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# Speed-up of reaction-diffusion fronts by a line of fast diffusion

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## Abstract

In these notes, we discuss a new model, proposed by H. Berestycki, J.-M. Roquejoffre and L. Rossi, to describe biological invasions in the plane when a strong diffusion takes place on a line. This model seems relevant to account for the effects of roads on the spreading of invasive species. In what follows, the diffusion on the line will either be modelled by the Laplacian operator, or the fractional Laplacian of order less than 1. Of interest to us is the asymptotic speed of spreading in the direction of the line, but also in the plane. For low diffusion, the line has no effect, whereas, past a threshold, the line enhances global diffusion in the plane and the propagation is directed by diffusion on the line. When the diffusion is the Laplacian, the global asymptotic speed of spreading on the line grows as the square root of the diffusion. In the other directions, the line of strong diffusion influences the spreading up to a critical angle, from which one recovers the classical spreading velocity. When the diffusion is the fractional Laplacian, the spreading on the line is exponential in time, and propagation in the plane is equivalent to that of a one-dimensional infinite planar front parallel to the line.

## 1 Introduction

The goal of these notes is to present a series of results for the large time behaviour of the following system of partial differential equations, coupling a diffusion equation on the real line to reaction-diffusion in the upper half plane:

$$\begin{cases} \partial_t u + Lu = -\mu u + v, & t > 0, x \in \mathbb{R}, y = 0 \\ \partial_t v - d\Delta v = f(v), & t > 0, x \in \mathbb{R}, y > 0 \\ -d\partial_y v = \mu u - v, & t > 0, x \in \mathbb{R}, y = 0, \end{cases} \quad (1.1)$$

completed with smooth nonnegative compactly supported initial conditions  $v(\cdot, \cdot, 0) = v_0$  and  $u(\cdot, 0) = u_0$ .

## 1.1 Assumptions

The constants  $\mu > 0$  and  $d > 0$  are given and positive. The nonlinear term accounting for the population growth in the field is a logistic type reaction-term. In other words, we will assume  $f$  to be smooth, concave, positive between  $v = 0$  and  $v = 1$ , with  $f(0) = f(1) = 0$ . This implies, in particular,  $f'(1) < 0 < f'(0)$ . In the sequel, only the values of  $v$  between 0 and 1 will be of interest to us. As for the operator  $L$ , we will consider two instances.

1.  $L = -D\partial_{xx}$ , where  $D > 0$  is a given constant. The interesting question will be what happens for large values of  $D$ , but we will see that we can treat the whole range of parameters.
2.  $L = (-\partial_{xx})^\alpha$ , with  $0 < \alpha < 1$ ; in other words the fractional Laplacian of order  $\alpha$ . Let us recall some features of that operator. For all  $u \in C_c^\infty(\mathbb{R}^N)$ , it is given by

$$(-\Delta)^\alpha u(x) = \mathcal{F}^{-1}(|\xi|^{2\alpha} \widehat{u}(\xi)) = \lim_{\epsilon \rightarrow 0} c_{N,\alpha} \int_{|y|>\epsilon} \frac{u(x+y) - u(x)}{|y|^{N+2\alpha}} dy.$$

The heat kernel  $p_\alpha(t, x)$  satisfies

$$\begin{cases} p_\alpha(t, x) = t^{-N/2\alpha} q_\alpha\left(\frac{x}{t^{1/2\alpha}}\right) \\ \lim_{|\eta| \rightarrow \infty} |\eta|^{N+2\alpha} q_\alpha(\eta) = c_{N,\alpha}, \quad [13] \end{cases} \quad (1.2)$$

and we have

$$p_\alpha(t, \cdot)(\xi) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}}), \quad q_\alpha(\xi) = \mathcal{F}^{-1}(e^{-|\xi|^{2\alpha}}).$$

Thus we may retrieve the well-known transition kernels for  $\alpha = 1$  and  $\alpha = \frac{1}{2}$ :

- if  $\alpha = \frac{1}{2}$ ,  $A = (-\Delta)^{1/2}$  and we have (Cauchy kernel)

$$p_{1/2}(t, x) = \frac{\Gamma(\frac{N+1}{2})}{\pi^{(N+1)/2} t^N (1 + \frac{|x|^2}{t^2})^{(N+1)/2}}.$$

- If  $\alpha = 1$ ,  $A = -\Delta$  and  $p_1(t, x) = \frac{1}{(4\pi t)^{N/2}} e^{-|x|^2/4t}$  (Gauss kernel). Notice that this case is a limiting one (the principal value in the definition of  $(-\Delta)^\alpha$  does not make sense anymore) and estimate (1.3) below does not hold.

Actually, the estimate that we will really use is the following, cruder inequality

$$\frac{B}{t^{N/2\alpha} (1 + |t^{-N/2\alpha} x|^{N+2\alpha})} \leq p_\alpha(t, x) \leq \frac{B}{t^{-N/2\alpha} (1 + |t^{-N/2\alpha} x|^{N+2\alpha})}, \quad (1.3)$$

which translates, for large values of  $t$  and  $x$ , into

$$\frac{B^{-1}t}{|x|^{N+2\alpha}} \leq p_\alpha(t, x) \leq \frac{Bt}{|x|^{N+2\alpha}}. \quad (1.4)$$

More examples are available, for instance, in Bony-Courrège-Priouret [14]; this paper (among many other properties) characterises the integral operators satisfying a maximum principle. See also a representation formula in Caffarelli-Silvestre [20], with spectacular applications to the regularity of nonlocal free boundaries - see for instance Caffarelli, Salsa, Silvestre [19] - and a surprising nonlinear comparison principle due to nonlocal diffusion (Constantin-Vicol [21]).

## 1.2 Motivation

System (1.1) was proposed by Berestycki, Roquejoffre and Rossi in [10] to describe biological invasions which are manifestly accelerated by transportation networks. Indeed, it has long been known that fast diffusion on roads can have a driving effect on the spread of epidemics. A classical example is the spread of the “Black death” plague in the middle of the 14th century, considered to be one of the most devastating in human history. This pandemic is known to have spread first along the silk road. After reaching the port of Marseilles, carried by merchant boats from Crimea, it spread northwards in Europe at a fast pace along the commercial roads connecting the cities that had fairs. It then also spread more slowly away from the roads, inland, bringing about a dramatic invasion. See, for instance, the account by [36]. More recently, it has been observed that invasive species such as the Processionary caterpillar of the pine tree in Europe, have been moving faster than anticipated. One plausible explanation is that enough individuals might have been carried on further distances than usual by vehicles travelling on roads going through infested areas. In the same vein, the invasion of the *Aedes albopictus* mosquito (also known as “Asian tiger mosquito”) is a concern of public health in Europe. The invasion by this insect is driven by roads. Rivers may accelerate the spread of plant pathologies. Another example of the effect of lines on propagation in open space comes from the observation of the population of wolves in the Western Canadian Forest. GPS observations reported by McKenzie et al. [34], 2012, suggest that wolves move and concentrate on seismic lines. These are straight lines (with a width of about 5m) used by oil exploration companies for testing of oil reservoirs.

The situation modelled in System (1.1). is that of a single species, which can move in a two-dimensional environment bounded by a line on which fast diffusion takes place, while reproduction and usual diffusion only occur outside this line. For the sake of simplicity, we will refer to the plane as “the field” and the line as “the road”. The density of this population in the field is the function  $v(t, x, y)$ , and the density on the road is  $u(t, x)$ . Exchanges of populations take place between the road and the field. It is assumed that the population in the field is subject to a logistic type of growth resulting in a Fisher-KPP type of reaction term  $f(v)$ . We assume that no such reaction occurs on the road. The diffusion coefficient in the field is represented by  $d$  and the diffusion on the road is represented by the operator  $L$ .

### 1.3 The question

What we will try to understand is how the level lines of the solution  $(u, v)$  to (1.1) spread as  $t \rightarrow +\infty$ . In other words, we look for a function  $R_\xi(t)$  such that, for every direction  $\xi \in S^1$ , and every constant  $c < 1$ ,

- If  $\xi = (\pm 1, 0)$  (behaviour on the road) we have

$$\lim_{t \rightarrow +\infty} \inf_{-R_{(\pm 1, 0)}(ct) \leq x \leq R_{(\pm 1, 0)}(ct)} u(t, x) = 1/\mu, \quad \lim_{t \rightarrow +\infty} \sup_{|x| \geq R_{(\pm 1, 0)}(c^{-1}t)} u(t, x) = 0 \quad (1.5)$$

- If  $\xi_2 > 0$  (in the field)

$$\lim_{t \rightarrow +\infty} \inf_{(x, y) = R_\xi(ct)\xi} v(t, x, y) = 1, \quad \lim_{t \rightarrow +\infty} \sup_{(x, y) = R_\xi(c^{-1}t)\xi} v(t, x, y) = 0. \quad (1.6)$$

One may, at first sight, wonder about the quantities 1 and  $1/\mu$ ; notice that they are global equilibria to (1.1). If (1.5)-(1.6) hold, we will say that the solution spreads like  $R_{(\pm 1, 0)}(t)$  on the road, and spreads like  $R_\xi(t)$  in the field. Note that there might be a discontinuity between the behaviour of  $R_\xi(t)$ ,  $\xi \neq (\pm 1, 0)$  and that of  $R_{(\pm 1, 0)}$ . This will especially be encountered in the case where  $L$  is the fractional Laplacian.

### 1.4 Related works

There have been a considerable number of works on propagation in heterogeneous media; see [5] for an exhaustive bibliography. Let us quote a few important contributions to models of the form

$$u_t - d\Delta u = f(x, u) \quad (1.7)$$

with  $f(x, \cdot)$  of the logistic type.

The unknown  $u(t, x)$  can be viewed as a population density, and the  $x$ -dependence in the function  $f$  account for how favourable to reproduction the environment is. Let us, by the way, point out that the assumptions on  $f$  are important, and that different sets of assumptions will imply different effects. Heuristically, the qualitative behaviour of  $u$  is easily deduced. Assume, to fix ideas, that  $f$  is  $x$ -independent and that  $f(1) = 0$ , so that  $u \equiv 1$  is a global equilibrium for (1.7). Then, the ODE

$$\dot{u} = f(u)$$

will push the solutions to the value 1, irrespective of their initial value, provided that it is nonzero. On the other hand, the heat equation

$$u_t - \Delta u = 0$$

will spread the initial datum and, in particular, will transform a nonzero, nonnegative (but not positive everywhere) initial datum into a positive function (with a possibly smaller maximum). The combination of these two effects will imply the development of a transition zone between the set  $\{u \sim 1\}$  and the set  $\{u \sim 0\}$ , and the question is thus how it will develop.

Model (1.7) has a long history, starting from a seminal work of Kolmogorov, Petrovskii and Piskunov [31], which is, together with a remarkable paper of Fisher [24] the first paper dealing with the issue. It treats the model

$$u_t - du_{xx} = f(u),$$

and proves that a solution of (1.4), starting from the Heaviside function at time  $t = 0$ , will converge for large time to a travelling wave with speed  $c_K$ , with

$$c_K := 2\sqrt{df'(0)},$$

From this work, the velocity  $c_K$  is often called the KPP speed. It turns out that, as we will see in Section 3, there is a shift in time which is nontrivial. Much later, an important set of results was established by Aronson-Weinberger [2], in a more general perspective. It asserts that, when  $f$  is  $x$ -independent in Model (1.7), one may take  $R_\xi(t) = c_K t$ .

If the  $x$ -dependence is periodic, a fundamental paper of Freidlin and Gärtner [27] computes the speed at which the level sets of  $u$  spread in each direction; the end result is a beautiful formula called the Freidlin-Gärtner formula. Due to its importance, several derivations have been given: Weinberger [37] with dynamical systems tools, Berestycki-Hamel-Nadin [6] with PDE tools. If the  $x$ -dependence in  $f$  has no particular structure, many tools have been introduced to study the large time behaviour of the level sets of  $u$ : local spreading velocities (Berestycki-Hamel-Nadireashvili [7], [8]), transition fronts (Berestycki-Hamel [4]), generalised eigenvalues (Berestycki-Rossi [12]) with an application to a sharp description of the one-dimensional situation (Berestycki-Nadin [9]).

System (1.1) presents a novel model of active heterogeneities, which will display new behaviours that we sum up as follows: even if there is no source term on the road, the overall propagation is always enhanced. In the case of a large standard diffusion, linear propagation holds - i.e. the level sets of  $u$  and  $v$  will asymptotically expand as a linear function of  $t$  but the spreading velocity can grow infinitely if the diffusion coefficient tends to infinity. In the case of an integral diffusion, the propagation is exponential in time.

## 1.5 Organisation of the notes

In the short Section 2, we will describe some basic features of the model and, in particular, a comparison principle that will be quite useful to us. In Section 3, we deal with the case where  $L$  is the standard Laplacian, and explain why the spreading velocity can grow indefinitely if the diffusion becomes arbitrarily large. In Section 4, we deal with the case where  $L$  is the fractional Laplacian, and prove exponential propagation in time. Section 5 is devoted to further questions raised by numerical simulations.

## 2 Elementary properties

If  $(u, v)$  is a solution of (1.1) with  $f \equiv 0$ , the quantity  $\|u(t, \cdot)\|_{L^1(\mathbb{R})} + \|v(t, \cdot)\|_{L^1(\mathbb{R} \times \mathbb{R}_+)}$  does not depend on  $t$ . To see this, suppose that  $u$  and  $v$  decay faster than some exponential functions at time  $t = 0$ . Anticipating on the next sections, we assert that this property still holds for  $t > 0$ , owing to parabolic estimates. We can therefore integrate by parts

the first two equations in (1.1) and we find:

$$\begin{aligned} \|u(T, \cdot)\|_{L^1(\mathbb{R})} - \|u(0, \cdot)\|_{L^1(\mathbb{R})} &= \int_0^T \int_{-\infty}^{+\infty} (v(t, x, 0) - \mu u(t, x)) dx dt, \\ \|v(T, \cdot)\|_{L^1(\mathbb{R} \times \mathbb{R}_+)} - \|v(0, \cdot)\|_{L^1(\mathbb{R} \times \mathbb{R}_+)} &= -d \int_0^T \int_{-\infty}^{+\infty} \partial_y v(t, x, 0) dx dt \\ &= \int_0^T \int_{-\infty}^{+\infty} (\mu u(t, x) - v(t, x, 0)) dx dt \\ &= -\|u(T, \cdot)\|_{L^1(\mathbb{R})} + \|u(0, \cdot)\|_{L^1(\mathbb{R})}, \end{aligned}$$

whence the result. So, in biological terms, our model is consistent with the conservation of the total population in the case of zero natality/mortality rate. The exchanges between the line and the open plane exactly compensate each other as is natural.

Another basic result is the Cauchy problem. Here, due to the regularising effects on the road and in the field, existence and uniqueness of a smooth solution is expected. However, this system, due to the 1D-2D coupling, is not standard and we do not know of any precise result of the literature that we could apply to it. The following proposition is proved in [10] when  $L = -D\partial_{xx}$ , and [23] when  $L = (-\partial_{xx})^\alpha$ .

**Proposition 2.1.** *Assume  $u(0, \cdot)$  and  $v(0, \cdot)$  to be compactly supported, nonnegative and continuous. The Cauchy problem for (1.1) admits a unique nonnegative, smooth solution.*

Less expected is that (1.1) admits a comparison principle. Recall that a subsolution (resp. supersolution) is a couple satisfying the system (in the distributional sense) with the  $=$  signs replaced by  $\leq$  (resp.  $\geq$ ) signs, which is also continuous up to  $t = 0$ .

**Proposition 2.2.** *Let  $(\underline{u}, \underline{v})$  and  $(\bar{u}, \bar{v})$  be respectively a subsolution bounded from above and a supersolution bounded from below of (1.1) satisfying  $\underline{u} \leq \bar{u}$  and  $\underline{v} \leq \bar{v}$  at  $t = 0$ . Then  $(\underline{u}, \underline{v}) \leq (\bar{u}, \bar{v})$  for  $t \leq T$ .*

A good way to see why this holds is to notice that the problem has the structure of a monotone system [30]. Such systems have the form

$$u_t - D\Delta u = F(x, u) \tag{2.1}$$

where  $u(t, x)$  has values in  $\mathbb{R}^m$ ,  $D$  is an  $m \times m$  diagonal matrix with positive entries and  $f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  smooth such that  $\partial_i f_j \geq 0$  if  $i \neq j$ . Indeed, extending  $v$  in an even fashion across the  $x$ -axis, system (1.1) reads

$$\begin{cases} \partial_t u + Lu = -\mu u + v, & t > 0, x \in \mathbb{R}, y = 0 \\ \partial_t v - d\Delta v = f(v) + 2d(\mu u - v)\delta_{y=0}, & t > 0, (x, y) \in \mathbb{R}^2 \end{cases}$$

which, if we treat the Dirac mass at  $y = 0$  as a smooth function, is exactly the form (2.1). As a consequence, if  $u_0 \leq 1/\mu$ ,  $v_0 \leq 1$ , this inequality will be preserved through the time-evolution. We will always assume this, without any further mention.

### 3 The case $L = -D\partial_{xx}$

We are going to compare the behaviour of (1.1) to that of the classical homogeneous model in the whole space. Let us recall its main features.

### 3.1 Homogeneous medium

We study

$$u_t - \Delta u = f(u), \quad u(0, \cdot) = u_0 \text{ nonnegative, compactly supported.} \quad (3.1)$$

In accordance with (1.5)-(1.6), for all direction  $\xi$  on the unit sphere, we are looking for a function  $R_\xi(t)$  such that

$$\lim_{t \rightarrow +\infty} \inf_{\{x = \rho\xi, 0 < \rho < R_\xi(ct)\}} u(t, x) > 0, \quad \text{and} \quad \lim_{t \rightarrow +\infty} \sup_{\{x = \rho\xi, \rho > R_\xi(c^{-1}t)\}} u(t, x) < 1.$$

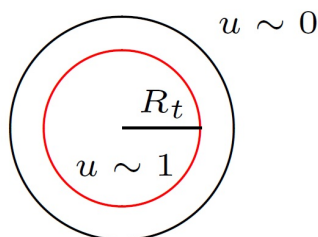


Figure 1: Transition between 0 and 1

**Theorem 3.1** (Aronson-Weinberger, [2]). *Let  $u(t, x)$  be the solution of (3.1). Then:*

- for all  $c > c_K$ , we have  $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u(t, x) = 0$ .
- For all  $c < c_K$ , we have  $\lim_{t \rightarrow +\infty} \inf_{|x| \leq ct} u(t, x) = 1$ .

Let us mention that an earlier version, valid for  $N = 1$ , is proved in [1]). In other words, we may take  $R_\xi(t) := R(t) = c_K t$ . In fact, a more precise (and surprisingly subtle) asymptotics holds: if we take

$$R(t) = c_K t - (N + 2)/c_K \ln(t)$$

then we have

$$0 < \liminf_{|x|=R(t)} u(t, x) \leq \limsup_{|x|=R(t)} u(t, x) < 1.$$

In other words, this last expression locates the transition zone up to  $O(1)$  terms. For  $N = 1$ , this fact is due to Bramson [15] (proof with probabilistic arguments, see a short deterministic proof in [28]), and to Gärtner [26] for  $N \geq 1$ .

PROOF OF THEOREM 3.1. Because it is short, we may give a full account of it.

(i) *Upper bound.* Assume the support of  $u_0$  to be contained in  $B(0, R)$ . Note that  $f(u) \leq f'(0)u$ , so that  $u(t, x) \leq \bar{u}(t, x)$  with

$$(\partial_t - \Delta - f'(0))\bar{u} = 0, \quad \bar{u}(0, x) = \mathbf{1}_{|x| \leq R(x)}.$$



So,

$$\begin{aligned} u(t, x) \leq \bar{u}(t, x) &= e^{f'(0)t} \int_{|x| \leq R} \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{N/2}} dy \\ &\leq e^{f'(0)t} \frac{C_\varepsilon}{(4\pi t)^{N/2}} e^{-(1-\varepsilon)|x|^2/4t} \end{aligned}$$

where  $C_\varepsilon$  is a constant that blows up as  $\varepsilon \rightarrow 0$ . So, in the end we have

$$u(t, x) \leq C_\varepsilon e^{(c_K^2 t - (1-\varepsilon)|x|^2/t)/4}$$

for every  $\varepsilon > 0$ ; consequently, if  $c > c_K$  and  $|x| = ct$ , we have indeed  $\lim_{t \rightarrow +\infty} \bar{u}(t, x) = 0$ .

(ii) *Lower bound.* Consider  $c < c_K$  and  $\delta \in (0, f'(0))$  such that

$$c < 2\sqrt{f'(0) - \delta}. \quad (3.2)$$

Thus the constant coefficient second order ODE

$$-\phi'' + c\phi' - (f'(0) - \delta)\phi = 0 \quad (3.3)$$

has two nonreal exponential solutions  $e^{(c \pm i\omega_\delta)x}$ , with

$$\omega_\delta = \frac{\pi}{2\sqrt{4(f'(0) - \delta) - c^2}}.$$

The real part is  $\phi_\delta(x) = e^{cx} \cos(\omega_\delta x)$ , its graph decays to 0 at  $-\infty$  in an oscillatory fashion. Take one positive arch, in other words consider

$$\underline{\phi}_\delta(x) = \phi_\delta(x) \text{ on } \left[-\frac{\pi}{2\omega_\delta}, \frac{\pi}{2\omega_\delta}\right], \quad \underline{\phi}_\delta(x) = 0 \text{ everywhere else.}$$

It is sufficient to prove that

$$\liminf_{t \rightarrow +\infty} u(t, x_1 - ct, x') = 1, \quad (3.4)$$

where  $x' = (x_2, \dots, x_N)$ . A similar argument will work for any other direction  $e$  of the unit sphere. Now, to prove (3.4), we are going to construct a compactly supported subsolution to the equation

$$-\Delta v + cv_{x_1} = f(v), \quad x \in \mathbb{R}^N. \quad (3.5)$$

Let  $R > 0$  and  $\lambda_1(R)$  be the first eigenvalue of the Dirichlet Laplacian in the ball of  $\mathbb{R}^{N-1}$  with centre 0 and radius  $R$ , we have  $\lim_{R \rightarrow +\infty} \lambda_1(R) = 0$ . Let  $\psi_R(x')$  be any eigenfunction.

We claim that, for  $\varepsilon > 0$  small enough and  $R > 0$  large enough, the function

$$\underline{u}(x) = \varepsilon \psi_R(x') \phi_\delta(x_1)$$

is a subsolution to (3.5); this can readily be checked by computation. Still by reducing  $\varepsilon$  we have  $\underline{u}(x) \leq u(t=1, x)$ , simply because the RHS is positive everywhere by virtue of the strong maximum principle - and  $\underline{u}$  is compactly supported. If  $u_-(t, x)$  is the solution to the Cauchy problem for (3.1) starting from  $\underline{u}$ , we have  $\partial_t u_- > 0$ , so  $u_-(t, \cdot)$  converges for large times, uniformly on compact sets, to a solution of (3.5).

So, it remains to prove that the only bounded nonzero solution of (3.5) is 1. For this we observe that any nonzero nonnegative bounded solution  $v(x)$  of (3.1) has to be bounded away from zero. Suppose indeed that this is not so: this would imply the existence of a contact point between a translate of  $\underline{u}(x)$  and  $v$ , a contradiction. This proves (3.4). •

**Remark 3.2.** Another way to prove Point (i) is to observe that, for any direction  $e$  on the unit sphere, and for any  $c > c_K$  and any  $c' \in (c_K, c)$ , the function

$$\bar{u}_e(x) = e^{rx \cdot e}, \quad r = \frac{c' - \sqrt{c'^2 - 4f'(0)}}{2},$$

is a super-solution to (3.5). Thus, if  $x = cte$  with  $c > c_K$ , we have  $\lim_{t \rightarrow +\infty} u(t, x) = 0$ . If  $e = e_1$ , the function  $u_e$  depends only on  $x_1$  and solves the ODE (3.3) with  $\delta = 0$ . It is sometimes called a plane wave solution to the linearised equation

$$u_t - \Delta u = f'(0)u.$$

This point of view will be useful in the sequel.

### 3.2 Including the line of fast diffusion

Theorem 3.1 is - as already said - not only a reference result. It can also be viewed as a benchmark that will help us to assess how important the role of the heterogeneity is. And, as a matter of fact, this role turns out to be quite important. Let us first recall the system under study:

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\mu u + v, & t > 0, x \in \mathbb{R}, y = 0 \\ \partial_t v - d\Delta v = f(v), & t > 0, (x, y) \in \Omega \\ -\partial_y v = \mu u - v, & t > 0, x \in \mathbb{R}, y = 0, \end{cases} \quad (3.6)$$

where we have denoted the upper half-plane by  $\Omega$ . Let us first explain what happens on the road itself.

**Theorem 3.3** (Berestycki-Roquejoffre-Rossi [10]). *(i). Spreading. There is an asymptotic speed of spreading  $c_* = c_*(\mu, d, D) > 0$  such that the following is true. Let  $(u, v)$  be a solution of (3.6) with a nonnegative, compactly supported initial datum  $(u_0, v_0) \not\equiv (0, 0)$ . Then:*

1. For all  $c > c_*$ , we have  $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} (u(x, t), v(x, y, t)) = (0, 0)$ .
2. For all  $c < c_*$ , we have  $\lim_{t \rightarrow +\infty} \inf_{|x| \leq ct} (u(x, t), v(x, y, t)) = (1/\mu, 1)$ .

*(ii). The spreading velocity. If  $d$  and  $\mu$  are fixed, and  $D$  varies in  $(0, +\infty)$ , the following holds true.*

1. If  $D \leq 2d$ , then  $c_*(\mu, d, D) = c_K$ .
2. If  $D > 2d$ , then  $c_*(\mu, d, D) > c_K$  and  $\lim_{D \rightarrow +\infty} c_*(\mu, d, D)/\sqrt{D}$  exists and is a positive real number.

In other words, in the vocabulary of (1.5)-(1.6), we may take  $R_{(\pm 1, 0)} = c_*(\mu, d, D)t$ . Note that, in the statement of Theorem 3.3, the convergence holds pointwise in  $y$ .

Just as in the proof of Theorem 3.1, one first has to list the possible steady states to (1.1). This is the role of the following - not completely trivial - proposition. We derive a Liouville-type result for stationary solutions of system (1.1).

$$\begin{cases} -DU'' = V(x, 0) - \mu U & x \in \mathbb{R} \\ -d\Delta V = f(V) & (x, y) \in \Omega \\ -d\partial_y V(x, 0) = \mu U(x) - V(x, 0) & x \in \mathbb{R}. \end{cases} \quad (3.7)$$

**Proposition 3.4.** *The unique nonnegative, bounded steady solutions of (1.1) are  $(U, V) \equiv (0, 0)$  and  $(U, V) \equiv (1/\mu, 1)$ .*

This being in hand, the first important step is a plane wave analysis for the linearised problem

$$\begin{cases} \partial_t u - D\partial_{xx}u = v(x, 0, t) - \mu u & x \in \mathbb{R}, t \in \mathbb{R} \\ \partial_t v - d\Delta v = f'(0)v & (x, y) \in \Omega, t \in \mathbb{R} \\ -d\partial_y v(t, x, 0) = \mu u(t, x) - v(t, x, 0) & x \in \mathbb{R}, t \in \mathbb{R}. \end{cases} \quad (3.8)$$

In other words, we look for solutions of the form

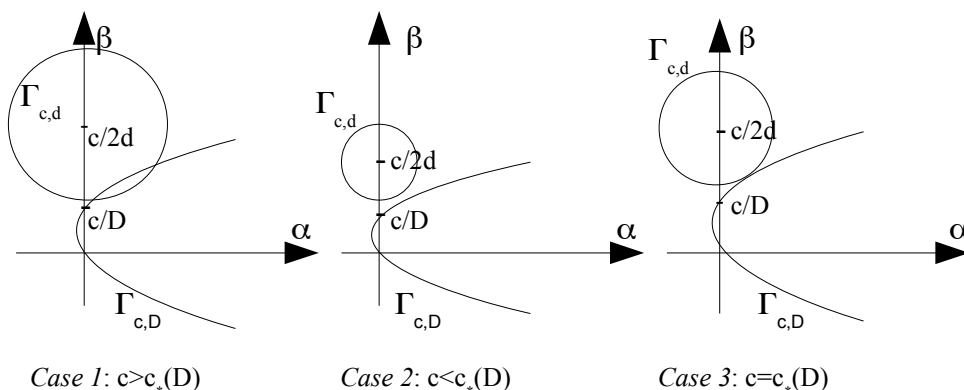
$$(u(t, x), v(t, x, y)) = (e^{\alpha(x+ct)}, \gamma e^{\alpha(x+ct)-\beta y}) \quad (3.9)$$

where  $\alpha$  and  $\gamma$  are positive constants and  $\beta$  is a real (not necessarily positive) constant. The system on  $(\alpha, \beta)$  - the unknown  $\gamma$  being trivially expressed as  $\gamma = \mu/(1 + d\beta)$  - reads

$$\begin{cases} -D\alpha^2 + c\alpha = -\frac{d\beta\mu}{(1 + d\beta)} \\ -d\alpha^2 + c\alpha = f'(0) + d\beta^2. \end{cases} \quad (3.10)$$

Notice that the second equation is void if  $c < c_K$ ; on the other hand, when  $c > c_K$  it simply represents the circle  $\Gamma_{c,d}$  centred at the point  $(0, c/2d)$  with radius  $\sqrt{c^2 - c_K^2}/(2d)$ . The first equation is that of a curve  $\Gamma_{c,D}$ , passing by the origin and the point  $(0, c/D)$ . Let  $\mathcal{G}_{c,D}$  and  $\mathcal{G}_{c,d}$  be the sets defined by the two equations in (3.10) respectively, with  $=$  signs replaced by  $\geq$ . Namely, the set  $\mathcal{G}_{c,d}$  is the closed disc with boundary  $\Gamma_{c,d}$ , while  $\mathcal{G}_{c,D}$  is the closed region bounded by  $\Gamma_{c,D}$  and containing the positive  $\beta$ -axis. This explains the dichotomy between  $D > 2d$  and  $D < 2d$ : in the first case, the centre of  $\Gamma_{c,d}$  is above  $\mathcal{G}_{c,D}$ , whereas, in the second case, it is inside. In other words, if  $D \leq 2d$ ,  $(\bar{u}(t, x), \bar{v}(t, x, y)) := (1, \mu)e^{c_K x}$  is a super-solution to (1.1), which explains why the spreading velocity is at most  $c_K$  in the case. In the case  $D > 2d$ , the geometric construction summarised in the next figure accounts for what is going on.

As for the lower bound, it one again relies on the construction of a compactly supported sub-solution, which is obtained in two steps: first, we bound the space in the  $y$  direction by a fence at which a Dirichlet condition is imposed: this gives a strip of width  $L$ . Second, by a Rouché-type argument we construct complex solutions  $(\phi_\delta(x), \psi_\delta(x, y))$  for  $c$  slightly less than  $c_*(D)$  and  $f'(0)$  replaced by  $f'(0) - \delta$ ,  $\delta > 0$  small. The function  $(\phi_\delta, \psi_\delta)$  is periodic in  $x$  and its positivity set is a periodic copy of bounded connected components. As in the proof of Theorem 3.1, we construct a subsolution  $(\underline{u}(x), \underline{v}(x, y))$  by restricting  $(\phi_\delta, \psi_\delta)$  to one connected component of its positivity set.


 Figure 2: The three cases when  $D > 2d$ 

Turn now to what happens in the field. Because the propagation is linear on the road, we may expect that it will be so in the field, and make (1.5)-(1.6) a little more precise by introducing the following notion. We say that (1.1) admits the *asymptotic expansion shape*  $\mathcal{W}$  if any solution  $(u, v)$  emerging from a compactly supported initial datum  $(u_0, v_0) \not\equiv (0, 0)$  satisfies

$$\forall \varepsilon > 0, \quad \lim_{t \rightarrow +\infty} \sup_{\substack{(x,y) \in \bar{\Omega} \\ \text{dist}(\frac{1}{t}(x,y), \mathcal{W}) > \varepsilon}} v(x, y, t) = 0, \quad (3.11)$$

$$\forall \varepsilon > 0, \quad \lim_{t \rightarrow +\infty} \sup_{\substack{(x,y) \in \bar{\Omega} \\ \text{dist}(\frac{1}{t}(x,y), \bar{\Omega} \setminus \mathcal{W}) > \varepsilon}} |v(x, y, t) - 1| = 0. \quad (3.12)$$

Roughly speaking, this means that the upper level sets of  $v$  look approximately like  $t\mathcal{W}$  for  $t$  large enough. Let us emphasise that the shape  $\mathcal{W}$  does not depend on the particular initial datum. If (1.5)-(1.6) hold with  $R_\xi$  linear, then the asymptotic expansion shape is given by

$$\mathcal{W} = \{(x, y) = \rho\xi : |\xi| = 1, 0 \leq \rho \leq R_\xi(1)\}.$$

In order to avoid that conditions (3.11), (3.12) are vacuously satisfied, as for the set  $\mathcal{W} = \mathbb{Q}^2 \cap \Omega$ , we require that the asymptotic expansion shape coincides with the closure of its interior. This condition automatically implies that the asymptotic expansion shape is unique when it exists. In the sequel, we will sometimes consider the polar coordinate system choosing the angle formed with the vertical axis. Namely, we will write points in the form  $r(\sin \vartheta, \cos \vartheta)$ . Here is the result.

**Theorem 3.5** (Berestycki-Roquejoffre-Rossi [11]). *The following properties hold true:*

1. (Spreading). *Problem (1.1) admits an asymptotic expansion shape  $\mathcal{W}$ .*
2. (Shape of  $\mathcal{W}$ ). *The set  $\mathcal{W}$  is convex and it is of the form*

$$\mathcal{W} = \{r(\sin \vartheta, \cos \vartheta) : -\pi/2 \leq \vartheta \leq \pi/2, 0 \leq r \leq w_*(\vartheta)\},$$

with  $w_* \in C^1([-\pi/2, \pi/2])$ , even and such that

$$w_* = c_K \quad \text{in } [0, \vartheta_0], \quad w'_* > 0 \quad \text{in } (\vartheta_0, \pi/2],$$

for some critical angle  $\vartheta_0 \in (0, \pi/2]$ . Moreover,  $\mathcal{W}$  contains the set

$$\underline{\mathcal{W}} := \text{Conv}((\overline{B}_{c_K} \cap \overline{\Omega}) \cup [-w_*(\pi/2), w_*(\pi/2)] \times \{0\}),$$

and the inclusion is strict if  $D > 2d$ .

3. (Directions with enhanced speed). If  $D \leq 2d$  then  $\vartheta_0 = \pi/2$ . Otherwise, if  $D > 2d$ ,  $\vartheta_0 < \pi/2$  and  $\vartheta_0$  is a strictly decreasing function of  $D$ .

Another way to state the spreading result of Theorem 3.5 is:

$$\lim_{t \rightarrow +\infty} v(ct \sin \vartheta, ct \cos \vartheta, t) = 0 \quad \text{if } c > w_*(\vartheta),$$

$$\lim_{t \rightarrow +\infty} v(ct \sin \vartheta, ct \cos \vartheta, t) = 1 \quad \text{if } 0 \leq c < w_*(\vartheta),$$

uniformly with respect to  $\vartheta \in [-\pi/2, \pi/2]$  and  $|c - w_*(\vartheta)| > \varepsilon$ , for any given  $\varepsilon > 0$ . The quantity  $w_*(\vartheta)$  represents the asymptotic speed of spreading in the direction forming the angle  $\vartheta$  with the vertical axis. Of course,  $w_*(\pm\pi/2)$  coincides with the speed  $c_*$  derived in Theorem 3.3.

If  $D \leq 2d$  then  $\mathcal{W} \equiv \overline{B}_{c_K} \cap \overline{\Omega}$ , that is, the road has no effect on the asymptotic speed of spreading, in any direction. Once again, in the case  $D > 2d$ , the spreading speed is enhanced, but - and this is a remarkable fact - only in some directions. More precisely, there is enhancement in all directions outside a cone around the normal to the road; the closer is the direction to the road, the higher is the speed. The opening  $2\vartheta_0$  of this cone can be characterised explicitly. The case  $D > 2d$  is summarized by Figure 3.

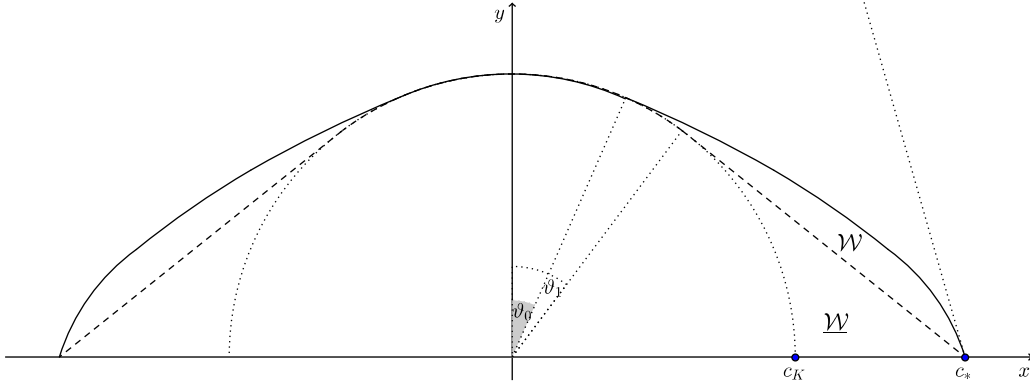


Figure 3: The sets  $\mathcal{W}$  (solid line) and  $\underline{\mathcal{W}}$  (dashed line) in the case  $D > 2d$ .

The set  $\underline{\mathcal{W}}$  has a very natural interpretation as the reachable set by moving with speed  $c_*$  on the road and  $c_K$  in the field. Indeed, the fastest trajectories subject to this constraint on the speed are obtained by moving on the road until time  $\lambda \in [0, 1]$  and then on a straight line in the field for the remaining time  $1 - \lambda$ . It follows that the reachable

set in time  $\leq 1$  starting from the origin is the convex hull of the union of the segment  $[-c_*, c_*] \times \{0\}$  and the half-disc  $\overline{B}_{c_K} \cap \overline{\Omega}$ , that is,  $\underline{\mathcal{W}}$ . It would have been tempting to think that the only influence of the road is through this type of trajectories. The fact that the asymptotic expansion shape is actually larger than this set shows that more subtle mechanisms are at work. The way we interpret it is that the presence of the road is felt even at large distances, through a modification of the tail of the population density. However, the effect of the road is felt only passed the critical angle  $\vartheta_0$ .

Some estimates on the size of  $\mathcal{W}$  can be derived. The inclusion  $\mathcal{W} \supset \underline{\mathcal{W}}$  yields

$$\vartheta_0 < \vartheta_1 := \arcsin \frac{c_K}{c_*},$$

and

$$\forall \vartheta \geq \vartheta_1, \quad w_*(\vartheta) > \frac{c_K c_*}{c_K \sin \vartheta + \sqrt{c_*^2 - c_K^2} \cos \vartheta}.$$

If we consider  $w_*$  and  $c_*$  as functions of  $D$ , with the other parameters frozen, we know from Theorem 3.3,  $c_*(D) \rightarrow \infty$  as  $D \rightarrow \infty$ . Hence, the above inequalities yield

$$\lim_{D \rightarrow \infty} \vartheta_0 = \lim_{D \rightarrow \infty} \vartheta_1 = 0, \quad \forall \vartheta > 0, \quad \liminf_{D \rightarrow \infty} w_*(\vartheta) \geq \frac{c_K}{\cos \vartheta}.$$

Since  $w_*(\vartheta) \leq c_K / \cos \vartheta$ , as it is readily seen by comparison with the tangent line  $y = c_K$ , we have the following

**Proposition 3.6.** *As functions of  $D$ , the quantities  $\vartheta_0$  and  $w_*$  satisfy*

$$\lim_{D \rightarrow \infty} \vartheta_0 = 0, \quad \forall \vartheta \in [-\pi/2, \pi/2], \quad \lim_{D \rightarrow \infty} w_*(\vartheta) = \frac{c_K}{\cos \vartheta}.$$

*That is, as  $D \rightarrow \infty$ , the sets  $\mathcal{W}$  invade the strip  $\mathbb{R} \times [0, c_K)$ .*

Theorem 3.5 is proved along the same lines as Theorem 3.3: we first construct plane waves in each direction  $\vartheta$ , which provides super-solutions. More precisely, take a unit vector  $\xi = (\xi_1, \xi_2)$ , with  $\xi_2 \geq 0$ . We look for exponential solutions of (3.8) moving in the direction  $\xi$  with a given speed  $c \geq 0$ . The solutions are sought for in the form

$$(u(t, x), v(t, x, y)) = (e^{-(\alpha, \beta) \cdot ((x, 0) - ct\xi)}, \gamma e^{-(\alpha, \beta) \cdot ((x, y) - ct\xi)}),$$

with  $c \geq 0$ ,  $\gamma > 0$  and  $\alpha, \beta \in \mathbb{R}$  (not necessarily positive). The velocity  $w_*(\vartheta)$  is the one that signals when this process is not possible anymore. And, at velocities which are just below  $w_*(\vartheta)$ , we construct complex plane waves, and the examination of the positivity sets of their real parts provide the required sub-solutions. Of course perturbation arguments have to be used all along the way. Note that, contrary to the situation of Theorem 3.3, the plane waves have to interact with the road, which creates nontrivial issues.

## 4 The case $L = (-\partial_{xx})^\alpha$

Given the results of the preceding section, it looks fairly obvious that the road will dramatically accelerate the propagation, but the question is how much. So, once again, before studying (1.1) on its own, it is useful to have in mind a 'worst case scenario', in other words what happens when the diffusion is fast everywhere. Having this example in hand, we can compare it with the situation under study.

## 4.1 Homogeneous media with fractional diffusion

A good benchmark is indeed the scalar reaction-diffusion equation in the whole space, that is

$$\begin{aligned} u_t + (-\Delta)^\alpha u &= f(u) \\ u(0, \cdot) &= u_0 \text{ Nonnegative, compactly supported} \end{aligned} \quad (4.1)$$

Here, it has since long been known by physicists, with the aid of formal arguments, that spreading is exponential, with exponent  $f'(0)/(N + 2\alpha)$ . And, indeed, it is not so difficult to believe it: just as in the standard Laplacian case, we have  $u(t, x) \leq \bar{u}(t, x)$  with

$$(\partial_t - \Delta - f'(0))\bar{u} = 0, \quad \bar{u}(0, x) = \mathbf{1}_{|x| \leq R}(x) \quad (4.2)$$

for  $R > 0$  large enough. So, if  $p_\alpha(t, x)$  is the fundamental solution of the fractional heat equation, we have

$$u(t, x) \leq \bar{u}(t, x) = e^{f'(0)t} \int_{|x| \leq R} p_\alpha(t, x - y) dy \leq \frac{Cte^{f'(0)t}}{|x|^{N+2\alpha}}.$$

Consequently, if  $c > f'(0)/(N + 2\alpha)$  and  $|x| = e^{ct}$ , we have indeed

$$\lim_{t \rightarrow +\infty} \bar{u}(t, x) = 0.$$

Then, the heuristic argument is straightforward: since 0 is, in some sense, the most unstable value of the equation  $\dot{u} = f(u)$  in the range  $(0, 1)$ , it is natural that the overall dynamics is driven by what happens at the small values of  $u$ . But, in this range, the equation very much looks like (4.2), hence the exponent  $f'(0)/(N + 2\alpha)$  is sharp. See [32]. And, as a matter of fact, such is the case. Interestingly enough the following result is the first rigorous one, to the best of our knowledge.

**Theorem 4.1** (Cabr e-Roquejoffre, [17], [18]). *Let  $u(t, x)$  be the solution of (4.1). Then:*

- For all  $c > \frac{f'(0)}{N + 2\alpha}$ , we have  $\lim_{t \rightarrow +\infty} \sup_{|x| \geq e^{ct}} u(t, x) = 0$ .
- For all  $c < \frac{f'(0)}{N + 2\alpha}$ , we have  $\lim_{t \rightarrow +\infty} \inf_{|x| \leq e^{ct}} u(t, x) = 1$ .

A more precise version, which is even valid in periodic media, is given in [16]. One could think that this type of exponential spreading is exceptional, in fact this is not the case. Accelerating solutions were, since [17], identified in many instances: integro-differential equations (Garnier [25]), and even equations with standard diffusion like (3.1), but with slowly decaying initial data (Hamel-Roques [29]).

As in the preceding section, let us explain why the lower bound holds. Essentially, we would like to turn the super-solution

$$\bar{u}(t) = e^{f'(0)t} p_\alpha(t, \cdot) * \mathbf{1}_{B_R}$$

into a subsolution, which is of course not true. However, a sequence of truncations/deformations of  $\bar{u}$  turns out to be a sequence of subsolutions, and this is what we choose to explain in the following lemma. It is not the most precise, but it holds when  $L$  is a much more general operator than the fractional Laplacian, the only condition being that it generates a Feller semigroup with estimates (1.4). The reader may check that it also holds in the standard diffusion case, provided that (1.4) is replaced by the standard Gaussian estimates.

**Lemma 4.2.** *Assume (1.3) holds. For every  $0 < \sigma < f'(0)/(N + 2\alpha)$ , there exists  $T_0 \geq 1$  and  $\varepsilon_0 \in (0, 1)$  depending only on  $N, \alpha, B, f'(0)$  and  $\sigma$ , for which the following holds. Given  $r_0 \leq 1$  and  $\varepsilon \leq \varepsilon_0$ , let  $a_0 > 0$  be defined by  $\varepsilon = a_0 |r_0|^{-(N+2\alpha)}$ , and let*

$$v_{r_0}(x) = \begin{cases} a_0 |x|^{-(N+2\alpha)} & \text{for } |x| \geq r_0 \\ \varepsilon & \text{for } |x| \leq r_0. \end{cases}$$

Then, the solution  $v$  of (4.1) with initial condition  $v_{r_0}$  satisfies

$$v(kT_0, x) \geq \varepsilon \quad \text{for } |x| \leq r_0 e^{\sigma k T_0}$$

and  $k \in \{0, 1, 2, 3, \dots\}$ .

PROOF. The lemma being of course true for  $k = 0$ , let us prove it for  $k = 1$ . Consider  $T_0 > 0$ , which will be chosen later depending only on  $n, \alpha, B, f'(0)$  and  $\sigma$ . Let  $\delta \in (0, 1)$  be so small that

$$\sigma < \frac{1}{2} \left( \sigma + \frac{f'(0)}{N + 2\alpha} \right) < \frac{1}{N + 2\alpha} \frac{f(\delta)}{\delta} < \frac{1}{N + 2\alpha} f'(0). \quad (4.3)$$

Define now  $0 < \varepsilon_0 < \delta$  by

$$\varepsilon_0 = \delta e^{-f'(0)T_0}. \quad (4.4)$$

Let  $r_0 \leq 1, \varepsilon \leq \varepsilon_0$  and  $w = e^{(f(\delta)/\delta)t} p_\alpha * v_{r_0}$ . This function satisfies

$$w_t + (-\Delta)^\alpha w = \frac{f(\delta)}{\delta} w, \quad w(0, \cdot) = v_{r_0},$$

and for  $t \leq T_0, 0 \leq w(t, \cdot) \leq e^{(f(\delta)/\delta)t} \varepsilon \leq e^{f'(0)T_0} \varepsilon_0 = \delta$ .

Since  $\delta^{-1} f(\delta) w \leq f(w)$  for  $w \leq \delta$ , we have that  $w$  is a subsolution of (4.1) in  $[0, T_0] \times \mathbb{R}^n$ . Thus,

$$v(T_0, \cdot) \geq w(T_0, \cdot) \geq \tilde{w}(T_0, \cdot) := e^{(f(\delta)/\delta)T_0} B^{-1}(p_\alpha(T_0, \cdot) * v_{r_0}) \quad \text{in } \mathbb{R}^n. \quad (4.5)$$

We have

$$v(T_0, \cdot) \geq w(T_0, \cdot) \geq \tilde{w}(T_0, \cdot) = e^{(f(\delta)/\delta)T_0} B^{-1}(p_\alpha(T_0, \cdot) * v_{r_0})(x) \quad (4.6)$$

$$\geq e^{(f(\delta)/\delta)T_0} c \frac{T_0}{T_0^{1+\frac{N}{2\alpha}} + 1} \frac{a_0}{|x|^{N+2\alpha}} \quad \text{for } |x| < r_0. \quad (4.7)$$

Let us define  $r_1 < 0$  by

$$e^{(f(\delta)/\delta)T_0} c \frac{T_0}{T_0^{1+\frac{N}{2\alpha}} + 1} \frac{a_0}{r_1^{N+2\alpha}} = \varepsilon. \quad (4.8)$$

Since  $a_0 = \varepsilon r_0^{N+2\alpha}$ , we get

$$r_1 = r_0 \left( c \frac{T_0}{T_0^{1+\frac{N}{2\alpha}} + 1} \right)^{1/(N+2\alpha)} e^{\frac{1}{N+2\alpha} \frac{f(\delta)}{\delta} T_0}.$$

Note that for  $T_0$  large, we have



$$\left( c \frac{T_0}{T_0^{1+\frac{N}{2\alpha}} + 1} \right)^{1/(N+2\alpha)} e^{\frac{1}{2}(\sigma + \frac{f'(0)}{N+2\alpha})T_0} \geq e^{\sigma T_0}$$

by the first inequality in (4.3). We choose  $T_0 \geq 1$ , depending only on  $n, \alpha, \beta, f'(0)$  and  $\sigma$ , to satisfy the previous inequality. By the second inequality in (4.3), we have

$$r_1 \leq r_0 e^{\sigma T_0} < r_0. \quad (4.9)$$

Now, since  $r_1 < r_0$ , (4.7) leads to  $v(T_0, x) \geq \tilde{w}(T_0, x) \geq \frac{a_1}{|x|^{N+2\alpha}}$  for  $|x| < r_1$ , and by (4.7)

and (4.8),  $a_1 = \varepsilon r_1^{N+2\alpha}$ . One may easily prove - see for instance Lemma 2.3 in [18] - that  $\tilde{w}$  is radially nondecreasing. So, (4.5) leads to  $v(T_0, x) \geq \tilde{w}(T_0, x) \geq \tilde{w}(T_0, r_1) = \varepsilon$  for  $|x| \leq r_1$ . Thus,  $v(T_0, \cdot) \geq \tilde{v}_{r_0}$  where  $\tilde{v}_{r_0}$  is given by the expression for  $v_{r_0}$  in the statement of the lemma, with  $(r_0, a_0)$  replaced by  $(r_1, a_1)$ . Note that  $r_1 \geq r_0 \geq 1$ .

Thus, we can repeat the argument above, now with initial time  $T_0$ , and get that

$$v(kT_0, x) \geq \varepsilon \quad \text{for} \quad |x| \leq r_k$$

for all  $k \in \{0, 1, 2, 3, \dots\}$ , with, by (4.9),

$$r_k \geq r_0 e^{\sigma(kT_0)}.$$

The statement of the lemma follows from these last two relations. •

## 4.2 Including the line of fast diffusion

Let us recall the system under study:

$$\begin{cases} \partial_t u + (-\partial_{xx})^\alpha u = -\mu u + v, & t > 0, x \in \mathbb{R}, y = 0 \\ \partial_t v - \Delta v = f(v), & t > 0, (x, y) \in \Omega \\ -\partial_y v = \mu u - v, & t > 0, x \in \mathbb{R}, y = 0. \end{cases} \quad (4.10)$$

Before giving the main result, let us recall that the limiting state can be characterised just as in the preceding section. Indeed, we have the following theorem :

**Theorem 4.3.** *Problem (4.10) admits  $(1/\mu, 1)$  as the unique positive bounded stationary solution of (4.10). The solution  $(u, v)$  to (4.10), starting from a nonnegative, compactly supported initial datum  $(u_0, v_0) \neq (0, 0)$ , satisfies*

$$(u(t, x), v(t, x, y)) \xrightarrow[t \rightarrow +\infty]{} (1/\mu, 1)$$

locally uniformly in  $(x, y) \in \mathbb{R} \times \mathbb{R}_+$ .

The issue is thus to track the invasion front, and this is done in the next two theorems, which are our main results on (4.10). The first one accounts for the road.

**Theorem 4.4** (Berestycki, Coulon, Roquejoffre, Rossi [3]). *Let  $(u, v)$  be the solution to (4.10) with  $(u_0, v_0) (\neq (0, 0))$  as nonnegative, compactly supported initial condition and  $\alpha \in (\frac{1}{4}, 1)$ . Set  $c_* := f'(0)/(1 + 2\alpha)$ . Then we have*

1. if  $c < c_*$  we have, pointwise in  $y$ :  $\lim_{t \rightarrow +\infty} \inf_{|x| \leq e^{ct}} (u(t, x), v(t, x, y)) > 0$ ,
2. if  $c > c_*$  we have, still pointwise in  $y$ :  $\lim_{t \rightarrow +\infty} \sup_{|x| \geq e^{ct}} (u(t, x), v(t, x, y)) = 0$ .

With a little more work, the first statement can be replaced by the more precise statement - see for instance [16] -  $\lim_{t \rightarrow +\infty} \inf_{|x| \leq e^{ct}} u(t, x) = 1/\mu$  and  $\lim_{t \rightarrow +\infty} \inf_{|x| \leq e^{ct}} v(t, x, y) = 1$ . We have to worry about what happens in the field.

**Theorem 4.5** ([3]). *Let  $(u, v)$  be the solution to (4.10) with  $(u_0, v_0) (\neq (0, 0))$  as in Theorem 4.4. For all  $\theta \in (0, \pi/2]$ , we have*

1. if  $c > c_K/\sin(\theta)$ ,  $\lim_{t \rightarrow +\infty} \sup_{|r| \geq ct} v(r \cos(\theta), |r| \sin(\theta), t) = 0$ ,
2. if  $0 < c < c_K/\sin(\theta)$ ,  $\lim_{t \rightarrow +\infty} \sup_{|r| \leq ct} v(r \cos(\theta), |r| \sin(\theta), t) > 0$ .

The speed of propagation is thus asymptotically equal to  $c_K/\sin(\theta)$ . When  $\theta$  is close to 0, this speed is infinite, which is consistent with Theorem 4.4. Notice again that the second statement of the theorem can be replaced by  $\lim_{t \rightarrow +\infty} \sup_{|r| \leq ct} v(r \cos(\theta), |r| \sin(\theta), t) = 1$ .

It is to be noticed that Theorem 4.5 is not at all surprising: the time scale of the invasion being so much shorter on the road than in the field, the situation is similar to that of a solution of  $u_t - \Delta u = f(u)$  in  $\Omega$  with the condition  $v(t, x, 0) \equiv 1$ . This is basically a one-dimensional problem in the  $y$  direction, and this is exactly what Theorem 4.5 is saying: the front is asymptotically planar, infinite in the  $x$ -direction. This fact is illustrated in the simulation given by Figure 4, extracted from the second author's PhD thesis [22]. So, everything reduces to proving Theorem 4.4. The upper bound relies on the analysis of the linearised equation:

$$\begin{cases} \partial_t u + (-\partial_{xx})^\alpha u = -\mu u + v, & t > 0, x \in \mathbb{R}, y = 0 \\ \partial_t v - \Delta v = f'(0)v, & t > 0, (x, y) \in \Omega \\ -\partial_y v = \mu u - v, & t > 0, x \in \mathbb{R}, y = 0. \end{cases} \quad (4.11)$$

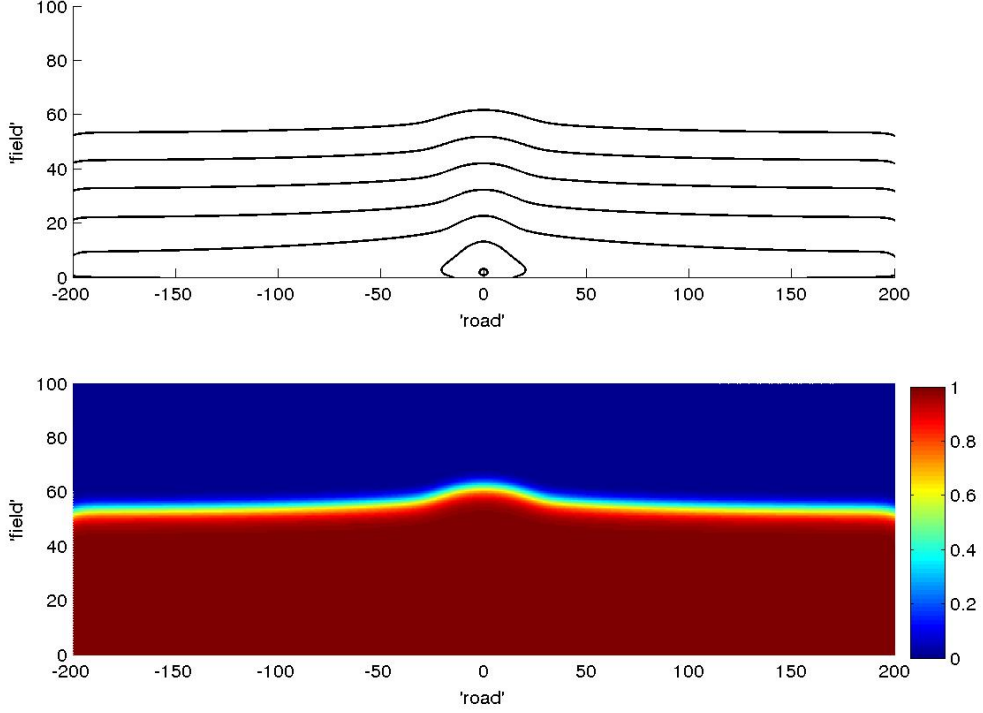
**Theorem 4.6** (Coulon [23]). *Assume  $(u(0, x), v(0, x, y)) = (u_0(x), 0)$  where  $u_0 \not\equiv 0$  is nonnegative, compactly supported. There exists a function  $R(t, x)$  and constants  $\delta > 0$ ,  $C > 0$  such that*

1. we have for  $x \neq 0$

$$\left| u(t, x) - \frac{8\alpha\mu \sin(\alpha\pi)\Gamma(2\alpha)\Gamma(3/2)}{\pi f'(0)^3} \frac{e^{f'(0)t}}{t^{3/2}|x|^{1+2\alpha}} \right| \leq R(t, x),$$

2. and the function  $R(t, x)$  is estimated as

$$0 \leq R(t, x) \leq C \left( e^{-\delta t} + \frac{e^{f'(0)t}}{|x|^{\min(1+4\alpha, 3)}} + \frac{e^{f'(0)t}}{|x|^{1+2\alpha} t^{5/2}} \right).$$


 Figure 4: Level sets of  $v$ 

This is a nontrivial Polya-type computation [35]. Note that, although we are at this stage only interested in what happens on the road, a full computation of the solution of (4.11) has to be carried out.

The lower bound is obtained by exploiting an idea of [16], which was initially devised to locate the level sets of the solutions of the scalar model (4.1) to  $O(1)$  precision. First, instead of considering model (4.10) in the whole half-space, we consider it in the strip  $\Omega_L = \mathbb{R} \times (0, L)$  and impose a Dirichlet condition at  $y = L$ , the idea being to let  $L \rightarrow +\infty$ . Then we construct a sub-solution to (4.10) in  $\mathbb{R} \times \Omega_L$ . Let us first give an idea of what is going on, and for this let us go back to model (4.1) with  $N = 1$ : if we believe that the level sets of the solution travel like  $e^{\lambda t}$ , set  $\xi = e^{\lambda t}x$  and

$$\tilde{u}(t, \xi) = u(t, e^{-\lambda t}\xi).$$

The equation for  $\tilde{u}$  is

$$\tilde{u}_t - \lambda \xi \tilde{u}_\xi + e^{-2\alpha \lambda t} (-\partial_{xx})^\alpha \tilde{u} = f(\tilde{u});$$

if we now ask  $\tilde{u}$  to be a good representation of  $u$  at large times (and so, in particular, to ask that it has bounded gradient and fractional Laplacian) - then it is a good idea to look for  $\tilde{u}$  as a solution to the profile equation

$$-\lambda \xi \tilde{u}_\xi = f(\tilde{u}), \tag{4.12}$$

an equation which has nontrivial solutions decaying like  $|\xi|^{-1/\lambda}$  as  $|\xi| \rightarrow +\infty$ . Now, the parameter  $\lambda$  is chosen to match  $\tilde{u}$  at infinity at  $t = 1$ , for instance; it can be readily

observed - just by convolution with the heat kernel - that

$$\frac{C^{-1}}{1 + |x|^{1+2\alpha}} \leq u(1, x) \leq \frac{C}{1 + |x|^{1+2\alpha}}. \quad (4.13)$$

This imposes  $\lambda = 1/(1 + 2\alpha)$ , and points towards a sub-solution to (4.12):

$$\underline{u}(t, x) = \frac{a}{1 + b(t)|x|^{1+2\alpha}}, \quad b(t) \sim_{t \rightarrow +\infty} e^{-f'(0)t}.$$

The idea proved flexible enough for periodic media, i.e. models of the form

$$u_t + (-\Delta)^\alpha u = \mu(x)u - u^2,$$

yielding results that are in sharp contrast with the standard Laplacian [27].

Now, to transpose this idea to model (4.10) we arrive at the following equation: Let us recall the system under study:

$$\begin{cases} -\lambda \xi u_\xi = -\mu u + v, & (\xi \in \mathbb{R}, y = 0) \\ -\lambda \xi v_\xi - v_{yy} = f(v), & (\xi, y) \in \Omega \\ -\partial_y v = \mu u - v, & \xi \in \mathbb{R}, y = 0. \end{cases} \quad (4.14)$$

And the change of variable  $\xi = e^{-\lambda\tau}$  reduces (4.14) to

$$\begin{cases} u_\tau = -\mu u + v, & (\tau \in \mathbb{R}, y = 0) \\ v_\tau - v_{yy} = f(v), & (\xi, y) \in \Omega \\ -\partial_y v = \mu u - v, & \xi \in \mathbb{R}, y = 0, \end{cases} \quad (4.15)$$

and this amounts to finding a time-global solution  $(u(\tau), v(\tau, y))$  of (4.15). Existence theorems can for instance be found in Matano [33], which is a good indication that this idea has some substance. However we need a more explicit solution, in order to control the remainders. So, we construct our sub-solution in two steps.

In what follows, for  $\lambda \in \mathbb{R}_+$  and  $x \neq 0$ , we define  $v_\lambda(x) := |x|^{-\lambda}$ .

**Lemma 4.7.** *Let  $g$  be a nonnegative function of class  $C^\infty(\mathbb{R})$ , with  $g'(0) > 0$ , and  $\sigma$  a positive constant. There exists a constant  $\tilde{\gamma} = g'(0)\sigma^{-1}$  such that for all  $\gamma \in [0, \tilde{\gamma}]$  the equation*

$$-\gamma x \psi'(x) = g(\psi(x)), \quad x \in \mathbb{R}, \quad (4.16)$$

*admits a subsolution  $\phi$  of class  $C^2(\mathbb{R})$ , smaller than 1, with the prescribed decay  $|x|^{-\sigma}$ , for large value of  $|x|$ . More precisely, there exists constants  $\beta > 0$ ,  $A_1 > 0$ ,  $A_2 > 0$ ,  $\varepsilon > 0$  and a constant  $D > 0$  depending on  $A_2$ ,  $\sigma$  and  $\varepsilon$  such that*

– for all  $|x| \geq A_2$ ,

$$\begin{aligned} -\gamma x \phi'(x) - g(\phi(x)) &\leq -\beta v_{\sigma+\varepsilon}(x), & -\phi''(x) &\leq D v_{\sigma+\varepsilon}(x), \\ (-\partial_{xx})^\alpha \phi(x) &\leq D \phi(x) \end{aligned} \quad (4.17)$$

– for all  $|x| \in (A_1, A_2)$ , the function  $x \mapsto -x\phi'(x)$  is smaller than  $\sigma A_2^{-\sigma}$  nondecreasing in  $|x|$  and thus

$$-\gamma x\phi'(x) - g(\phi(x)) \leq -\beta v_{\sigma+\varepsilon}(A_2). \quad (4.18)$$

– for all  $|x| \leq A_1$ ,  $\phi(x) = \phi(A_1)$ .

Choose now  $L$  such that

$$L > \max \left( 2, \pi \left( \frac{f'(0)}{1+2\alpha} - \gamma \right)^{-1/2} \right). \quad (4.19)$$

We want our subsolution to have the algebraic decay  $|\xi|^{-(1+2\alpha)}$  for large values of  $|\xi|$ . Since  $L > \pi f'(0)^{-1/2}$ , we apply Lemma 4.7 with

$$g(s) = f(s) - \left( \frac{\pi}{L} \right)^2 s \quad \text{and} \quad \sigma = 1 + 2\alpha. \quad (4.20)$$

Let us define

$$\underline{V}(\xi, y) = \begin{cases} \phi(\xi) \sin \left( \frac{\pi}{L} y + h \right) & \text{if } 0 \leq y < L \left( 1 - \frac{h}{\pi} \right) \\ 0 & \text{if } y \geq L \left( 1 - \frac{h}{\pi} \right) \end{cases} \quad \text{and} \quad \underline{U}(\xi) = c_h \phi(\xi), \quad (4.21)$$

where

$$h \in \left( 0, \arctan \left( \frac{\pi}{L} \right) \right) \quad \text{and} \quad c_h = \min \left( \frac{\sin(h)}{2(\tilde{\gamma}\sigma + \mu + k)}, \frac{\sin(h)\phi(A_2)A_2^\sigma}{4\gamma\sigma} \right), \quad (4.22)$$

and  $\phi, A_1, A_2, \tilde{\gamma}$  are given by Lemma 4.7.

**Lemma 4.8.**  $(\underline{V}(\xi, y), \underline{U}(\xi))$  is a subsolution to (4.15).

Carefully adjusting the constants by comparison with  $(u(t=1, \cdot), v(t=1, \cdot, \cdot))$ , then letting  $L \rightarrow +\infty$ , end the proof of Theorem 4.5.

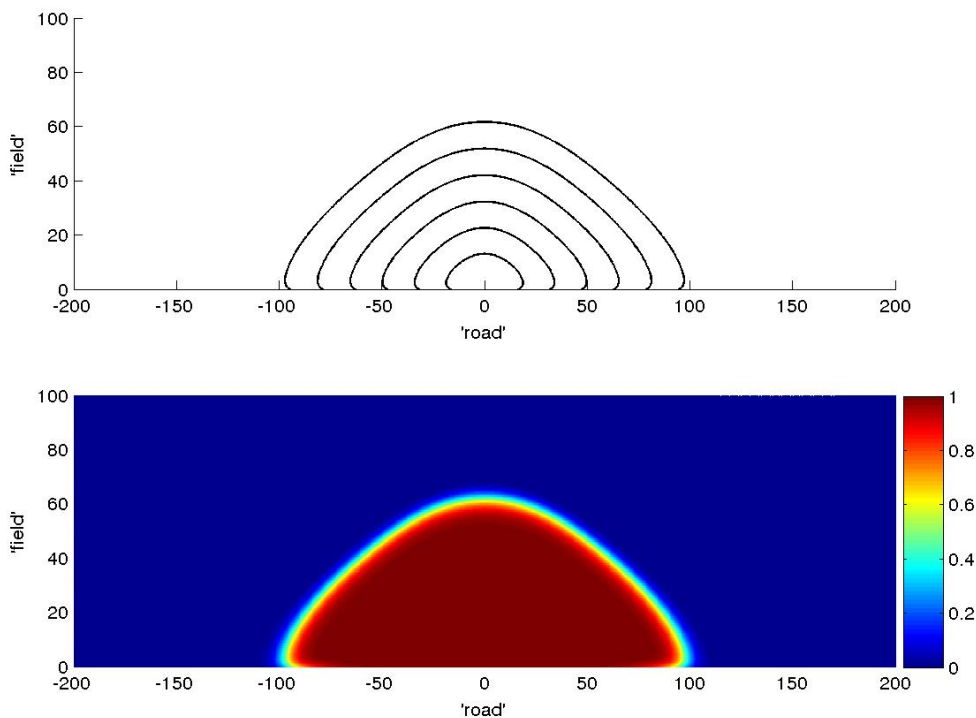
## 5 Further questions

It would be quite interesting to know more qualitative properties of the solutions, as well as more precise asymptotics. The second author carried out, in her PhD thesis [22], numerical simulations that are reported here.

### 5.1 When $L = -D\partial_{xx}$ : refined description of the expansion set

In polar coordinates, the asymptotic expansion set is given by the interior of a curve  $\vartheta \mapsto w_*(\vartheta)$ , the function  $w_*$  being an increasing function of its argument. However, the asymptotic expansion set is only defined up to  $o(t)$  terms, which can cause some perturbations at smaller scales. Let us look at the following first simulation.

The figure gives the shape of the level sets of value 0, 5 of  $v$ , solution to (3.6), for  $d = 1$  and  $D = 10$ , at successive times  $t = 10, 15, \dots, 35$ , and the display of the density  $v$  in the field at time  $t = 35$ . The level sets displayed on this figure are even and decreasing

Figure 5: Level sets of  $v$ 

in  $|x|$  functions. A first check of validity is the speed of propagation in the direction normal to the road, and it corresponds indeed to the standard KPP velocity. Looking it more closely, this reveals that the level sets seem to be circular in a sector whose axis is normal to the road. The shape of the level sets we obtained corresponds to the set  $\mathcal{W}$  of Section 3. The shape of the level sets of  $v$  is almost similar to the one described of Theorem 3.5. There is, however, a particular phenomenon in a neighbourhood of the road. Indeed, for  $y \in [0, Y_D(t)]$  where  $Y_D$  is a function that may depend on time and the diffusion coefficient  $D$ ,  $\partial_y v$  seems to be positive. At first sight, this is surprising, even though not incompatible with Theorem 3.5. This calls for more simulations.

The figure shows, in particular, the tangent lines to the level set of value 0,5 of  $v$ , at  $y = 0$ . The angle between the tangent and the normal to the road is equal to 77.4 degrees for  $D = 50$ , 83.1 degrees for  $D = 100$ , 86.6 degrees for  $D = 500$  and 87 degrees for  $D = 1000$ . This slope therefore seems to decrease as  $D$  tends to infinity. Let us tempt a heuristic explanation: for large  $D$ , the overall dynamics is driven by the road. And so, the term  $v - \mu u$  in the equation for  $u$  in (3.6) should act as a source term, hence should be positive. This implies  $v_y > 0$ . Of course this is only a heuristic explanation, but it is substantiated by Figure 7. Notice a striking analogy with a positive reaction term, as, for instance, flame propagation theory (see [38]). A mathematical proof of that fact, at least for very large values of  $D$ , should be investigated.

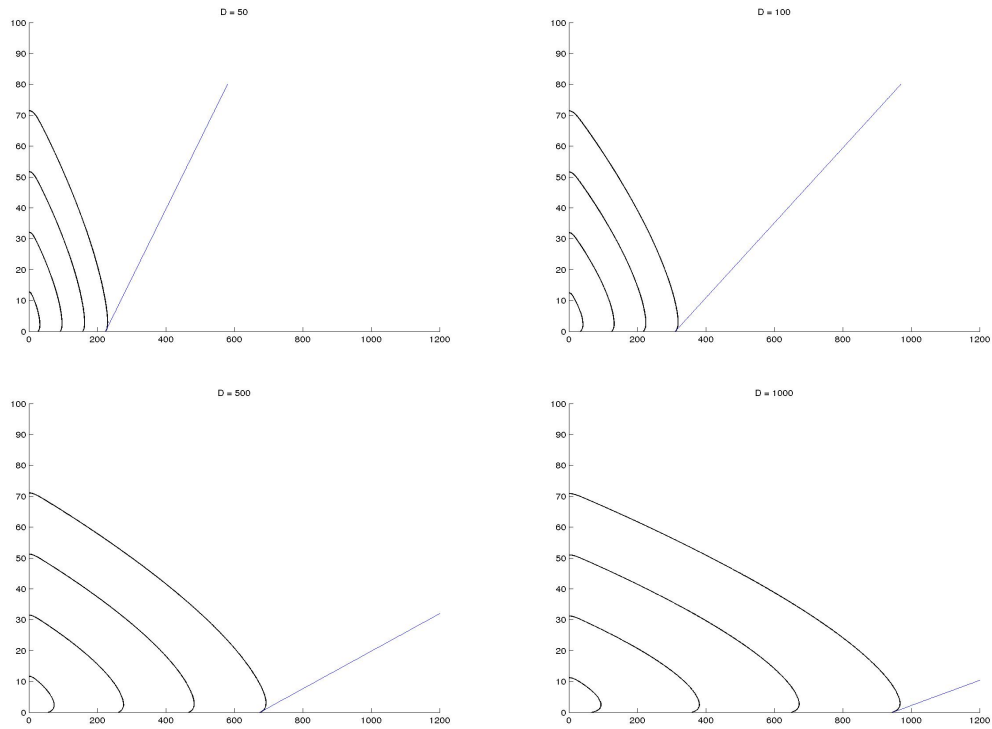


Figure 6: Level sets of value 0,5 of  $v$  at successive times  $t = 10, 20, 30, 40$  and the tangent line to the level set at  $y = 0$  and at time  $t = 40$  for the values  $D = 50, D = 100, D = 500$  and  $D = 1000$  (from left to right). The  $x$  axis and  $y$  axis do not have the same scale.

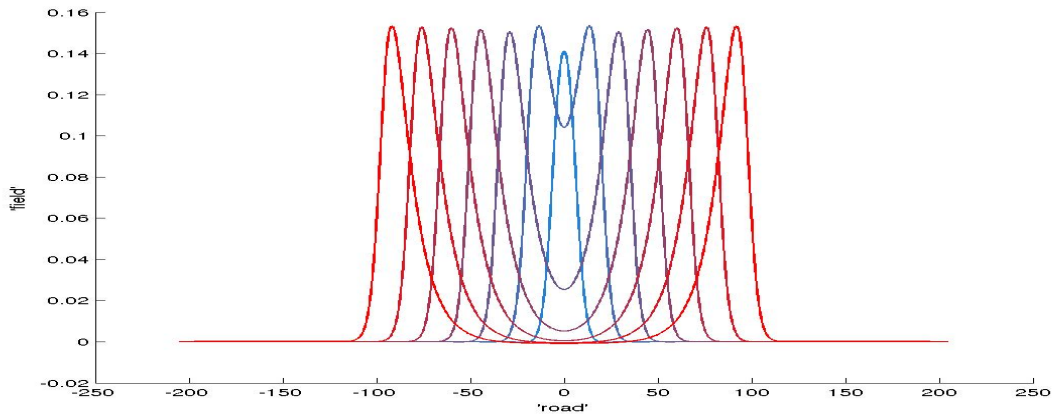


Figure 7: Display of  $v - \mu u$  for  $D = 10$ , at successive times  $t = 5, 15, \dots, 35$ , with a colour graduation from blue to red.

## 5.2 When $L = (-\partial_{xx})^\alpha$ : sharp location of the level sets on the road

The preceding section shows that, on the road, the location of the level set  $\{u(t, x) = 1/2\}$  cannot be precisely  $e^{f'(0)t/(1+2\alpha)}$ . Indeed, it moves slower than that of the linear system (4.11), which moves like  $e^{f'(0)t/(1+2\alpha)}/t^{3/(2(1+2\alpha))}$ . This raises the question of whether this is the correct asymptotics, because such is not - as already mentioned in Section 3 - the case in general. This is investigated in the following simulation that concerns the rescaled problem satisfied by  $\tilde{v}$  and  $\tilde{u}$  defined on  $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ , by

$$\tilde{v}(\tilde{x}, y, t) = v(e^{lt}t^{-m}\tilde{x}, y, t) \quad \text{and} \quad \tilde{u}(\tilde{x}, t) = u(e^{lt}t^{-m}\tilde{x}, t),$$

with  $l = \frac{1}{1+2\alpha}$  and  $m \geq 0$  the constant that we want to study.

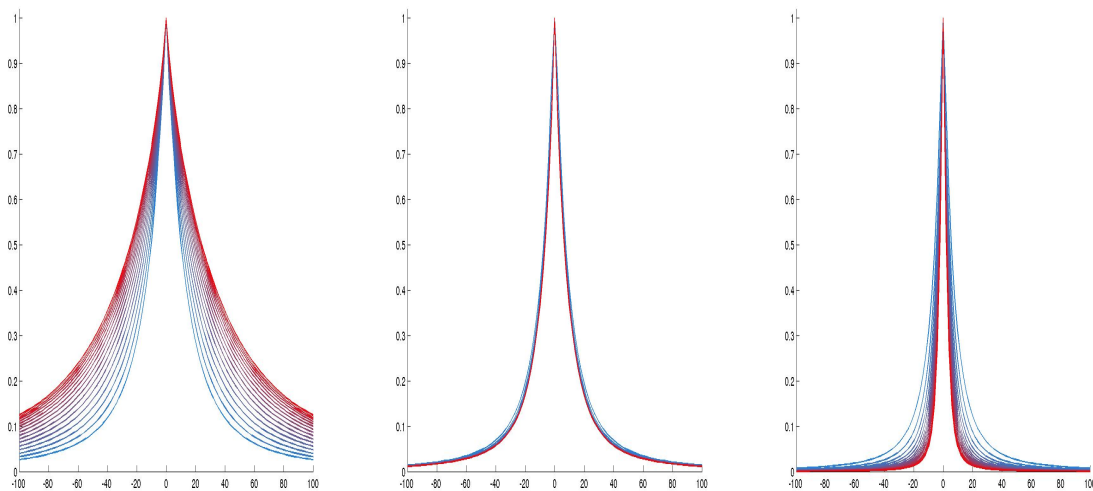


Figure 8: Evolution of the density  $\tilde{u}$  with  $\alpha = 0,5$ , for  $m = 0$  (on the left),  $m = \frac{3}{2(1+2\alpha)}$  (in the center) and  $m = \frac{3}{1+2\alpha}$  (on the right), at successive times  $t = 30, 40, 50, \dots, 200$  with a colour graduation from blue to red.

The left side of Figure 8, that concerns  $m = 0$ , shows that the level sets move faster than  $e^{t/(1+2\alpha)}$ , whereas the right side, that concerns  $m = \frac{3}{1+2\alpha}$ , shows that the level sets move slower than  $t^{-3/(1+2\alpha)}e^{t/(1+2\alpha)}$ . The center of Figure 8 concerns the particular choice  $m = \frac{3}{2(1+2\alpha)}$ , suggested by the upper bound of Theorem 4.6. On compact sets, the rescaled density  $\tilde{u}$  seems to converge to a function that does not move in time.

## References

- [1] D. G. Aronson and H. F. Weinberger. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In *Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974)*, volume 446, pages 5–49. Springer, Berlin, 1975.
- [2] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.*, 30(1):33–76, 1978.



- [3] H. Berestycki, A.-C. Coulon, J.-M. Roquejoffre, and L. Rossi. Exponential front propagation in a Fisher-KPP model driven by a line of integral diffusion. *in preparation*, 2014.
- [4] H. Berestycki and F. Hamel. Generalized transition waves and their properties. *Comm. Pure Appl. Math.*, 65:592–648, 2012.
- [5] H. Berestycki and F. Hamel. *Reaction-diffusion equations and propagation phenomena*. Applied Mathematical Sciences, 2014.
- [6] H. Berestycki, F. Hamel, and N. Nadin. Asymptotic spreading in heterogeneous diffusive excitable media. *J. Funct. Analysis*, 255:2146–2189, 2008.
- [7] H. Berestycki, F. Hamel, and N. Nadirashvili. The speed of propagation for KPP type problems. I. Periodic framework. *J. European Math. Soc.*, 7:173–213, 2005.
- [8] H. Berestycki, F. Hamel, and N. Nadirashvili. The speed of propagation for KPP type problems. II. General domains. *J. Amer. Math. Soc.*, 23(1):1–34, 2010.
- [9] H. Berestycki and N. Nadin. Spreading speeds for one-dimensional monostable reaction-diffusion equations. *J. Math. Phys.*, 53(11), 2012.
- [10] H. Berestycki, J.-M. Roquejoffre, and L. Rossi. The influence of a line with fast diffusion on Fisher-KPP propagation. *J. Math. Biol.*, 66(4-5):743–766, 2013.
- [11] H. Berestycki, J.-M. Roquejoffre, and L. Rossi. The shape of expansion induced by a line with fast diffusion in Fisher-KPP propagation. [arXiv:1402.1441](https://arxiv.org/abs/1402.1441), 2014.
- [12] H. Berestycki and L. Rossi. Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains. *Comm. Pure Appl. Math.*, 2014.
- [13] R. M. Blumenthal and R. K. Gettoor. Some theorems on stable processes. *Trans. Amer. Math. Soc.*, 95:263–273, 1960.
- [14] J.-M. Bony, P. Courrège, and P. Priouret. Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 2):369–521 (1969), 1968.
- [15] M. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. *Memoirs of the Amer. Math. Soc.*, 44, 1983.
- [16] X. Cabré, A.-C. Coulon, and J.-M. Roquejoffre. Propagation in Fisher-KPP type equations with fractional diffusion in periodic media. *C. R. Math. Acad. Sci. Paris*, 350(19-20):885–890, 2012.
- [17] X. Cabré and J.-M. Roquejoffre. Front propagation in Fisher-KPP equations with fractional diffusion. *C. R. Math. Acad. Sci. Paris*, 347:1361–1366, 2009.
- [18] X. Cabré and J.-M. Roquejoffre. The influence of fractional diffusion in Fisher-KPP equations. *Comm. Math. Phys.*, 320(3):679–722, 2013.
- [19] L. Caffarelli, S. Salsa, and L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.*, 171(2):425–461, 2008.
- [20] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.
- [21] P. Constantin and V. Vicol. Nonlinear maximum principles for dissipative linear nonlocal operators and applications. *Geom. Funct. Anal.*, 22:1289–1321, 2012.
- [22] A.-C. Coulon. *Fast front propagation in reaction-diffusion models with integral diffusion*. Toulouse University PhD thesis, 2014.
- [23] A.-C. Coulon. The heat kernel in a parabolic model with a line of fast diffusion. *in preparation*, 2014.
- [24] R.A. Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7:355–369, 1937.
- [25] J. Garnier. Accelerating solutions in integro-differential equations. *SIAM J. Math. Anal.*, 43(4):1955–1974, 2011.

- [26] J. Gartner. Location of wave fronts for the multi-dimensional KPP equation and Brownian first exit densities. *Math. Nachr.*, 105:317–351, 1982.
- [27] J. Gartner and M. I. Freidlin. The propagation of concentration waves in periodic and random media. *Dokl. Akad. Nauk SSSR*, 249:521–525, 1979.
- [28] F. Hamel, J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. A short proof of the logarithmic Bramson correction in Fisher-KPP equations. *Netw. Heterog. Media*, pages 275–289, 2013.
- [29] F. Hamel and L. Roques. Fast propagation for KPP equations with slowly decaying initial conditions. *J. Differential Equations*, 249(7):1726–1745, 2010.
- [30] M.V. Hirsch. Stability and convergence in strongly monotone dynamical systems. *J. Reine Angew. Math.*, 383:1–53, 1988.
- [31] A.N. Kolmogorov, I.G. Petrovskii, and N.S. Piskunov. Etude de l'équation de diffusion avec accroissement de la quantité de matière, et son application à un problème biologique. *Bjul. Moskovskogo Gos. Univ.*, 17:1–26, 1937.
- [32] R. Mancinelli, D. Vergni, and A. Vulpiani. Front propagation in reactive systems with anomalous diffusion. *Phys. D*, 185(3-4):175–195, 2003.
- [33] H. Matano. Existence of nontrivial unstable sets for equilibria of strongly order preserving systems. *J. Fac. Sci. Tokyo*, 1984.
- [34] H.W. McKenzie, E.H. Merrill, R.J. Spiteri, and M.A. Lewis. How linear features alter predator movement and the functional response. *Interface focus*, 2(2):205–216, 2012.
- [35] G. Polya. On the zeros of an integral function represented by Fourier's integral. *Messenger of Math.*, 52:185–188, 1923.
- [36] A. Siegfried. *Itinéraires des contagions, épidémies et idéologies*. A. Colin, Paris, 1960.
- [37] H.F. Weinberger. On spreading speeds and traveling waves for growth and migration in periodic habitat. *J. Math. Biol.*, 45:511–548, 2002.
- [38] F.A. Williams. *Combustion theory*. Benjamin Cummings, 1985.