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Anomalous diffusion phenomena: A kinetic approach

Antoine Mellet*

Abstract

In this talk, we review some aspects of the derivation of fractional diffusion equations from kinetic equations and in particular some applications to the description of anomalous energy transport in FPU chains. This is based on joint works with N. Ben Abdallah, L. Cesbron, S. Merino, S. Mischler, C. Mouhot and M. Puel

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1 Introduction

1.1 Particles transport and diffusion equations

The issue at the center of this talk is the approximation of particles transport by diffusion type equations. Consider for example a cloud of particles interacting with a background such as a gas of (light) electrons in a plasma, interacting with the (heavier) ions and atoms.

- At the **microscopic** level, the gas is constituted of N particles with positions $X_i(t)$ and velocity $V_i(t)$ satisfying an ODE system of the form:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = F + \text{collisions} + \text{noise} + \cdots \end{cases}$$

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We note immediately that the electrons in a plasma may collide with themselves or with other atoms and ions. In this talk however, we will not consider collisions of the particles with each others, but only their interactions with the background. As a consequence, the equations considered here will be linear.

- In the **kinetic description** of particle transport, the particles distribution function $f(t, x, v)$ (interpreted as the probability of finding a particle at position x with velocity v) solves a Vlasov type equation:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v(Ff) = \frac{1}{\varepsilon} Q(f)$$

where F models body forces (e.g. electro-magnetic field), and $Q(f)$ takes into account all other interactions with the background (typically collisions or noise).

The parameter ε is the **Knudsen number** (ratio of the mean free path of the particles and the macroscopic length scale) which measures the importance of the term $Q(f)$ compared to the transport terms. It plays an essential role in the derivation of a macroscopic model.

- The goal is now to derive a **macroscopic** model describing the motion of the electrons. More precisely, we look for an equation describing the evolution of macroscopic quantities such as the density $\rho(x, t) = \int_{\mathbb{R}^N} f(x, v, t)\, dv$ and the momentum $j(x, t) = \int_{\mathbb{R}^N} v\, f(x, v, t)\, dv$. This derivation is the main goal of the study of **hydrodynamic limits of Kinetic equations**, and is the main topic of this talk. It is classically done by assuming that the Knudsen number ε is small and by rescaling the equation appropriately to observe the behavior of the particles over a long time.

The study of hydrodynamic limits for Boltzmann equation (derivation of the equations of fluid dynamics) goes back to the early work of Maxwell and Boltzmann (see also Hilbert's 6th problem) and will not be discussed in this talk. Instead, we are concerned with the diffusion approximation of linear transport equations. Classical references for this are Conwell [11], Rode [32], Bensoussan-Lions-Papanicolaou [7], Bardos-Santos-Sentis [2] among others. The papers of Golse-Poupaud [16] and Degond-Goudon-Poupaud [13] are also particularly relevant to this talk.

1.2 Example 1: the Vlasov-Fokker-Planck equation

We now give a first simple and classical example of diffusion approximation for kinetic equations by considering the the Vlasov-Fokker-Planck equation.

The Fokker-Planck operator is often use to model the interactions of the electrons with the background in Plasma physic. It is given by

$$\mathcal{L}(f) := \Delta_v f + \mathrm{div}(vf)$$

and it corresponds to the following microscopic equation for the velocity of the particles

$$\dot{V}_i = -V_i + \text{Brownian noise}$$

(this equation is usually referred to as Langevin equation). We immediately make the following classical observations:

- The thermodynamical equilibriums are given by Maxwell's distribution:

$$\ker \mathcal{L} = \mathrm{Span}\{M\}, \qquad M(v) = \frac{1}{(2\pi)^{N/2}} e^{-|v|^2/2}$$

- $\mathcal{L}$ has a **spectral gap**:

$$-\langle \mathcal{L}(f), f \rangle_{M^{-1}} \geq \lambda ||f||^2_{L^2_{M^{-1}}} \qquad \text{if } \int f\, dv = 0$$

We now consider the VFP equation in the small Knudsen number regime and at time scale of the order of ε^{-1}:

$$\varepsilon\, \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \mathcal{L}(f^\varepsilon). \qquad (1)$$

Then it can be proved that f^ε converges weakly to a function $\rho(x,t)M(v)$ where the density ρ solves a diffusion equation:

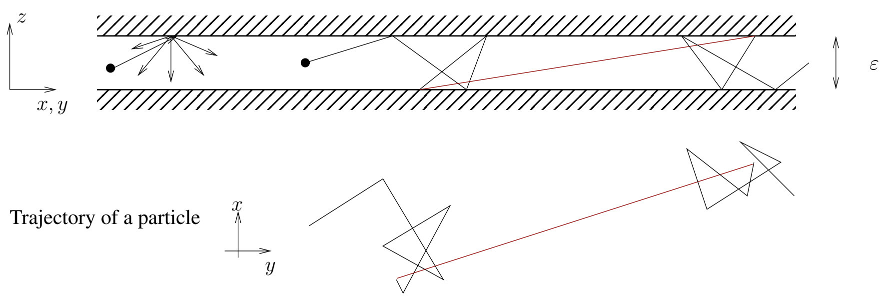
$$\partial_t \rho - \mathrm{div}_x(D \nabla_x \rho) = 0, \qquad D = -\int v \otimes \mathcal{L}^{*-1}(v) M(v)\, dv$$

Remarks:

- The scaling in (1) corresponds to a time scale ε^{-1} which is "just right" in order to observe the diffusion process: Longer time scale would lead to ρ being a stationary solution of the diffusion equation, while under smaller time scale, ρ would still be equal to the initial density function.
- Note that $\mathcal{L}^{*-1}(v)$ exists (and $D < \infty$) because of the spectral gap property of $\mathcal{L}$ and the fact that $v \in L^2_M$. One of the goal of this talk will be to discuss situations were one of these facts fails.

1.3 Example 2: transport between two parallel plates

We now discuss a second example in which classical diffusion typically fails to take place: Consider a rarefied gas confined between two parallel plates separated by a thin gap of size $\varepsilon \ll 1$.



Neglecting the interactions of the particles with each other (rarefied gas assumption), we write the following kinetic equation:

$$\partial_t f + v \cdot \nabla_{x,y} f + w \partial_z f = 0, \quad z \in (0, \varepsilon)$$

which we supplement with Maxwell boundary conditions along the plates $z = 0$ and $z = 1$. For instance, we can take

$$f(x, y, 1, v, w) = M(v, w) \int_{w>0} f(x, y, 1, v', w')|w'|dv'\,dw' \qquad \text{for } w < 0.$$

Before studying the limit as ε goes to zero, we rescale the z variable (so that $z \in (0, 1)$) and the time variable (as in the previous example, we choose a time scale corresponding to time of order ε^{-1}). We thus obtain:

$$\varepsilon \partial_t f + v \cdot \nabla_{x,y} f + \frac{1}{\varepsilon} w \partial_z f = 0, \qquad z \in (0, 1).$$

With this standard diffusion scaling, one can show that $f^\varepsilon \to \rho(t, x)M(v, w)$ where ρ solves

$$\partial_t \rho - \mathrm{div}_{x,y} \Big(D \nabla_{x,y} \rho \Big) = 0.$$

with

$$D = \int \frac{v \otimes v}{w} M(v, w)\,dv\,dw.$$

But we immediately notice that if $M(v, 0) \neq 0$, then this integral has a non-integrable singularity at $w = 0$ and thus $D = \infty$. This was observed for instance by Babovsky, Bardos and Platkowsky [1] who justified rigorously the asymptotic above in the case where $D < \infty$ (for instance by truncating M near $w = 0$). In general, though, this standard diffusion limit fails because of the grazing collision with the plates: particles traveling nearly tangentially to the plates can travel in straight lines for a very long time before hitting the opposite plate and changing direction.

In order to obtain an equation for the limiting density, we thus need to use a different time scale. A result in the general case ($M(v, 0) \neq 0$) was first obtained by Börgers-Greengard-Thomann [9] using a probabilistic approach and then recovered by F. Golse [15] using a purely analytic approach. The main result is that if M is the Maxwellian distribution function, then the appropriate time scale is given by

$$\varepsilon |\ln \varepsilon| \partial_t f + \varepsilon v \cdot \nabla_{x,y} f + \frac{1}{\varepsilon} w \partial_z f = 0.$$

and the limiting density satisfies a diffusion equation:

$$\partial_t \rho - \kappa \Delta_{x,y} \rho = 0$$

with $\kappa < \infty$. In this example we thus obtain a regular diffusion equation, but with an anomalous time scale.

More recently, we proved with Benjamin Texier that if we increase the distribution of grazing particles, for instance by taking

$$M(v, w) = |w|^{-\sigma} \chi_{B_1}(v, w) \qquad \sigma \in (0, 1)$$

then the appropriate scaling is

$$\varepsilon^{2-\sigma} \partial_t f + \varepsilon v \cdot \nabla_{x,y} f + w \partial_z f = 0$$

and the density ρ solves

$$\partial_t \rho + \kappa(-\Delta)^{1-\sigma/2} \rho = 0.$$

where the fractional laplacian $(-\Delta)^s$ can be defined, for instance, by

$$\mathcal{F}((-\Delta)^s f)(\xi) = |\xi|^{2s}\mathcal{F}(f)(\xi).$$

This is a first simple example in which the standard diffusion fails and a fractional diffusion equation is obtained instead. We can interpret the difference between standard diffusion and anomalous diffusion in terms of the trajectories of the particles. When a particle is re-emitted by the plate with near tangential velocity, it will travel for a long distance in a straight line before hitting the opposite plate and change direction again. Because of these long jumps, the trajectories approach a Lévy flight rather than a Brownian motion.

2 Anomalous diffusion limit for the linear Boltzmann equation

In this second part of the talk, we consider the linear Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f)$$

where Q is a linear integral operator of the form:

$$\begin{aligned} Q(f) &= \int_{\mathbb{R}^N} \Big[\sigma(v,v')f(v') - \sigma(v',v)f(v)\Big]\, dv' \\ &= K(f) - \nu(v)f \end{aligned}$$

with $\nu(v) = \displaystyle\int_{\mathbb{R}^N} \sigma(v',v)\, dv'$ (called the collision frequency).

The properties of such an operator Q are classical. In particular,

- Q is conservative:
$$\int_{\mathbb{R}^N} Q(f)\, dv = 0 \qquad \text{for all } f$$
- Under reasonable assumptions on σ, it can be shown that there exists an **equilibrium distribution function** $F(v) > 0$ such that $\displaystyle\int_{\mathbb{R}^N} F(v)\, dv = 1$ and
$$\ker(Q) = \text{Span}\,\{F(v)\}.$$
 It is often assumed that $F(v) = M(v)$, but we will be mostly interested in situation where this is not the case. We will however always assume that $\int_{\mathbb{R}^N} v\, F(v)\, dv = 0$ (or, in cases where this integral is not well defined, that F is an even function).
- $-Q$ is a non-negative operator in $L^2_{F^{-1}}$:
$$\boxed{-\int Q(f)\, f\, \frac{1}{F}\, dv \geq 0.}$$

Furthermore, Q satisfies important spectral gap properties:

- If $0 < \nu_1 \leq \nu(v) \leq \nu_2$ for all v, then

$$-\int Q(f)\, f\, \frac{dv}{F} \geq c_0 \int |f - \rho F|^2\, \frac{1}{F(v)} dv.$$

- For more general ν, under some assumptions on σ, we have

$$-\int Q(f)\, f\, \frac{dv}{F} \geq c_0 \int |f - \Pi f|^2\, \frac{\nu(v)}{F}\, dv$$

where $\Pi(f) = \dfrac{\displaystyle\int \nu f\, dv}{\displaystyle\int \nu F\, dv} F.$

The second inequality is of course weaker than the first one when the collision frequency ν is degenerate. It implies in particular, that the operator $Q^{\star -1}$ is well defined on $L^2(\frac{F}{\nu}dv)$.

We now, once again, consider the small Knudsen number/long time asymptotic:

$$\varepsilon\, \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon).$$

Then we have the following classical result (see for instance Degond-Goudon-Poupaud '00):

Theorem 2.1. *Under appropriate assumptions on σ and F, the solution $f^\varepsilon(x, v, t)$ converges weakly to $\rho(x, t)F(v)$ where ρ solves*

$$\partial_t \rho - \nabla_x \cdot (D \nabla_x \rho) = 0$$

with

$$D = -\int Q^{\star -1}(v) \otimes v\, F(v)\, dv.$$

The proof of such a result requires in particular that $Q^{\star -1}(v)$ exists, that is $v \in L^2(\nu^{-1} F\, dv)$:

$$\int_{\mathbb{R}^N} |v|^2 \frac{F(v)}{\nu(v)}\, dv < \infty$$

So this diffusion limit breaks down ($D = \infty$), for instance, in the following situations:

- If

$$\frac{F(v)}{\nu(v)} \sim \frac{1}{|v|^{N+\alpha}} \quad \text{as } |v| \to \infty$$

  with $\alpha < 2$ (in this case the integral above diverges for the large $|v|$).

- If $F(v) = M(v)$ is a Maxwellian distribution (or another nice function) and

$$\nu(v) \sim |v|^{N+\beta} \quad \text{as } |v| \to 0$$

  with $\beta > 2$ (in this case the integral above diverges because of the small $|v|$).

In the next sections, we explore those two situations

2.1 Heavy-tail distribution functions

First, we assume that the operator Q is such that

- the thermodynamical equilibrium $F(v)$ satisfies

$$F(v) \sim \frac{\kappa_0}{|v|^{\alpha+N}} \qquad \text{as } |v| \to \infty$$

  for some $\alpha > 0$.

- the collision frequency satisfies

$$\nu(v) \sim \nu_0 |v|^{\beta} \qquad \text{as } |v| \to \infty$$

  for some $\beta < \alpha$.

We recall that if $\beta > 2 - \alpha$, then

$$\int |v|^2 \frac{F(v)}{\nu(v)}\, dv < \infty$$

and the usual diffusion limit of Theorem 2.1 holds with the scaling

$$\varepsilon^2\, \partial_t f^\varepsilon + \varepsilon\, v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon)$$

When this fails, we have instead the following result:

Theorem 2.2 (Mellet-Mischler-Mouhot [29]). *Assume $\alpha > 0$ and $\beta < \min\{\alpha; 2-\alpha\}$ and define*

$$\gamma := \frac{\alpha - \beta}{1 - \beta} \in (0, 2).$$

Then the solution f^ε of

$$\varepsilon^\gamma\, \partial_t f^\varepsilon + \varepsilon\, v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon)$$

converges weakly to $\rho(t,x)\, F(v)$ with

$$\begin{aligned} &\partial_t \rho + \kappa\, (-\Delta_x)^{\gamma/2} \rho = 0 \qquad && \textit{in } (0,\infty) \times \mathbb{R}^N, \\ &\rho(0,.) = \rho_0 && \textit{in } \mathbb{R}^N, \end{aligned}$$

This result was proved in [29] using Laplace-Fourier Transform. The diffusion coefficient κ can be computed explicitly. It only depends on N, α, and $\lim_{|v|\to\infty} |v|^{N+\alpha+\beta} F(v)/\nu(v)$. This is very different from the usual case where the diffusion coefficient involves an integral of some moments of F.

This theorem was later extended using a different method to handle space dependent coefficients in the collision operator:

Theorem 2.3 (Mellet [27]). *Under similar assumptions, but if $\nu = \nu(x,v)$, then a similar result holds, but $(-\Delta_x)^{\gamma/2}$ is replaced by $\mathcal{L}$, elliptic operator of order γ defined by the singular integral:*

$$\mathcal{L}(\rho) = P.V. \int_{\mathbb{R}^N} g(x,y) \frac{\rho(x) - \rho(y)}{|x-y|^{N+\gamma}}\, dy$$

with

$$g(x,y) = \nu_0(x)\nu_0(y) \int_0^\infty z^\gamma e^{-z \int_0^1 \nu_0((1-s)x + sy)\, ds}\, dz.$$

This theorem was proved in [27] using appropriate test functions.

2.2 Degenerate collision frequency

Next, we consider the case where

- $F(v) = M(v)$
- there exists $\beta > 2$ such that

$$\nu(v) \sim |v|^{N+\beta} \quad \text{as } |v| \to 0$$

In that case also, the standard diffusion limit leads to $D = \infty$. We can then prove:

Theorem 2.4 (Ben Abdallah-Mellet-Puel [6]). *Let*

$$\gamma = \frac{\beta + 2N}{\beta + N - 1}.$$

The solution of

$$\varepsilon^\gamma \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon)$$

converges weakly to $\rho(x,t)M(v)$ *where* ρ *solves*

$$\partial_t \rho + \kappa\, (-\Delta_x)^{\gamma/2} \rho = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^N,$$
$$\rho(0,.) = \rho_0 \qquad \text{in } \mathbb{R}^N$$

We note that in this case, the range of power γ is limited to $(1,2)$. As before, κ only depends on β, N and $\lim_{|v|\to 0} |v|^{N+\beta} M(v)/\nu(v)$.

3 Applications

In this section, we will present two applications of these results.

3.1 Application 1: The Vlasov-Lévy-Fokker-Planck equation

The first application comes from Plasma physic. First, we recall that the classical Fokker-Planck operator

$$\mathcal{L}(f) := \Delta_v f + \mathrm{div}(vf)$$

corresponds to Langevin equation

$$\dot{V}_i = -V_i + \text{Brownian white noise}$$

If we consider instead the microscopic equation

$$\dot{V}_i = -V_i + \text{Lévy white noise}$$

then we get the Lévy-Fokker-Planck operator

$$\mathcal{L}^s(f) := -(-\Delta_v)^s f + \mathrm{div}(vf)$$

This operator is sometime introduced in plasma physic, because one can check that the thermodynamical equilibrium function, solution of $\mathcal{L}^s(F) = 0$, is given by

$$F(v) = \mathcal{F}^{-1}(e^{-|\xi|^{2s}})$$

and satisfies in particular

$$F(v) \sim \frac{1}{|v|^{N+2s}} \qquad \text{as } |v| \to \infty.$$

As the classical Fokker-Planck operator, this operator is positive:

$$-\int_{\mathbb{R}^N} \mathcal{L}^s(f) \frac{f}{F} \, dv \geq 0$$

and its spectral gap properties have been studied in particular by Gentil-Imbert [14].

Because of the heavy-tail equilibrium function, we are in the framework discussed previously, where the usual diffusion limit fails due to the large velocity particles. We can however prove:

Theorem 3.1 (Cesbron-Mellet-Trivisa [10])**.** *The solution of*

$$\varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \mathcal{L}^s(f^\varepsilon), \qquad x \in \mathbb{R}^N,\ v \in \mathbb{R}^N,\ t > 0$$

converges weakly in $L^\infty(0, T, L^2_{F^{-1}}(\mathbb{R}^{2N}))$ *to* $\rho(x,t)F(v)$ *with* ρ *solution of*

$$\partial_t \rho + (-\Delta_x)^s \rho = 0 \quad \textit{in } \mathbb{R}^N \times (0, \infty)$$

This result is proved using test functions of the form $\phi^\varepsilon(x, v, t) = \varphi(x - \varepsilon v, t)$.

3.2 Application 2: Heat transport in FPU-β chains

We now turn to the last part of this talk in which we discuss an application of this anomalous diffusion phenomena to the modeling of heat transport in chains of oscillators.

At the microscopic level, a solid crystal is described as a set of atoms that oscillate around given equilibrium positions ($\mathbb{Z}^n$ for simplicity). The interactions between these atoms are described by a Hamiltonian that typically takes into account nearest neighbor interactions and a confining potential:

$$H(p,q) = \sum_{i \in \mathbb{Z}^n} \frac{1}{2} p_i^2 + \sum_{|i-j|=1} V(q_i - q_j) + \sum_{i \in \mathbb{Z}^n} U(q_i)$$

where (q_i, p_i) are the displacement and momentum of atom $i \in \mathbb{Z}^n$. This model leads to a very large system of ODE for $(q_i, p_i)_{i \in \mathbb{Z}^n}$. Importantly, in insulating crystals, heat is transported by the vibrations of the lattice thanks to the coupling potential V.

At the macroscopic level, Fourier's law claims that the heat flux is proportional to the gradient of the temperature:

$$J = -\kappa(T) \nabla_x T$$

The question is thus wether it is possible to derive such a law from the microscopic model via some hydrodynamic scaling limit ($r = \varepsilon q$, $t = \varepsilon^\alpha \tau$).

This is a very difficult and largely open problem. In this talk, we address a related question, by using kinetic theory as an intermediary step. The idea behind this analysis goes back to Debye [12] and Peierls [31]. One possible motivation for the introduction of a kinetic equation in this context is to describe the vibrations of the lattice, responsible for heat transport, as a gas of **interacting phonons** whose distribution function solves the **Boltzmann Phonon Equation**. Spohn [34] made the derivation of this kinetic equation from the hamiltonian dynamic more precise by using the Wigner transform of the displacement field. This equation is reminiscent of the classical Boltzmann equation for gas dynamic, but the velocity variable is replaced by a wave vector $k \in \mathbb{R}/\mathbb{Z}$ and the kinetic energy is replaced by the dispersion relation $\omega(k)$.

Our goal in this final part of this talk will be to derive some kind of Fourier's law from the Boltzmann Phonon Equation in the particular framework of the FPU-β chain. This model got its name from Fermi, Pasta and Ulam who investigated (numerically) the approach to thermal equilibrium for chains of coupled atoms whose dynamic can be described by the simple Hamiltonian:

$$H(p,q) = \sum_{i\in\mathbb{Z}} \frac{1}{2} p_i^2 + \sum_{i\in\mathbb{Z}} V(q_{i+1} - q_i).$$

When V is quadratic ($V(r) = \gamma r^2/2$), we get a linear model which leads to infinite conductivity. So Fermi, Pasta and Ulam considered two cases:

$$V(r) = \gamma \frac{r^2}{2} + \alpha \frac{r^3}{3} \quad \text{(FPU-}\alpha\text{ chain)}$$

and

$$V(r) = \gamma \frac{r^2}{2} + \beta \frac{r^4}{4} \quad \text{(FPU-}\beta\text{ chain)}$$

After the numerical experiments of Fermi-Pasta-Ulam (1955), there were a considerable amount of work devoted to the further study of these chain (see in particular Lepri-Livi-Politi [20], [21], [22], [23], [24], [25]. All these studies points to the divergence of the heat conductivity and anomalous scaling of heat transport.

More recently, these problems have been approached using probabilistic tools (by considering a harmonic potential perturbed by a stochastic noise preserving momentum and energy). See in particular Basile-Bernardin-Olla [3, 4], Basile-Olla-Spohn [5], Jara-Komorowski-Olla [18, 19], Olla [30], Bernardin-Gonçalves-Jara [8].

The kinetic approach, which is at the core of our analysis was first developed by Peierls (1929) and made more precise by Spohn [34] and (in the context of the FPU-β chain) by Lukkarinen-Spohn [26].

The Boltzmann phonon equation [34] reads:

$$\partial_t W(t,x,k) + \omega'(k)\partial_x W(t,x,k) = C(W) \quad (t,x,k) \in (0,\infty)\times\mathbb{R}\times\mathbb{T}$$

where k is the wave-number in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\omega(k)$ is the dispersion relation (determined by the **harmonic** part of the potential) and $C(W)$ is the collision operator (determined by the **anharmonic** perturbation)

The function W is the limit of the rescaled *Wigner transform* of the displacement field q (it can also be interpreted as a density distribution function of phonons). It is worth mentioning that the total energy of the crystal, which in the microscopic model is given by

$$E = \sum_{i\in\mathbb{Z}} \frac{1}{2} p_i^2 + V_h(q)$$

can be expressed, with the Wigner transform as

$$E = \int_{\mathbb{R}} \int_{\mathbb{T}} \omega(k) W(x,k,t)\, dk\, dx.$$

As noted above, the dispersion relation is determined by the **harmonic** part of the potential. When we have

$$V_h(q) = \frac{1}{2}\sum_{i\in\mathbb{Z}}(q_{i+1} - q_i)^2$$

then we get

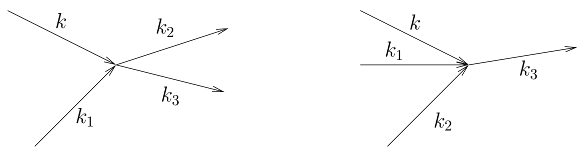
$$\omega(k) = |\sin(\pi k)|.$$

Note in particular that $\omega(0) = 0$.

The collision operator is determined by the anharmnonic potential. For the Quartic potential (FPU-β chain) we have (see Spohn [34]):

$$\begin{aligned} C(W) = \sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} \int dk_1 \int dk_2 \int dk_3 F(k, k_1, k_2, k_3)^2 \\ \times \delta(k + \sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 k_3)\delta(\omega + \sigma_1\omega_1 + \sigma_2\omega_2 + \sigma_3\omega_3) \\ \times [W_1 W_2 W_3 + W(\sigma_1 W_2 W_3 + W_1\sigma_2 W_3 + W_1 W_2\sigma_3)] \end{aligned}$$

Borrowing to our usual understanding of the Boltzmann operator for gas dynamics, we can think of the various terms in the operator as describing different type of interactions between phonons:



However, we note that the number of phonon is not a priori preserved during these interactions (splitting and merging of the phonons is possible). As in the Boltzmann operator, The Dirac masses account for the preservation of momentum (mod $\mathbb{Z}$) and energy. For the first type of interaction (two phonons $\to$ two phonons), we must have $k_3 = k + k_1 - k_2$ and

$$\omega(k) + \omega(k_1) = \omega(k_2) + \omega(k + k_1 - k_2)$$

while for the second type of interactions (three phonons $\to$ one phonons), we must have $k_3 = k + k_1 + k_2$ and

$$\omega(k) + \omega(k_1) + \omega(k_2) = \omega(k + k_1 + k_2).$$

However, this second equation has only trivial solutions, so there are no contributions of three phonons merger/split in the operator C. This fact will have very important consequences below. In particular, we note that

- this simplification implies that the "number" of phonons is conserved. This is not however a physical quantity.
- For the cubic potential (FPU-α chain), the corresponding (quadratic) operator vanishes completely (no three phonons interactions).

In view of this simplification, we see that we can rewrite the collision operator as

$$\begin{aligned} C(W) = \int dk_1 \int dk_2 \int dk_3 F(k,k_1,k_2,k_3)^2 \\ \times \delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3) \\ \times [W_1W_2W_3 + WW_2W_3 - WW_1W_3 - WW_1W_2]. \end{aligned}$$

Finally, we note that the collision kernel is given by (for nearest neighbors coupling)

$$F(k,k_1,k_2,k_3)^2 = C_0 \prod_{i=0}^{3} \frac{\sin^2(\pi k_i)}{\omega(k_i)} = C_0 \prod_{i=0}^{3} |\sin(\pi k_i)|.$$

The properties of the operator C are discussed in [34] and [26]. In particular, we have:

- Conserved quantities:

  $$\int_{\mathbb{T}} \omega(k) C(W)\, dk = 0, \qquad \int_{\mathbb{T}} C(W)\, dk = 0$$

  The equation conserves the energy and the "number of phonons" (not physical - due to the lack of three phonons merger/split). Note that the momentum is only conserved modulo integers (umklapp process).
- Entropy inequality

  $$\int_{\mathbb{T}^1} W^{-1} C(W)\, dk \geq 0$$
- Using this entropy inequality, Lukkarinen and Spohn [26] prove that the stationary solutions form a two parameters family of functions:

  $$W_{\alpha,\beta}(k) = \frac{1}{\beta\omega(k)+\alpha},$$

  where the parameter α is due to the symmetry of the operator (lack of three phonons merger/split).

Our main result will be the derivation of a fractional diffusion equation from the linearized Boltzmann phonon equation. More precisely, we will be studying the behavior of small perturbation of a thermodynamical equilibrium

$$W_0(k) = \frac{\overline{T}}{\omega(k)}.$$

We must thus introduce the linearized operator

$$\begin{aligned} L(f) &:= W_0^{-1} DC(W_0)(W_0 f) \\ &= \overline{T}^2 \omega(k) \int\int\int \delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3) \\ &\qquad [\omega_3 f_3 + \omega_2 f_2 - \omega_1 f_1 - \omega f]\, dk_1\, dk_2\, dk_3 \end{aligned}$$

which we can write in the form

$$L(f) = K(f) - Vf$$

where K is an integral operator and V a positive multiplicative constant. The properties of this operator, which we recall below, are investigated in [26]:

- The kernel of L is given by Ker L = span $\{1, \omega^{-1}\}$.
- The operator L satisfies $\int L(f)\, dk = 0$, and $\int \omega^{-1} L(f)\, dk = 0$.
- $L : L^2(\mathbb{T}, V(k)\, dk) \to L^2(\mathbb{T}, V(k)^{-1}\, dk)$ is a bounded self-adjoint operator.
$$L(f) = K(f) - Vf$$
with K compact operator and $V(k) > 0$.
- We have
$$-\int_{\mathbb{T}} L(f) f\, dk \geq 0 \quad \text{(the } L^2 \text{ norm is decreasing).}$$
and
$$-\int_{\mathbb{T}} L(f) f\, dk \geq c_0 \int_{\mathbb{T}} V(k) |f - \Pi(f)|^2\, dk$$
where $\Pi(f)$ is the orthogonal projection of f onto $\ker(L)$.
- and finally (this is perhaps the most important result of [26]), $V(k)$ is degenerate when $k \to 0$:
$$V(k) \sim V_0 |k|^{5/3} \qquad \text{as } k \to 0.$$

So, this is similar to the situation described in the first part of this talk (linear Boltzmann operator with degenerate collision frequency). In particular one can check that the standard diffusion limit for linearized Boltzmann equation would fail. However, L is not a linear Boltzmann equation, but a linearized Boltzmann operator. In particular the operator K does not have a sign (no maximum principle). Furthermore, the kernel of L is too big because it contains the mode ω^{-1}. Formally we thus expect

$$f^\varepsilon(x, k, t) \sim T(x, t) + S(x, t)\omega^{-1}(k).$$

But we also have $\int\int |f^\varepsilon|^2\, dk\, dx < \infty$ (provided the corresponding norm is finite at time $t = 0$) so we expect to find that $S = 0$ since $\omega(k)^{-1}$ is not square integrable (recall that $\omega(k) \sim |k|$). This fact, however, is not easy to establish rigorously (and we will need to get a rate of convergence to zero for this term).

The main theorem of [28] is then:

Theorem 3.2 (Mellet-Merino [28]). *Let f^ε be a solution of*

$$\varepsilon^{8/5} \partial_t f^\varepsilon + \varepsilon \omega'(k) \partial_x f^\varepsilon = L(f^\varepsilon)$$

with initial data $f_0 \in L^2(\mathbb{R} \times \mathbb{T})$. Then, up to a subsequence,

$$f^\varepsilon(t, x, k) \rightharpoonup T(t, x) \qquad L^\infty((0, \infty); L^2(x, k))\text{-weak}$$

where T solves the fractional diffusion equation

$$\partial_t T + \frac{\kappa}{T^{6/5}} (-\Delta_x)^{4/5} T = 0 \qquad \text{in } (0, \infty) \times \mathbb{R}$$

$\kappa \in (0, \infty)$ *with initial condition*

$$T(0, x) = \frac{1}{\langle V \rangle} \int_0^1 V f_0(t, x, k)\, dk.$$

This equation corresponds to an anomalous Fourier's law of order $3/5$:

$$j = -\kappa(\overline{T})\nabla(-\Delta)^{-\frac{1}{5}}T$$

which is consistent with some (but not all) numerical findings (see below for more discussion about this).

Sketch of the proof. We briefly describe the proof:

- It relies on the Laplace-Fourier Transform Method introduced in [29].
- A crucial point is to show that the projection of f^ε onto the singular part of the kernel of L goes to zero fast enough. In fact, we can obtain the following expansion:

  $$f^\varepsilon(t,x,k) = T^\varepsilon(t,x) + \varepsilon^{\frac{3}{5}}S^\varepsilon(t,x)\omega(k)^{-1} + \varepsilon^{\frac{4}{5}}h^\varepsilon(t,x,k)$$

  where T^ε, S^ε and h^ε are bounded in appropriate functional spaces.
- Projecting onto the constant mode of the kernel, we then get the following equation

  $$\partial_t T + \kappa_1(-\Delta)^{4/5}T + \kappa_2(-\Delta)^{1/2}S = 0.$$

  We see that the competition between the smallness ($\varepsilon^{3/5}$) of the S term and its singularity at $k = 0$ leads to a term of order 1 in the T equation.
- In order to get an equation for S, we now project onto the singular mode of the kernel, and we get:

  $$\kappa_2(-\Delta)^{1/2}T + \kappa_3(-\Delta)^{1/5}S = 0.$$

  Note that there is no $\partial_t S$ in this equation. This can be explained by the fact that the quantity S diffuses faster than T, and so at the time scale that we are considering here, it has already reached equilibrium. A similar phenomenon was first observed by S. Hittmeir and S. Merino [17] in the context of hydrodynamic limits for a Linear BGK equation with degenerate collision frequency.
- We can now eliminate S from the first equation using the second one and get the equation for T:

  $$\partial_t T + \kappa(-\Delta)^{4/5}T = 0 \quad \text{with } \kappa = \kappa_1 - \frac{\kappa_2^2}{\kappa_3} > 0$$

  We note in particular that the S term disappeared in the limit but nevertheless had an effect at the macroscopic scale, by reducing the value of the diffusion coefficient. □

In conclusion, we have derived an anomalous Fourier's law from the **linearized** BPE for the FPU-β chain. But it is important to remember that the BPE itself is formally derived as a weak perturbation limit of the Hamiltonian dynamics (for small quartic perturbation of the quadratic potential). There is no reason to believe that the two limits should commute, and it is thus not clear that the scaling of this Fourier law is consistent with the scaling of the microscopic Hamiltonian dynamics. In particular other approaches, that do not rely on the kinetic description, lead to different powers (see Spohn [35], Bernardin-Gonçalves-Jara [8], Jara-Komorowski-Olla [18]). We also point out that in higher dimensions numerical simulations as well as theoretical arguments point to anomalous behavior in dimension 2 and normal diffusion in dimension 3.

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