Séminaire Laurent Schwartz

EDP et applications

Année 2012-2013

30 Years of Calderón's Problem

Gunther Uhlmann*

Abstract

In this article we survey some of the most important developments since the 1980 paper of A.P. Calderón in which he proposed the problem of determining the conductivity of a medium by making voltage and current measurements at the boundary.

1 Introduction

In 1980 A. P. Calderón published a short paper entitled "On an inverse boundary value problem" [21]. This pioneer contribution motivated many developments in inverse problems, in particular in the construction of "complex geometrical optics" solutions of partial differential equations to solve several inverse problems. We survey some of these developments in this paper.

The problem that Calderón considered was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is known as *Electrical Impedance Tomography* (EIT). Calderón was motivated by oil prospection. In the 40's he worked as an engineer for Yacimientos Petroliferos Fiscales (YPF), the state oil company of Argentina, and he thought about this problem then although he did not publish his results until many years later. For use of electrical methods in geophysical prospection see [121]. EIT also arises in medical imaging given that human organs and tissues have quite different conductivities [63]. One potential application is the early diagnosis of breast cancer [123]. The conductivity of a malignant breast tumor is typically 0.2 mho which is significantly higher than normal tissue which has been typically measured at 0.03 mho. Another application is to monitor pulmonary functions [57]. We now describe more precisely the mathematical problem. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary (many of the results we will describe are valid for domains with Lipschitz boundaries). The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation for the potential is given by

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega \tag{1}$$

---

*Department of Mathematics, University of Washington, Seattle, WA 98195, USA, and Fondation de Sciences Mathématiques de Paris. email: gunther@math.washington.edu

since, by Ohm's law, $\gamma \nabla u$ represents the current flux. Given a potential $f \in H^{1/2}(\partial\Omega)$ on the boundary the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned}
\nabla \cdot (\gamma \nabla u) &= 0 \text{ in } \Omega, \\
u\big|_{\partial\Omega} &= f.
\end{aligned} \tag{2}$$

The Dirichlet to Neumann map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu} \right) \bigg|_{\partial\Omega} \tag{3}$$

where ν denotes the unit outer normal to $\partial\Omega$. The inverse problem is to determine γ knowing Λ_γ. It is difficult to find a systematic way of prescribing voltage measurements at the boundary to be able to find the conductivity. Calderón took instead a different route. Using the divergence theorem we have

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f \, dS \tag{4}$$

where dS denotes surface measure and u is the solution of (2). In other words $Q_\gamma(f)$ is the quadratic form associated to the linear map $\Lambda_\gamma(f)$, and to know $\Lambda_\gamma(f)$ or $Q_\gamma(f)$ for all $f \in H^{1/2}(\partial\Omega)$ is equivalent. $Q_\gamma(f)$ measures the energy needed to maintain the potential f at the boundary. Calderón's point of view is that if one looks at $Q_\gamma(f)$ the problem is changed to finding enough solutions $u \in H^1(\Omega)$ of the equation (1) in order to find γ in the interior. These are the complex geometrical optics (CGO) solutions considered in this paper.

Because of limitations of space we cannot give a complete list of references. See [116], [117], [65] for other recent survey papers and references therein.

2   Boundary Determination

Kohn and Vogelius proved the following identifiability result at the boundary [73].

Theorem 2.1. *Let $\gamma_i \in C^\infty(\overline{\Omega})$ be strictly positive. Assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then*

$$\partial^\alpha \gamma_1 \big|_{\partial\Omega} = \partial^\alpha \gamma_2 \big|_{\partial\Omega}, \qquad \forall |\alpha|.$$

This settled the identifiability question for the non-linear problem in the real-analytic category. They extended the identifiability result to piecewise real-analytic conductivities in [74].

Sketch of proof of Theorem 2.1. We outline an alternative proof to the one given by Kohn and Vogelius of Theorem 2.1. In the case $\gamma \in C^\infty(\overline{\Omega})$ we know, by another result of Calderón [22], that Λ_γ is a classical pseudodifferential operator of order 1. Let (x', x^n) be coordinates near a

point $x_0 \in \partial\Omega$ so that the boundary is given by $x^n = 0$. The function $\lambda_\gamma(x', \xi')$ denotes the full symbol of Λ_γ in these coordinates. It was proved in [111] that

$$\lambda_\gamma(x', \xi') = \gamma(x', 0)|\xi'| + a_0(x', \xi') + r(x', \xi') \tag{5}$$

where $a_0(x', \xi')$ is homogeneous of degree 0 in ξ' and is determined by the normal derivative of γ at the boundary and tangential derivatives of γ at the boundary. The term $r(x', \xi')$ is a classical symbol of order -1. Then $\gamma\big|_{\partial\Omega}$ is determined by the principal symbol of Λ_γ and $\frac{\partial\gamma}{\partial x^n}\big|_{\partial\Omega}$ by the principal symbol and the term homogeneous of degree 0 in the expansion of the full symbol of Λ_γ. More generally the higher order normal derivatives of the conductivity at the boundary can be determined recursively. In [82] one can find a general approach to the calculation of the full symbol of the Dirichlet to Neumann map that applies to more general situations. We note that this gives also a reconstruction procedure. We first can reconstruct γ at the boundary since $\gamma\big|_{\partial\Omega}|\xi'|$ is the principal symbol of Λ_γ (see (5)). In other words in coordinates (x', x^n) so that $\partial\Omega$ is locally given by $x^n = 0$ we have

$$\gamma(x', 0)a(x') = \lim_{s \to \infty} e^{-is\langle x', \omega'\rangle} \frac{1}{s} \Lambda_\gamma(e^{is\langle x', \omega'\rangle} a(x'))$$

with $\omega' \in \mathbb{R}^{n-1}$ and $|\omega'| = 1$ and a a smooth and compactly supported function. In a similar fashion, using (5), one can find $\frac{\partial\gamma}{\partial\nu}\big|_{\partial\Omega}$ by computing the principal symbol of $(\Lambda_\gamma - \gamma\big|_{\partial\Omega}\Lambda_1)$ where Λ_1 denotes the Dirichlet to Neumann map associated to the conductivity 1. The other terms can be reconstructed recursively in a similar fashion. We also observe, by taking an appropriate cut-off function a above, that this procedure is *local*, that is we only need to know the DN map in an open set of the boundary to determine the Taylor series of the conductivity in that open set. This method also leads to stability estimates at the boundary [111].

Theorem 2.2. *Suppose that γ_1 and γ_2 are C^∞ functions on $\overline{\Omega} \subseteq \mathbb{R}^n$ satisfying*

  i) $0 < 1/E \le \gamma_i \le E$

  ii) $\|\gamma_i\|_{C^2(\overline{\Omega})} \le E$

Given any $0 < \sigma < \frac{1}{n+1}$, there exists $C = C(\Omega, E, n, \sigma)$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \le C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2, -1/2} \tag{6}$$

and

$$\left\|\frac{\partial\gamma_1}{\partial\nu} - \frac{\partial\gamma_2}{\partial\nu}\right\|_{L^\infty(\partial\Omega)} \le C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2, -1/2}^\sigma. \tag{7}$$

This result implies that we don't need the conductivity to be smooth to determine the conductivity and its normal derivative at the boundary. In the case γ is continuous on $\overline{\Omega}$ we can determine γ at the boundary by using the stability estimate (6) and an approximation argument since for (6) we only need condition i). In the case that $\gamma \in C^2(\overline{\Omega})$ we can determine, knowing the DN map, γ and its normal derivative at the boundary using the estimate (7) above and an approximation argument. For other results and approaches to boundary determination of the conductivity see [3], [16], [86], [91]. In one way or another the boundary determination involves testing the DN map against highly oscillatory functions at the boundary.

2.1 Complex geometrical optics solutions with a linear phase

Calderón considered in [21] harmonic functions of the form $e^{x \cdot \rho}, \rho \in \mathbb{C}^n, \rho \cdot \rho = 0$ in the study of the linearized problem at a constant conductivity, Sylvester and Uhlmann [109, 110] constructed in dimension $n \geq 2$ complex geometrical optics (CGO) solutions of the conductivity equation for C^2 conductivities that behave like Calderón exponential solutions for large frequencies. This can be reduced to constructing solutions in the whole space (by extending $\gamma = 1$ outside a large ball containing Ω) for the Schrödinger equation with potential. We describe this more precisely below. Let $\gamma \in C^2(\mathbb{R}^n)$, γ strictly positive in $\mathbb{R}^n$ and $\gamma = 1$ for $|x| \geq R$, $R > 0$. Let $L_\gamma u = \nabla \cdot \gamma \nabla u$. Then we have

$$\gamma^{-1/2} L_\gamma(\gamma^{-1/2}) = \Delta - q \tag{8}$$

where

$$q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}. \tag{9}$$

Therefore, to construct solutions of $L_\gamma u = 0$ in $\mathbb{R}^n$ it is enough to construct solutions of the Schrödinger equation $(\Delta - q)u = 0$ with q of the form (9). The next result proven in [109, 110] states the existence of complex geometrical optics solutions for the Schrödinger equation associated to any bounded and compactly supported potential.

Theorem 2.3. *Let $q \in L^\infty(\mathbb{R}^n)$, $n \geq 2$, with $q(x) = 0$ for $|x| \geq R > 0$. Let $-1 < \delta < 0$. There exists $\epsilon(\delta)$ and such that for every $\rho \in \mathbb{C}^n$ satisfying*

$$\rho \cdot \rho = 0$$

and

$$\frac{\|(1 + |x|^2)^{1/2} q\|_{L^\infty(\mathbb{R}^n)} + 1}{|\rho|} \leq \epsilon$$

there exists a unique solution to

$$(\Delta - q)u = 0$$

of the form

$$u = e^{x \cdot \rho}(1 + \psi_q(x, \rho)) \tag{10}$$

with $\psi_q(\cdot, \rho) \in L^2_\delta(\mathbb{R}^n)$. *Moreover* $\psi_q(\cdot, \rho) \in H^2_\delta(\mathbb{R}^n)$ *and for* $0 \leq s \leq 2$ *there exists* $C = C(n, s, \delta) > 0$ *such that*

$$\|\psi_q(\cdot, \rho)\|_{H^s_\delta} \leq \frac{C}{|\rho|^{1-s}}. \tag{11}$$

Here

$$L^2_\delta(\mathbb{R}^n) = \{f; \int (1 + |x|^2)^\delta |f(x)|^2 dx < \infty\}$$

with the norm given by $\|f\|^2_{L^2_\delta} = \int (1 + |x|^2)^\delta |f(x)|^2 dx$ and $H^m_\delta(\mathbb{R}^n)$ denotes the corresponding Sobolev space. Note that for large $|\rho|$ these solutions behave like Calderón's exponential solutions $e^{x \cdot \rho}$. The equation for ψ_q is given by

$$(\Delta + 2\rho \cdot \nabla)\psi_q = q(1 + \psi_q). \tag{12}$$

The equation (12) is solved by constructing an inverse for $(\Delta + 2\rho \cdot \nabla)$ and solving the integral equation

$$\psi_q = (\Delta + 2\rho \cdot \nabla)^{-1}(q(1 + \psi_q)) \tag{13}$$

in the appropriate spaces. If 0 is not a Dirichlet eigenvalue for the Schrödinger equation we can also define the DN map

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

where u solves

$$(\Delta - q)u = 0; \quad u|_{\partial\Omega} = f.$$

More generally we can define the set of Cauchy data for the Schrödinger equation. Let $q \in L^\infty(\Omega)$. We define the Cauchy data as the set

$$\mathcal{C}_q = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \right\}, \tag{14}$$

where $u \in H^1(\Omega)$ is a solution of

$$(\Delta - q)u = 0 \text{ in } \Omega. \tag{15}$$

We have $\mathcal{C}_q \subseteq H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. If 0 is not a Dirichlet eigenvalue of $\Delta - q$, then in fact $\mathcal{C}_q$ is a graph, namely

$$\mathcal{C}_q = \{(f, \Lambda_q(f)) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)\}.$$

Complex geometrical optics for first order equations and systems under different regularity assumptions of the coefficients have been constructed in [92], [93], [32], [101], [100], [79]. For the case of the magnetic Schrödinger operator unique identifiability of the magnetic field and the electrical potential was shown in [79] assuming that both the electrical potential and magnetic potential are both just bounded. Applications of CGO solutions to hybrid problems are in [20], [10], [28], [60].

2.2 The Calderón problem in dimension $n \geq 3$

In this section we summarize some of the basic theoretical results for Calderón's problem in dimension three or higher. The identifiability question was resolved in [109] for smooth enough conductivities. The result is

Theorem 2.4. *Let $\gamma_i \in C^2(\overline{\Omega})$, γ_i strictly positive, $i = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

In dimension $n \geq 3$ this result is a consequence of a more general result.

Theorem 2.5. *Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then $q_1 = q_2$.*

We now show that Theorem 2.5 implies Theorem 2.4. Using (8) we have

$$\mathcal{C}_{q_i} = \left\{ \left(f, \left(\frac{1}{2} \gamma_i^{-1/2} \Big|_{\partial\Omega} \frac{\partial\gamma_i}{\partial\nu} \Big|_{\partial\Omega} \right) f + \gamma_i^{-1/2} \Big|_{\partial\Omega} \Lambda_{\gamma_i} \left(\gamma^{-1/2} \Big|_{\partial\Omega} f \right) \right), \quad f \in H^{1/2}(\partial\Omega) \right\}.$$

Then we conclude $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ using the the boundary identifiability result of Kohn and Vogelius [73] and its extension [111].

Proof of Theorem 2.5. Let $u_i \in H^1(\Omega)$ be a solution of

$$(\Delta - q_i)u_i = 0 \text{ in } \Omega, \qquad i = 1, 2.$$

Then using the divergence theorem we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = \int_{\partial\Omega} \left(\frac{\partial u_1}{\partial\nu} u_2 - u_1 \frac{\partial u_2}{\partial\nu} \right) dS. \tag{16}$$

Now it is easy to prove that if $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then the LHS of (16) is zero

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0. \tag{17}$$

Now we extend $q_i = 0$ in Ω^c. We take solutions of $(\Delta - q_i)u_i = 0$ in $\mathbb{R}^n$ of the form

$$u_i = e^{x \cdot \rho_i}(1 + \psi_{q_i}(x, \rho_i)), \qquad i = 1, 2 \tag{18}$$

with $|\rho_i|$ large, $i = 1, 2$, with

$$\rho_1 = \frac{\eta}{2} + i \left(\frac{k + l}{2} \right) \tag{19}$$

$$\rho_2 = -\frac{\eta}{2} + i \left(\frac{k - l}{2} \right)$$

and $\eta, k, l \in \mathbb{R}^n$ such that

$$\eta \cdot k = k \cdot l = \eta \cdot l = 0 \qquad (20)$$
$$|\eta|^2 = |k|^2 + |l|^2.$$

Condition (21) guarantees that $\rho_i \cdot \rho_i = 0$, $i = 1, 2$. Substituting (18) into (17) we conclude

$$\widehat{(q_1 - q_2)}(-k) = -\int_\Omega e^{ix \cdot k}(q_1 - q_2)(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2})dx. \qquad (21)$$

Now $\|\psi_{q_i}\|_{L^2(\Omega)} \leq \frac{C}{|\rho_i|}$. Therefore by taking $|l| \to \infty$ we obtain

$$\widehat{\chi_\Omega(q_1 - q_2)}(k) = 0 \qquad \forall\, k \in \mathbb{R}^n$$

concluding the proof. Theorem 2.4 has been extended to conductivities having $3/2$ derivatives in some sense in [97], [17]. Uniqueness for conormal conductivies in $C^{1+\epsilon}$ was shown in [35]. Recently Haberman and Tataru in a very nice article [44] have extended the uniqueness result to C^1 conductivities or small in the $W^{1,\infty}$ norm. It is an open problem whether uniqueness holds in dimension $n \geq 3$ for Lipschitz or less regular conductivities. Theorem 2.5 was extended to potentials in $L^{n/2}$ and small potentials in the Fefferman-Phong class in [27]. For conormal potentials with strong singularities so that the potential is not in $L^{n/2}$, for instance almost a delta function of an hypersurface, uniqueness was shown in [35].

For the case of the magnetic Schrödinger operator unique identifiability of the magnetic field and the electrical potential was shown in [79] assuming that both the electrical potential and magnetic potential are both just bounded improving the regularity assumed in [93] and [108]. The important case of Maxwell's equations was considered in [95], [96] for C^2 electromagnetic parameters and in [26] for C^1 coefficients.

Using the CGO solutions the following stability estimates were proven in [2].

Theorem 2.6. *Let* $n \geq 3$. *Suppose that* $s > \frac{n}{2}$ *and that* γ_1 *and* γ_2 *are* C^∞ *conductivities on* $\overline{\Omega} \subseteq \mathbb{R}^n$ *satisfying*

i) $0 < 1/E \leq \gamma_j \leq E, j = 1, 2.$

ii) $\|\gamma_j\|_{H^{s+2}(\Omega)} \leq E, j = 1, 2.$

Then there exists $C = C(\Omega, E, n, s)$ *and* $0 < \sigma < 1$ $(\sigma = \sigma(n, s))$ *such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\big(|\log\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2,-1/2}|^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2,-1/2}\big) \qquad (22)$$

where $\|\quad\|_{1/2,-1/2}$ *denotes the operator norm as operators from* $H^{1/2}(\partial\Omega)$ *to* $H^{-1/2}(\partial\Omega)$.

Stability estimates using the method of [44] were proven in [24] for $C^{1,\epsilon}, \epsilon > 0$ conductivities. Notice that this is logarithmic type stability estimates indicates that the problem is severely ill-posed. Mandache [84] has shown that this estimate is optimal up to to the value of the exponent. There is the question of whether under some additional a-priori condition one can improve this logarithmic type stability estimate. Alessandrini and Vessella [4] have shown that this is indeed the case and one has a Lipschitz type stability estimate if the conductivity is piecewise constant with jumps on a finite number of domains. Rondi [99] has subsequently shown that the constant in the estimate grows exponentially with the number of domains. It is conjectured, and this is supported by numerical experiments, that the stability estimate should be "better" near the boundary and gets increasingly worse as one penetrated deeper into the domain (Theorem 2.2 shows that at the boundary we have Lipschitz type stability estimate.) This type of depth dependence stability estimate has been proved in [88] for the case of some electrical inclusions. Theorem 2.6 is a consequence of Theorem 2.2 and the following result.

Theorem 2.7. *Assume 0 is not a Dirichlet eigenvalue of $\Delta - q_i$, $i = 1, 2$. Let $s > n/2$, $n \geq 3$ and*

$$\|q_j\|_{H^s(\Omega)} \leq M.$$

Then there exists $C = C(\Omega, M, n, s)$ and $0 < \sigma < 1$ ($\sigma = \sigma(n, s)$) such that

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\big(|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{1/2,-1/2}|^{-\sigma} + \|\Lambda_{q_1} - \Lambda_{q_2}\|_{1/2,-1/2}\big). \tag{23}$$

Theorem 2.7 also applies to the Helmholtz equation with the potential of the form $q = -k^2 c(x)$ where $c(x)$ denotes the speed of sound and k the frequency. It is proven in [89] that the stability estimates changes from logarithmic from low frequencies to Lipschitz for high frequencies. The same has been observed in [59] for the case that the potential is of the form $q(x) - k^2$.

The complex geometrical optics solution of Theorems 2.4 and 2.5 were also used by A. Nachman [86] and R. Novikov [94] to give a reconstruction procedure of the conductivity from Λ_γ. Reconstruction for C^1 conductivities was done in [34].

3 The Partial Data Problem

In several applications in EIT one can only measure currents and voltages on part of the boundary. Substantial progress has been made recently on the problem of whether one can determine the conductivity in the interior by measuring the DN map on part of the boundary. We review here the article [67]. The paper [20] used the method of Carleman estimates with a linear weight to prove that, roughly speaking, knowledge of the DN map in "half" of the boundary is enough to determine uniquely a C^2 conductivity. The regularity assumption on the conductivity was relaxed to $C^{\frac{3}{2}+\epsilon}, \epsilon > 0$ in [68]. Stability estimates for the uniqueness result of [20] were given in [45]. Stability estimates for the magnetic Schrödinger operator with partial data in the setting

of [20] can be found in [114]. The result [20] was substantially improved in [67]. The latter paper contains a global identifiability result where it is assumed that the DN map is measured on any open subset of the boundary of a strictly convex domain for all functions supported, roughly, on the complement. We state the theorem more precisely below. The key new ingredient is the construction of a larger class of CGO solutions than the ones considered in Section 2.1. Let $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$, where $\mathrm{ch}\,(\Omega)$ denotes the convex hull of Ω. Define the front and the back faces of $\partial\Omega$ by

$$F(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu \le 0\},\ \ B(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu > 0\}.$$

The main result of [67] is the following:

Theorem 3.1. *Let $n > 2$. With Ω, x_0, $F(x_0)$, $B(x_0)$ defined as above, let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials and assume that there exist open neighborhoods $\widetilde{F}, \widetilde{B} \subset \partial\Omega$ of $F(x_0)$ and $B(x_0) \cup \{x \in \partial\Omega;\ (x - x_0) \cdot \nu = 0\}$ respectively, such that*

$$\Lambda_{q_1} u = \Lambda_{q_2} u \ in\ \widetilde{F},\ for\ all\ u \in H^{1/2}(\partial\Omega) \cap \mathscr{E}'(\widetilde{B}). \tag{24}$$

Then $q_1 = q_2$.

Here $\mathscr{E}'(\widetilde{B})$ denotes the space of compactly supported distributions in $\widetilde{B}$. The proof of this result uses Carleman estimates for the Laplacian with limiting Carleman weights (LCW). The Carleman estimates allow one to construct, for large τ, a larger class of CGO solutions for the Schrödinger equation than previously used. These have the form

$$u = e^{\tau(\phi + i\psi)}(a + r), \tag{25}$$

where $\nabla\phi \cdot \nabla\psi = 0, |\nabla\phi|^2 = |\nabla\psi|^2$ and ϕ is the LCW. Moreover a is smooth and non-vanishing and $\|r\|_{L^2(\Omega)} = O(\frac{1}{\tau}), \|r\|_{H^1(\Omega)} = O(1)$. Examples of LCW are the linear phase $\phi(x) = x \cdot \omega$, $\omega \in S^{n-1}$, used previously, and the non-linear phase $\phi(x) = \ln|x - x_0|$, where $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$ which was used in [67]. Any conformal transformation of these would also be a LCW. A characterization of all the LCW in $\mathbb{R}^n$, $n > 2$, was given in [31]. In two dimensions any harmonic function is a LCW [119]. The CGO solutions used in [67] are of the form

$$u(x, \tau) = e^{\log|x - x_0| + id(\frac{x - x_0}{|x - x_0|}, \omega)}(a + r) \tag{26}$$

where x_0 is a point outside the convex hull of Ω, ω is a unit vector and $d(\frac{x - x_0}{|x - x_0|}, \omega)$ denote distance. We take directions ω so that the distance function is smooth for $x \in \overline{\Omega}$. For more details see [67].

It is an open problem in dimension $n > 2$ whether if we take Dirichlet data supported in an open subset of the boundary and the Neumann data is measured in the same set one can determine uniquely the potential. This was shown in [5] if the potential is known near the

boundary. Isakov [58] proved such a uniqueness result in dimension three or higher for the case when the complement of the open set where the measurements are made is an open subset of a plane or a sphere. The methods of [67] and Isakov [58] and the results of those papers were extended in [64]. See [65] for a recent survey. the case of partial data on a slab was studied in [83]. The DN map with partial data for the magnetic Schrödinger operator was studied in [30], [69], [114], [76], [29]. The case of the polyharmonic operator was considered in [77]. We also mention that in [38] (resp. [61]) CGO approximate solutions are concentrated near planes (resp. spheres) and provided some local results related to the local DN map. It would be very interesting to extend the partial data result to systems. See [102] for Dirac systems, [25] for Maxwell and [53] for elasticity. Using methods of hyperbolic geometry similar to [61] it is shown in [47] that one can reconstruct inclusions from the *local* DN map using CGO solutions that decay exponentially inside a ball and grow exponentially outside. These are called *complex spherical waves*. A numerical implementation of this method has been done in [47]. The construction of complex spherical waves can also be done using the CGO solutions constructed in [67]. This was done in [118] in order to detect elastic inclusions, in [119] to detect inclusions in the two dimensional case for a large class of systems with inhomogeneous background, and in [103] for the case of inclusions contained in a slab.

Stability estimates for the result of [67] were proven in [23] and a reconstruction method proposed in [87].

4 The Calderón Problem in Two Dimensions

Astala and Päivärinta [6], in a seminal contribution, have extended significantly the uniqueness result of [85] for conductivities having two derivatives in an appropriate sense and the result of [18] for conductivities having one derivative in appropriate sense, by proving that any L^∞ conductivity in two dimensions can be determined uniquely from the DN map. We remark that the method of [85] and [18] uses, CGO solutions, and the $\overline{\partial}$ method. The proof of [6] relies also on construction of CGO solutions for the conductivity equation with L^∞ coefficients and the $\overline{\partial}$ method. This is done by transforming the conductivity equation to a quasi-regular map. Let $\mathcal{D}$ be the unit disk in the plane. Then we have

Lemma 4.1. *Assume $u \in H^1(\mathcal{D})$ is real valued and satisfies the conductivity equation on $\mathcal{D}$. Then there exists a function $v \in H^1(\mathcal{D})$, unique up to a constant, such that $f = u + iv$ satisfies the Beltrami equation*

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{27}$$

where $\mu = (1 - \gamma)/(1 + \gamma)$. Conversely, if $f \in H^1(\mathcal{D})$ satisfies (27) with a real-valued μ, then $u = Ref$ and $v = Imf$ satisfy

$$\nabla \cdot \gamma \nabla u = 0 \quad and \quad \nabla \cdot \frac{1}{\gamma} \nabla v = 0, \tag{28}$$

respectively, where $\gamma = (1 - \mu)/(1 + \mu)$.

Let us denote $\kappa = ||\mu||_{L^\infty} < 1$. Then (27) means that f is a quasi-regular map. The function v is called the γ-harmonic conjugate of u and it is unique up to a constant. Astala and Päivärinta consider the μ-Hilbert transform $\mathcal{H}_\mu : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ that is defined by

$$\mathcal{H}_\mu : u\big|_{\partial\Omega} \mapsto v\big|_{\partial\Omega}$$

and show that the DN map Λ_γ determines $\mathcal{H}_\mu$ and vice versa. Below we use the complex notation $z = x_1 + ix_2$. Moreover, for the equation (27), it is shown that for every $k \in \mathbb{C}$ there are complex geometrical optics solutions of the Beltrami equation that have the form

$$f_\mu(z, k) = e^{ikz} M_\mu(z, k), \tag{29}$$

where

$$M_\mu(z, k) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty. \tag{30}$$

More precisely, they prove that:

Theorem 4.2. *For each $k \in \mathbb{C}$ and for each $2 < p < 1 + 1/\kappa$ the equation (27) admits a unique solution $f \in W_{loc}^{1,p}(\mathbb{C})$ of the form (29) such that the asymptotic formula (30) holds true.*

In the case of non-smooth coefficients the function M_μ grows sub-exponentially in k. Astala and Päivärinta introduce the "transport matrix" to deal with this problem. Using a result of Bers connecting pseudoanalytic functions with quasi-regular maps they show that this matrix is determined by the Hilbert transform H_μ and therefore the DN map. Then they use the transport matrix to show that Λ_γ determines uniquely γ. See [6] for more details. Logarithmic type stability estimates for Hölder conductivities of positive exponent have been given in [13].

4.1 Bukhgeim's Result

In an important breakthrough, Bukhgeim [19] proved that a potential in $L^p(\Omega), p > 2$ can be uniquely determined from the set of Cauchy data as defined in (14). An earlier result [107] gave this for a generic class of potentials. As before, if two potentials q_1, q_2 have the same set of Cauchy data, we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0 \tag{31}$$

where u_i, $i = 1, 2$, are solutions of the Schrödinger equation. Assume now that $0 \in \Omega$. Bukhgeim takes CGO solutions of the form

$$u_1(z, k) = e^{z^2 k}(1 + \psi_1(z, k)), \quad u_2(z, k) = e^{-\bar{z}^2 k}(1 + \psi_2(z, k)) \tag{32}$$

where $z, k \in \mathbb{C}$ and we have used the complex notation $z = x_1 + ix_2$. Moreover ψ_1 and ψ_2 decay uniformly in Ω, in an appropriate sense, for $|k|$ large. Note that the weight $z^2 k$ in the

exponential is a limiting Carleman weight since it is a harmonic function but it is singular at 0 since its gradient vanishes there. Substituting (32) into (31) we obtain

$$\int_\Omega e^{2i\tau x_1 x_2}(q_1 - q_2)(1 + \psi_1 + \psi_2 + \psi_1\psi_2)dx = 0.$$

Now using the decay of ψ_i in τ, $i = 1, 2$, and applying stationary phase (the phase function $x_1 x_2$ that has a non-degenerate critical point at 0) we obtain $q_1(0) = q_2(0) = 0$ in Ω. Of course we can do this at any point of Ω proving the result. This result also shows that complex conductivities can be determined uniquely from the DN map. Francini has shown in [33] that this was the case for conductivities with small imaginary part. It also implies unique determination of a potential from the fixed energy scattering amplitude in two dimensions.

Stability estimates for potentials in $W^{\epsilon,p}, p > 2$ were proven in [15].

4.2 Partial Data Problem in 2D

It is shown in [49] that for a two dimensional bounded domain the Cauchy data for the Schrödinger equation measured on an arbitrary open subset of the boundary determines uniquely the potential. This implies, for the conductivity equation, that if one measures the current fluxes at the boundary on an arbitrary open subset of the boundary produced by voltage potentials supported in the same subset, one can determine uniquely the conductivity. The paper [49] uses Carleman estimates with weights which are harmonic functions with non-degenerate critical points to construct appropriate complex geometrical optics solutions to prove the result. We describe this more precisely below. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain which consists of N smooth closed curves γ_j, $\partial\Omega = \cup_{j=1}^N \gamma_j$. As before we define the set of Cauchy data for a bounded potential q by:

$$\widehat{C_q} = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \mid (\Delta - q)u = 0 \text{ on } \Omega, \ u \in H^1(\Omega) \right\}. \tag{33}$$

Let $\Gamma \subset \partial\Omega$ be a non-empty open subset of the boundary. Denote $\Gamma_0 = \partial\Omega \setminus \overline{\Gamma}$. The main result of [49] gives global uniqueness by measuring the Cauchy data on Γ. Let $q_j \in C^{2+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let q_j be complex-valued. Consider the following sets of Cauchy data on Γ:

$$\mathcal{C}_{q_j} = \left\{ \left(u|_\Gamma, \frac{\partial u}{\partial \nu}\Big|_\Gamma \right) \mid (\Delta - q_j)u = 0 \text{ in } \Omega, \ u|_{\Gamma_0} = 0, \ u \in H^1(\Omega) \right\}, \quad j = 1, 2. \tag{34}$$

Theorem 4.3. *Assume* $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. *Then* $q_1 = q_2$.

Using Theorem 4.3 one concludes immediately, as a corollary, the following global identifiability result for the conductivity equation (2). This result uses that knowledge of the Dirichlet-to-Neumann map on an open subset of the boundary determines γ and its first derivatives on Γ (see [72], [111]).

Corollary 4.4. *With some $\alpha > 0$, let $\gamma_j \in C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$, be non-vanishing functions. Assume that*

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \, on \, \Gamma \, for \, all \, f \in H^{1/2}(\Gamma), \, supp \, f \subset \Gamma.$$

Then $\gamma_1 = \gamma_2$.

It is easy to see that Theorem 4.3 implies the analogous result to [67] in the two dimensional case. Notice that Theorem 4.3 does not assume that Ω is simply connected.

The two dimensional case has special features since one can construct a much larger set of complex geometrical optics solutions than in higher dimensions. On the other hand, the problem is formally determined in two dimensions and therefore more difficult. The proof of Theorem 4.3 is based on the construction of appropriate complex geometrical optics solutions by Carleman estimates with degenerate weight functions.

For this result it is used in [49] a more general class of CGO solutions than the ones constructed by Bukhgeim, since we would like to have the imaginary part of the phase vanish on Γ. So we consider more general holomorphic functions with non-degenerate critical points as phases. Let the function $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^2(\overline{\Omega})$ be holomorphic in Ω and $\operatorname{Im} \Phi|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0$. Notice that this implies $\nabla\varphi \cdot \nu = 0$ on $\partial\Omega \setminus \widetilde{\Gamma}$. We denote the set of critical points of Φ by

$$\mathcal{H} = \{z \in \overline{\Omega} | \partial_z \Phi(z) = 0\}.$$

We assume that Φ has a finite number of non-degenerate critical points in $\overline{\Omega}$, that is $\partial_z^2 \Phi(z) \neq 0$, $z \in \mathcal{H}$.

The CGO solutions used in [49] of

$$(\Delta - q)u = 0 \quad \text{in } \Omega; \quad u|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0 \tag{35}$$

are of the form

$$u(x) = e^{\tau\Phi(z)}(a(z) + a_0(z)/\tau) + e^{\tau\overline{\Phi(z)}}\overline{(a(z) + a_1(z)/\tau)} + e^{\tau\varphi}u_1 + e^{\tau\varphi}u_2. \tag{36}$$

The functions $a, a_0, a_1 \in C^2(\overline{\Omega})$ are holomorphic in Ω and $\operatorname{Re} a|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0$. Moreover

$$\|u_j\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right), \quad \tau \to \infty, j = 1, 2. \tag{37}$$

This method has been extended to a general class of second order elliptic equations in [50], [51] and with measurements in disjoint sets in [52]. A similar result to Theorem 4.3 was proven in [54] for the Neumann-to-Dirichlet map which is more physical in some situations. For other developments see the survey paper [56].

5   Anisotropic Conductivities

Anisotropic conductivities depend on direction. Muscle tissue in the human body is an important example of an anisotropic conductor. For instance cardiac muscle has a conductivity of 2.3 mho in the transverse direction and 6.3 in the longitudinal direction. The conductivity in this case is represented by a positive definite, smooth, symmetric matrix $\gamma = (\gamma^{ij}(x))$ on Ω. Under the assumption of no sources or sinks of current in Ω, the potential u in Ω, given a voltage potential f on $\partial\Omega$, solves the Dirichlet problem

$$
\begin{cases}
\sum\limits_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(\gamma^{ij}\dfrac{\partial u}{\partial x_j}\right) = 0 \text{ on } \Omega \\
\qquad\qquad u|_{\partial\Omega} = f.
\end{cases}
\tag{38}
$$

The DN map is defined by

$$
\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}
\tag{39}
$$

where $\nu = (\nu^1, \ldots, \nu^n)$ denotes the unit outer normal to $\partial\Omega$ and u is the solution of (38). The inverse problem is whether one can determine γ by knowing Λ_γ. The map Λ_γ doesn't determine γ uniquely. This observation is due to L. Tartar (see [72] for an account). Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\psi|_{\partial\Omega} = Id$ where Id denotes the identity map. We have

$$
\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma
\tag{40}
$$

where

$$
\widetilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|}\right) \circ \psi^{-1}.
\tag{41}
$$

Here $D\psi$ denotes the (matrix) differential of ψ, $(D\psi)^T$ its transpose and the composition in (41) is to be interpreted as multiplication of matrices. We have then a large number of conductivities with the same DN map: any change of variables of Ω that leaves the boundary fixed gives rise to a new conductivity with the same electrostatic boundary measurements. The question is then whether this is the only obstruction to unique identifiability of the conductivity. In two dimensions this has been shown for $L^\infty(\Omega)$ conductivities in [7]. This is done by reducing the anisotropic problem to the isotropic one by using isothermal coordinates [108] and using Astala and Päivärinta's result in the isotropic case [6]. Earlier results were for C^3 conductivities using the result of Nachman [85] and for Lipschitz conductivities in [106] using the techniques of [18].

In three dimensions, as was pointed out in [82], this is a problem of geometrical nature and makes sense for general compact Riemannian manifolds with boundary. Let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$
\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right)
\tag{42}
$$

where (g^{ij}) is the matrix inverse of the matrix (g_{ij}). Let us consider the Dirichlet problem associated to (42)

$$\Delta_g u = 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = f. \tag{43}$$

We define the DN map in this case by

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^i g^{ij} \frac{\partial u}{\partial x_j} \sqrt{\det g}\,\big|_{\partial\Omega} \tag{44}$$

The inverse problem is to recover g from Λ_g. We have

$$\Lambda_{\psi^* g} = \Lambda_g \tag{45}$$

where ψ is any C^∞ diffeomorphism of $\overline{M}$ which is the identity on the boundary. As usual $\psi^* g$ denotes the pull back of the metric g by the diffeomorphism ψ. In the case that M is an open, bounded subset of $\mathbb{R}^n$ with smooth boundary, it is easy to see ([82]) that for $n \geq 3$

$$\Lambda_g = \Lambda_\gamma \tag{46}$$

where

$$(g_{ij}) = (\det \gamma^{kl})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \quad (\gamma^{ij}) = (\det g_{kl})^{1/2} (g_{ij})^{-1}. \tag{47}$$

In the two dimensional case there is an additional obstruction since the Laplace-Beltrami operator is conformally invariant. More precisely we have

$$\Delta_{\alpha g} = \frac{1}{\alpha} \Delta_g$$

for any function α, $\alpha \neq 0$. Therefore we have, for $n = 2$,

$$\Lambda_{\alpha(\psi^* g)} = \Lambda_g \tag{48}$$

for any smooth function $\alpha \neq 0$ so that $\alpha|_{\partial M} = 1$. Lassas and Uhlmann ([80]) proved that (45) is the only obstruction to unique identifiability of the conductivity for real-analytic manifolds in dimension $n \geq 3$. In the two dimensional case they showed that (48) is the only obstruction to unique identifiability for smooth Riemannian surfaces. Moreover these results assume that the DN map is measured only on an open subset of the boundary. We state the two basic results. Let Γ be an open subset ∂M. We define for f, $\operatorname{supp} f \subseteq \Gamma$

$$\Lambda_{g,\Gamma}(f) = \Lambda_g(f)|_\Gamma.$$

Theorem 5.1 ($n = 2$). *Let (M, g) be a compact Riemannian surface with boundary. Let $\Gamma \subseteq \partial M$ be an open subset. Then $\Lambda_{g,\Gamma}$ determines uniquely the conformal class of (M, g) up to isometry.*

Theorem 5.2 ($n \geq 3$). *Let (M, g) be a real-analytic compact, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$ be real-analytic and assume that g is real-analytic up to Γ. Then $\Lambda_{g,\Gamma}$ determines uniquely (M, g) up to an isometry.*

The article [98] shows that one can determine for simple manifolds in two dimensions Λ_g if one knows the boundary distance function. This lead to the solution of the boundary rigidity problem in two dimensions.

In [14] another proof was given of Theorem 5.2. Einstein manifolds are real-analytic in the interior and it was conjectured by Lassas and Uhlmann that they were uniquely determined up to isometry by the DN map. This was proven in [39]. Notice that these results don't assume any condition on the topology of the manifold except for connectedness. An earlier result of [82] assumed that (M, g) was strongly convex and simply connected and $\Gamma = \partial M$ in both results. Theorem 5.2 was extended in [81] to non-compact, connected real-analytic manifolds with boundary. These results were extended to differential forms in [75].

5.0.1 The Calderón Problem on Manifolds

The invariant form on a Riemannian manifold with boundary (M, g) for an isotropic conductivity β is given by

$$\mathrm{div}_g(\beta \nabla_g) u = 0 \tag{49}$$

where div_g (resp. ∇_g) denotes divergence (resp. gradient) with respect to the Riemannian metric g. This includes the case considered by Calderón with g the Euclidean metric, the anisotropic case by taking $g^{ij} = \gamma^{ij} \beta$ and $\beta = \sqrt{\det g}$. It was shown in [106] for bounded domains of Euclidean space in two dimensions that the isometric class of (β, g) is determined uniquely by the DN map associated to (49). In two dimensions, when the metric g is known, it is proven in [46] that one can uniquely determine the conductivity β. Guillarmou and Tzou [41] have shown that a potential is uniquely determined for the Schrödinger equation $\Delta_g - q$, with Δ_g the Laplace-Beltrami operator associated to the metric g, generalizing the result of [46]. This result has been extended to connections in [42] and to general elliptic Systems on vector bundles in [1].

In dimension $n \geq 3$ it is an open problem whether one can determine the isotropic conductivity β from the corresponding DN map associated to (49). As before one can consider the more general problem of recovering the potential q from the DN map associated to $\Delta_g - q$. We review below the progress that has been made on this problem based on [31].

5.1 Complex geometrical optics on manifolds

We review the recent construction of complex geometrical optics construction for a class of Riemannian manifolds based on [31]. In this paper those Riemannian manifolds which admit limiting Carleman weights, were characterized. All such weights in Euclidean space were listed in Theorem 1.13.

Theorem 5.3. *If (M, g) is an open manifold having a limiting Carleman weight, then some conformal multiple of the metric g admits a parallel unit vector field. For simply connected manifolds, the converse is also true.*

Locally, a manifold admits a parallel unit vector field if and only if it is isometric to the product of an Euclidean interval and another Riemannian manifold. Thus, if (M, g) has an LCW φ, one can choose local coordinates in such a way that $\phi(x) = x_1$ and

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$

where c is a positive conformal factor. Conversely, any metric of this form admits $\varphi(x) = x_1$ as a limiting weight. In the case $n = 2$, limiting Carleman weights in (M, g) are exactly the harmonic functions with non-vanishing differential. Let us now introduce the class of manifolds which admit limiting Carleman weights and for which one can prove uniqueness results. For this we need the notion of simple manifolds.

Definition 5.4. *A manifold (M, g) with boundary is simple if ∂M is strictly convex, and for any point $x \in M$ the exponential map $\exp_x$ is a diffeomorphism from some closed neighborhood of 0 in $T_x M$ onto M.*

Definition 5.5. *A compact manifold with boundary (M, g), of dimension $n \geq 3$, is admissible if it is conformal to a submanifold with boundary of $\mathbb{R} \times (M_0, g_0)$ where (M_0, g_0) is a compact simple $(n-1)$-dimensional manifold.*

Examples of admissible manifolds include the following:

1. Bounded domains in Euclidean space, in the sphere minus a point, or in hyperbolic space. In the last two cases, the manifold is conformal to a domain in Euclidean space via stereographic projection.

2. More generally, any domain in a locally conformally flat manifold is admissible, provided that the domain is appropriately small. Such manifolds include locally symmetric 3-dimensional spaces, which have parallel curvature tensor so their Cotton tensor vanishes.

3. Any bounded domain M in $\mathbb{R}^n$, endowed with a metric which in some coordinates has the form
$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$
   with $c > 0$ and g_0 simple, is admissible.

4. The class of admissible metrics is stable under C^2-small perturbations of g_0.

The first inverse problem involves the Schrödinger operator

$$\mathcal{L}_{g,q} = \Delta_g - q,$$

where q is a smooth complex valued function on (M, g). We make the standard assumption that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g,q}$ in M. Then the Dirichlet problem

$$\begin{cases} \mathcal{L}_{g,q}u = 0 \text{ in } M, \\ \quad u = f \text{ on } \partial M \end{cases}$$

has a unique solution for any $f \in H^{1/2}(\partial M)$, and we may define the DN map

$$\Lambda_{g,q} : f \mapsto \partial_\nu u|_{\partial M}.$$

Given a fixed admissible metric, one can determine the potential q from boundary measurements.

Theorem 5.6. *Let (M, g) be admissible, and let q_1 and q_2 be two smooth functions on M. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.*

This result was known previously in dimensions $n \geq 3$ for the Euclidean metric [109] and for the hyperbolic metric [60]. It has been generalized to Maxwell's equations in [66].

References

[1] Albin, P, Guillarmou, C., Tzou, L. and Uhlmann, G., Inverse boundary problems for systems in two dimensions, to appear *Annales Institut Henri Poincaré*.

[2] Alessandrini, G., Stable determination of conductivity by boundary measurements, *App. Anal.*, **27** (1988), 153–172.

[3] Alessandrini, G., Singular solutions of elliptic equations and the determination of conductivity by boundary measurements, *J. Diff. Equations*, **84** (1990), 252-272.

[4] Alessandrini, G. and Vessella, S., Lipschitz stability for the inverse conductivity problem, *Adv. in Appl. Math.*, **35** (2005), 207–241.

[5] Ammari, H. and Uhlmann, G., Reconstruction of the potential from partial Cauchy data for the Schrödinger equation, *Indiana Univ. Math. J.*, **53** (2004), 169-183.

[6] Astala, K. and Päivärinta, L., Calderón's inverse conductivity problem in the plane. *Annals of Math.*, **163** (2006), 265-299.

[7] Astala, K., Lassas, M. and Päiväirinta, L., Calderón's inverse problem for anisotropic conductivity in the plane, *Comm. Partial Diff. Eqns.*, **30** (2005), 207–224.

[8] Bal, G., Ren, K., Uhlmann, G, and Zhou, T., Quantitative thermo-acoustics and related problems, *Inverse Problems*, **27** (2011), 055007.

[9] Bal, G. and Uhlmann, G., Inverse diffusion theory of photoacoustics, *Inverse Problems*, **26** (2010), 085010.

[10] Bal, G. and Uhlmann, G., Reconstructions for some coupled-physics inverse problems, *Applied Mathematics Letters*, **25** (2012), 1030-1033.

[11] Bal, G. and Uhlmann, G., Reconstructions of coefficients in scalar second-order elliptic equations from knowledge of their solutions, to appear *Comm. Pure Appl. Math.*

[12] Barceló, B., Barceló, J.A., and Ruiz, A., Stability of the inverse conductivity problem in the plane for less regular conductivities, J. Differential Equations, **173** (2001), 231-270.

[13] Barceló, J.A., Faraco, D. and Ruiz, A., Stability of Calderón's inverse problem in the plane, *Journal des Mathématiques Pures et Appliquées*, **88** (2007), 522-556.

[14] Belishev, M. I., The Calderón problem for two-dimensional manifolds by the BC-method, *SIAM J. Math. Anal.*, **35** (2003), 172–182.

[15] Blaasten, E, Stability and uniqueness for the inverse problem of the Schrödinger equation with potentials in $W^{p,\epsilon}$, arXiv:1106.0632.

[16] Brown, R., Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result, *J. Inverse Ill-Posed Probl.*, **9** (2001), 567–574.

[17] Brown, R. and Torres, R., Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in $L^p, p > 2n$, *J. Fourier Analysis Appl.*, **9** (2003), 1049-1056.

[18] Brown, R. and Uhlmann, G., Uniqueness in the inverse conductivity problem with less regular conductivities in two dimensions, *Comm. PDE*, **22** (1997), 1009-10027.

[19] Bukhgeim, A., Recovering the potential from Cauchy data in two dimensions, *J. Inverse Ill-Posed Probl.*, **16** (2008), 19-34.

[20] Bukhgeim, A. and Uhlmann, G., Recovering a potential from partial Cauchy data, *Comm. PDE*, **27** (2002), 653-668.

[21] Calderón, A. P., On an inverse boundary value problem, *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pp. 65–73, Soc. Brasil. Mat., Rio de Janeiro, 1980.

[22] Calderón, A. P., Boundary value problems for elliptic equations. Outlines of the joint Soviet-American symposium on partial differential equations, 303-304, Novisibirsk (1963).

[23] Caro, P., Dos Santos Ferreira, D. and Ruiz, A., Stability estimates for the Radon transform with restricted data and applications, arXiv:1211.1887 (2012).

[24] Caro, P., Garcia, A. and Reyes, J.M., Stability of the Calderón problem for less regular conductivities, *J. Differential Equations* **254** (2013), 469–492.

[25] Caro, P., Ola, P. and Salo, M., Inverse boundary value problem for Maxwell equations with local data, *Comm. PDE*, **34** (2009), 1425-1464.

[26] Caro, P. and Zhou, T., On global uniqueness for an IBVP for the time-harmonic Maxwell equations, to appear *Anal & PDE*, arXiv:1210.7602.

[27] Chanillo S., A problem in electrical prospection and a *n*-dimensional Borg-Levinson theorem, *Proc. AMS*, **108** (1990), 761–767.

[28] Chen, J. and Yang, Y., Quantitative photo-acoustic tomography with partial data, *Inverse Problems*, **28** (2012), 115014.

[29] Chung, F., A partial data result for the magnetic Schrödinger operator, preprint, arXiv:1111.6658.

[30] Dos Santos Ferreira, D., Kenig, C.E., Sjöstrand, J. and Uhlmann, G., Determining a magnetic Schrödinger operator from partial Cauchy data, *Comm. Math. Phys.*, **271** (2007), 467–488.

[31] Dos Santos Ferreira, D., Kenig, C.E., Salo, M., and Uhlmann, G., Limiting Carleman weights and anisotropic inverse problems, *Inventiones Math.*, **178** (2009), 119-171.

[32] Eskin, G., Ralston, J., On the inverse boundary value problem for linear isotropic elasticity, *Inverse Problems*, **18** (2002), 907–921.

[33] Francini, E., Recovering a complex coefficient in a planar domain from the Dirichlet-to-Neumann map, *Inverse Problems*, **16** (2000), 107–119.

[34] Garcia, A. and Zhang, G., Reconstruction from boundary measurements for less regular conductivities, preprint, arXiv:1212.0727.

[35] Greenleaf, A., Lassas, M. and Uhlmann, G., The Calderón problem for conormal potentials, I: Global uniqueness and reconstruction, *Comm. Pure Appl. Math*, **56** (2003), 328–352.

[36] Greenleaf, A., Lassas, M. and Uhlmann, G., Anisotropic conductivities that cannot be detected in EIT, *Physiolog. Meas.* (special issue on Impedance Tomography), **24** (2003), 413-420.

[37] Greenleaf, A., Lassas, M. and Uhlmann, G., On nonuniqueness for Calderón's inverse problem, *Math. Res. Lett.*, **10** (2003), 685-693.

[38] Greenleaf, A. and Uhlmann, G., Local uniqueness for the Dirichlet-to-Neumann map via the two-plane transform, *Duke Math. J.*, **108** (2001), 599-617.

[39] Guillarmou, C. and Sá Barreto, A., Inverse problems for Einstein manifolds, *Inverse Problems and Imaging*, **3** (2009), 1-15.

[40] Guillarmou, C. and Tzou, L., Calderón inverse problem on Riemann surfaces, *Proceedings of CMA*, **44** (2009), 129-142. Volume for the AMSI/ANU workshop on Spectral Theory and Harmonic Analysis.

[41] Guillarmou, C. and Tzou, L., Calderón inverse problem with partial data on Riemann surfaces, *Duke Math. J.*, **158** (2011), 83-120.

[42] Guillarmou, C. and Tzou, L, Identification of a connection from Cauchy data space on a Riemann surface with boundary, *Geometric and Functional Analysis (GAFA)*, **21** (2011), 393-418.

[43] Hähner, P., A periodic Faddeev-type solution operator, *J. Differential Equations*, **128** (1996), 300–308.

[44] Haberman, B. and Tataru, D., Uniqueness in Calderón's problem with Lipschitz conductivities, to appear *Duke Math. J.*

[45] Heck, H. and Wang, J.-N., Stability estimates for the inverse boundary value problem by partial Cauchy data, *Inverse Problems*, **22** (2006), 1787–1796.

[46] Henkin, G. and Michel, V., Inverse conductivity problem on Riemann surfaces, *J. Geom. Anal.*, **18** (2008), 1033–1052.

[47] Ide, T., Isozaki, H., Nakata S., Siltanen, S. and Uhlmann, G., Probing for electrical inclusions with complex spherical waves, *Comm. Pure and Applied Math.*, **60** (2007), 1415-1442.

[48] Ikehata, M., The enclosure method and its applications, Chapter 7 in "Analytic extension formulas and their applications" (Fukuoka, 1999/Kyoto, 2000), *Int. Soc. Anal. Appl. Comput.*, Kluwer Acad. Pub., **9** (2001), 87-103.

[49] Imanuvilov, O., Uhlmann, G. and Yamamoto, M., The Calderón problem with partial data in two dimensions, *Journal AMS*, **23** (2010), 655-691.

[50] Imanuvilov, O., Uhlmann, G. and Yamamoto, M., On determination of second order operators from partial Cauchy data, *Proceedings National Academy of Sciences.*, **108** (2011), 467-472.

[51] Imanuvilov, O., Uhlmann, G. and Yamamoto, M., Partial data for general second order elliptic operators in two dimensions, *Publ. Research Insti. Math. Sci.*, **48** (2012), 971-1055.

[52] Imanuvilov, O., Uhlmann, G. and Yamamoto, M., Inverse boundary problem with Cauchy data on disjoint sets, *Inverse Problems*, **27** (2011), 085007.

[53] Imanuvilov, O., Uhlmann, G. and Yamamoto, M., On reconstruction of Lamé coefficients from partial Cauchy data in three dimensions, *Inverse Problems*, **28** (2012), 125002.

[54] Imanuvilov, O., Uhlmann, G. and Yamamoto, M., Inverse boundary value problem by partial data for the Neumann-to-Dirichlet map in two dimensions, preprint, arXiv:1210.1255.

[55] Imanuvilov, O. and Yamamoto, M., Inverse boundary value for Schrödinger equation in two dimensions, arXiv:1211.1419v1.

[56] Imanuvilov, O. and Yamamoto, M., Uniqueness for inverse boundary problems by Dirichlet-to-Neumann map on arbitrary subboundaries, preprint, arXiv:1303.2159.

[57] Isaacson, D., Newell, J. C., Goble, J. C. and Cheney M., Thoracic impedance images during ventilation, *Annual Conference of the IEEE Engineering in Medicine and Biology Society*, **12** (1990), 106–107.

[58] Isakov, V., On uniqueness in the inverse conductivity problem with local data, *Inverse Problems and Imaging*, **1** (2007), 95-105.

[59] Isakov, V., Nakamura, G., Uhlmann, G. and Wang, J.-N., Increasing stability of the inverse boundary problem for the Schröedinger equation, to appear Contemp. Math., arXiv:1302.0940.

[60] Isozaki, H., Inverse spectral problems on hyperbolic manifolds and their applications to inverse boundary value problems in Euclidean space, *Amer. J. Math.*, **126** (2004), 1261–1313.

[61] Isozaki, H. and Uhlmann, G., Hyperbolic geometric and the local Dirichlet-to-Neumann map, *Advances in Math.* **188** (2004), 294-314.

[62] Jordana, J., Gasulla, J. M. and Paola's-Areny, R., Electrical resistance tomography to detect leaks from buried pipes, *Meas. Sci. Technol.*, **12** (2001), 1061-1068.

[63] Jossinet, J., The impedivity of freshly excised human breast tissue, *Physiol. Meas.*, **19** (1998), 61-75.

[64] Kenig, C. and Salo, M., The Calderón problem with partial data on manifolds and applications, preprint, arXiv:1211.1054.

[65] Kenig, C. and Salo, M., Recent progress in the Calderón problem with partial data, preprint, arXiv:1302.4218.

[66] Kenig, C., Salo, M. and Uhlmann, G., Inverse problems for the anisotropic Maxwell equations", *Duke Math. J.*, **157** (2011), 369-419.

[67] Kenig, C., Sjöstrand, J. and Uhlmann, G., The Calderón problem with partial data, *Annals of Math.*, **165** (2007), 567-591.

[68] Knudsen, K., The Calderón problem with partial data for less smooth conductivities, *Comm. Partial Differential Equations*, **31** (2006), 57–71.

[69] Knudsen, K. and Salo, M., Determining nonsmooth first order terms from partial boundary measurements, *Inverse Problems and Imaging*, **1** (2007), 349-369.

[70] Kocyigit, I., Acoustic-electric tomography and CGO solutions with internal data, *Inverse Problems*, **28** (2012), 125004.

[71] Kohn, R., Shen, H., Vogelius, M. and Weinstein, M., Cloaking via change of variables in Electrical Impedance Tomography, *Inverse Problems* **24** (2008), 015016 (21pp).

[72] Kohn, R. and Vogelius, M., Identification of an unknown conductivity by means of measurements at the boundary, in *Inverse Problems, SIAM-AMS Proc.*, **14** (1984).

[73] Kohn, R. and Vogelius, M., Determining conductivity by boundary measurements, *Comm. Pure Appl. Math.*, **37** (1984), 289–298.

[74] Kohn, R. and Vogelius, M., Determining conductivity by boundary measurements II. Interior results, *Comm. Pure Appl. Math.*, **38** (1985), 643–667.

[75] Krupchyk, K., Lassas, M. and Uhlmann, G., Inverse problems for differential forms on Riemannian manifolds with boundary, *Comm. PDE.*, **36** (2011), 1475-1509.

[76] Krupchyk, K., Lassas, M. and Uhlmann, G., Inverse problems with partial data for the magnetic Schrödinger operator in an infinite slab and on a bounded domain *Comm. Math. Phys.*, **312** (2012), 87-126.

[77] Krupchyk, K., Lassas, M. and Uhlmann, G., Inverse boundary value problems for the polyharmonic operator, *Journal Functional Analysis*, **262** (2012), 1781-1801.

[78] Krupchyk, K., Lassas, M. and Uhlmann, G, Determining a first order perturbation of the biharmonic operator by partial boundary measurements, to appear *Transactions AMS*.

[79] Krupchyk, K., Uhlmann, G, Determining a magnetic Schrödinger operator with a bounded magnetic potential from boundary measurements, preprint, arXiv:1206.4727.

[80] Lassas, M. and Uhlmann, G., Determining a Riemannian manifold from boundary measurements, *Ann. Sci. École Norm. Sup.*, **34** (2001), 771–787.

[81] Lassas, M., Taylor, M. and Uhlmann, G., The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary, *Comm. Geom. Anal.*, **11** (2003), 207-222.

[82] Lee, J. and Uhlmann, G., Determining anisotropic real-analytic conductivities by boundary measurements, *Comm. Pure Appl. Math.*, **42** (1989), 1097–1112.

[83] Li, X. and Uhlmann, G., Inverse problems on a slab, *Inverse Problems and Imaging*, **4** (2010), 449-462.

[84] Mandache, N., Exponential instability in an inverse problem for the Schrödinger equation, *Inverse Problems*, **17** (2001), 1435–1444.

[85] Nachman, A., Global uniqueness for a two-dimensional inverse boundary value problem, *Ann. of Math.*, **143** (1996), 71-96.

[86] Nachman, A., Reconstructions from boundary measurements, *Ann. of Math.*, **128** (1988), 531–576.

[87] Nachman, A. and Street, B., Reconstruction in the Calderón problem with partial data, *Comm. PDE*, 35 (2010), 375-390.

[88] Nagayasu, S., Uhlmann, G. and Wang, J.-N., Depth dependent stability estimate in electrical impedance tomography, *Inverse Problems*, **25** (2009), 075001.

[89] Nagayasu, S., Uhlmann, G. and Wang, J.-N., Reconstruction of penetrable obstacles in acoustics, *SIAM J. Math. Anal.*, **43** (2011), 189-211.

[90] Nagayasu, S, Uhlmann, G. and Wang, J.-N., Increasing stability for the acoustic equation, *Inverse Problems*, **29** (2013), 020012.

[91] Nakamura, G. and Tanuma, K., Local determination of conductivity at the boundary from the Dirichlet-to-Neumann map, *Inverse Problems*, **17** (2001), 405–419.

[92] Nakamura G. and Uhlmann, G., Global uniqueness for an inverse boundary value problem arising in elasticity, *Invent. Math.*, **118** (1994), 457–474. Erratum: *Invent. Math.*, **152** (2003), 205–207.

[93] Nakamura, G., Sun, Z. and Uhlmann, G., Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field, *Math. Annalen*, **303** (1995), 377–388.

[94] Novikov R. G., Multidimensional inverse spectral problems for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, *Funktsionalny Analizi Ego Prilozheniya*, **22** (1988), 11-12, Translation in *Functional Analysis and its Applications*, **22** (1988) 263–272.

[95] Ola, P., Päivärinta, L. and Somersalo, E., An inverse boundary value problem in electrodynamics, *Duke Math. J.*, **70** (1993), 617–653.

[96] Ola, P. and Somersalo, E. , Electromagnetic inverse problems and generalized Sommerfeld potentials, *SIAM J. Appl. Math.*, **56** (1996), 1129-1145

[97] Päivärinta, L., Panchenko, A. and Uhlmann, G., Complex geometrical optics for Lipschitz conductivities, *Revista Matematica Iberoamericana*, **19** (2003), 57-72.

[98] Pestov, L. and Uhlmann, G., Two dimensional simple Riemannian manifolds with boundary are boundary distance rigid,*Annals of Math.*, **161** (2005), 1089-1106.

[99] Rondi, L., A remark on a paper by G. Alessandrini and S. Vessella: "Lipschitz stability for the inverse conductivity problem" [Adv. in Appl. Math. 35 (2005), 207–241], *Adv. in Appl. Math.*, **36** (2006), 67–69.

[100] Salo, M., Semiclassical pseudodifferential calculus and the reconstruction of a magnetic field, *Comm. PDE*, **31** (2006), 1639-1666.

[101] Salo, M., Inverse problems for nonsmooth first order perturbations of the Laplacian, *Ann. Acad. Sci. Fenn. Math. Diss.*, **139** (2004), 67 pp.

[102] Salo, M. and Tzou, L., Inverse problems with partial data for a Dirac system: a Carleman estimate approach, *Advances in Math.*, **225** (2010), 487-513.

[103] Salo, M. and Wang, J.-N. , Complex spherical waves and inverse problems in unbounded domains, *Inverse Problems* **22** (2006), 2299–2309.

[104] Siltanen, S., Müller, J. L. and Isaacson, D., A direct reconstruction algorithm for electrical impedance tomography, *IEEE Transactions on Medical Imaging*, **21** (2002), 555-559.

[105] Somersalo, E., Isaacson, D. and Cheney, M., A linearized inverse boundary value problem for Maxwell's equations, *Journal of Comp. and Appl. Math.*, **42** (1992),123-136.

[106] Sun, Z. and Uhlmann, G., Anisotropic inverse problems in two dimensions, *Inverse Problems*, **19** (2003), 1001-1010.

[107] Sun, Z. and Uhlmann, G., Generic uniqueness for an inverse boundary value problem, *Duke Math. Journal*, **62** (1991), 131–155.

[108] Sylvester, J., An anisotropic inverse boundary value problem, *Comm. Pure Appl. Math.*, **43** (1990), 201–232.

[109] Sylvester, J. and Uhlmann, G., A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.*, **125** (1987), 153–169.

[110] Sylvester, J. and Uhlmann, G., A uniqueness theorem for an inverse boundary value problem in electrical prospection, *Comm. Pure Appl. Math.*, **39** (1986), 92–112.

[111] Sylvester, J. and Uhlmann, G., Inverse boundary value problems at the boundary – continuous dependence, *Comm. Pure Appl. Math.*, **41** (1988), 197–221.

[112] Sylvester, J. and Uhlmann, G., *Inverse problems in anisotropic media*, *Contemp. Math.*, **122** (1991), 105–117.

[113] Takuwa, H., Uhlmann, G. and Wang, J.-N., Complex geometrical optics solutions for anisotropic equations and applications, *Journal of Inverse and Ill Posed Problems*, **16** (2008), 791-804. **29** (1998), 116–133.

[114] Tzou, L., Stability estimates for coefficients of magnetic Schrödinger equation from full and partial measurements, *Comm. PDE*, **33** (2008), 161-184.

[115] Uhlmann, G., Calderón's problem and electrical impedance tomography, *Inverse Problems*, 25th Anniversary Volume, **25** (2009), 123011 (39pp.)

[116] Uhlmann, G., Editor of *Inside Out II: Inverse Problems and Applications*, MSRI Publications **60**, Cambridge University Press (2012).

[117] Uhlmann, G., Developments in inverse problems since Calderón's foundational paper, Chapter 19 in "Harmonic Analysis and Partial Differential Equations", *University of Chicago Press* (1999), 295-345, edited by M. Christ, C. Kenig and C. Sadosky.

[118] Uhlmann, G. and Wang, J.-N., Complex spherical waves for the elasticity system and probing of inclusions, *SIAM J. Math. Anal.*, **38** (2007), 1967–1980.

[119] Uhlmann, G. and Wang, J.-N., Reconstruction of discontinuities in systems, *SIAM J. Appl. Math.*, **28** (2008), 1026-1044.

[120] Uhlmann, G., Wang, J.-N and Wu, C. T., Reconstruction of inclusions in an elastic body, *Journal de Mathématiques Pures et Appliquées*, **91** (2009), 569-582.

[121] Zhdanov, M. S. Keller, G. V., The geoelectrical methods in geophysical exploration, *Methods in Geochemistry and Geophysics*, **31** (1994), Elsevier.

[122] Zhou, T., Reconstructing electromagnetic obstacles by the enclosure method, *Inverse Problems and Imaging*, **4** (2010), 547-569.

[123] Zou, Y. and Guo, Z, A review of electrical impedance techniques for breast cancer detection, *Med. Eng. Phys.*, **25** (2003), 79-90.