Séminaire Laurent Schwartz

EDP et applications

Année 2013-2014

Zaher Hani, Benoit Pausader, Nikolay Tzvetkov, and Nicola Visciglia

Growing Sobolev norms for the cubic defocusing Schrödinger equation

Séminaire Laurent Schwartz — EDP et applications (2013-2014), Exposé n^o XVI, 11 p.

<http://slsedp.cedram.org/item?id=SLSEDP_2013-2014____A16_0>

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2013-2014
Exposé n° XVI, 1-11

GROWING SOBOLEV NORMS FOR THE CUBIC DEFOCUSING SCHRÖDINGER EQUATION

ZAHER HANI, BENOIT PAUSADER, NIKOLAY TZVETKOV AND NICOLA VISCIGLIA

1. Statement of the result

1.1. This text is based on a talk given by the third author on February 11, 2014 at the seminar Laurent Schwartz, École Polytechnique, Paris. It aims to describe results of the authors on the long time behavior of NLS on product spaces with a particular emphasis on the existence of solutions with growing higher Sobolev norms.

1.2. We start by presenting a classical result of Bourgain. Consider the Cauchy problem associated with the defocusing NLS

$$(i\partial_t + \Delta - |u|^2)u = 0, \quad u|_{t=0} = u_0(x), \quad t \in \mathbb{R}, \;\; x \in \mathbb{R} \times \mathbb{T}^2. \tag{1.1}$$

The following quantity (energy) is conserved by the flow of (1.1)

$$E(u) = \|u\|^2_{H^1(\mathbb{R}\times\mathbb{T}^2)} + \frac{1}{2}\|u\|^4_{L^4(\mathbb{R}\times\mathbb{T}^2)} .$$

Therefore there is a global control on the H^1 norm of the solutions of (1.1). The following result is a consequence of the analysis of [2].

Theorem 1.1. *Let $s \geq 1$. For every $u_0 \in H^s(\mathbb{R} \times \mathbb{T}^2)$ there is a unique solution of (1.1) in $C(\mathbb{R}; H^s(\mathbb{R} \times \mathbb{T}^2))$. In addition there is a positive constant A (independent of s, t and u_0) such that*

$$\|u(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^2)} \lesssim (1 + |t|)^{A(s-1)} \tag{1.2}$$

(the implicit constant is independent of t but depends on u_0 and s).

The global existence part of Theorem 1.1 is a consequence of the fact that for every $u_0 \in H^s$ there is a time τ depending only on $E(u_0)$ (this is the key point) such that (1.1) has a unique solution on $[0, \tau]$. Thanks to the conservation of E, one then can iterate this local statement to get a global solution $u(t)$. The control of the H^s norm then comes from the following bound on $u(t)$,

$$\|u(t + \tau)\|^2_{H^s} \leq \|u(t)\|^2_{H^s} + C\|u(t)\|^{2-2\gamma}_{H^s}, \quad t \in \mathbb{R}, \tag{1.3}$$

for some positive γ. The above estimate with $\gamma = 0$ is a consequence of the so called tame estimates in Sobolev spaces. The refinement with $\gamma > 0$ follows from a smoothing property in the Bourgain spaces, where the solutions of (1.1) belong. Once we get (1.3), we readily get (1.2) by observing that the sequence $\alpha_n = \|u(n\tau)\|^2_{H^s}$ satisfy $\alpha_{n+1} \leq \alpha_n + C\alpha_n^{1-\gamma}$ which in turn implies $\alpha_n \lesssim n^{1/\gamma}$.

1.3. In [3], the author asks whether one can find, for some $s > 1$, an H^s global solution of the cubic defocusing NLS such that the H^s norm of this solution does not remain bounded as t goes to infinity ? If yes can we quantify the growth ? In [9], we present a partial solution of the problem. More precisely, we have the following statement.

Theorem 1.2. *Let us fix $s \geq 30$ and $\varepsilon > 0$. Then there exists $u_0 \in H^s(\mathbb{R} \times \mathbb{T}^2)$ such that $\|u_0\|_{H^s} < \varepsilon$ and such that the corresponding solution of*

$$(i\partial_t + \Delta - |u|^2)u = 0, \quad u|_{t=0} = u_0(x), \quad t \in \mathbb{R}, \ x \in \mathbb{R} \times \mathbb{T}^2,$$

obtained in Theorem 1.1 satisfies

$$\limsup_{t \to \infty} \|u(t)\|_{H^s} = +\infty .$$

The proof of the above result is a combination of a modified scattering on a product space with an analysis of a resonant system initiated in a work by Colliander-Keel-Staffilani-Takaoka-Tao [7]. By using a refinement by Guardia-Kaloshin [11], we can give some quantification of the growth (for instance faster than any power of $\ln \ln(t)$). The limitation $s \geq 30$ is only technical in order to use simple exponents in the proof and could be lowered by simple modifications.

2. Modified scattering for NLS on product spaces

2.1. We first present a general result. Consider the Cauchy problem

$$(i\partial_t + \Delta - |u|^2)u = 0, \quad u|_{t=0} = u_0, \tag{2.1}$$

where now $u(t) : \mathbb{R}^n \times M \to \mathbb{C}$, M being a compact riemannian manifold.

Let $n \geq 2$. Using a vector valued Strichartz estimate (i.e. exploiting only the x dispersion) we can obtain that for every u_0 which is small in a suitable Sobolev space, there is a unique global solution $u(t)$ (in a suitable class) of (2.1) and a function v_0 in the initial data class such that

$$u(t) = e^{it\Delta}(v_0) + o(1), \quad t \gg 1,$$

in the initial data norm.

For $n = 1$ one expects a modified scattering (take data independent of the M variable), i.e. the free evolution should be replaced by a dynamics with a more involved asymptotic behavior.

2.2. From now on, we will only consider the case $n = 1$ and M a torus. Consider therefore the Cauchy problem

$$(i\partial_t + \Delta - |u|^2)u = 0, \quad u|_{t=0} = u_0, \tag{2.2}$$

where $u(t) : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$, $d = 1, 2, 3, 4$ and $\Delta = \Delta_{\mathbb{R}\times\mathbb{T}^d}$. Consider the "resonant part" of (2.2)

$$i\partial_t G(t) = \mathcal{R}[G(t), G(t), G(t)], \tag{2.3}$$

where the nonlinearity is given by

$$\mathcal{F}_{\mathbb{R}\times\mathbb{T}^d} \mathcal{R}[G, G, G](\xi, p) = \sum_{\substack{p+r=q+s \\ |p|^2+|r|^2=|q|^2+|s|^2}} \widehat{G}(\xi, q)\overline{\widehat{G}(\xi, r)}\widehat{G}(\xi, s), \tag{2.4}$$

where $\widehat{G}(\xi, p) = \widehat{G}_p(\xi) = \mathcal{F}_{\mathbb{R}\times\mathbb{T}^d} G(\xi, p)$ is the Fourier transform of G at $(\xi, p) \in \mathbb{R} \times \mathbb{Z}^d$. The dependence on ξ is merely parametric, the system (2.4) is none other than the resonant system for the cubic NLS equation on $\mathbb{T}^d$. The summation constraint in the right hand-side of (2.4) can be seen as follows : for a fixed $p \in \mathbb{Z}^d$ one sums over all $(q, r, s) \in \mathbb{Z}^{3d}$ such that (p, q, r, s) forms a (possibly degenerate) rectangle. Let also notice that it makes no sense to talk about focusing or defocusing case at the level of the resonant system (2.3) since a complex conjugation changes the sign in front of the nonlinearity.

We next introduce several function spaces relevant for our analysis and needed to state the modified scattering results. We define a weak norm

$$\|F\|_Z^2 := \sup_{\xi \in \mathbb{R}} \left[1 + |\xi|^2\right]^2 \sum_{p \in \mathbb{Z}^d} [1 + |p|]^2 \, |\widehat{F}(\xi, p)|^2 \, .$$

It is of crucial importance that Z is a conserved quantity for the resonant system. It is expected that if u is a solution of (2.2) then $e^{-it\Delta}u(t)$ remains bounded in Z.

Fix $N \geq 30$. We define a strong norm

$$\|F\|_S := \|F\|_{H^N_{x,y}} + \|xF\|_{L^2_{x,y}}$$

and an even stronger one (but only in x !)

$$\|F\|_{S^+} := \|F\|_S + \|(1 - \partial_{xx})^4 F\|_S + \|xF\|_S.$$

For data in S^+, the solution is expected to grow slowly in S^+. The difference between the true solution and the solution of the resonant system is supposed to decay in S (after factorizing the free evolution).

An important feature dictating the choice of the S and S^+ norms is the following property they should satisfy. If F is supported in $\{x : |x| > t^\alpha\}$, $\alpha > 0$ then there exists $\beta > 0$ such that

$$\|F\|_S \lesssim t^{-\beta} \|F\|_{S^+}.$$

A similar property should hold if F contains only x frequencies $\gtrsim t^\alpha$.

We have the hierarchy $S^+ \subset S \subset Z \subset H^1_{x,y}$ and we have a useful Gagliardo-Nirenberg type bound

$$\|F\|_Z \lesssim \|F\|^{\frac{1}{4}}_{L^2_{x,y}} \|F\|^{\frac{3}{4}}_S \, .$$

We also have the basic dispersive bound

$$\|e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}} F\|_{L^\infty_x H^1_y} \lesssim (1 + |t|)^{-\frac{1}{2}} \|F\|_Z + (1 + |t|)^{-\frac{5}{8}} \|F\|_S \, . \tag{2.5}$$

The proof of (2.5) follows by “summing-up with respect to the transverse variable” the classical $1d$ dispersive bound

$$\|e^{it\partial_x^2} f\|_{L^\infty_x} \lesssim (1 + |t|)^{-\frac{1}{2}} \|\widehat{f}\|_{L^\infty} + (1 + |t|)^{-\frac{3}{4}} \|xf\|_{L^2} \, .$$

A bound similar to (2.5) holds if we replace $L^\infty_x H^1_y$ by $L^\infty_x H^s_y$, where Z should be replaced by Z^s defined by

$$\|F\|^2_{Z^s} := \sup_{\xi \in \mathbb{R}} \sum_{p \in \mathbb{Z}^d} [1 + |p|]^{2s} \, |\widehat{F}(\xi, p)|^2$$

and S is replaced by S^s defined by $\|F\|_{S^s} := \|F\|_{H^{N+s}_{x,y}} + \|xF\|_{L^2_{x,y}}$.

Unfortunately, for solutions of the resonant system, the Z^s norms, $s > 1$ are not bounded uniformly in time. This forces us to work with the low regularity (in y) space Z, where we put the contributions with a critical decay in t. This essentially explains the appearance of Strichartz estimates on $\mathbb{T}^d$ in our analysis.

2.3. We are now in position to state our modified scattering results.

Theorem 2.1 (modified scattering). *Let $1 \leq d \leq 4$. There exists $\varepsilon > 0$ such that if $U_0 \in S^+$ satisfies*

$$\|U_0\|_{S^+} \leq \varepsilon,$$

and if $U(t)$ solves the cubic defocusing NLS posed on $\mathbb{R} \times \mathbb{T}^d$ with initial data U_0, then $U \in C((0, +\infty); S)$ exists globally and exhibits modified scattering to its resonant dynamics in the following sense: there exists $G_0 \in S$ such that if $G(t)$ is the solution of the resonant system with initial data $G(0) = G_0$, then

$$\|e^{-it\Delta_{\mathbb{R}\times\mathbb{T}^d}} U(t) - G(\pi \ln t)\|_S \to 0 \qquad as \quad t \to +\infty.$$

Moreover

$$\|U(t)\|_{L_x^\infty H_y^1} \lesssim (1 + |t|)^{-\frac{1}{2}}.$$

If in addition we assume that the number N involved in the definition of S is large enough (for instance $N = s + 30$) then we also have the bound

$$\|U(t)\|_{L_x^\infty H_y^s} \lesssim (1 + |t|)^{-\frac{1}{2}+\delta}, \quad \delta > 0,\ s > 1\,.$$

Theorem 2.2 (existence of modified wave operator). *Let $1 \leq d \leq 4$. There exists $\varepsilon > 0$ such that if $G_0 \in S^+$ satisfies*

$$\|G_0\|_{S^+} \leq \varepsilon,$$

and $G(t)$ solves the resonant system with initial data G_0, then there exists $U \in C((0, \infty); S)$ a solution of the cubic defocusing NLS, posed on $\mathbb{R} \times \mathbb{T}^d$ such that

$$\|e^{-it\Delta_{\mathbb{R}\times\mathbb{T}^d}} U(t) - G(\pi \ln t)\|_S \to 0 \ \ as \ \ t \to +\infty.$$

In particular

$$\|U(t) - e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}} G(\pi \ln t)\|_{H^N(\mathbb{R}\times\mathbb{T}^d)} \to 0 \ \ as \ \ t \to +\infty.$$

It is worth mentioning that a slight modification of the proof of Theorems 1.1 and 1.2 shows that similar statements hold if $\mathbb{T}^d$ is replaced by the sphere $S^d, d = 2, 3$ (with a suitably modified resonant system).

Our analysis combines techniques from high regularity small data theory for quasilinear dispersive equations on $\mathbb{R}$ and tools from the theory of large data low regularity dispersive equations on the Torus. The former helps us propagate smoothness in Fourier space and the latter to control the remaining resonant term in an economical fashion once we have spent all the leverage provided by our tools to peel off the non resonant terms.

We decided to present our results in the case of a defocusing nonlinearity since in this case there are no results on blowup in finite time. However, it is worth mentioning that results similar to Theorems 2.1, 2.2, 1.2 hold in the focusing case as well, where "small" finite energy solutions are also global (at least for $d \leq 3$).

Let us also mention that in Theorem 2.1 the modification with respect to the free evolution is more involved than a "phase correction" which typically occurs in previous works on modified scattering.

2.4. We now show how Theorem 2.2 implies the existence of solutions with growing Sobolev norms. We take initial data of the cubic defocusing NLS on $\mathbb{R} \times \mathbb{T}^d$ of the form

$$G_0(x,y) = \varepsilon \mathcal{F}_{\mathbb{R}}^{-1}(\varphi)(x) g(y), \quad x \in \mathbb{R},\ y \in \mathbb{T}^d,$$

with $\varphi \in C_0^\infty(\mathbb{R})$ real valued. The solution $G(t)$ to the resonant system with initial data $G_0(x,y)$ as above is given in the Fourier space by

$$\widehat{G}_p(t,\xi) = \varphi(\xi) a_p(\varphi(\xi)^2 t), \quad a_p(0) = \mathcal{F}_{\mathbb{T}^d}(g)(p),$$

where the vector $a = (a_p)_{p \in \mathbb{Z}^d}$ solves the *resonant equation*

$$i\partial_t a_p(t) = \sum_{\substack{p+r=q+s \\ |p|^2+|r|^2=|q|^2+|s|^2}} a_q(t)\overline{a_r(t)} a_s(t)\,.$$

In particular, if $\varphi = 1$ on an open interval I, then $\widehat{G}_p(t,\xi) = a_p(t)$ for all $t \in \mathbb{R}$ and $\xi \in I$. We can therefore deduce Theorem 1.2 from Theorem 2.2 by applying the following result which follows from some elaborations on the works [7, 8, 11].

Theorem 2.3 (growth for the resonant equation)**.** *Define*

$$\|(a_p)\|_{h_p^s}^2 := \sum_{p \in \mathbb{Z}^d} \left[1 + |p|^2\right]^s |a_p|^2.$$

Let $d \geq 2$ and $s > 1$. There exists global solutions to the resonant equation in $C(\mathbb{R}; h_p^s)$ such that

$$\sup_{t>0} \|a(t)\|_{h_p^s} = \infty.$$

More precisely, for any $\varepsilon > 0$, there exists a solution $a(t) \in C(\mathbb{R}; h_p^s)$ such that for some sequence of times $t_k \to \infty$ we have that

$$\|a(0)\|_{h_p^s} \leq \varepsilon, \qquad \|a(t_k)\|_{h_p^s} \gtrsim \exp(c(\log t_k)^{\frac{1}{2}})$$

for some $c > 0$.

It is worth mentioning that the solutions we construct also satisfy the upper bound

$$\|a(t)\|_{h_p^s} \lesssim \exp(c(\log t)^{\frac{1}{2}})\,.$$

Unfortunately they do not belong to $h_p^{s'}$ for $s' > s$.

3. On the proof of the main results

3.1. Our first purpose is to explain where the resonant system comes from. Let $U(t)$ be a solution of the cubic defocusing NLS, posed on $\mathbb{R} \times \mathbb{T}^d$. Then $F(t) = e^{-it\Delta}U(t)$ solves

$$i\partial_t F(t) = \mathcal{N}^t[F(t), F(t), F(t)],$$

where the trilinear form $\mathcal{N}^t$ is defined by

$$\mathcal{N}^t[F,G,H] := e^{-it\Delta_{\mathbb{R}\times\mathbb{T}^d}}\Big(e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}}F \cdot e^{-it\Delta_{\mathbb{R}\times\mathbb{T}^d}}\overline{G} \cdot e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}}H\Big).$$

Now, we can compute the Fourier transform of the last expression which leads to the identity

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,p) = \sum_{p-q+r-s=0} e^{it\left[|p|^2-|q|^2+|r|^2-|s|^2\right]} \mathcal{I}^t[\widehat{F_q, G_r}, H_s](\xi),$$

where

$$\mathcal{I}^t[f,g,h] := \mathcal{U}(-t)\Big(\mathcal{U}(t)f\overline{\mathcal{U}(t)g}\mathcal{U}(t)h\Big), \quad \mathcal{U}(t) = \exp(it\partial_x^2).$$

One verifies that

$$\widehat{\mathcal{I}^t[f,g,h]}(\xi) = \int_{\mathbb{R}^2} e^{it2\eta\kappa}\widehat{f}(\xi-\eta)\overline{\widehat{g}}(\xi-\eta-\kappa)\widehat{h}(\xi-\kappa)d\kappa d\eta.$$

Thus one may also write

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,p) = \sum_{p-q+r-s=0} e^{it\left[|p|^2-|q|^2+|r|^2-|s|^2\right]} \int_{\mathbb{R}^2} e^{it2\eta\kappa}\widehat{F}_q(\xi-\eta)\overline{\widehat{G}_r}(\xi-\eta-\kappa)\widehat{H}_s(\xi-\kappa)d\kappa d\eta\,.$$

A formal stationary phase argument ($t \gg 1$) suggests to define $\mathcal{R}$ as

$$\mathcal{F}\mathcal{R}[F,G,H](\xi,p) := \sum_{\substack{p+r=q+s\\ |p|^2+|r|^2=|q|^2+|s|^2}} \widehat{F}_q(\xi)\overline{\widehat{G}_r}(\xi)\widehat{H}_s(\xi)\,.$$

Therefore one expects that the nonlinearity can be decomposed as follows

$$\mathcal{N}^t[F,G,H] = \frac{\pi}{t}\mathcal{R}[F,G,H] + \text{ better term}$$

Recall that the resonant system is precisely $i\partial_t F = \mathcal{R}[F,F,F]$. Observe that one can eliminate the $1/t$ factor by introducing the slow time $\ln(t)$. This explains the appearance of the $\ln(t)$ in the statements of Theorems 2.1, 2.2. This slow time of the resonant system allows us to convert rough Gronwall type bounds for the resolvent system to slow polynomial growth bounds in the context of the original equation.

We have a remarkable Leibniz rule for $\mathcal{I}^t[f,g,h]$, namely

$$Z\mathcal{I}^t[f,g,h] = \mathcal{I}^t[Zf,g,h] + \mathcal{I}^t[f,Zg,h] + \mathcal{I}^t[f,g,Zh], \quad Z \in \{ix, \partial_x\}. \tag{3.1}$$

A similar property holds for the whole nonlinearity $\mathcal{N}^t[F,G,H]$, where Z can also be ∂_{y_j}. This property is related to the idea of Klainerman vector fields, used in similar problems for the wave equation. Thanks to (3.1), we can treat the multiplication by x as a derivation and therefore reduce the estimates in S to L^2 bounds via some standard Littlewood-Paley analysis.

The basic strategy in estimating the nonlinearity is to use $1d$ dispersive estimates for fixed frequencies of the periodic variable and then sum-up the pieces. In many cases (when we have S norms as outputs), we use the simple but useful bound

$$\Big\| \sum_{p-q+r-s=0} c^1_q c^2_r c^3_s \Big\|_{l^2_p} \lesssim \min_{\sigma\in\mathfrak{S}_3} \|c^{\sigma(1)}\|_{l^2_p}\|c^{\sigma(2)}\|_{l^1_p}\|c^{\sigma(3)}\|_{l^1_p}. \tag{3.2}$$

In the remaining cases (with the low regularity Z norm as an output), we use multi-linear Strichartz estimates on the torus, in order to sum-up the pieces.

3.2. We next present a basic bound. Using the inequality (3.2), the energy bound

$$\|\mathcal{I}^t[f^a, f^b, f^c]\|_{L^2_x} \lesssim \min_{\sigma\in\mathfrak{S}_3} \|f^{\sigma(a)}\|_{L^2_x}\|e^{it\partial_{xx}}f^{\sigma(b)}\|_{L^\infty_x}\|e^{it\partial_{xx}}f^{\sigma(c)}\|_{L^\infty_x}$$

and the dispersive bound

$$\|e^{it\partial_{xx}}f\|_{L^\infty_x} \lesssim |t|^{-\frac{1}{2}}\|f\|^{\frac{1}{2}}_{L^2_x}\|xf\|^{\frac{1}{2}}_{L^2_x},$$

we get the following basic bound

$$\|\mathcal{N}^t[F,G,H]\|_S \lesssim (1+|t|)^{-1}\|F\|_S\|G\|_S\|H\|_S. \tag{3.3}$$

Therefore, being optimistic we may hope to apply modified scattering techniques. The bound (3.3) is not very useful alone but it may become sufficient if one of the functions F, G, H has a better decay, for instance if it is localized at high frequencies (in terms of $t \gg 1$) or away of the origin in the physical space (again in terms of t, e.g the region $|x| > t^{\frac{1}{100}}$).

3.3. We next present two propositions allowing to estimate the nonlinearity. Set

$$\|F\|_{X_T} := \sup_{0\le t\le T}\big(\|F(t)\|_Z + \langle t\rangle^{-\delta}\|F(t)\|_S + \langle t\rangle^{1-3\delta}\|\partial_t F(t)\|_S\big),$$

$$\|F\|_{X_T^+} := \|F\|_{X_T} + \sup_{0\le t\le T}\big(\langle t\rangle^{-5\delta}\|F(t)\|_{S^+} + \langle t\rangle^{1-7\delta}\|\partial_t F(t)\|_{S^+}\big),$$

where $\delta \in (0, 10^{-3})$ is fixed. We have the following statements.

Proposition 3.1. *For $T \ge 1$, we can decompose the nonlinearity as*

$$\mathcal{N}^t[F(t), G(t), H(t)] = (\frac{\pi}{t}\mathcal{R} + \mathcal{E}^t)[F(t), G(t), H(t)],$$

with the bounds

$$\Big\| \int_{T/2}^T \mathcal{E}^t[F(t), G(t), H(t)]dt\Big\|_S \lesssim T^\delta\|F\|_{X_T}\|G\|_{X_T}\|H\|_{X_T}$$

and

$$\Big\| \int_{T/2}^T \mathcal{E}^t[F(t), G(t), H(t)]dt\Big\|_Z \lesssim T^{-\delta}\|F\|_{X_T}\|G\|_{X_T}\|H\|_{X_T}$$

uniformly in $T \ge 1$.

The statement of Proposition 3.1 says that if the remainder $\mathcal{E}^t$ has inputs bounded in Z and slightly growing in S then $\mathcal{E}^t$ reproduces the same growth in S and even decays in Z (the effect of the stationary phase).

In order to estimate the source term in the fixed point argument allowing to construct a modified wave operator, we also need the following statement.

Proposition 3.2. *In the context of the previous proposition, if we assume in addition*

$$\|F\|_{X_T^+} + \|G\|_{X_T^+} + \|H\|_{X_T^+} \le 1,$$

then we also have

$$\Big\| \int_{T/2}^T \mathcal{E}^t[F(t), G(t), H(t)]dt\Big\|_S \lesssim T^{-2\delta}.$$

Proposition 3.2 has the spirit of Proposition 3.1 where the couple (Z, S) is "lifted" to (S, S^+).

3.4. We now present several reductions used in the proof of Propositions 3.1, 3.2.

In order to make a first reduction, we perform a decomposition of the nonlinearity

$$\sum_{A,B,C-\text{dyadic}} \mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)],$$

where Q_A, Q_B, Q_C are Littlewood-Paley projectors in the x variable. If we look for decay estimates for $t \sim T$, $T \gg 1$, then in the regime $\max(A,B,C) \geq T^{\frac{1}{6}}$ we can exchange frequency localization to decay. In the case when two inputs have high frequencies, we can simply conclude by using energy estimates while in the case when the highest frequency is much higher than the others, we invoke the bilinear refinements of the Strichartz estimate on $\mathbb{R}$ (see [4, 6]). The summation in the y frequencies is done via the rough bound (3.2). Thus we may suppose that the x frequencies of F, G, H are $\lesssim T^{\frac{1}{6}}$.

A second reduction next allows to eliminate the fast time oscillations. For that purpose, we split the nonlinearity as

$$\mathcal{N}^t[F,G,H] = \Pi^t[F,G,H] + \widetilde{\mathcal{N}}^t[F,G,H],$$

with

$$\mathcal{F}\widetilde{\mathcal{N}}^t[F,G,H](\xi,p) = \sum_{\substack{p+r=q+s \\ |p|^2+|r|^2\neq|q|^2+|s|^2}} e^{it\left[|p|^2-|q|^2+|r|^2-|s|^2\right]} \mathcal{I}^t[\widehat{F_q, G_r}, H_s](\xi). \tag{3.4}$$

Recall that

$$\mathcal{I}^t[\widehat{f,g,h}](\xi) = \int_{\mathbb{R}^2} e^{it2\eta\kappa} \widehat{f}(\xi-\eta)\overline{\widehat{g}}(\xi-\eta-\kappa)\widehat{h}(\xi-\kappa)d\kappa d\eta.$$

In order to bound $\widetilde{\mathcal{N}}^t[F,G,H]$, we distinguish two cases in the (κ,η) integration :

1. If $|\eta\kappa| \lesssim T^{-\frac{1}{4}}$ then we integrate by parts in t thanks to the oscillation $e^{it(|p|^2-|q|^2+|r|^2-|s|^2)}$ (normal form reduction). More precisely for any expression $E(t)$ one can write

$$e^{it\omega}E(t) = \partial_t\Big(\frac{e^{it\omega}\ E(t)}{i\omega}\Big) - \frac{e^{it\omega}}{i\omega}\partial_t E(t)\,, \tag{3.5}$$

where $\omega = |p|^2-|q|^2+|r|^2-|s|^2$. Here it is crucial that the denominator does not approach zero, namely $|\omega| \geq 1$ because ω is an integer different from zero. We substitute (3.5) applied with $E(t) = \mathcal{I}^t[\widehat{F_q, G_r}, H_s](\xi)$ in (3.4). The first term of (3.5) gives decaying contributions to (3.4) (with respect to time integration) thanks to the basic bound presented above. The contributions of the second term in (3.5) decay too. Indeed, if the time derivative hits one of the inputs F, G, H then we can again conclude by the basic bound. If the time derivative hits the oscillating factor $e^{it2\eta\kappa}$ then one also gets an additional decay thanks to the restriction $|\eta\kappa| \lesssim T^{-\frac{1}{4}}$.

2. If $|\eta\kappa| \gtrsim T^{-\frac{1}{4}}$ then we reduce matters to integrations by parts in κ (or in η or in $\kappa+\eta$) thanks the oscillation $e^{it2\eta\kappa}$. These integrations by parts are efficient since on the support of the integration $|\eta| \gtrsim T^{-\frac{5}{12}}$ (recall that thanks to the first reduction $|\kappa| \leq T^{\frac{1}{6}}$). Of course before performing the integration by parts one should also localize the inputs in the physical space. This is possible because the contributions having at least one input supported away from the origin in the physical space are estimated once again by the basic bound (3.3) presented above.

3.5. We now present the analysis on the resonant set which is the main novelty in our analysis. We use the following bound :

$$\|R[a^1,a^2,a^3]\|_{l^2_p} \lesssim \min_{\tau\in\mathfrak{S}_3} \|a^{\tau(1)}\|_{l^2_p}\|a^{\tau(2)}\|_{h^1_p}\|a^{\tau(3)}\|_{h^1_p}, \tag{3.6}$$

where R is $\mathcal{R}$ liberated from the ξ dependence, namely

$$R[a^1,a^2,a^3] = \sum_{\substack{p+r=q+s\\ |p|^2+|r|^2=|q|^2+|s|^2}} a^1_q\overline{a^2_r}a^3_s\,.$$

The proof of (3.6) for $d=2,3$ uses the multi-linear Strichartz estimates on the torus of Bourgain [1] (the case $d=4$ is particularly hard and one needs to invoke the more recent works [5, 10]). It is worth comparing (3.6) with the rough bound (3.2). In estimate (3.2) we have an output in l^1_p which (for $d>1$) is a stronger norm compared to h^1_p which is the output of (3.6). This improvement is possible thanks to the additional constraint $|p|^2+|r|^2=|q|^2+|s|^2$ in the summation defining $R[a^1,a^2,a^3]$ (compare to (3.2), where one only has the constraint $p+r=q+s$).

We next state the key proposition allowing to estimate the contributions of the resonant level set.

Proposition 3.3. *Set*

$$\|F\|_{\tilde{Z}_t} := \|F\|_Z + (1+|t|)^{-\delta}\|F\|_S\,.$$

Then we have

$$\|\Pi^t[F^a,F^b,F^c]\|_S \lesssim (1+|t|)^{-1}\sum_{\sigma\in\mathfrak{S}_3}\|F^{\sigma(a)}\|_{\tilde{Z}_t}\cdot\|F^{\sigma(b)}\|_{\tilde{Z}_t}\cdot\|F^{\sigma(c)}\|_S$$

and

$$\|\Pi^t[F,G,H] - \frac{\pi}{t}\mathcal{R}[F,G,H]\|_S \lesssim (1+|t|)^{-1-20\delta}\|F\|_{S^+}\|G\|_{S^+}\|H\|_{S^+}.$$

Let us give the proof of the first part of the above proposition. By a soft argument, we estimate

$$\|\Pi^t[F^a,F^b,F^c](x)\|_{L^2_{x,y}}$$

by

$$C\Big\|\sum_{\substack{p+r=q+s\\ |p|^2+|r|^2=|q|^2+|s|^2}} |e^{it\partial_{xx}}F^a_q(x)|\cdot|e^{-it\partial_{xx}}F^b_r(x)|\cdot|e^{it\partial_{xx}}F^c_s(x)|\ \Big\|_{L^2_{x,p}}$$

and by the Strichartz bound (3.6), we can continue as follows

$$\lesssim \min_{j\in\{a,b,c\}}\|e^{it\partial_{xx}}F^j_p(x)\|_{L^2_{x,p}}\prod_{k\neq j}\left[\sup_x\sum_{p\in\mathbb{Z}^d}\left[1+|p|^2\right]|e^{it\partial_{xx}}F^k_p(x)|^2\right]^{\frac12}.$$

Applying an abstract transfer principle, we deduce the claimed estimates in S thanks to our dispersive bound

$$\sup_{x\in\mathbb{R}}\sum_{p\in\mathbb{Z}^d}\left[1+|p|^2\right]|e^{it\partial_{xx}}F_p(x)|^2 \lesssim \langle t\rangle^{-1}\Big(\|F\|_Z^2+\langle t\rangle^{-\frac14}\|F\|_S^2\Big)$$

(recall (2.5)). The transfer principle is in the same spirit as the one allowing to passe from

$$\|f_1f_2f_3\|_{L^2} \leq \min_{\sigma\in\mathfrak{S}_3} \|f_{\sigma_1}\|_{L^2}\|f_{\sigma_2}\|_{L^\infty}\|f_{\sigma_3}\|_{L^\infty}$$

to

$$\|f_1f_2f_3\|_{H^s} \lesssim \sum_{\sigma\in\mathfrak{S}_3} \|f_{\sigma_1}\|_{H^s}\|f_{\sigma_2}\|_{L^\infty}\|f_{\sigma_3}\|_{L^\infty}.$$

Observe that in the proof of the basic bound (3.3), we already need such a transfer principle.

For the second estimate of Proposition 3.3, in addition to the previous analysis we use a soft stationary phase argument.

The Strichartz bound (3.6) also gives the estimate

$$\|\mathcal{R}[F^a,F^b,F^c]\|_Z \lesssim \|F^a\|_Z\|F^b\|_Z\|F^c\|_Z. \tag{3.7}$$

Estimate (3.7) is very useful in the local and global study of the resonant system.

3.6. The existence of a modified wave operator now follows by a fix point argument for

$$F(t)\mapsto -i\int_t^\infty \Big\{\mathcal{N}^\sigma[F+G,F+G,F+G]-\frac{\pi}{\sigma}\mathcal{R}[G(\sigma),G(\sigma),G(\sigma)]\Big\}d\sigma, \tag{3.8}$$

where F is such that

$$\sup_{t\geq 0}\Big((1+|t|)^\delta\|F(t)\|_S+(1+|t|)^{2\delta}\|F(t)\|_Z+(1+|t|)^{1-\delta}\|\partial_tF(t)\|_S\Big)<\infty.$$

Thanks to our estimates in S^+ (Proposition 3.2), we have that

$$\int_t^\infty \mathcal{E}^\sigma[G(\sigma),G(\sigma),G(\sigma)]d\sigma,\quad G(t)\in S^+,$$

decays like $(1+|t|)^{-\delta}$ in S and like $(1+t)^{-2\delta}$ in Z.

Thanks to our estimates we can reproduce this information and construct F. As usual in such perturbative arguments, the main point is to evaluate the contributions to (3.8) linear in F. Let us explain how we estimate the linear term $\mathcal{N}^t[F(t),G(t),G(t)]$. Using Proposition 3.1 and (3.7) we can write

$$(1+t)\|\mathcal{N}^t[F(t),G(t),G(t)]\|_Z \lesssim \|G(t)\|_Z^2\|F(t)\|_Z + \text{better}$$

and

$$(1+t)\|\mathcal{N}^t[F(t),G(t),G(t)]\|_S \lesssim \|G(t)\|_{\tilde{Z}_t}^2\|F(t)\|_S + \|F(t)\|_{\tilde{Z}_t}\|G(t)\|_{\tilde{Z}_t}\|G(t)\|_S + \text{better}.$$

One can conclude. This essentially explains how we proceed in the proof of Theorem 2.2

3.7. The proof of Theorem 2.1 follows similar lines. Roughly speaking one gets bounds in the strong norm S^+ and convergence in the weaker norm S. There is however an important additional ingredient concerning the estimates of the solutions of

$$\partial_tF(t)=\mathcal{N}^t[F(t),F(t),F(t)]=(\frac{\pi}{t}\mathcal{R}+\mathcal{E}^t)[F(t),F(t),F(t)] \tag{3.9}$$

in the norm Z. For that purpose one multiplies (3.9) with the multiplier giving the Z conservation of the resonant system. This allows to get rid of the singular term in the right hand-side of (3.9). Consequently, even if Theorem 2.1 is a small data result its proof is not perturbative since it uses the conservation law of the resonant system.

3.8. We make several concluding remarks. A very nice reference concerning the modified scattering for the cubic NLS on $\mathbb{R}$ is the work by Kato-Pusateri [12], where one finds a new proof of the classical result by Ozawa [13]. One may wish to see the result of Theorem 2.1 as a sort of transverse stability of the resonant dynamics. There are certainly other non explored issues as for instance considering more general manifolds as a spatial domain or the NLS with a partial harmonic confinement. One may also try to prove Theorem 1.2 for lower values of s. In this context it can be mentioned that Theorem 1.2 would hold for any $s > 1$ if one finds a solution of the resonant equation growing in h_p^s ($s \in (1, 30)$) and belonging to $C(\mathbb{R}; h_p^{30})$. Unfortunately, as already mentioned, the solutions constructed in Theorem 2.3 do not have this property.

References

[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, *Geom. Funct. Anal.* **3**, (1993), 107–156.

[2] J. Bourgain, On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE, *Internat. Math. Res. Notices* 1996, no. 6, 277–304.

[3] J. Bourgain, Problems in Hamiltonian PDE's, *Geom. Funct. Anal.*, **2000**. (Special volume, Part I), 32–56.

[4] J. Bourgain, Refinements of Strichartz inequality and applications to 2D-NLS with critical nonlinearity, *Int. Math. Res. Not.*, (1998), 253-283.

[5] J. Bourgain, Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces. *Israel J. Math.*, 193 (2013), no. 1, 441–458.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global well-posedness for Schrödinger equations with derivative, *SIAM J. Math. Anal.*, 33 (2001), 649–669.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Invent. Math.*, 181 (2010), no. 1, 39–113.

[8] Z. Hani, Long-time strong instability and unbounded orbits for some periodic nonlinear Schödinger equations, *Arch. Rat. Mech. Anal.*, to appear (DOI: 10.1007/s00205-013-0689-6).

[9] Z. Hani, B. Pausader. N. Tzvetkov. N. Visciglia, Modified scattering for the cubic Schrödinger equation on product spaces and applications, Preprint 2013.

[10] S. Herr, D. Tataru, and N. Tzvetkov, Strichartz estimates for partially periodic solutions to Schrödinger equations in 4*d* and applications, *J. Ang. Math.*, to appear, doi:10.1515/crelle-2012-0013.

[11] M. Guardia and V. Kaloshin, Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation, *J. Eur. Math. Soc*, to appear.

[12] J. Kato and F. Pusateri, A new proof of long range scattering for critical nonlinear Schrödinger equations, *J. Diff. Int. Equ.*, Vol. 24, no. 9–10 (2011).

[13] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension, *Comm. Math. Phys.*, 139 (1991), pp. 479–493.

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York NY 10012
E-mail address: hani@cims.nyu.edu

Université Paris-Nord
E-mail address: pausader@math.univ-paris13.fr

Université Cergy-Pontoise
E-mail address: nikolay.tzvetkov@u-cergy.fr

Universita di Pisa
E-mail address: viscigli@dm.unipi.it