Séminaire Laurent Schwartz

EDP et applications

Année 2013-2014

Resonant averaging for weakly nonlinear stochastic Schrödinger equations

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0   Introduction: weak turbulence

Probably the weak turbulence (WT) originated in the work by Rudolf Peierls [Pei97]. Modern state of affairs may be found in [ZLF92, Naz11]. The method of WT applies to various hamiltonian PDE. E.g., to the cubic NLS equation, which is the topic of my lecture:

$$\dot{u} - i\Delta u + i|u|^2 u = 0, \qquad x \in \mathbb{T}_L^d = \mathbb{R}^d/(L\mathbb{Z}^d).$$

The WT deals with small solutions of this equation. That is, with solutions of order one of the rescaled equation:

$$\dot{u} - i\Delta u + \varepsilon^2 \rho\, i|u|^2 u = 0, \qquad x \in \mathbb{T}_L^d; \quad \rho = \text{const.} \qquad \text{(NLS)}$$

Consider the exponential basis of the space of L-periodic functions

$$\{e_{\mathbf{k}}(x) = e^{i\mathbf{k}\cdot x}, \mathbf{k} \in \mathbb{Z}_L^d\}, \quad \mathbb{Z}^d/L = \mathbb{Z}_L^d$$

It is formed by eigenfunctions of $-\Delta$,

$$-\Delta e_{\mathbf{k}} = \lambda_{\mathbf{k}} e_{\mathbf{k}}; \quad \lambda_{\mathbf{k}} = |\mathbf{k}|^2, \quad \mathbf{k} \in \mathbb{Z}_L^d.$$

So there is plenty of exact resonances in the spectrum of the operator $-\Delta$, which corresponds to the linearised at zero equation (NLS). This is the prerequisite for the WT.

We decompose u in the Fourier series, $u(t,x) = \sum u_{\mathbf{k}}(t) e_{\mathbf{k}}(x)$, and write (NLS) as

$$\dot{u}_{\mathbf{k}} + i\lambda_{\mathbf{k}} u_{\mathbf{k}} = -\varepsilon^2 \rho i \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3}, \qquad \mathbf{k} \in \mathbb{Z}_L^d. \qquad (0.1)$$

This is a hamiltonian system with the Hamiltonian

$$\mathcal{H}^4 = \frac{\rho}{4} \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3} \bar{u}_{\mathbf{k}_4} .$$

In the WT they study solutions for (0.1) with a "typical" initial data $u(0) = u^0$, during "long" time. Time is so long that "solutions approach an invariant measure of the equation". They claim that *for large values of time only resonant terms in (0.1) are important.* For $t \gg 1$, they decompose solutions in asymptotical series in ε and study this series as

$$\varepsilon \to 0, \qquad L \to \infty .$$

They go to a limit, by replacing sums $\sum_{\mathbf{k} \in \mathbb{Z}_L^d}$ by the integrals $\int_{\mathbf{k} \in \mathbb{R}^d}$. In particular, they study under that limit properly scaled quantities $\frac{1}{2}|u_{\mathbf{k}}(t)|^2$, and claim that

$$\frac{1}{2}\langle |u_{\mathbf{k}}(t)|^2 \rangle \sim |\mathbf{k}|^{-\varkappa}, \qquad \varkappa > 0 , \qquad \text{(KZ spectrum)}$$

if $|\mathbf{k}|$ belongs to certain "inertial range". Here "$\langle \cdot \rangle$" indicates some averaging. The function $|\mathbf{k}| \mapsto \frac{1}{2}\langle |u_{\mathbf{k}}(t)|^2 \rangle$ calls the *energy spectrum of a solution*, and its specific form above is called the *Kolmogorov-Zakharov (KZ) spectrum.*

Certainly (KZ spectrum) cannot be true for all solution of (NLS). Say, since due to the KAM theory the equation has time-quasiperiodic solutions which stay analytic in x uniformly in t. So we must assume that u_0 is random, and then try to prove (KZ spectrum) for typical u_0, or to incorporate in the averaging $\langle \cdot \rangle$ the ensemble-averaging.

It is not quite clear in what order we should send $\varepsilon \to 0$ and $L \to \infty$. So it may be better to talk not about the limit of WT, but about WT limits (although physicists do not do that).

Alternatively the method of the WT may be applied in the stochastic setting. Following Zakharov - L'vov (see [ZL75, CFG08]) let us consider solutions of the NLS equation with small damping and small random force:

$$\dot{u} - i\Delta u + \varepsilon^2 \rho\, i|u|^2 u = -\nu(-\Delta + 1)^p u + \sqrt{\nu}\, \langle \text{rand. force} \rangle, \quad x \in \mathbb{T}_L^d, \qquad (0.2)$$

where $\varepsilon, \nu \ll 1$. Here ν is the inverse time-scale of the forced oscillations and ε is their (small) amplitude. The scaling "$\sqrt{\nu}$ in front of the random force" is natural since under this scaling solutions of the equation stay of order one when $t \to \infty$ and $\nu \to 0$. The physicists impose some relation between ν and ε, depending

on the physical nature of the problem for which the equation is a model. The random force usually is

$$\sum_{\mathbf{k}\in\mathbb{Z}_L^d} b_{\mathbf{k}}\frac{d}{dt}\boldsymbol{\beta}^{\mathbf{k}}(t)e^{i\mathbf{k}\cdot x}, \quad b_{\mathbf{k}}>0 \text{ and } b_{\mathbf{k}}\to 0 \text{ fast at } |\mathbf{k}|\to\infty\,,$$

where $\{\boldsymbol{\beta}^{\mathbf{k}}(t)\}$ are independent standard complex Wiener processes. It is a physical postulate, now rigorously proved under some restriction on the equation, that when $t\to\infty$ solutions of (0.2) converge in distribution to a stationary measure $\mu_{\varepsilon,\nu}$ of the equation (which is a measure in the space, formed by functions $u(x)$):

$$\mathcal{D}u(t)\rightharpoonup\mu_{\varepsilon,\nu} \qquad \text{as } t\to\infty$$

(here and below $\rightharpoonup$ signifies the weak convergence of measures). The measure $\mu_{\varepsilon,\nu}$ is a "statistical equilibrium of the equation". The fact that the convergence above holds for each solution of the equation is called *the mixing*. See discussion in [KN13] (we note that the existing proof of the mixing applies only to damped/driven NLS equations with at most cubic nonlinearities; e.g. it does not apply to (0.2), where the nonlinearity $|u|^2u$ is replaced by $|u|^4u$).

Similar to the deterministic case, Zakharov - L'vov write the equation in the Fourier presentation:

$$\dot{u}_{\mathbf{k}}+i\lambda_{\mathbf{k}}u_{\mathbf{k}}=-\varepsilon^2\rho\,i\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}}u_{\mathbf{k}_1}u_{\mathbf{k}_2}\bar{u}_{\mathbf{k}_3}-\nu(\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}}+\sqrt{\nu}\,b_{\mathbf{k}}\dot{\boldsymbol{\beta}}^{\mathbf{k}}(t), \quad \mathbf{k}\in\mathbb{Z}_L^d.$$

Zakharov - L'vov decompose solutions of (0.2) and/or its stationary measure in series in a suitable small parameter, built from ε, ν and L, and study this series as

$$\text{this small parameter}\to 0, \qquad L\to\infty.$$

Again, for this limit only resonant terms of the equation are important, and again it is unclear in which order the two limits should be taken.

We choose $\varepsilon^2=\nu$ – this is within the bounds, usually imposed in physics, see [Naz11]. It is illuminating to pass to the slow time $\tau=\nu t$ and write the (0.2) as

$$u_\tau-i\nu^{-1}\Delta u+i\rho|u|^2u=-(-\Delta+1)^p u+\langle\text{rand. force}\rangle', \quad x\in\mathbb{T}_L^d. \qquad \text{(ZL)}$$

This is the equation I will discuss, mostly following my work with Alberto Maiocchi [KM13b, KM13a].

We suggest to study the WT limits (at least, some of them) by splitting the limiting process in two steps:

I) prove that when $\nu \to 0$, main characteristics of solutions u^ν and of the stationary measure for (ZL) have limits of order one, described by certain well-posed *effective equation.*

II) Show that main characteristics of solutions for the effective equation and of its stationary measure have non-trivial limits of order one when $L \to \infty$ and $\rho = \rho(L)$ is a suitable function of L.

Step I) is rigorously made in [KM13b], and I discuss it below. I stress that the results of Step I) along cannot justify the predictions of WT since the (KZ spectrum) cannot hold when the period L is fixed and finite. At the end of the lecture, following [KM13a], I will show that a heuristic argument *a-la* WT with a suitable choice of the function $\rho(L)$ in the equation (ZL) leads in the limit of $L \to \infty$ to a Kolmogorov-Zakharov type kinetic equation and to a (KZ spectrum).

1 Averaging for PDEs without resonances

In my works [KP08, Kuk10, Kuk13] I studied the long-time behaviour of solutions for perturbed hamiltonian PDEs without strong resonances. Namely, in [KP08, Kuk10] I considered perturbed integrable equations like

$$\dot{u} - iu_{xx} + i|u|^2 u = \nu(u_{xx} - u) + \sqrt{\nu}\,\langle \text{rand. force}\rangle, \quad x \in S^1,$$

and in [Kuk13] – perturbed linear equations like

$$\dot{u} + i(-\Delta + V(x))u + i\nu|u|^2 u = -\nu(-\Delta + 1)^p u + \sqrt{\nu}\,\langle \text{rand. force}\rangle, \quad x \in \mathbb{T}^d, \ (1.1)$$

where $p \in \mathbb{N}$ and $V(x)$ is such that there are no resonances in the spectrum of $-\Delta + V(x)$. [1] The key idea to study these equations with small $\nu > 0$ was suggested in [Kuk10]: describe the long-time behaviour of the actions $\{\frac{1}{2}|z_{\mathbf{k}}|^2, \mathbf{k} \in \mathbb{Z}_L^d\}$ of solutions, using certain auxiliary Effective Equation. The latter is some well posed quasilinear SPDE with a non-local nonlinearity. It turned out that for eq. (1.1) without resonances, the Effective Equation is linear and does not depend on the Hamiltonian term $\nu i|u|^2 u$. Situation changes if we add a non-linear dissipation and consider the equation

$$\dot{u} + i(-\Delta + V(x))u + i\nu|u|^2 u = -\nu C|u|^{2q}u - \nu(-\Delta + 1)^p u + \sqrt{\nu}\,\langle \text{rand. force}\rangle.$$

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[1] This is a mild restriction which holds for typical potentials $V(x)$.

Now the effective equation is non-linear, see [Kuk13].

2 Averaging for PDEs with resonances

Now I pass to the results of [KM13b, KM13a]. There we apply the method of [KP08, Kuk10, Kuk13] to the equation of Zakharov-L'vov (0.2) with $\varepsilon^2 = \nu$, written using the slow time $\tau = \nu t$:

$$u_\tau - i\nu^{-1}\Delta u + i\rho|u|^2 u = -(-\Delta + 1)^p u + \langle \text{rand. force}\rangle'. \qquad \text{(ZL)}$$

We have to impose some restrictions on p and d to make the equation well posed. E.g., $p = 1$ if $d \leq 3$ (if $p > 1$, then d may be bigger than 3).

We write $u(\tau, x) = \sum_{\mathbf{k} \in \mathbb{Z}_L^d} u_\mathbf{k}(\tau)e^{i\mathbf{k}\cdot x}$ and re-write the equation in terms of the Fourier coefficients $u_\mathbf{k}$:

$$\frac{d}{d\tau}u_\mathbf{k} + i\lambda_\mathbf{k}\nu^{-1}u_\mathbf{k} = -i\rho \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1}u_{\mathbf{k}_2}\bar{u}_{\mathbf{k}_3} - (\lambda_\mathbf{k} + 1)^p u_\mathbf{k} + b_\mathbf{k}\frac{d}{d\tau}\boldsymbol{\beta}^\mathbf{k}(\tau), \quad (2.1)$$

where $\mathbf{k} \in \mathbb{Z}_L^d$. We wish to control the asymptotic behaviour of the actions $\frac{1}{2}|u_\mathbf{k}|^2(\tau)$ and of other characteristics of solutions via suitable effective equation. The Effective Equation for (2.1) may be derived through the *interaction representation*, i.e. by transition to the fast rotating variables $a_\mathbf{k}$:

$$a_\mathbf{k}(\tau) = e^{i\nu^{-1}\lambda_\mathbf{k}\tau}v_\mathbf{k}(\tau), \qquad \mathbf{k} \in \mathbb{Z}_L^d.$$

Note that

$$|a_\mathbf{k}(\tau)| \equiv |v_\mathbf{k}(\tau)|. \qquad (2.2)$$

In these variables eq. (2.1) reeds

$$\frac{d}{d\tau}a_\mathbf{k} = -(\lambda_\mathbf{k} + 1)^p a_\mathbf{k} + b_\mathbf{k}\, e^{i\nu^{-1}\lambda_\mathbf{k}\tau}\frac{d}{d\tau}\beta^\mathbf{k}(\tau)$$
$$- i\rho \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} a_{\mathbf{k}_1}a_{\mathbf{k}_2}\bar{a}_{\mathbf{k}_3} \exp\left(-i\nu^{-1}\tau(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2} - \lambda_{\mathbf{k}_3} - \lambda_\mathbf{k})\right).$$

The terms, constituting the nonlinearity, oscillate fast as ν goes to zero, unless the sum of the eigenvalues in the second line vanishes. So only the terms for which this sum equals zero contribute to the limiting dynamics. The processes $\{\tilde{\boldsymbol{\beta}}^\mathbf{k}(\tau), \mathbf{k} \in \mathbb{Z}_L^d\}$ such that $\frac{d}{d\tau}\tilde{\boldsymbol{\beta}}^\mathbf{k}(\tau) = e^{i\nu^{-1}\lambda_\mathbf{k}\tau}\frac{d}{d\tau}\beta^\mathbf{k}(\tau)$ also are standard independent complex

Wiener processes. Accordingly, the effective equation should be the following damped/driven hamiltonian system

$$\frac{d}{d\tau}v_{\mathbf{k}} = -(\lambda_{\mathbf{k}}+1)^p v_{\mathbf{k}} - R_{\mathbf{k}}(v) + b_{\mathbf{k}}\frac{d}{d\tau}\tilde{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d, \qquad \text{(Eff.Eq.)}$$

where $R_{\mathbf{k}}(v)$ is the resonant part of the hamiltonian nonlinearity:

$$R_{\mathbf{k}}(v) = i\rho \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k} \\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}|^2}} v_{\mathbf{k}_1}v_{\mathbf{k}_2}\bar{v}_{\mathbf{k}_3}.$$

It is easy to see that $R(v)$ is the hamiltonian vector field $R = i\nabla\mathcal{H}_{\text{res}}^4$, where $\mathcal{H}_{\text{res}}^4$ is the resonant part of the Hamiltonian $\mathcal{H}^4$:

$$\mathcal{H}_{\text{res}}^4 = \frac{\rho}{4} \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4 \\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}_4|^2}} v_{\mathbf{k}_1}v_{\mathbf{k}_2}\bar{v}_{\mathbf{k}_3}\bar{v}_{\mathbf{k}_4}.$$

The lemma below comprises some important properties of $\mathcal{H}_{\text{res}}^4$ and of (Eff.Eq):

Lemma 2.1. *1)* $\mathcal{H}_{res}^4$ *has two convex quadratic integrals of motion,* $H_0 = \sum |v_{\mathbf{k}}|^2$ *and* $H_1 = \sum(|v_{\mathbf{k}}|^2|\mathbf{k}|^2)$.
2) The hamiltonian vector-field $i\nabla\mathcal{H}_{res}^4(v)$ *is Lipschitz in sufficiently smooth Sobolev spaces.*
3) (Eff.Eq) is well posed in sufficiently smooth Sobolev spaces.

Due to the homogeneity of $\mathcal{H}_{\text{res}}^4$ and the property 1) of the lemma, the hamiltonian equation with the Hamiltonian $\mathcal{H}_{\text{res}}^4$ is similar to the space-periodic 2d Euler equation, and the (Eff.Eq.) is similar to the 2d Navier-Stokes equations on a 2-torus.

Recall that the actions of a solution $u^\nu(\tau)$ are $\{I_{\mathbf{k}}^\nu(\tau) = \frac{1}{2}|u_{\mathbf{k}}^\nu(\tau)|^2, \ \mathbf{k} \in \mathbb{Z}_L^d\}$.

Theorem 2.2. *Let* $\{u_{\mathbf{k}}^\nu(\tau)\}$ *and* $\{v_{\mathbf{k}}(\tau)\}$ *be solutions of* (0.2) *and of (Eff.Eq) with the same initial data. Then, for each* $\mathbf{k}$ *and for* $0 \le \tau \le 1$,

$$\mathcal{D}I_{\mathbf{k}}^\nu(\tau) \rightharpoonup \mathcal{D}\tfrac{1}{2}|v_{\mathbf{k}}(\tau)|^2 \ \text{as } \nu \to 0.$$

Does the effective equation control the angles $\varphi_{\mathbf{k}} = \arg u_{\mathbf{k}} =: \varphi(u_{\mathbf{k}})$? No, instead it controls the angles of of the a-variables $a_{\mathbf{k}}^\nu(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau}v_{\mathbf{k}}^\nu(\tau)$, which fast rotate compare to the angles $\varphi_{\mathbf{k}}$.

Now consider a stationary measure μ_{st}^ν for (0.2). Let $u_{st}^\nu(\tau) = (u_{st\,\mathbf{k}}^\nu(\tau), \mathbf{k} \in \mathbb{Z}_L^d)$ be a corresponding stationary solution, i.e.

$$\mathcal{D}(u_{st}^\nu(\tau)) \equiv \mu^\nu.$$

Theorem 2.3. *Every sequence $\nu'_j \to 0$ has a subsequence $\nu_j \to 0$ such that*

$$\mathcal{D}(I_{\mathbf{k}}(u^{\nu_j})) \rightharpoonup \mathcal{D}(I_{\mathbf{k}}(v(\tau)) \quad as \quad \nu_j \to 0\,,$$

for each $\mathbf{k}$, where $v(\tau)$ is a stationary solution for the (Eff.Eq)

But is the limit unique? And what we know about the phases $\varphi_{\mathbf{k}}(u^{\nu}_{st}(\tau))$ of the stationary solutions $u^{\nu}_{st}(\tau)$? If the effective equation is mixing, the answer is given by the next result:

Theorem 2.4. *Let (Eff.Eq) has a unique stationary measure m_0. Then*

$$\mu^{\nu} \rightharpoonup m_0 \quad as \quad \nu \to 0.$$

So, if in addition equation (0.2) has a unique stationary measure and is mixing, then for any its solution $u^{\nu}(\tau)$ we have

$$\lim_{\nu \to 0} \lim_{\tau \to \infty} \mathcal{D}\big(u^{\nu}(\tau)\big) = m_0.$$

But when (Eff.Eq) has a unique stationary measure? This is the case when the dimension d is not too high in terms of the exponent p which defines the dumping:

Theorem 2.5. *1) Let $p = 1$. Then (Eff.Eq) has a unique stationary measure if $d \leq 3$.*
2) Take any d. Then (Eff.Eq) has a unique stationary measure if $p \geq p_d$ for a suitable $p_d \geq 0$.

3   Limit $L \to \infty$ for the (Eff.Eq) on the physical level of accuracy.

Consider the (Eff.Eq) with a bit more general damping:

$$\frac{d}{d\tau}v_{\mathbf{k}} = -R_{\mathbf{k}}(v) - \gamma_{\mathbf{k}}v_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}^d_L,$$

$\gamma_{\mathbf{k}} = (a|\mathbf{k}|^m + b)$. Consider the moments of its solutions

$$M^{\mathbf{k}_1,\dots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1},\dots\mathbf{k}_{n_1+n_2}}(\tau) = \mathbf{E}(v_{\mathbf{k}_1}\dots v_{\mathbf{k}_{n_1}}\bar{v}_{\mathbf{k}_{n_1+1}}\dots\bar{v}_{\mathbf{k}_{n_1+n_2}}).$$

In [KM13a] we study the behaviour of the moments as $L \to \infty$ under the two heuristic assumptions, traditionally used in the WT (see in [ZLF92]):

i) *Quasi-Gaussian approximation:*

$$M_{\mathbf{l_3},\mathbf{l_4}}^{\mathbf{l_1},\mathbf{l_2}} \sim M_{\mathbf{l_1}}^{\mathbf{l_1}} M_{\mathbf{l_2}}^{\mathbf{l_2}} (\delta_{\mathbf{l_1}}^{\mathbf{l_3}} + \delta_{\mathbf{l_1}}^{\mathbf{l_4}})(\delta_{\mathbf{l_2}}^{\mathbf{l_3}} + \delta_{\mathbf{l_2}}^{\mathbf{l_4}}) \,,$$

and similar for higher order moments.

ii) *Quasi stationary approximation for equations in the chain of moment equations.*

Denote

$$n_{\mathbf{k}} = L^d M_{\mathbf{k}}^{\mathbf{k}}/2 \,, \qquad \tilde{b}_{\mathbf{k}} = L^{d/2} b_{\mathbf{k}} \,.$$

The former is the normalised action (or the normalised energy) of the wave-vector $\mathbf{k}$. Accepting the two hypotheses above we get for the vector-function $(n_{\mathbf{k}}(\tau), \mathbf{k} \in \mathbb{Z}_L^d)$ the Kolmogorov-Zakharov (KZ) kinetic equations:

Theorem 3.1. *When $L \to \infty$ we have*

$$\frac{d}{d\tau} n_{\mathbf{k}} = -2\gamma_{\mathbf{k}} n_{\mathbf{k}} + \tilde{b}_{\mathbf{k}}^2$$

$$+4 \frac{\rho^2}{L} \int_\Gamma d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \frac{f_{\mathbf{k}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}} \tag{3.1}$$

$$\times (n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3}).$$

Here Γ is the resonant surface,

$$\Gamma = \{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in \mathbb{R}^{3d} : \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} + \mathbf{k}_3, \ |\mathbf{k}_1|^2 + |\mathbf{k}_2|^2 = |\mathbf{k}|^2 + |\mathbf{k}_3|^2\},$$

$\gamma_{\mathbf{k}} = (a|\mathbf{k}|^m + b)$ and $f_{\mathbf{k}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is some bounded smooth function, constructed in terms of the normal frame in $\mathbb{R}^{3d}$ to the surface Γ at a point $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$.

Because of the dissipation and the noise in the effective equation (Eff.Eq.), the kinetic equation (3.1) is "better" then usually the kinetic equations in WT are: the divisor in the integrand has no zeroes and growths to infinity with $|\mathbf{k}|$. In particular, analyzing equations (3.1) in the way, usual for the WT, we obtain their homogeneous time-independent solutions with the KZ spectra.

I recall that $\gamma_k = a|k|^m + b$. Looking for solutions of (KZ) such that $n_{\mathbf{k}}$ depends only on $|\mathbf{k}|$ and arguing *a-la* Zakharov (see in [ZLF92]) we find the following:

i) if $0 < a \ll b \ll 1$, then

$$n_{\mathbf{k}} \sim |\mathbf{k}|^{-d+2/3} \,, \quad \text{or} \ \ n_{\mathbf{k}} \sim |\mathbf{k}|^{-d}.$$

ii) if $0 < b \ll a \ll 1$, then

$$n_{\mathbf{k}} \sim |\mathbf{k}|^{-\frac{m+3d-2}{3}} \,, \quad \text{or} \ \ n_{\mathbf{k}} \sim |\mathbf{k}|^{-\frac{m+3d}{3}} \,.$$

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