lesssim \|\langle x\rangle^{\gamma} u\|_{L^2(\mathbb{R}^d)}$. This finally gives

$$\begin{aligned}\|V(x)a(x,\xi)^w u\|^2_{L^2(\mathbb{R}^d)} &\lesssim \|\langle x\rangle^{\gamma} u\|^2_{L^2(\mathbb{R}^d)} \lesssim \|u\|^2_{L^2(\mathbb{R}^d)} + \||x|^{\gamma} u\|^2_{L^2(\mathbb{R}^d)} \\ &\lesssim \|u\|^2_{L^2(\mathbb{R}^d)} + \|Vu\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

according to the lower bound in Assumption (2.2). □

Proof of Theorem 2.1. We start with the case $\mu \leq -1$. We have then

$$\|(Q_0-\mu)u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \geq \mathrm{Re}\langle(Q_0-\mu)u, u\rangle_{L^2(\mathbb{R}^d)} \geq -\mu\langle u, u\rangle_{L^2(\mathbb{R}^d)} = |\mu|\|u\|^2_{L^2(\mathbb{R}^d)},$$

so that Estimate (2.3) holds for $\mu \leq -1$.

Next, let us prove that the operator Q_0 on $L^2(\mathbb{R}^d)$ has a compact resolvent. For $u \in \mathscr{C}_c^\infty(\mathbb{R}^d)$, we have, with $V = W^{1/2}$,

$$2\|Q_0 u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \geq \mathrm{Re}\langle Q_0 u, u\rangle_{L^2(\mathbb{R}^d)} + \mathrm{Im}\langle Q_0 u, u\rangle_{L^2(\mathbb{R}^d)} = \|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \|Vu\|^2_{L^2(\mathbb{R}^d)}.$$

Since $\mathscr{C}_c^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, this also holds for all $u \in D(Q_0)$. As V is non-negative, we have

$$D(Q_0) \subset H_V^1(\mathbb{R}^d) := \{u \in H^1(\mathbb{R}^d), Vu \in L^2(\mathbb{R}^d)\}.$$

Thanks to (2.2), the injection $H_V^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is compact. Thus $D(Q_0)$ injects compactly in $L^2(\mathbb{R}^d)$, and the operator Q_0 on $L^2(\mathbb{R}^d)$ has a compact resolvent. This implies in particular that $\mathrm{Sp}(Q_0)$ is constituted only by eigenvalues with finite multiplicity.

Assume now that there exists $\mu \in \mathbb{R} \cap \mathrm{Sp}(Q_0)$ and let $u \in D(Q_0)$ be an associated eigenfunction. Then we have $(Q_0 - \mu)u = 0$ and in particular

$$0 = \mathrm{Im}\langle (Q_0 - \mu)u, u\rangle_{L^2(\mathbb{R}^d)} = \langle Wu, u\rangle_{L^2(\mathbb{R}^d)}.$$

Hence, we have $u = 0$ in $L^2(\mathbb{R}^d)$ since $W > 0$ almost everywhere under assumption (2.2). This yields a contradiction, proving that $\mathbb{R} \cap \mathrm{Sp}(Q_0) = \varnothing$.

Let $\mu_0 > 0$ be given. On the compact set $[-1, \mu_0]$ the resolvent $(Q_0 - \mu)^{-1}$ is a continuous (hence bounded) function since $\mathbb{R} \cap \mathrm{Sp}(Q_0) = \varnothing$. As a consequence, Estimate (2.3) is now proven to hold for all $\mu \in (-\infty, \mu_0]$.

We are going now to study the most substantial case where $\mu > \mu_0$, but we may keep in mind that we can freely choose the large fixed constant μ_0. We set

$$\nu = \mu^{1/2}, \quad Q_\nu = Q_0 - \nu^2, \tag{2.4}$$

and study the asymptotics when $\nu \to +\infty$. From the above remarks, we have only to prove the estimate (2.3) for $\nu \geq \nu_0$, where ν_0 can be chosen arbitrarily large. First of all, we note that

$$\|Q_\nu u\|_{L^2(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^d)} \geq \mathrm{Im}\langle Q_\nu u, u\rangle_{L^2(\mathbb{R}^d)} = \langle Wu, u\rangle_{L^2(\mathbb{R}^d)} \geq C_1^{-1} \langle |x|^{2\gamma} u, u\rangle_{L^2(\mathbb{R}^d)}, \tag{2.5}$$

which will be used several times during the proof. In particular, this estimate provides the right scale in the region $|x| \geq \nu^{1/(2\gamma+1)}$, according to the lower bound in Assumption (2.2). Next, we split the phase space in two different regions.

The propagative region. Let $\chi \in \mathscr{C}_c^\infty(\mathbb{R}^+; [0,1])$, such $\chi = 1$ on $[1/2, 3/2]$ and $\chi = 0$ on $[0, 1/4]$. Let $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^d; [0,1])$ such that $\phi(x) = \frac{1}{2}$ if $|x| \leq \frac{1}{2}$ and $\phi(x) = 0$ if $|x| \geq 1$. We define

$$\psi(x, \xi) = \int_{-\infty}^{0} \phi(x + \tau\xi) d\tau \in \mathscr{C}^\infty\big(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})\big),$$

which is bounded on $\mathbb{R}^d \times \big(\mathbb{R}^d \setminus B(0, \frac{1}{4})\big)$ since ϕ is compactly supported. We set

$$m_\nu(x, \xi) = \chi\Big(\frac{|\xi|^2}{\nu^2}\Big)\psi\Big(\frac{x}{\nu^{1/(2\gamma+1)}}, \frac{\xi}{\nu}\Big) \in S\Big(1, \frac{|dx|^2}{\nu^{2/(2\gamma+1)}} + \frac{|d\xi|^2}{\nu^2}\Big),$$

where each seminorm of the symbol m_ν is bounded above independently of $\nu \geq 1$; in particular, we get that m_ν^w is bounded on $L^2(\mathbb{R}^d)$ with $\sup_{\nu \geq 1} \|m_\nu^w\|_{\mathcal{L}(L^2)} < +\infty$. Next, we have

$$\begin{aligned} 2\,\mathrm{Re}\langle Q_\nu u, i m_\nu^w u\rangle_{L^2(\mathbb{R}^d)} &= \big\langle \big[i\big(|\xi|^2 - \nu^2\big)^w, m_\nu^w\big] u, u\big\rangle_{L^2(\mathbb{R}^d)} + 2\,\mathrm{Re}\langle V^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \big\{|\xi|^2 - \nu^2, m_\nu\big\}^w u, u\rangle_{L^2(\mathbb{R}^d)} + 2\,\mathrm{Re}\langle V^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{2.6}$$

since the symbol $|\xi|^2 - \nu^2$ is quadratic. Moreover, we can compute

$$\begin{aligned} \frac{1}{2}\big\{|\xi|^2 - \nu^2, m_\nu\big\} &= \xi \cdot \partial_x m_\nu = \chi(|\xi|^2/\nu^2)\xi \cdot \frac{\partial}{\partial x}\Big(\psi\Big(\frac{x}{\nu^{1/(2\gamma+1)}}, \frac{\xi}{\nu}\Big)\Big) \\ &= \int_{-\infty}^{0} \nu^{-1/(2\gamma+1)} \chi(|\xi|^2/\nu^2)(\xi \cdot d\phi)(x\nu^{-1/(2\gamma+1)} + \tau\xi\nu^{-1}) d\tau \\ &= \int_{-\infty}^{0} \nu\nu^{-1/(2\gamma+1)} \chi(|\xi|^2/\nu^2) \frac{d}{d\tau}\Big(\phi(x\nu^{-1/(2\gamma+1)} + \tau\xi\nu^{-1})\Big) d\tau \\ &= \nu^{2\gamma/(2\gamma+1)} \chi(|\xi|^2/\nu^2)\phi(x\nu^{-1/(2\gamma+1)}) \end{aligned}$$

since $|\xi\nu^{-1}|^2 \geq 1/4$ on $\mathrm{supp}\,\chi(|\xi|^2/\nu^2)$. Hence, we obtain

$$\begin{aligned} \big\{|\xi|^2 - \nu^2, m_\nu\big\} &= 2\nu^{2\gamma/(2\gamma+1)} \chi(|\xi|^2/\nu^2)\phi(x\nu^{-1/(2\gamma+1)}) \\ &\geq \begin{cases} \nu^{2\gamma/(2\gamma+1)} & \text{if } ||\xi|^2\nu^{-2} - 1| \leq 1/2 \text{ and } |x|\nu^{-1/(2\gamma+1)} \leq 1/2, \\ 0 & \text{on } T^*\mathbb{R}^d. \end{cases} \end{aligned}$$

Moreover we have,

$$2\nu^{2\gamma/(2\gamma+1)}\chi(|\xi|^2/\nu^2)\phi(x\nu^{-1/(2\gamma+1)}) \in S\Big(\nu^{2\gamma/(2\gamma+1)}, \frac{|dx|^2}{\nu^{2/(2\gamma+1)}} + \frac{|d\xi|^2}{\nu^2}\Big).$$

As a consequence, using the sharp Gårding inequality in (2.6) yields

$$\begin{aligned} C\|Q_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \geq \nu^{2\gamma/(2\gamma+1)} &\Big(\Big(\chi_0(|\xi|^2\nu^{-2}-1)\chi_0(|x|^2\nu^{-2/(2\gamma+1)})\Big)^w u, u\Big)_{L^2(\mathbb{R}^d)} \\ &- |2\,\mathrm{Re}\langle V^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}| - C\nu^{-2/(2\gamma+1)}\|u\|^2_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{2.7}$$

where, for some $\varepsilon_0 \in (0, 1/8)$,

$$\begin{cases} \chi_0 \in \mathscr{C}_c^\infty(\mathbb{R}; [0,1]) \text{ is such that } \{\chi_0 = 1\} = [-\varepsilon_0, \varepsilon_0], \text{ and} \\ \{\chi_0 = 0\} = [-2\varepsilon_0, 2\varepsilon_0]^c, \ \{0 < \chi_0(t) < 1\} = \{\varepsilon_0 < |t| < 2\varepsilon_0\}. \end{cases} \tag{2.8}$$

Next, we check the term $2\,\mathrm{Re}\langle V^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}$. We have

$$\big|2\,\mathrm{Re}\langle V^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\big| = \big|2\,\mathrm{Re}\langle Vu, Vm_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\big| \leq 2\|Vu\|_{L^2(\mathbb{R}^d)}\|Vm_\nu^w u\|_{L^2(\mathbb{R}^d)}.$$

Recalling that $m_\nu \in S(1, \frac{|dx|^2}{\nu^{2/(2\gamma+1)}} + \frac{|d\xi|^2}{\nu^2}) \subset S(1, |dx|^2 + |d\xi|^2)$ as $\nu \geq 1$, we may apply Lemma 2.2 to obtain

$$\big|\mathrm{Re}\langle V^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\big| \lesssim \|Vu\|_{L^2(\mathbb{R}^d)}\big(\|u\|_{L^2(\mathbb{R}^d)} + \|Vu\|_{L^2(\mathbb{R}^d)}\big),$$

where the constant involved is uniform w.r.t. ν. Next, using (2.5), we obtain

$$\begin{aligned} \big|2\,\mathrm{Re}\langle V^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\big| &\lesssim \|u\|^{3/2}_{L^2(\mathbb{R}^d)}\|Q_\nu u\|^{1/2}_{L^2(\mathbb{R}^d)} + \|Q_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \|u\|^2_{L^2(\mathbb{R}^d)} + \|Q_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.9}$$

Combining this estimate with (2.7), we have, for $\nu \geq \nu_0$ and ν_0 large enough,

$$\begin{aligned} &C\|u\|^2_{L^2(\mathbb{R}^d)} + C\|Q_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \\ &\qquad \geq \nu^{2\gamma/(2\gamma+1)}\Big(\Big(\chi_0(|\xi|^2\nu^{-2}-1)\chi_0(|x|^2\nu^{-2/(2\gamma+1)})\Big)^w u, u\Big)_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.10}$$

The elliptic region. We now check the regions where $|\xi|^2 \ll \nu^2$ or $|\xi|^2 \gg \nu^2$. Let $\varepsilon_0 \in (0, 1/2)$; we consider a function $\theta \in \mathscr{C}^\infty(\mathbb{R}; [-1,1])$ such that

$$\theta(\sigma) = \begin{cases} 1 & \text{for } \sigma \geq 1 + 2\varepsilon_0, \\ \{0 < \theta < 1\} & \text{for } \sigma \in (1+\varepsilon_0, 1+2\varepsilon_0), \\ 0 & \text{for } 1-\varepsilon_0 \leq \sigma \leq 1+\varepsilon_0, \\ \{-1 < \theta < 0\} & \text{for } \sigma \in (1-2\varepsilon_0, 1-\varepsilon_0), \\ -1 & \text{for } \sigma \leq 1-2\varepsilon_0, \end{cases} \tag{2.11}$$

and hence

$$\forall \sigma \in \mathbb{R}, \quad (\sigma - 1)\theta(\sigma) \geq |\theta(\sigma)|(\sigma+1)\frac{\varepsilon_0}{2+\varepsilon_0}. \tag{2.12}$$

A consequence of (2.12) is that, with $c_0 = \varepsilon_0/(2+\varepsilon_0)$, we have

$$\forall \xi \in \mathbb{R}^d, \forall \nu \geq 1, \quad (|\xi|^2 - \nu^2)\theta(|\xi|^2\nu^{-2}) \geq c_0|\theta(|\xi|^2\nu^{-2})|\big(|\xi|^2 + \nu^2\big). \tag{2.13}$$

We compute

$$\begin{aligned} &\mathrm{Re}\langle Q_\nu u, \theta(|\xi|^2\nu^{-2})^w u\rangle_{L^2(\mathbb{R}^d)} \\ &\quad = \big((|\xi|^2 - \nu^2)^w u, \theta(|\xi|^2\nu^{-2})^w u\big)_{L^2(\mathbb{R}^d)} + \mathrm{Re}\big(iV^2 u, \theta(|\xi|^2\nu^{-2})^w u\big)_{L^2(\mathbb{R}^d)} \\ &\quad \geq c_0\Big(\Big((|\xi|^2+\nu^2)|\theta(|\xi|^2\nu^{-2})|\Big)^w u, u\Big)_{L^2(\mathbb{R}^d)} - \Big|\mathrm{Re}\big(iV^2 u, \theta(|\xi|^2\nu^{-2})^w u\big)_{L^2(\mathbb{R}^d)}\Big|. \end{aligned}$$

Following (2.9), we have

$$\left|\mathrm{Re}\langle iV^2 u, \theta(|\xi|^2\nu^{-2})^w u\rangle_{L^2(\mathbb{R}^d)}\right| \lesssim \|u\|^2_{L^2(\mathbb{R}^d)} + \|Q_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)},$$

so that we finally obtain, as $\theta(|\xi|^2\nu^{-2})^w$ is bounded on $L^2(\mathbb{R}^d)$,

$$C\|u\|^2_{L^2(\mathbb{R}^d)} + C\|Q_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \geq \left(\left((|\xi|^2+\nu^2)|\theta(|\xi|^2\nu^{-2})|\right)^w u, u\right)_{L^2(\mathbb{R}^d)}. \tag{2.14}$$

Patching the estimates together. Combining (2.5), (2.10) and (2.14), we obtain the following estimate

$$\begin{aligned} C\|u\|^2_{L^2(\mathbb{R}^d)} + C\|Q_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \geq \|Vu\|^2_{L^2(\mathbb{R}^d)} + \nu^2\left(\left(|\theta(|\xi|^2\nu^{-2})|\right)^w u, u\right)_{L^2(\mathbb{R}^d)} \\ + \nu^{2\gamma/(2\gamma+1)}\left(\left(\chi_0(|\xi|^2\nu^{-2}-1)\chi_0(|x|^2\nu^{-2/(2\gamma+1)})\right)^w u, u\right)_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.15}$$

Since χ_0 is given and satisfies (2.8), we define now $\theta(\sigma) = \mathrm{sign}(\sigma-1)\left(1-\chi_0(\sigma-1)\right)$. The function θ is smooth and satisfies (2.11) so that (2.12) holds. We note that

$$|\theta(\sigma)| + \chi_0(\sigma-1) = 1,$$

since $|1-\chi_0(\sigma-1)| + \chi_0(\sigma-1) = 1-\chi_0(\sigma-1)+\chi_0(\sigma-1) = 1$. As a consequence, we write

$$\begin{aligned} 1 &= |\theta(|\xi|^2\nu^{-2})| + \chi_0(|\xi|^2\nu^{-2}-1) \\ &= |\theta(|\xi|^2\nu^{-2})| + \chi_0(|x|^2\nu^{-2/(2\gamma+1)})\chi_0(|\xi|^2\nu^{-2}-1) + \left(1-\chi_0(|x|^2\nu^{-2/(2\gamma+1)})\right)\chi_0(|\xi|^2\nu^{-2}-1), \end{aligned}$$

and hence

$$\begin{aligned} \nu^{2\gamma/(2\gamma+1)} &\leq \nu^{2\gamma/(2\gamma+1)}|\theta(|\xi|^2\nu^{-2})| \\ &\quad + \nu^{2\gamma/(2\gamma+1)}\chi_0(|x|^2\nu^{-2/(2\gamma+1)})\chi_0(|\xi|^2\nu^{-2}-1) + \nu^{2\gamma/(2\gamma+1)}(1-\chi_0(|x|^2\nu^{-2/(2\gamma+1)})). \end{aligned}$$

Since the symbols on both sides of the inequality belong to the class $S\left(\nu^{2\gamma/(2\gamma+1)}, \frac{|dx|^2}{\nu^{2/(2\gamma+1)}} + \frac{|d\xi|^2}{\nu^2}\right)$, we can apply Gårding's inequality. Note that the gain in the pseudodifferential calculus for symbols in this class is given by $\nu^{-1/(2\gamma+1)}\nu^{-1} = \nu^{-(2\gamma+2)/(2\gamma+1)}$. This gives, for $\nu \geq \nu_0$ and ν_0 large enough,

$$\begin{aligned} \nu^{2\gamma/(2\gamma+1)}\left(\left(\chi_0(|\xi|^2\nu^{-2}-1)\chi_0(|x|^2\nu^{-2/(2\gamma+1)})\right)^w u, u\right)_{L^2(\mathbb{R}^d)} + \nu^{2\gamma/(2\gamma+1)}\left\langle\left(|\theta(|\xi|^2\nu^{-2})|\right)^w u, u\right\rangle_{L^2(\mathbb{R}^d)} \\ + \left(\nu^{2\gamma/(2\gamma+1)}(1-\chi_0(|x|^2\nu^{-2/(2\gamma+1)}))u, u\right)_{L^2(\mathbb{R}^d)} \geq \frac{1}{2}\nu^{2\gamma/(2\gamma+1)}\|u\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Next, we note that, because of the properties of χ_0, given in (2.8) we find

$$\nu^{2\gamma/(2\gamma+1)}(1-\chi_0(|x|^2\nu^{-2/(2\gamma+1)})) \leq CV(x)^2,$$

according to the lower bound in Assumption (2.2). Using the last two inequalities together with (2.15) gives

$$C\|u\|^2_{L^2(\mathbb{R}^d)} + C\|Q_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \geq \nu^{2\gamma/(2\gamma+1)}\|u\|^2_{L^2(\mathbb{R}^d)}, \tag{2.16}$$

which concludes the proof of the theorem, dividing by $\|u\|_{L^2(\mathbb{R}^d)}$ and taking $\nu \geq \nu_0$ with ν_0 large enough. □

3 The scaling argument

First, we prove the following lemma, which is a consequence of Theorem 2.1 together with a scaling argument (using the homogeneity of W). We define on $L^2(\mathbb{R}^d)$ (below, we shall take $d = n'$) the operator

$$\tilde{P}_{\lambda,\omega} = -\Delta - \omega + i\lambda W. \tag{3.1}$$

Lemma 3.1. *Let $\gamma > 0$ be given and W be an homogeneous function of order 2γ on $\mathbb{R}^d$ with $W(x) > 0$ for $x \in \mathbb{S}^{d-1}$. Then, there exists $C > 0$ such that for all $u \in \mathscr{C}_c^2(\mathbb{R}^d)$, for all $\lambda > 0$ and for all $\omega \in \mathbb{R}$, we have*

$$C\|\tilde{P}_{\lambda,\omega}u\|_{L^2(\mathbb{R}^d)} \geq \lambda^{1/(\gamma+1)}\left(1 + \mathbf{1}_{\mathbb{R}_+}(\omega)\left(\frac{\omega}{\lambda^{1/(\gamma+1)}}\right)^{\gamma/(2\gamma+1)}\right)\|u\|_{L^2(\mathbb{R}^d)}.$$

Remark 3.2. Note that this lemma does not use either λ large, or $0 \leq \omega \leq \lambda^2$, a feature due to the homogeneity of W.

Proof of Lemma 3.1. First, we remark that for all $\alpha > 0$, the operator

$$\begin{array}{rcl} T_\alpha : L^2(\mathbb{R}^d) & \to & L^2(\mathbb{R}^d) \\ u(x) & \mapsto & \alpha^{\frac{d}{2}}u(\alpha x) \end{array}$$

is an isometry, with inverse $(T_\alpha)^{-1} = T_{\alpha^{-1}}$. Using that W is homogeneous of order 2γ, we have

$$T_\alpha \tilde{P}_{\lambda,\omega}(T_\alpha)^{-1} = -\alpha^{-2}\Delta - \omega + i\lambda W(\alpha x) = -\alpha^{-2}\Delta - \omega + i\alpha^{2\gamma}\lambda W(x).$$

Choosing then $\alpha = \lambda^{-\frac{1}{2(\gamma+1)}}$, we have $\alpha^{-2} = \alpha^{2\gamma}\lambda = \lambda^{1/(\gamma+1)}$ so that with Q_0 given by (2.1)

$$T_\alpha \tilde{P}_{\lambda,\omega}(T_\alpha)^{-1} = \lambda^{1/(\gamma+1)}\left(Q_0 - \omega\lambda^{-1/(\gamma+1)}\right), \qquad \alpha = \lambda^{-1/2(\gamma+1)}.$$

Since the assumption on W in Lemma 3.1 implies Assumption (2.2), we can apply Theorem 2.1. This yields (still with $\alpha = \lambda^{-1/2(\gamma+1)}$),

$$\begin{aligned} \|\tilde{P}_{\lambda,\omega}u\|_{L^2(\mathbb{R}^d)} = \|T_\alpha\tilde{P}_{\lambda,\omega}(T_\alpha)^{-1}T_\alpha u\|_{L^2(\mathbb{R}^d)} &= \lambda^{1/(\gamma+1)}\left\|\left(Q_0 - \omega\lambda^{-1/(\gamma+1)}\right)T_\alpha u\right\|_{L^2(\mathbb{R}^d)} \\ &\geq C_0^{-1}\lambda^{1/(\gamma+1)}\left(1 + 1_{\mathbb{R}_+}(\omega)\left(\frac{\omega}{\lambda^{1/(\gamma+1)}}\right)^{\gamma/(2\gamma+1)}\right)\|T_\alpha u\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

concluding the proof of the lemma since T_α is an isometry. □

4 The reduction to a problem on M' by Fourier transform, and end of proof of Theorem 1.6

After a Fourier transform in the x'' variable, Theorem 1.6 reduces to the following result.

Theorem 4.1. *Assume* (1.15), (1.17) *and define the operator acting on $L^2(M')$*

$$P_{\lambda,\omega} = -\Delta_{M'} - \omega + i\lambda b, \quad D(P_{\lambda,\omega}) = H^2(M'). \tag{4.1}$$

Then, there exist $C > 0$ and $\lambda_0 > 0$ such that for all $u \in H^2(M')$, for all $\lambda \geq \lambda_0$ and for all $\omega \leq \lambda^2$, we have

$$\|P_{\lambda,\omega}u\|_{L^2(M')} \geq C\lambda^{1/(\gamma+1)}\|u\|_{L^2(M')}. \tag{4.2}$$

Proof that Theorem 4.1 ⇒ Theorem 1.6. We perform a Fourier transform in the variable $x'' \in \mathbb{T}^{n''}$, and write $u(x', x'') = \sum_{k\in\mathbb{Z}^{n''}} \hat{u}_k(x')e^{ik\cdot x''}$. Then, for $u \in H^2(M)$, we have with P_λ defined in (1.4) and $P_{\lambda,\omega}$ in (4.1),

$$(P_\lambda u)(x', x'') = \sum_{k\in\mathbb{Z}^{n''}}\left((-\Delta_{M'} + |k|^2 - \lambda^2 + i\lambda b)\hat{u}_k\right)(x')e^{ik\cdot x''} = \sum_{k\in\mathbb{Z}^{n''}}\left(P_{\lambda,\lambda^2-|k|^2}\hat{u}_k\right)(x')e^{ik\cdot x''},$$

as $b = b(x')$ does not depend on the x''-variable. Finally, as a consequence of Theorem 4.1, we have $\|P_{\lambda,\lambda^2-|k|^2}w\|_{L^2(M')} \geq C\lambda^{1/(\gamma+1)}\|w\|_{L^2(M')}$ where $C > 0$ does not depend on k. This yields

$$\|P_\lambda u\|^2_{L^2(M)} = (2\pi)^{n''}\sum_{k\in\mathbb{Z}}\|P_{\lambda,\lambda^2-|k|^2}\hat{u}_k\|^2_{L^2(M')} \geq C^2\lambda^{2/(\gamma+1)}\|u\|^2_{L^2(M)},$$

which proves Theorem 1.6. □

We now want to use Lemma 3.1 in a neighbourhood of $\{y'\} \times \mathbb{T}^1$ and to patch estimates together to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\chi_0 \in \mathscr{C}_c^\infty(B(y', \varepsilon_0); [0,1])$ such that $\chi_0 = 1$ in a neighbourhood of y' and set $\chi_1 = 1 - \chi_0 \in \mathscr{C}^\infty(M')$. On the one hand, we have, with $P_{\lambda,\omega}$ given by (4.1),

$$C\|P_{\lambda,\omega}\chi_1 u\|_{L^2(M')} \geq \lambda \|\chi_1 u\|_{L^2(M')}, \tag{4.3}$$

since, according to Assumption (1.17), b is bounded from below on $\operatorname{supp}(\chi_1)$ and hence

$$\begin{aligned}\lambda\|\chi_1 u\|^2_{L^2(M')} &\leq C\lambda\langle b\chi_1 u, \chi_1 u\rangle_{L^2(M')} = C \operatorname{Im}\langle P_{\lambda,\omega}\chi_1 u, \chi_1 u\rangle_{L^2(M')}\\ &\leq C\|P_{\lambda,\omega}\chi_1 u\|_{L^2(M')}\|\chi_1 u\|_{L^2(M')}.\end{aligned}$$

On the other hand, we have

$$\|P_{\lambda,\omega}\chi_0 u\|^2_{L^2(M')} = \|P_{\lambda,\omega}\chi_0 u\|^2_{L^2(B(y',\varepsilon_0))}.$$

According to Assumption (1.16), we can extend by homogeneity in the variable $x' - y'$ the function b from $B_{\mathbb{R}^{n'}}(y', \varepsilon_0)$ to the whole $\mathbb{R}^{n'}$ as an homogeneous function W of degree 2γ on $\mathbb{R}^{n'}$ with $W(x') > 0$ for $|x' - y'| = 1$. Hence, we have $P_{\lambda,\omega}\chi_0 = \tilde{P}_{\lambda,\omega}\chi_0$, where $\tilde{P}_{\lambda,\omega}$ is given by (3.1). We may then apply Lemma 3.1 to the operator $\tilde{P}_{\lambda,\omega}$ in $\mathbb{R}^{n'}$. This yields, for some $C > 0$,

$$\begin{aligned}C\|P_{\lambda,\omega}\chi_0 u\|^2_{L^2(M')} &= C\|\tilde{P}_{\lambda,\omega}\chi_0 u\|^2_{L^2(\mathbb{R}^{n'})}\\ &\geq \lambda^{2/(\gamma+1)}\left(1 + 1_{\mathbb{R}_+}(\omega)\left(\frac{\omega}{\lambda^{1/(\gamma+1)}}\right)^{2\gamma/(2\gamma+1)}\right)\|\chi_0 u\|^2_{L^2(M')}.\end{aligned} \tag{4.4}$$

We now want to estimate the remainder term

$$\begin{aligned}\|[P_{\lambda,\omega}, \chi_1]u\|_{L^2(M')} = \|[P_{\lambda,\omega}, \chi_0]u\|_{L^2(M')} &= \|[-\Delta_{M'}, \chi_0]u\|_{L^2(M')}\\ &\leq \|(\Delta_{M'}\chi_0)u\|_{L^2(M')} + 2\|\nabla_{x'}\chi_0 \cdot \nabla_{x'}u\|_{L^2(M')}.\end{aligned} \tag{4.5}$$

For this, we take $\psi = \psi(x') \in \mathscr{C}_c^\infty(B(0, \varepsilon_0); [0,1])$ such that $\psi = 1$ on $\operatorname{supp}(\nabla_{x'}\chi_0)$ and $\psi = 0$ in a neighbourhood of 0. We compute

$$\begin{aligned}\operatorname{Re}\langle P_{\lambda,\omega}u, \psi^2 u\rangle_{L^2(M')} &= \operatorname{Re}\langle(-\Delta_{M'} - \omega)u, \psi^2 u\rangle_{L^2(M')} + \overbrace{\operatorname{Re}\langle i\lambda\psi^2 bu, u\rangle_{L^2(M')}}^{=0}\\ &= \operatorname{Re}\langle -\Delta_{M'}u, \psi^2 u\rangle_{L^2(M')} - \omega\|\psi u\|^2_{L^2(M')},\end{aligned} \tag{4.6}$$

with $\Delta_{M'} = \Delta_{\mathbb{R}^{n'}}$ on $\operatorname{supp}(\psi)$. Moreover, we have

$$\begin{aligned}\operatorname{Re}\langle -\Delta_{\mathbb{R}^{n'}}u, \psi^2 u\rangle_{L^2(M')} &= \langle\nabla_{x'}u, \psi^2\nabla_{x'}u\rangle_{L^2(M')} + \operatorname{Re}\langle\nabla_{x'}u, u\nabla_{x'}\psi^2\rangle_{L^2(M')}\\ &= \langle\nabla_{x'}u, \psi^2\nabla_{x'}u\rangle_{L^2(M')} + \sum_{j=1}^{n'}\operatorname{Re}\langle iD_{x_j}u, u\partial_{x_j}\psi^2\rangle_{L^2(M')}\\ &= \langle\nabla_{x'}u, \psi^2\nabla_{x'}u\rangle_{L^2(M')} - \sum_{j=1}^{n'}\frac{i}{2}\langle[D_{x_j}, \partial_{x_j}\psi^2]u, u\rangle_{L^2(M')}\\ &= \langle\nabla_{x'}u, \psi^2\nabla_{x'}u\rangle_{L^2(M')} - \frac{1}{2}\langle(\Delta_{\mathbb{R}^{n'}}\psi^2)u, u\rangle_{L^2(M')},\end{aligned}$$

since the two operators D_{x_j} and $\partial_{x_j}\psi^2$ are selfadjoint. We have thus

$$\begin{aligned}\|\psi\nabla_{x'}u\|^2_{L^2(M')} = \operatorname{Re}\langle\left(-\Delta_{\mathbb{R}^{n'}} - \omega + i\lambda b\right)u, \psi^2 u\rangle_{L^2(M')} &+ \overbrace{\operatorname{Re}\langle\left(\omega - i\lambda b\right)u, \psi^2 u\rangle_{L^2(M')}}^{=\omega\|\psi u\|^2_{L^2(M')}}\\ &+ \frac{1}{2}\langle(\Delta_{\mathbb{R}^{n'}}\psi^2)u, u\rangle_{L^2(M')},\end{aligned}$$

and consequently we obtain

$$\|\psi\nabla_{x'}u\|^2_{L^2(M')} \le \|P_{\lambda,\omega}u\|_{L^2(M')}\|u\|_{L^2(M')} + C_1\|u\|^2_{L^2(M')} + \omega\|\psi u\|^2_{L^2(M')}.$$

As a result, we can estimate the commutator of (4.5) by

$$\|[P_{\lambda,\omega},\chi_0]u\|^2_{L^2(M')} \le C_2\left(\|P_{\lambda,\omega}u\|_{L^2(M')}\|u\|_{L^2(M')} + \|u\|^2_{L^2(M')} + \omega\|\psi u\|^2_{L^2(M')}\right). \tag{4.7}$$

Now, we have $\|P_{\lambda,\omega}(\chi_j u)\|^2_{L^2(M')} \le 2\|[P_{\lambda,\omega},\chi_j]u\|^2_{L^2(M')} + 2\|\chi_j P_{\lambda,\omega}u\|^2_{L^2(M')}$, so that

$$2\|P_{\lambda,\omega}u\|^2_{L^2(M')} \ge \|P_{\lambda,\omega}\chi_0 u\|^2_{L^2(M')} + \|P_\lambda\chi_1 u\|^2_{L^2(M')} - 4\|[P_{\lambda,\omega},\chi_0]u\|^2_{L^2(M')}, \tag{4.8}$$

which, combined with the estimates (4.3), (4.4) and (4.7), yields

$$\begin{aligned} C_3\left(\|P_{\lambda,\omega}u\|^2_{L^2(M')} + \|u\|^2_{L^2(M')}\right) \ge \lambda^{2/(\gamma+1)}\left(1 + 1_{\mathbb{R}_+}(\omega)\left(\frac{\omega}{\lambda^{1/(\gamma+1)}}\right)^{2\gamma/(2\gamma+1)}\right)\|\chi_0 u\|^2_{L^2(M')} \\ + \lambda^2\|\chi_1 u\|^2_{L^2(M')} - c_1\omega\|\psi u\|^2_{L^2(M')}, \end{aligned} \tag{4.9}$$

where c_1 is a fixed positive constant.

In the régime $\omega \le 0$ (or, more generally, $\omega \le \omega_0$ for any given ω_0), this suffices to prove (4.2).

Let us now study the régime $\omega \ge 0$. We notice that, for $\omega \le \lambda^2$, $\lambda \ge 1$,

$$\begin{aligned} \lambda^{2/(\gamma+1)}\left(1 + \left(\frac{\omega}{\lambda^{1/(\gamma+1)}}\right)^{2\gamma/(2\gamma+1)}\right) &= \lambda^{2/(\gamma+1)} + \lambda^{\frac{2}{\gamma+1} - \frac{2\gamma}{(2\gamma+1)(\gamma+1)}}\omega^{2\gamma/(2\gamma+1)} \\ &\le \lambda^{2/(\gamma+1)} + \lambda^{\frac{2}{\gamma+1} - \frac{2\gamma}{(2\gamma+1)(\gamma+1)} + \frac{4\gamma}{2\gamma+1}} = \lambda^{2/(\gamma+1)} + \lambda^2 \le 2\lambda^2, \end{aligned}$$

and that, for all $v \in L^2(M')$ we have

$$\|\psi v\|^2_{L^2(M')} \le C_4\|v\|^2_{L^2(M')} = C_4\|(\chi_0+\chi_1)v\|^2_{L^2(M')} \le 2C_4\|\chi_0 v\|^2_{L^2(M')} + 2C_4\|\chi_1 v\|^2_{L^2(M')}.$$

This, together with (4.9) then yields

$$C_5\left(\|P_{\lambda,\omega}u\|^2_{L^2(M')} + \|u\|^2_{L^2(M')}\right) \ge \lambda^{2/(\gamma+1)}f(\lambda,\omega)\|(\chi_0+\chi_1)u\|^2_{L^2(M')} = \lambda^{2/(\gamma+1)}f(\lambda,\omega)\|u\|^2_{L^2(M')},$$

where

$$f(\lambda,\omega) = 1 + \left(\frac{\omega}{\lambda^{1/(\gamma+1)}}\right)^{2\gamma/(2\gamma+1)} - c_0\,\omega\lambda^{-\frac{2}{\gamma+1}}. \tag{4.10}$$

with a fixed positive constant c_0. According to Lemma 4.2, there exists $\lambda_0 > 0$ and $\delta := c_0^{-(2\gamma+1)} > 0$, such that for all $\lambda \ge \lambda_0$ and $\omega \in [0, \delta\lambda^2]$, we have $f(\lambda,\omega) \ge 1$. As a consequence, (4.2) is satisfied in this régime. Finally, suppose that $\delta\lambda^2 \le \omega \le \lambda^2$, where $\delta = c_0^{-(2\gamma+1)}$. In this régime, the estimate (4.2) is a direct consequence of the usual (stronger) geometric control estimate (see Lemma 4.3 below). □

The above proof relies on the following two lemmata, the first of which is elementary.

Lemma 4.2. $\forall c_0 > 0, \forall\lambda > 0, \forall\omega \in [0, c_0^{-(2\gamma+1)}\lambda^2]$, *we have* $f(\lambda,\omega) \ge 1$, *where* $f(\lambda,\omega)$ *is defined in* (4.10).

The next lemma states the estimate associated to the geometric control condition (which is satisfied by the set $M' \setminus \{y'\}$ in M'). It is very classical and we refer e.g. to [RT74, Leb96, AL14], or [LL14, Lemma 5.1] for a simple self contained proof.

Lemma 4.3. *Let* $\delta > 0$, *and suppose that* b *satisfies* (1.16)-(1.17). *Then, there exist* $\lambda_0 > 0$ *and* $C > 0$ *such that for all* $\lambda \ge \lambda_0$, *for all* $\omega \in [\delta\lambda^2, \lambda^2]$, *we have*

$$\|P_{\lambda,\omega}u\|_{L^2(M')} \gtrsim \lambda\|u\|_{L^2(M')}.$$

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Some time after the article [LL14] was submitted, N. Burq and C. Zuily [BZ15] managed to weaken some of the assumptions of our Theorem 1.5.

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