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Bérenger/Maxwell with Discontinuous Absorptions: Existence, Perfection, and No Loss

Laurence Halpern * Jeffrey Rauch †‡

Abstract

We analyse Bérenger's split algorithm applied to the system version of the two dimensional wave equation with absorptions equal to Heaviside functions of x_j , $j = 1, 2$. The methods form the core of the analysis [11] for three dimensional Maxwell equations with absorptions not necessarily piecewise constant. The split problem is well posed, has no loss of derivatives (for divergence free data in the case of Maxwell), and is perfectly matched.

Keywords. PML. well posedness, loss of derivatives, perfect matching.

AMS Subject Classification. 65M12, 65M55, 30E10.

1 Absorbing strategy and corners

Bérenger's algorithm is an absorbing layer method for computing approximate solutions of Maxwell's equations in vacuum on \mathbb{R}^d with $d = 2, 3$. It can be applied to other equations but was designed for Maxwell and excels in that context.

One is interested in the values of the solution only in a compact set. One chooses a rectangle \mathbf{R} larger, but ideally not too much larger. The rectangle \mathbf{R} is inside a larger one, \mathbf{S} as in Figure 1.

The boundary of \mathbf{S} is not a physical boundary. At the boundary one must impose artificial boundary conditions that are designed to be as transparent

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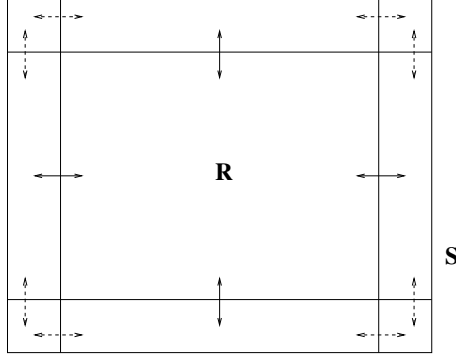


Figure 1: Geometry of Bérenger's algorithm

to outgoing waves as possible. This poses a serious difficulty at the corners of \mathbf{S} as the boundary conditions on the adjacent faces are usually different. The absorbing layer strategy aims to reduce the intensity of the waves that reach the exterior boundary. This places less demands on the transparent boundary condition. In the rectangle \mathbf{R} one solves Maxwell's equations. In the region between the two rectangles one solves a dissipative equation that represents an absorbing layer. The unknowns in \mathbf{R} are an electric field and magnetic field so six scalar functions. For Bérenger's absorbing layer, the unknown in the layer has more components than the original equation.

This paper as well as its predecessor, [10], do not address the external boundary. Take $\mathbf{S} = \mathbb{R}^d$. The outer boundary from Figure 1 recedes to infinity as in Figure 2.

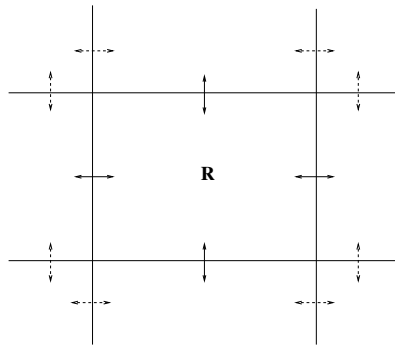


Figure 2: Geometry with $\mathbf{S} = \mathbb{R}^d$. No external boundary.

A great advantage of Bérenger algorithm is that it explicitly treats the internal corners in \mathbf{R} . The problem discussed here is to show that the algorithm of Bérenger defines a well posed problem and is perfectly matched at the boundaries of \mathbf{R} , including corners. *Contrary to popular belief, it satisfies estimates without loss of derivatives for the divergence free solutions of Maxwell's equations.*

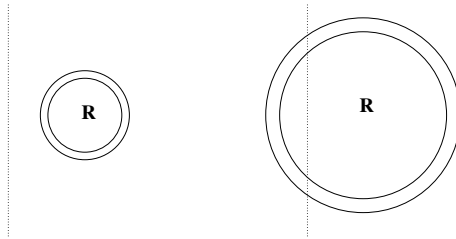


Figure 3: Exact initial data and later solution. A perfectly matched layer would show the same result to the right of the dotted boundary

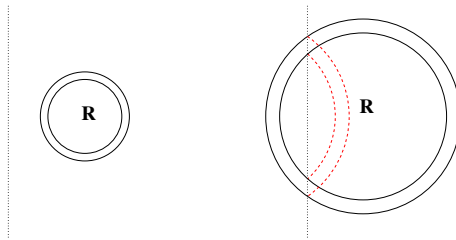


Figure 4: An imperfectly matched layer would have errors of reflection

Perfect matching (following [2]) means that when used to compute solutions with sources in \mathbf{R} , the computed solution in \mathbf{R} is equal to the exact solution as in Figure 3. The surrounding medium does not pollute the solution in \mathbf{R} with any reflections at the boundaries or diffracted waves from the corners. It is an unreasonable demand that is met nevertheless.

To describe the proof with a minimum of technical obstructions, we discuss the case $d = 2$ and make two additional simplifications. The first is to take $\mathbf{R} =]0, \infty[^2$ equal to the positive quadrant, as in Figure 5. The origin is the unique corner in this geometry.

The second is to replace Maxwell's equations by the system analogue of the

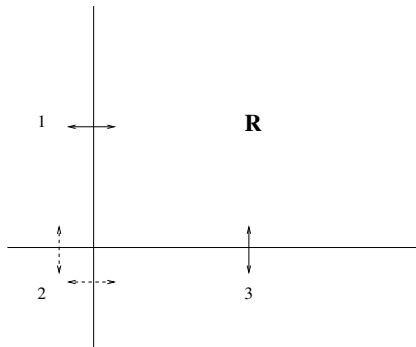


Figure 5: Simplified geometry. One corner, no external boundary.

wave equation,

$$L(\partial_t, \partial_x) := \partial_t + \sum_j A_j \partial_j := \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2. \quad (1.1)$$

Acknowledgement. Thanks to G. Métivier for suggestions leading to cleaner estimates.

2 Bérenger's splitting

The objective is to approximate in \mathbf{R} the values of a solution \mathcal{U} to an equation on \mathbb{R}^{1+2} ,

$$L(\partial_t, \partial_x)\mathcal{U} = \left(\partial_t + A_1 \partial_1 + A_2 \partial_2 \right) \mathcal{U} = F, \quad \mathcal{U} = F = 0 \quad \text{for } t < 0. \quad (2.1)$$

The source term F is supported strictly inside \mathbf{R} , that is, there is a bounded open ω with $\bar{\omega} \subset \mathbf{R}$ and

$$\text{supp } F \subset [0, \infty[\times \bar{\omega}. \quad (2.2)$$

Bérenger's algorithm introduces a \mathbb{C}^2 valued function V on $\mathbb{R}_t \times \mathbf{R}$ and a \mathbb{C}^4 valued function $\tilde{U} := (U^1, U^2)$ on the complement, $\mathbb{R}_t \times (\mathbb{R}^2 \setminus \bar{\mathbf{R}})$. The U^j take values in \mathbb{C}^2 . The equation for V is simply the original equation

$$LV = F, \quad \text{in } \mathbb{R} \times \mathbf{R}, \quad V = 0 \quad \text{for } t < 0. \quad (2.3)$$

The equations for \tilde{U} on $\mathbb{R} \times (\mathbb{R}^2 \setminus \mathbf{R})$ are written $\tilde{L}\tilde{U} = 0$ and are called the *split equations*. They are

$$\begin{aligned} (\partial_t + \sigma_1(x_1))U^1 + A_1\partial_1(U^1 + U^2) &= 0, \\ (\partial_t + \sigma_2(x_2))U^2 + A_2\partial_2(U^1 + U^2) &= 0. \end{aligned} \tag{2.4}$$

The first equation has the ∂_1 derivatives and the other has the ∂_2 terms. Where the σ_j vanish, the sum $U^1 + U^2$ satisfies the original equation.

The functions σ_j called *absorption coefficients* are nonnegative, bounded and supported in $] -\infty, 0]$. The absorptions vanish in \mathbf{R} . The original algorithm proposed by Bérenger made the choice

$$\sigma_j := c\mathbf{1}_{]-\infty, 0]}, \quad c > 0 \tag{2.5}$$

equal to a constant multiple of the indicator function of $] -\infty, 0]$, see [6]. In this case the operator on the left of (2.4) has constant coefficients in each of the three subdomains of the complement of \mathbf{R} indicated as domains 1,2, and 3 in Figure 5. As the coefficients are discontinuous across the coordinate axes, solutions will not be smooth across the axes.

The differential equations (2.3) in $\mathbb{R} \times \mathbf{R}$ and (2.4) on $\mathbb{R}_t \times (\mathbb{R}^2 \setminus \mathbf{R})$ are linked by a transmission condition that requires that the function

$$U := \begin{cases} V & \text{for } x \in \mathbf{R} \\ U^1 + U^2 & \text{for } x \in \mathbb{R}^2 \setminus \mathbf{R} \end{cases} \tag{2.6}$$

is continuous across the coordinate axes. This asserts equality of V and $U^1 + U^2$ across the segments indicated with solid arrows in the Figure 5 and continuity of $U^1 + U^2$ across the segments indicated with dotted arrows. These transmissions are insured by constructing $U(t) \in H^1(\mathbb{R}^2)$ for which traces from the two sides of segments on the axis are equal.

The miraculous behavior of Bérenger's algorithm [6, 7, 8] is that the discontinuities of $\sigma_j(x_j)$ produce no reflections at all. For σ_j that are sufficiently smooth or when there is only one non zero σ_j this is proved in [10]. The present paper addresses the case of the discontinuous absorptions (2.5). §9 proves that perfection for Bérenger follows from the well posedness of the transmission problem.

The well posedness of the transmission problem with more than one discontinuous absorption has remained open for more than twenty years. The method presented here also works in the case of smooth absorptions. In all of these cases the transmission problem has no loss of derivatives for

divergence free solutions of Maxwell. The proofs of [13] , [10], for differentiable σ_j lose derivatives. So do the constant coefficient problems in regions 1,2,3, see Arbarbanel and Gottlieb [1] . Our result without loss seems in contradiction. The explanation of this apparent paradox, is that the loss of derivatives from [1] does not occur for divergence free solutions of Maxwell's equations. A key element is the construction of a divergence free condition that is propagated by the split equations.

For the wave equation system, the result is the following.

Definition 2.1 Denote by Ω the complement of the axes in \mathbb{R}^2

$$\Omega := \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \neq 0 \right\}.$$

Theorem 2.2 There are positive constants C and λ_0 so that if F is a distribution on $\mathbb{R} \times \mathbb{R}^2$ supported in $[0, \infty[\times \bar{\omega}$ and for $\lambda > \lambda_0$

$$\{F_t, \nabla F\} \in e^{\lambda t} L^2(\mathbb{R}; L^2(\mathbb{R}^2)),$$

there is a unique triple V, U^1, U^2 with V supported in $[0, \infty[\times \bar{\mathbf{R}}$, U^j supported in $[0, \infty[\times \bar{\Omega} \setminus \mathbf{R}$,

$$V, U^j, V_t, U_t^j, \nabla_x(V|_{\mathbf{R}}), \nabla_x(U^j|_{\Omega \setminus \mathbf{R}}) \in e^{\lambda t} L^2(\mathbb{R}; L^2(\mathbb{R}^2)),$$

and satisfying the Béranger transmission problem in the sense that (2.3) and (2.4) are satisfied and U defined in (2.6) belongs to $e^{\lambda t} L^2(\mathbb{R}; H^1(\mathbb{R}^2))$. In addition with $\|\cdot\|$ denoting norm in $L^2(\mathbb{R}^2)$,

$$\begin{aligned} & \int e^{-2\lambda t} (\lambda^2 \|V, U^j\|^2 + \|\partial_t V, \partial_t U^j\|^2 + \|\nabla_x(V|_{\mathbf{R}}, (U^1 + U^2)|_{\Omega \setminus \mathbf{R}})\|^2) dt \\ & \leq C \int e^{-2\lambda t} (\|\partial_t F(t)\|^2 + \|\nabla F(t)\|^2) dt. \end{aligned}$$

Both sides estimate H^1 norms. There is almost no loss of derivatives. One has time derivatives of U^j but x derivatives of only $U^1 + U^2$.

The majority of the paper is devoted to the proof of existence. The solution is constructed by solving, with good estimates, the Laplace transformed equations. From the Laplace transformed system a scalar second order elliptic equation resembling the Helmholtz equation is extracted.

Open problems. i. The estimates for the Laplace transform are weaker than those in the Hille-Phillips Theorem of semigroup theory. For the wave

equation with elliptic generator, the proof in [10] shows that there a C^0 semigroup on L^2 . Analogous estimates for divergence free Maxwell are not known. **ii.** If the source term F vanishes for $t \geq T$ it is not known is the resulting free evolution of the Berenger system is uniformly bounded in time.

3 Reduction to a single unknown u

In the next Lemma, the equations involving $L(\tau, \partial_x)$ and $\tilde{L}(\tau, \partial_x)$ come from Laplace transformation in time of the Bérenger equations on $\mathbb{R} \times \mathbf{R}$ and $\mathbb{R} \times (\Omega \setminus \mathbf{R})$ respectively.

Lemma 3.1 *Suppose that v and u^1, u^2 are distributions on \mathbf{R} and $\Omega \setminus \mathbf{R}$ respectively, $f \in L^2(\mathbf{R})$ with $\text{supp } f$ disjoint from $\partial\mathbf{R}$, and $\text{Re } \tau > 0$. Suppose that*

$$L(\tau, \partial)v = f, \quad \text{and,} \quad \tilde{L}(\tau, \partial)(u^1, u^2) = 0 \quad \text{in } \Omega \setminus \mathbf{R}. \quad (3.1)$$

Define $u = v$ in \mathbf{R} and $u = u^1 + u^2$ in $\Omega \setminus \mathbf{R}$.

i. Then on Ω one has

$$\tau u + \frac{\tau}{\tau + \sigma_1(x_1)} A_1 \partial_1 u + \frac{\tau}{\tau + \sigma_2(x_2)} A_2 \partial_2 u = f. \quad (3.2)$$

ii. Conversely if $u \in \mathcal{D}'(\Omega)$ satisfies (3.2), then setting $v = u$ on \mathbf{R} and by defining u^1 and u^2 on $\Omega \setminus \mathbf{R}$ by the analogue of (2.4),

$$\begin{aligned} (\tau + \sigma_1(x_1))u^1 + A_1 \partial_1 u &= 0, \\ (\tau + \sigma_2(x_2))u^2 + A_2 \partial_2 u &= 0. \end{aligned} \quad (3.3)$$

yields (3.1).

Proof of Lemma. i. Since the functions σ_j vanish in \mathbf{R} , equation (3.2) is identical to the equation $Lv = f$ on \mathbf{R} . On $\mathbb{R}^2 \setminus \overline{\mathbf{R}}$ the function f vanishes. In that domain, multiply the first equation in (3.3) by $\tau + \sigma_2(x_2)$ and the second by $\tau + \sigma_1(x_1)$ to find

$$\begin{aligned} (\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))u^1 + (\tau + \sigma_2(x_2))A_1 \partial_1 u &= 0, \\ (\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))u^2 + (\tau + \sigma_1(x_1))A_2 \partial_2 u &= 0. \end{aligned} \quad (3.4)$$

Add to find

$$(\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))u + (\tau + \sigma_2(x_2))A_1 \partial_1 u + (\tau + \sigma_1(x_1))A_2 \partial_2 u = 0. \quad (3.5)$$

Multiplying by $\tau/[(\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))]$ yields (3.2).

ii. Immediate. \square

Remark 3.1 **i.** *The u^j are one derivative less smooth than u .* **ii.** *Equation (3.2) holds only on the complement of the axes since we have multiplied and divided by factors $\tau + \sigma_j(x_j)$ that are not smooth on the axes.*

4 A reduced scalar wave equation

For $\operatorname{Re} \tau > 0$ define a scalar divergence form second order elliptic operator p with smooth bounded coefficients on the open set Ω by

$$p(\tau, x, \partial) := \partial_1 \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} \partial_1 + \partial_2 \frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} \partial_2. \quad (4.1)$$

The corresponding quadratic form and Dirichlet integral are

$$\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} \xi_1^2 + \frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} \xi_2^2, \quad \int \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} |\partial_1 \psi|^2 + \frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} |\partial_2 \psi|^2 \right) dx.$$

The coefficients are constant in each of the four components of Ω .

Lemma 4.1 **i.** *For $\operatorname{Re} \tau > 0$, $p(\tau, x, \partial)$ maps $H^1(\mathbb{R}^2)$ to $H^{-1}(\mathbb{R}^2)$.*

ii. *If $u \in \mathcal{D}'(\Omega)$ satisfies (3.2) with $f \in L^2(\mathbb{R}^2)$ supported in $\bar{\omega}$, then on Ω*

$$\begin{aligned} & (\tau + \sigma_1(x_1)) (\tau + \sigma_2(x_2)) u - p(\tau, x, \partial) u = \\ & \frac{(\tau + \sigma_1(x_1)) (\tau + \sigma_2(x_2))}{\tau^2} L \left(-\tau, \frac{\tau}{\tau + \sigma_1(x_1)} \partial_1, \frac{\tau}{\tau + \sigma_2(x_2)} \partial_2 \right) f. \end{aligned} \quad (4.2)$$

iii. *For $\operatorname{Re} \tau > 0$, the coefficients satisfy,*

$$\left| \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} - 1 \right| + \left| \frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} - 1 \right| \leq \frac{C}{|\tau|}. \quad (4.3)$$

iv. *If $u \in H^1(\mathbb{R}^2)$ satisfies (4.2) on Ω then it satisfies (4.2) on \mathbb{R}^2 . In this case, the triple v, u^1, u^2 defined in terms of u as in Lemma 3.1 is a solution of the transformed Béranger transmission problem.*

Proof. **i.** This is a consequence of the facts that the σ_j are bounded and nonnegative.

ii. Since $A_j^2 = I$ and $A_1A_2 + A_2A_1 = 0$ it follows that

$$\begin{aligned} L(\tau, \xi)^2 &= (\tau^2 + |\xi|^2) I + 2\tau(\xi_1A_1 + \xi_2A_2) \\ &= (\tau^2 + |\xi|^2) I + 2\tau(L(\tau, \xi) - \tau I) \\ &= (|\xi|^2 - \tau^2)I + 2\tau L(\tau, \xi). \end{aligned}$$

Therefore,

$$\begin{aligned} -L\left(\tau, \frac{\tau}{\tau + \sigma_1}\xi_1, \frac{\tau}{\tau + \sigma_2}\xi_2\right)^2 &= \\ \left(\tau^2 - \left(\frac{\tau}{\tau + \sigma_1}\xi_1\right)^2 - \left(\frac{\tau}{\tau + \sigma_2}\xi_2\right)^2\right)I - 2\tau L\left(\tau, \frac{\tau}{\tau + \sigma_1}\xi_1, \frac{\tau}{\tau + \sigma_2}\xi_2\right). \end{aligned}$$

Consequently one has the following identity among piecewise constant coefficient partial differential operators on Ω ,

$$\begin{aligned} -L\left(\tau, \frac{\tau}{\tau + \sigma_1}\partial_1, \frac{\tau}{\tau + \sigma_2}\partial_2\right)^2 &= \\ \left(\tau^2 - \left(\frac{\tau}{\tau + \sigma_1}\right)^2\partial_1^2 - \left(\frac{\tau}{\tau + \sigma_2}\right)^2\partial_2^2\right)I - 2\tau L\left(\tau, \frac{\tau}{\tau + \sigma_1}\partial_1, \frac{\tau}{\tau + \sigma_2}\partial_2\right). \end{aligned}$$

Multiplying (3.2) by $-L$ yields

$$\begin{aligned} L\left(\tau, \frac{\tau}{\tau + \sigma_1}\partial_1, \frac{\tau}{\tau + \sigma_2}\partial_2\right)^2 u &= L\left(\tau, \frac{\tau}{\tau + \sigma_1}\partial_1, \frac{\tau}{\tau + \sigma_2}\partial_2\right) f - 2\tau f \\ &= L\left(-\tau, \frac{\tau}{\tau + \sigma_1}\partial_1, \frac{\tau}{\tau + \sigma_2}\partial_2\right) f \end{aligned}$$

Multiplying by

$$\frac{(\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))}{\tau^2}$$

yields (4.2).

iii. Estimate

$$\left|\frac{\tau + \sigma_2}{\tau + \sigma_1} - 1\right| = \left|\frac{\sigma_2 - \sigma_1}{\tau + \sigma_1}\right| \leq \frac{\|\sigma_2 - \sigma_1\|_{L^\infty}}{|\tau + \sigma_1|} \leq \frac{\|\sigma_2 - \sigma_1\|_{L^\infty}}{|\tau|}.$$

iv. **Step I.** Define

$$g := \frac{(\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))}{\tau^2} L\left(-\tau, \frac{\tau}{\tau + \sigma_1(x_1)}\partial_1, \frac{\tau}{\tau + \sigma_2(x_2)}\partial_2\right) f. \quad (4.4)$$

Since $u \in H^1(\mathbb{R}^2)$, $f \in L^2$, and, the support of f is disjoint from $\partial\Omega$, it follows that,

$$\left((\tau + \sigma_1)(\tau + \sigma_2) - p \right) u - g \in H^{-1}(\mathbb{R}^2). \quad (4.5)$$

Since the equation holds on Ω , the support of the left hand side of (4.5) is contained in $\{x : x_1 x_2 = 0\}$.

Step II. Show that the support intersects the $\{x_1 = 0\}$ axis at most at the origin.

Since $u \in H^1(\mathbb{R}^2)$, the derivatives $\partial_j u \in L^2$. Because the coefficients are bounded, this implies that

$$\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u, \quad \frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} A_2 \partial_2 u \in L^2(\mathbb{R}^2).$$

Equation (4.2) expresses

$$\partial_1 \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right)$$

as a sum of terms each in $L^2([0, \infty[_{x_1} ; H^{-1}(\mathbb{R}_{x_2}))$. Therefore,

$$\partial_1 \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right) \in L^2([0, \infty[_{x_1} ; H^{-1}(\mathbb{R}_{x_2})).$$

Denote by χ_{\pm} the characteristic function of $\{x \in \mathbb{R}^2 : \pm x_1 > 0\}$. Since $u \in H^1(\mathbb{R}^2)$, it follows that in the sense of distributions on \mathbb{R}^2 ,

$$\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u = \chi_+ \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u + \chi_- \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u, \quad (4.6)$$

and

$$\begin{aligned} \partial_1 \left(\chi_+ \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right) = \\ \chi_+ \partial_1 \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right) + \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \Big|_{x_1=0^+} \delta(x_1). \end{aligned} \quad (4.7)$$

The trace on the right is a well defined element of $H^{-1/2}(\mathbb{R}_{x_2})$. From $x_1 < 0$ one has

$$\begin{aligned} \partial_1 \left(\chi_- \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right) = \\ \chi_- \partial_1 \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right) - \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \Big|_{x_1=0^-} \delta(x_1). \end{aligned} \quad (4.8)$$

The transmission condition on the parts $x_2 \neq 0$ of the x_1 -axis implies that

$$\text{supp} \left[\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \Big|_{x_1=0^+} - \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \Big|_{x_1=0^-} \right] \subset \{(0, 0)\}.$$

Since the only element of $H^{-1/2}(\mathbb{R})$ supported at the origin is 0, it follows that

$$\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \Big|_{x_1=0^+} - \frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \Big|_{x_1=0^-} = 0. \quad (4.9)$$

Summing (4.8) and (4.7) using (4.9) proves the distribution derivative

$$\begin{aligned} \partial_1 \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right) = \\ \chi_- \partial_1 \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right) + \chi_+ \partial_1 \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} A_1 \partial_1 u \right). \end{aligned}$$

The distribution derivative is just the ordinary derivative computed on each side of the axis.

If χ^\pm denote the characteristic functions of $\{x \in \mathbb{R}^2 : \pm x_2 > 0\}$, an analogous computation shows that

$$\begin{aligned} \partial_2 \left(\frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} A_2 \partial_2 u \right) = \\ \chi^- \partial_2 \left(\frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} A_2 \partial_2 u \right) + \chi^+ \partial_2 \left(\frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} A_2 \partial_2 u \right). \end{aligned}$$

These identities imply that the differential equation is satisfied on $\mathbb{R}^2 \setminus 0$. Therefore the difference between the left and right hand sides is an element of $H^{-1}(\mathbb{R}^2)$ that is supported at 0. It therefore vanishes.

Step III. Since $u \in H^1(\mathbb{R}^2)$ its traces from above and below on $\{x_2 = 0\}$ are equal. This proves the desired transmission condition for (v, u^1, u^2) on the positive and negative x_1 -axis. An analogous argument works for the x_2 -axis. \square

5 Reduced equation estimate

We seek u in $H^1(\mathbb{R}^2)$ for sources F with values $H^1(\mathbb{R}^2)$. Since g is given in terms of first derivatives of F that yields $g \in L^2$. Therefore estimate (5.1) is the desired regularity.

Proposition 5.1 *There are constants C, M so that for all $g \in L^2(\mathbb{R}^2)$ and τ with $\operatorname{Re} \tau > M$ there is one and only one solution $u \in H^1(\mathbb{R}^2)$ to*

$$\left((\tau + \sigma_1(x_1)) (\tau + \sigma_2(x_2)) - p(\tau, x, \partial) \right) u = g.$$

In addition

$$(\operatorname{Re} \tau) \|u\| + \|\nabla u\| \leq C \|g\|. \quad (5.1)$$

Proof. It suffices to prove (5.1) as an *a priori* estimate. The estimate is proved by analysing the real and imaginary parts of the identity

$$\begin{aligned} (u, g) &= ((\tau + \sigma_1(x_1)) (\tau + \sigma_2(x_2)) u, u) + \\ &\quad \int \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} |\partial_1 u|^2 + \frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} |\partial_2 u|^2 \right) dx. \end{aligned} \quad (5.2)$$

The integral is not far from its unperturbed value

$$\left| \int \left(\frac{\tau + \sigma_2(x_2)}{\tau + \sigma_1(x_1)} |\partial_1 u|^2 + \frac{\tau + \sigma_1(x_1)}{\tau + \sigma_2(x_2)} |\partial_2 u|^2 \right) dx - \|\nabla u\|^2 \right| \leq C \frac{\|\nabla u\|^2}{|\tau|}. \quad (5.3)$$

One can take $C = \sup |\sigma_1(x_1) - \sigma_2(x_2)|$.

Extract the information from the imaginary part of (5.2). Compute using $\operatorname{Im} \tau^2 = 2 \operatorname{Im} \tau \operatorname{Re} \tau$,

$$\begin{aligned} \operatorname{Im} \left((\tau + \sigma_1(x_1)) (\tau + \sigma_2(x_2)) \right) &= \operatorname{Im} \tau^2 + (\sigma_1(x_1) + \sigma_2(x_2)) \operatorname{Im} \tau \\ &= \operatorname{Im} \tau \left(2 \operatorname{Re} \tau + \sigma_1(x_1) + \sigma_2(x_2) \right). \end{aligned}$$

Therefore

$$\operatorname{Im} \left((\tau + \sigma_1(x_1)) (\tau + \sigma_2(x_2)) u, u \right) = \operatorname{Im} \tau \int (2 \operatorname{Re} \tau + \sigma_1(x_1) + \sigma_2(x_2)) |u|^2 dx.$$

Taking absolute values and noting that $2 \operatorname{Re} \tau + \sigma_1(x_1) + \sigma_2(x_2) \geq 2 \operatorname{Re} \tau$ yields using (5.2) and (5.3)

$$\begin{aligned} 2 |\operatorname{Im} \tau| \operatorname{Re} \tau \|u\|^2 &\leq \left| \operatorname{Im} \left((\tau + \sigma_1(x_1)) (\tau + \sigma_2(x_2)) u, u \right) \right| \\ &\leq \|u\| \|g\| + \frac{C \|\nabla u\|^2}{|\tau|}. \end{aligned}$$

This is used to estimate $|\operatorname{Im} \tau| \|u\|$. Precisely multiply by $|\operatorname{Im} \tau|/2 \operatorname{Re} \tau$ to find

$$|\operatorname{Im} \tau|^2 \|u\|^2 \leq \|u\| \|g\| \frac{|\operatorname{Im} \tau|}{2 \operatorname{Re} \tau} + \|\nabla u\|^2 \frac{C |\operatorname{Im} \tau|}{2 |\tau| \operatorname{Re} \tau}. \quad (5.4)$$

Next extract the information in the real part of (5.2). Compute the real part of the zero order term beginning with

$$\begin{aligned} \operatorname{Re}\left((\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))\right) &= \operatorname{Re} \tau^2 + (\sigma_1 + \sigma_2)\operatorname{Re} \tau + \sigma_1\sigma_2 \\ &\geq \operatorname{Re} \tau^2 = (\operatorname{Re} \tau)^2 - (\operatorname{Im} \tau)^2. \end{aligned}$$

Therefore the real part of (5.2) using (5.3) yields

$$(\operatorname{Re} \tau)^2 \|u\|^2 + \|\nabla u\|^2 \leq \|u\| \|g\| + (\operatorname{Im} \tau)^2 \|u\|^2 + \frac{C \|\nabla u\|^2}{|\tau|}$$

Use (5.4) to find

$$(\operatorname{Re} \tau)^2 \|u\|^2 + \|\nabla u\|^2 \leq \|u\| \|g\| \left(1 + \frac{|\operatorname{Im} \tau|}{2 \operatorname{Re} \tau}\right) + \left(1 + \frac{|\operatorname{Im} \tau|}{2 \operatorname{Re} \tau}\right) \frac{C \|\nabla u\|^2}{|\tau|}. \quad (5.5)$$

Choose $M > 1$ so that for $\operatorname{Re} \tau > M$ one has

$$\left(1 + \frac{|\operatorname{Im} \tau|}{2 \operatorname{Re} \tau}\right) \frac{C + 1}{|\tau|} \leq \frac{1}{10}. \quad (5.6)$$

Then,

$$(\operatorname{Re} \tau)^2 \|u\|^2 + \|\nabla u\|^2 \leq \frac{1}{10} \|\tau u\| \|g\| + \frac{1}{10} \|\nabla u\|^2. \quad (5.7)$$

The differential equation expresses

$$\tau^2 u = pu - [\tau(\sigma_1 + \sigma_2) + \sigma_1\sigma_2]u + g$$

Taking the L^2 scalar product with u shows that for $|\tau| > 1$,

$$\|\tau u\|^2 \lesssim \|\nabla u\|^2 + \|\tau u\| \|u\| + \|u\| \|g\|$$

Therefore

$$\|\tau u\|^2 \lesssim \|\nabla u\|^2 + \|u\|^2 + \|g\|^2, \quad \|\tau u\| \lesssim \|\nabla u\| + \|u\| + \|g\|.$$

Inject in (5.7) to find with constants C changing from line to line

$$\begin{aligned} (\operatorname{Re} \tau)^2 \|u\|^2 + \|\nabla u\|^2 &\leq \frac{1}{10} \|\nabla u\|^2 + C \|g\| (\|\nabla u\| + \|u\| + \|g\|) \\ &\leq \frac{1}{5} \|\nabla u\|^2 + C (\|u\|^2 + \|g\|^2) \end{aligned}$$

For $\operatorname{Re} \tau$ large this yields the desired estimate. \square

6 Laplace Transform, Paley-Wiener, and Plancherel

Recall the Paley-Wiener-Plancherel characterization of Laplace transforms of square integrable functions.

The Laplace transform of a distribution F supported in $t \geq 0$ and so that $e^{-\lambda t} F \in L^1(\mathbb{R})$ for $\lambda > M$, is defined for $\operatorname{Re} \tau > M$ by

$$\widehat{F}(\tau) := \int e^{-\tau t} F(t) dt.$$

It is defined and holomorphic in a half space $\operatorname{Re} \tau > M$. Our functions F take values in a Hilbert space H .

Theorem 6.1 *The Laplace transforms of functions F supported in $t \geq 0$ and so that $e^{-Mt} F \in L^2(\mathbb{R}; H)$ are exactly the functions $G(\tau)$ holomorphic in $\operatorname{Re} \tau > M$ with values in H and so that*

$$\sup_{\lambda > M} \int_{\operatorname{Re} \tau = \lambda} \|\widehat{F}(\tau)\|_H^2 |d\tau| < \infty.$$

In this case $\widehat{F}(\tau)$ has trace at $\operatorname{Re} \tau = M$ that is square integrable and

$$\int e^{-2Mt} \|F(t)\|_H^2 dt = \sup_{\lambda > M} \int_{\operatorname{Re} \tau = \lambda} \|\widehat{F}(\tau)\|_H^2 |d\tau| = \int_{\operatorname{Re} \tau = M} \|\widehat{F}(\tau)\|_H^2 |d\tau|.$$

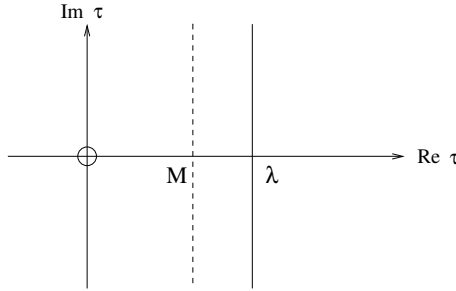


Figure 6: Laplace transform contours

With an eye to applying Theorem 6.1 to the solution of the transformed equation (4.2), we estimate the right hand member g in (4.4).

Since the support of \widehat{F} lies strictly inside \mathbf{R} , the σ_j vanish identically on the support of \widehat{F} . Therefore, $g = h(\tau, x) L(-\tau, \partial_1, \partial_2) \widehat{F}$, $h(\tau, x)$ is bounded on $\{\operatorname{Re} \tau \geq 1\}$ so,

$$\|g\| \lesssim \|\tau \widehat{F}\| + \|\nabla \widehat{F}\|.$$

Using this in (5.1) yields

$$(\operatorname{Re} \tau)^2 \|u\|^2 + \|\nabla u\|^2 \lesssim \|\tau \widehat{F}\|^2 + \|\nabla \widehat{F}\|^2.$$

Integrate to find

$$\begin{aligned} \int_{\operatorname{Re} \tau = \lambda} (\lambda^2 \|u(\tau)\|^2 + \|\nabla u(\tau)\|^2) |d\tau| &\lesssim \int_{\operatorname{Re} \tau = \lambda} (\|\tau \widehat{F}\|^2 + \|\nabla \widehat{F}\|^2) |d\tau| \\ &= \int_{\operatorname{Re} \tau = \lambda} (\|\widehat{F}_t\|^2 + \|\nabla \widehat{F}_t\|^2) |d\tau| \\ &= \int e^{-2\lambda t} (\|F_t(t)\|^2 + \|\nabla F(t)\|^2) dt. \end{aligned}$$

Therefore if $e^{-\lambda t}\{F_t, \nabla_x F\} \in L^2(\mathbb{R}; L^2(\mathbb{R}^2))$ for some $\lambda > M$, then the function $u(\tau)$, clearly holomorphic, satisfies the necessary and sufficient condition from Theorem 6.1, perhaps with a larger M since (5.1) is known only for λ larger than a constant that could be larger than M .

Applying Theorem 6.1 shows that there is a unique function $U(t)$ supported in $t > 0$ so that $e^{-\lambda t}U$ is square integrable in time with values in $H^1(\mathbb{R}^2)$ and whose Laplace transform in $u(\tau)$. In addition one has the estimate

$$\begin{aligned} \lambda^2 \int e^{-2\lambda t} \|U(t)\|^2 dt + \int e^{-2\lambda t} \|\nabla_x U(t)\|^2 dt \\ \lesssim \int e^{-2\lambda t} (\|\partial_t F(t)\|^2 + \|\nabla F(t)\|^2) dt. \end{aligned} \tag{6.1}$$

The constant in \lesssim is independent of λ large.

7 Reduced equations to Bérenger split equations

From the function $U(t)$ constructed in the last section, we construct a solution of the Bérenger split system. The split system requires a function $V(t)$ on \mathbf{R} and split solutions U^1 and U^2 defined for $x \in \Omega \setminus \mathbf{R}$.

Define

$$V := U|_{\mathbf{R}}.$$

Estimates for V are direct consequences of (6.1).

The split solutions U^j are the unique solutions of (2.4) with U^j vanishing for $t < 0$. The Laplace transforms satisfy

$$\widehat{U}^j = \frac{-1}{\tau + \sigma_j(x_j)} A_j \partial_j \widehat{U} = \frac{-1}{\tau + \sigma_j(x_j)} A_j \partial_j u.$$

Estimate (5.1) implies

$$\|\widehat{U}^j\| \lesssim \frac{1}{|\tau|} \|\nabla u\|, \quad \|\tau \widehat{U}^j\| \lesssim \|\nabla u\|, \quad (7.1)$$

Plancherel's theorem yields with constant independent of λ large,

$$\int e^{-2\lambda t} \|\partial_t U^j(t)\|^2 dt \leq C \int e^{-2\lambda t} \|\nabla u(t)\|^2 dt.$$

To estimate x -derivatives of the U^j , we must be careful because the U^j are typically discontinuous across both the x_j axes in $\mathbb{R}^2 \setminus \overline{\mathbf{R}}$. On the other hand equations (3.3) express spatial derivatives in terms of time derivative away from those axes. These x derivatives are the distribution derivative in regions 1,2, and 3 from Figure 2. These are exactly the open components of $\Omega \setminus \mathbf{R}$ yielding,

$$\int e^{-2\lambda t} \|\nabla(U^1 + U^2)|_{\Omega \setminus \mathbf{R}}\|^2 dt \leq C \int e^{-2\lambda t} \|\partial_t U^j(t)\|^2 dt.$$

This yields the symmetric estimate without loss of derivatives for the primitive variables of Béranger

$$\begin{aligned} \int e^{-2\lambda t} \|\nabla_x(V, (U^1 + U^2)|_{\Omega \setminus \mathbf{R}})(t)\|^2 dt \leq \\ C \int e^{-2\lambda t} \left(\|\partial_t F(t)\|^2 + \|\nabla F(t)\|^2 \right) dt. \end{aligned} \quad (7.2)$$

Remark 7.1 *In the case of Maxwell's equations, A_j is not invertible and to find $\partial_j \widehat{U}^j$ one must use in addition the divergence equations.*

It remains to estimate the L^2 norms of $V, U^1, U^2, V_t, U_t^1, U_t^2$. Using the differential equations satisfied by V, U^1, U^2 yields

$$\begin{aligned} \int e^{-2\lambda t} \|\partial_t V, (\partial_t + \sigma_1(x_1))U^1, (\partial_t + \sigma_2(x_2))U^2\|^2 dt \leq \\ C \int e^{-2\lambda t} \left(\|\partial_t F(t)\|^2 + \|\nabla F(t)\|^2 \right) dt. \end{aligned} \quad (7.3)$$

Integration in time yields for λ large,

$$\lambda^2 \int e^{-2\lambda t} \|V, U^1, U^2\|^2 dt \lesssim \int e^{-2\lambda t} \left(\|\partial_t F(t)\|^2 + \|\nabla F(t)\|^2 \right) dt. \quad (7.4)$$

This completes the existence part of the proof of Theorem 2.2.

8 Proof of uniqueness

Proof. Suppose that $0 < T < \infty$ and V, U^1, U^2 is a solution for the Bérenger split system on $-\infty < t < T$ that vanishes for $t < 0$ and so that V and U^j belong to $H^1(]0, T[\times \mathbf{R})$ and $H^1(]0, T[\times (\Omega \setminus \mathbf{R}))$ respectively. It suffices to prove that V and U^j vanish on for $0 < t < T_1$ for arbitrary $T_1 \in]0, T[$.

Choose $\chi \in C^\infty(\mathbf{R})$ supported in $] - \infty, T[$ and equal to 1 on $] - \infty, T_1]$. It suffices to show that χV and $\chi(U^1, U^2)$ vanish for $t \leq T$. Compute on \mathbf{R} and $\Omega \setminus \mathbf{R}$ respectively,

$$L(\partial_t, \partial_x)(\chi(t)V) = \chi'(t)V, \quad \tilde{L}(\partial_t, \partial_x)\left(\chi(t)(U^1, U^2)\right) = \chi'(t)(U^1, U^2).$$

Define two functions valued in $H^1(\mathbf{R}^2)$

$$\underline{U} := \begin{cases} \chi V & \text{for } x \in \mathbf{R} \\ \chi(U^1 + U^2) & \text{for } x \in \Omega \setminus \mathbf{R} \end{cases}, \quad W := \begin{cases} \chi' V & \text{for } x \in \mathbf{R} \\ \chi'(U^1 + U^2) & \text{for } x \in \Omega \setminus \mathbf{R} \end{cases}.$$

Then derive for the Laplace transform $\underline{u}(\tau)$ of \underline{U} ,

$$L\left(\tau, \frac{\tau}{\tau + \sigma_1(x_1)}\partial_1, \frac{\tau}{\tau + \sigma_1(x_2)}\partial_2\right)\underline{u} = \widehat{W}.$$

Repeat the derivation of (4.2) to find

$$(\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))\underline{u} - p(\tau, x, \partial)\underline{u} = \gamma(\tau),$$

$$\gamma(\tau) := \frac{(\tau + \sigma_1(x_1))(\tau + \sigma_2(x_2))}{\tau^2} L\left(-\tau, \frac{\tau}{\tau + \sigma_1(x_1)}\partial_1, \frac{\tau}{\tau + \sigma_2(x_2)}\partial_2\right)\widehat{W}.$$

In the derivation of (4.2) the multiplication by the discontinuous function $\tau^{-2}\Pi_j(\tau + \sigma_j(x_j))$ was justified by the fact that f had support disjoint from the discontinuities. This time it is because the discontinuous function is bounded and the second factor $L\widehat{W}$ is an element of L^2 .

Since W is supported in $T_1 \leq t \leq T$ it follows that

$$\begin{aligned} \int_{\operatorname{Re} \tau = \lambda} \|\gamma\|^2 |d\tau| &= \int_{T_1}^T e^{-2\lambda t} \left(\|W_t(t)\|^2 + \|\nabla W(t)\|^2 \right) dt \\ &\leq e^{-2\lambda T_1} \int_{T_1}^T \left(\|W_t(t)\|^2 + \|\nabla W(t)\|^2 \right) dt = C e^{-2\lambda T_1}. \end{aligned}$$

with C independent of λ when $\lambda \rightarrow \infty$. The basic estimate for the reduced equation together with Plancherel's theorem implies that

$$\int e^{-2\lambda t} \|\underline{U}(t)\|^2 dt \leq C e^{-2\lambda T_1}, \quad \lambda \rightarrow \infty.$$

It follows that $\underline{U} = 0$ for $t \leq T_1$. So V and $U^1 + U^2$ vanish for $t \leq T_1$.

For U^j use for $t \leq T_1$,

$$\left(\partial_t + \sigma_j(x_j)\right)U^j = -A_j \partial_j \underline{U} = 0.$$

Since $U^j = 0$ for $t < 0$ it follows that $U^j = 0$ for $t < T_1$, completing the proof of uniqueness. \square

9 Proof of Perfection

Theorem 9.1 *The Bérenger split transmission problem is perfectly matched, that is, the solution \mathcal{U} to (2.1) and the solution V, U^1, U^2 of the split equations (2.3) (2.4) satisfy $\mathcal{U}|_{\mathbf{R}} = V$.*

Proof. The proof below is from §3.5 of [10]. Prove that the Laplace transforms are equal, that is $\widehat{\mathcal{U}}|_{\mathbf{R}} = \widehat{V}$ for $\text{Re } \tau \gg 1$.

The literature of perfect matching has a persistent theme of complex changes of variables, see for example [9] that inspired us. The proof below relies on the fact that the complex transformation is real when τ is real and that allows us to prove equality of Laplace transforms for those τ .

Since the Maxwell equations and the Bérenger split problems are well posed with at most exponential growth, the two Laplace transforms are holomorphic. Therefore analytic continuation shows that it suffices to prove that

$$\widehat{\mathcal{U}}(\tau)|_{\mathbf{R}} = \widehat{V}(\tau), \quad 1 \ll \tau \in \mathbb{R}. \quad (9.1)$$

Define U to be equal to V on \mathbf{R} and $U^1 + U^2$ on $\Omega \setminus \mathbf{R}$ one has

$$L\left(\tau, \frac{\tau}{\tau + \sigma_1(x_1)} \partial_1, \frac{\tau}{\tau + \sigma_2(x_2)} \partial_2\right) \widehat{U} = \widehat{F}.$$

For real τ , introduce new variables $X(x) = (X_1, X_2, X_3)$, suppressing the dependence on τ , by

$$\frac{dX_j}{dx_j} = \frac{\tau + \sigma_j(x_j)}{\tau}, \quad X_j(0) = 0.$$

The change of variable is equal to the identity on \mathbf{R} .

Then,

$$\frac{\partial \widehat{\mathcal{U}}(X(x))}{\partial x_j} = \frac{\partial \widehat{\mathcal{U}}}{\partial X_j} \Big|_{X(x)} \frac{dX_j}{dx_j} = \frac{\tau + \sigma_j(x_j)}{\tau} \frac{\partial \widehat{\mathcal{U}}}{\partial X_j} \Big|_{X(x)}$$

Therefore since F is supported where $X = x$ one has

$$\begin{aligned} L\left(\tau, \frac{\tau}{\tau + \sigma_1(x_1)} \partial_1, \frac{\tau}{\tau + \sigma_2(x_2)} \partial_2\right) \left(\widehat{\mathcal{U}}(X(x))\right) &= \left(L(\tau, \partial) \widehat{\mathcal{U}}\right) \Big|_{X(x)} \\ &= \widehat{F}(X(x)) = \widehat{F}(x). \end{aligned}$$

Thus, $\widehat{\mathcal{U}}(X(x))$ satisfies the differential equation defining \widehat{U} so by uniqueness

$$\widehat{\mathcal{U}}(X(x)) = \widehat{U}(x).$$

For $x \in \mathbf{R}$, $X(x) = x$ so $\widehat{\mathcal{U}}(x) = \widehat{U}(x)$, the desired relation (9.1). \square

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