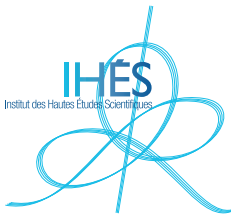

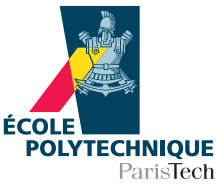

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BLOW UP AND NEAR SOLITON DYNAMICS FOR THE L^2 CRITICAL GKDV EQUATION

YVAN MARTEL, FRANK MERLE, AND PIERRE RAPHAËL

Abstract. These notes present the main results of [22, 23, 24] concerning the mass critical (gKdV) equation $u_t + (u_{xx} + u^5)_x = 0$ for initial data in H^1 close to the soliton. These works revisit the blow up phenomenon close to the family of solitons in several directions: definition of the stable blow up and classification of all possible behaviors in a suitable functional setting, description of the minimal mass blow up in H^1, construction of various exotic blow up rates in H^1, including grow up in infinite time.

1. Introduction

In these notes, we present a series of recent works [22, 23, 24] on the description of the long time dynamics of solutions of the L^2-critical generalized Korteweg–de Vries equation (gKdV)

$$\text{(gKdV)} \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

The main objective of these works is to determine all possible behaviors of solutions of (1) for initial data close in H^1 to the solitons. It turns out that the answer is rather subtle since to provide a satisfactory classification and in particular, to understand the "stable" blow up (in a sense to clarify later), H^1 is not exactly the right setting. For H^1 initial data satisfying a suitable space decay property (see the definition of the set $\mathcal{A}$ in (13)), we prove that only three possible behaviors can occur:

(Blowup) The solution blows up in finite time $T > 0$ with blow up rate $\frac{1}{T-t}$.

(Soliton) The solution is global and locally converges to a soliton in large time.

(Exit) The solution defocuses and eventually exits any small neighborhood of the solitons.

We also clarify the minimal mass blow up dynamics in H^1, proving existence and uniqueness of a minimal mass blow up solution. This minimal mass solution happens to be a universal object in the problem since it appears in the behavior of all solutions in the (Exit) case. We present a new and general approach to both:

- construct the minimal mass solution ;
- prove the universality of the (Exit) case and get "no-return" lemmas based on the properties of the minimal mass solution.

Finally, we exhibit a large variety of "exotic" blow up rates (including grow up in infinite time) for initial data in H^1 which do not satisfying the decay assumption. This proves the necessity of such an assumption to obtain stable blow up.

We will also give various references concerning previous works related to these questions on other nonlinear dispersive equations.

1.1. **Preliminary information on critical (gKdV).** The Cauchy problem for (1) is locally well posed in the energy space H^1 from Kenig, Ponce and Vega [10]: given $u_0 \in H^1$, there exists a unique[1] maximal solution $u(t)$ of (1) in $C([0,T),H^1)$ with either $T=+\infty$, or $T<+\infty$ and then $\lim_{t\to T}\|u_x(t)\|_{L^2}=+\infty$ (blow up). Moreover, for H^1 solutions, mass and energy are conserved by the flow: $\forall t\in[0,T)$,

$$M(u(t))=\int u^2(t)=M_0,\quad E(u(t))=\frac{1}{2}\int u_x^2(t)-\frac{1}{6}\int u^6(t)=E_0,$$

where $M_0=M(u_0)$, $E_0=E(u_0)$. Recall also that the scaling symmetry ($\lambda>0$)

$$u_\lambda(t,x)=\lambda^{\frac{1}{2}}u(\lambda^3 t,\lambda x)$$

leaves invariant the L^2 norm so that the problem is *mass critical*.

The family of travelling wave solutions

$$u(t,x)=\lambda_0^{-\frac{1}{2}}Q\left(\lambda_0^{-1}(x-\lambda_0^{-2}t-x_0)\right),\quad (\lambda_0,x_0)\in\mathbb{R}_+^*\times\mathbb{R},$$

with

$$Q(x)=\left(\frac{3}{\cosh^2(2x)}\right)^{\frac{1}{4}},\quad Q''+Q^5=Q,\quad E(Q)=0, \tag{2}$$

plays a distinguished role in the analysis. It is well-known that, from a variational argument [45], H^1 initial data with subcritical mass $\|u_0\|_{L^2}<\|Q\|_{L^2}$ generate global and bounded solutions ($T=+\infty$).

The study of singularity formation for H^1 initial data with mass close to the minimal mass

$$\|Q\|_{L^2}\le\|u_0\|_{L^2}<\|Q\|_{L^2}+\alpha^*,\quad \alpha^*\ll 1, \tag{3}$$

has been initiated in a series of works by Martel and Merle [17, 18, 26, 19, 21, 20] where mainly two new tools were introduced: (1) monotonicity formula and localized virial identities, (2) Liouville type theorems to classify asymptotic solutions. In particular, the first proof of blow up in finite or infinite time is obtained for initial data

$$u_0\in H^1 \text{ with (3) and } E(u_0)<0. \tag{4}$$

The proof is indirect and based on a classification argument [17, 26]. A related result [19] says that if $u(t)$ blows up in finite or infinite time T with (3), then it admits near blow up time a decomposition of the form

$$u(t,x)=\frac{1}{\lambda^{\frac{1}{2}}(t)}(Q+\varepsilon)\left(t,\frac{x-x(t)}{\lambda(t)}\right)\quad\text{with}\quad \varepsilon(t)\to 0\ \text{ in }\ L^2_{\rm loc}\ \text{ as }\ t\to T. \tag{5}$$

Finally, in [20], for initial data

$$u_0 \text{ satisfying (4) and } \int_{x'>x}u_0^2(x')dx'<\frac{C}{x^6}\text{ for } x>0, \tag{6}$$

blow up is proved to occur in finite time T and the following upper bound is proved for a sequence $t_n\to T$:

$$\|u_x(t_n)\|_{L^2}\le\frac{C(u_0)}{T-t_n}. \tag{7}$$

[1] in a certain sense

Concerning the minimal mass problem $\|u_0\|_{L^2} = \|Q\|_{L^2}$, assuming in addition the following decay $\int_{x'>x} u_0^2(x')dx' < \frac{C}{x^3}$ for $x > 0$, it was proved in [21] that the solution is global and does not blow up in infinite time.

1.2. **Blow up in H^1 for the L^2 critical NLS.** We now draw a parallel between L^2 critical gKdV and NLS equations

$$\text{(NLS)} \quad \begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{N}} u = 0, \\ u_{|t=0} = u_0 \end{cases} \qquad (t,x) \in [0,T) \times \mathbb{R}^N \tag{8}$$

which display a similar structure. The solitary wave Q_{NLS} is the unique (up to translation) H^1 nonnegative solution of $\Delta Q_{\text{NLS}} - Q_{\text{NLS}} + Q_{\text{NLS}}^{1+\frac{4}{N}} = 0$ (in dimension one, it is the same function Q as before). Solutions of (NLS) with initial data in H^1 with $\|u_0\|_{L^2} < \|Q_{\text{NLS}}\|_{L^2}$ are global and bounded ([45]).

For $u_0 \in H^1$ with minimal mass $\|u_0\|_{L^2} = \|Q_{\text{NLS}}\|_{L^2}$, Merle [25] proved that the only blow up solution (up to the symmetries of the equation) is the explicit one

$$S_{\text{NLS}}(t,x) = \frac{1}{t^{N/2}} e^{-i(\frac{|x|^2}{4t} - \frac{1}{t})} Q_{\text{NLS}}\left(\frac{x}{t}\right). \tag{9}$$

For negative energy initial data close to solitons, double log correction to self similarity for stable blow up was conjectured from numerics by Landman, Papanicolou, Sulem and Sulem [41]. A family of such solutions was then rigorously constructed by Perelman in dimension one, [42].

In contrast, as shown by Bourgain and Wang [2] (see also Krieger, Schlag [14]), there are blow up solutions of the type (9) ($\|u(t)\|_{H^1} \sim \frac{c}{T-t}$ at the blow up time) with mass strictly larger than the mass of Q_{NLS}. Such solutions correspond to an *unstable threshold dynamics* as recently proved by Merle, Raphaël, Szeftel [32].

Next, the program developed by Merle and Raphaël [27, 28, 29, 6, 36, 30, 31] for the mass critical nonlinear Schrödinger equation in dimensions $1 \le N \le 5$ (any dimension, up to a spectral assumption) has led to a complete description of the stable blow up scenario for small super critical mass H^1 initial data

$$\|Q_{\text{NLS}}\|_{L^2} < \|u_0\|_{L^2} < \|Q_{\text{NLS}}\|_{L^2} + \alpha^*, \quad \alpha^* \ll 1. \tag{10}$$

In particular, an H^1 open set of solutions is exhibited where solutions blow up in finite time at the so-called log–log speed:

$$\|\nabla u(t)\|_{L^2} \sim C^* \sqrt{\frac{\log|\log(T-t)|}{T-t}}. \tag{11}$$

Moreover, nonpositive energy solutions belong to this set of generic blow up. Finally, under (10), the quantization of the focused mass at blow up is proved

$$|u(t)|^2 \rightharpoonup \|Q_{\text{NLS}}\|_{L^2}^2 \delta_{x=x(T)} + |u^*|^2, \quad u^* \in L^2. \tag{12}$$

Natural analogies have been made between mass critical problems and *energy critical* problems for which similar results now exist. For the energy critical wave map problem, after the pioneering work of Rodnianski and Sterbenz [43], a complete description of a generic finite time blow up dynamics (log correction to the self similar speed) was given by Raphaël and Rodnianski [38], while *unstable regimes* with different speeds were constructed by Krieger,

Schlag, Tataru [15]. See also Merle, Raphaël, Rodnianski [33] for the Schrödinger map system and Raphaël, Schweyer [39] for the parabolic harmonic heat flow.

2. Statement of the results for L^2 critical (gKdV)

Consider for $0 < \alpha_0 \ll 1$, the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y>0} y^{10}\varepsilon_0^2 < 1 \right\}. \tag{13}$$

2.1. Nonpositive energy blow up in $\mathcal{A}$.

Theorem 1 (Blow up for nonpositive energy solutions in $\mathcal{A}$, [22]). *Let $0 < \alpha_0 \ll 1$. Let $u_0 \in \mathcal{A}$. If $E(u_0) \leq 0$ and u_0 is not a soliton, then $u(t)$ blows up in finite time T and there exists $\ell_0 = \ell_0(u_0) > 0$ such that*

$$\|u_x(t)\|_{L^2} \sim \frac{\|Q'\|_{L^2}}{\ell_0(T-t)} \quad \text{as } t \to T. \tag{14}$$

Moreover, there exist $\lambda(t)$, $x(t)$ and $u^* \in H^1$, $u^* \neq 0$, such that

$$u(t,x) - \frac{1}{\lambda^{\frac{1}{2}}(t)} Q\left(\frac{x - x(t)}{\lambda(t)}\right) \to u^* \quad \text{in} \quad L^2 \quad \text{as} \quad t \to T, \tag{15}$$

$$\lambda(t) \sim \ell_0(T-t), \quad x(t) \sim \frac{1}{\ell_0^2(T-t)} \quad \text{as} \quad t \to T, \tag{16}$$

Comments on Theorem 1

1. Blow up speed and stable blow up. An important feature of Theorem 1 is the derivation of the blow up speed for $u_0 \in \mathcal{A}$ with non positive energy:

$$\|u_x(t)\|_{L^2} \sim \frac{C(u_0)}{T-t} \tag{17}$$

which implies in particular that $x(t) \to +\infty$ as $t \to T$. The concentrating soliton and the remainder term u^* thus split spatially. Observe that the blow up speed is far above the scaling blow up law which would be for (gKdV): $\|u_x(t)\|_{L^2} \sim c/(T-t)^{\frac{1}{3}}$ (see [33], [39] for a similar gap phenomenon in energy critical geometrical problems).
To complemenent Theorem 1, we claim that the set of initial data in $\mathcal{A}$ which led to the blow up (14)–(16) is open in the H^1 topology (see [22]). We thus call *stable blow up* such behavior.

2. Decay assumption on the right. Let us stress the importance of the decay assumption on the right in space for the initial data, which was already fundamental in the earlier works [20], [21]. Indeed, in contrast with the NLS equation, the universal dynamics can not be seen in H^1 and an additional assumption of decay to the right is required (see Theorem 4 below). Note however, that we do not claim sharpness in the y^{10} weight in Theorem 1.

3. Dynamical characterization of Q: Recall from the variational characterization of Q that $E(u_0) \leq 0$ implies $\|u_0\|_{L^2} > \|Q\|_{L^2}$, unless $u_0 \equiv Q$ up to scaling and translation symmetries. Theorem 1 therefore recovers the dynamical classification of Q as the unique global zero energy solution in $\mathcal{A}$, like for the mass critical (NLS), see [31].

2.2. **Minimal mass blow up.**

Theorem 2 (Existence and uniqueness of the minimal mass blow up element, [23])**.**
(i) Existence. *There exists a solution* $S(t) \in \mathcal{C}((0,+\infty), H^1)$ *of* (1) *with minimal mass*

$$\|S(t)\|_{L^2} = \|Q\|_{L^2}$$

which blows up backward at the origin:

$$\|S(t)\|_{H^1} \sim \frac{\|Q'\|_{L^2}}{t} \quad as \quad t \downarrow 0, \tag{18}$$

$$S(t,x) - \frac{1}{t^{\frac{1}{2}}} Q\left(\frac{x + \frac{1}{t} + \overline{c}t}{t}\right) \to 0 \quad in \quad L^2 \quad as \quad t \downarrow 0$$

where $\overline{c}$ *is a universal constant. Moreover,* S *is smooth and exponentially decays at the right in space:*

$$\forall x \geq 1, \quad S(1,x) \leq e^{-Cx}. \tag{19}$$

(ii) Uniqueness. *Let* $u_0 \in H^1$ *with* $\|u_0\|_{L^2} = \|Q\|_{L^2}$ *and assume that the corresponding solution* $u(t)$ *to* (1) *blows up in finite time. Then*

$$u \equiv S$$

up to the symmetries of the flow.

Observe that $S(t)$ blows up with the same speed as the negative energy blow up obtained in Theorem 1. However, the minimal mass blow up is by essence unstable by perturbation since for example initial data $S_\varepsilon(0) = (1-\varepsilon)S(0)$, for $\varepsilon > 0$, has subcritical mass and thus leads to a global, bounded solution. Theorem 2 shows that the decay assumption to the right in [21] is essential and that the (unique) minimal blow up solution has slow decay to the left[2].

2.3. **Classification of the dynamics in $\mathcal{A}$.** Define the L^2 modulated tube around the soliton manifold:

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \quad \text{with} \quad \inf_{\lambda_0 > 0,\ x_0 \in \mathbb{R}} \left\| u - \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\}. \tag{20}$$

Here α_0, α^* are universal constants with

$$0 < \alpha_0 \ll \alpha^* \ll 1. \tag{21}$$

Theorem 3 (Rigidity of the dynamics in $\mathcal{A}$, [22, 23])**.** *For* $u_0 \in \mathcal{A}$*, only three scenarios are possible:*

(Blow up) *For all* $t \in [0,T)$, $u(t) \in \mathcal{T}_{\alpha^*}$ *and the solution blows up in finite time* $T < +\infty$ *in the regime described by Theorem 1* (14), (15), (16).

(Soliton) *The solution is global, for all* $t \geq 0$, $u(t) \in \mathcal{T}_{\alpha^*}$*, and there exist* $\lambda_\infty > 0$ *and* $x(t)$ *such that*

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot + x(t)) \to Q \quad in\ H^1_{\text{loc}}\ as\ t \to +\infty, \tag{22}$$

$$|\lambda_\infty - 1| \leq o_{\alpha_0 \to 0}(1), \quad x(t) \sim \frac{t}{\lambda_\infty^2} \quad as \quad t \to +\infty. \tag{23}$$

[2] remember that it blows up backwards in time.

(Exit) *There exists* $t^* \in (0,T)$ *such that* $u(t^*) \notin \mathcal{T}_{\alpha^*}$. *Let* $t_u^* \gg 1$ *be the corresponding exit time. Then there exist* $\tau^* = \tau^*(\alpha^*)$ *(independent of* u*) and* (λ_u^*, x_u^*) *such that*

$$\left\| (\lambda_u^*)^{\frac{1}{2}} u\,(t_u^*, \lambda_u^* x + x_u^*) - S(\tau^*, x) \right\|_{L^2} \leq \delta_I(\alpha_0),$$

where $\delta_I(\alpha_0) \to 0$ *as* $\alpha_0 \to 0$.

The exit time t_u^* in Theorem 3 is defined as follows:

$$t^* = \sup\{0 < t < T;\ \text{such that } \forall t' \in [0,t],\ u(t) \in \mathcal{T}_{\alpha^*}\}. \tag{24}$$

In view of the universality of S as a attractor to all exiting solutions, and in continuation of Theorem 3, it is an important open problem to understand the behavior of $S(t)$ as $t \to +\infty$. For the mass critical (NLS), $S_{\rm NLS}(t)$ scatters as $t \to \infty$. For (gKdV), scattering of $S(t)$ as $t \to +\infty$ is an open problem[3]. We conjecture that $S(t)$ actually scatters, and because scattering is an open in L^2 property (see [11]), we obtain the corollary:

Corollary 1 ([23])**.** *Assume that* $S(t)$ *scatters as* $t \to +\infty$. *Then any solution in the* (Exit) *scenario is global for positive time and scatters as* $t \to +\infty$.

Related rigidity theorems near the solitary wave were recently obtained by Nakanishi and Schlag [34], [35] and Krieger, Nakanishi and Schlag [13], for super critical wave and Schrödinger equations using the invariant set methods of Beresticky and Cazenave [1], the Kenig and Merle concentration compactness approach [9], the classification of minimal dynamics [4], [5], and a further "no return" lemma in the (Exit) regime. In the analogue of the (Exit) regime, this lemma shows that the solution cannot come back close to solitons and in fact scatters. In critical situations, such an analysis is more delicate and incomplete, see [13], and both the blow up statements and the no return lemma in [34], [35] rely on a specific algebraic structure - the virial identity - which does not exist for (gKdV).

2.4. **Exotic blow up rates for initial data with slow decay.** Now we produce a wide range of different blow up rates, including grow up in infinite time, for initial data $u_0 \notin \mathcal{A}$ having slow decay on the right. In particular, the blow up rate $\frac{1}{(T-t)}$, which is universal in $\mathcal{A}$, is not valid anymore for such initial data.

Theorem 4 (Exotic blow up rates, [24])**.**
(i) Blow up in finite time: *for any* $\nu > \frac{11}{13}$, *there exists* $u \in C((0,T_0), H^1)$ *solution of* (1) *blowing up at* $t = 0$ *with*

$$\|u_x(t)\|_{L^2} \sim t^{-\nu} \quad as \quad t \to 0^+. \tag{25}$$

(ii) Grow up in infinite time: *there exists* $u \in C([T_0, +\infty), H^1)$ *solution of* (1) *blowing up at* $+\infty$ *with*

$$\|u_x(t)\|_{L^2} \sim e^t \quad as \quad t \to +\infty. \tag{26}$$

For any $\nu > 0$, *there exists* $u \in C([T_0, +\infty), H^1)$ *solution of* (1) *blowing up at* $+\infty$ *with*

$$\|u_x(t)\|_{L^2} \sim t^{\nu} \quad as \quad t \to +\infty. \tag{27}$$

Moreover, such solutions can be taken arbitrarily close in H^1 *to solitons.*

[3] by scattering for (gKdV), we mean that there exists a solution $v(t,x)$ to the Airy equation $\partial_t v + v_{xxx} = 0$ such that $\lim_{t\to+\infty} \|S(t) - v(t)\|_{L^2} = 0$.

As one can see from the proof of Theorem 4 in [24], the grow up rate is directly related to the peculiar behavior of the initial data on the right. In particular, other type of blow up speeds can be produced by similar arguments by changing the tail of the initial data. A similar phenomenon was observed for global in time growing up solutions to the parabolic energy critical harmonic heat flow by Gustafson, Nakanishi and Tsai [7]. In [7], an explicit formula on the growth of the solution at infinity is given directly in terms of the initial data. Continuums of blow up rates were also observed in pioneering works by Krieger, Schlag and Tataru [15], [16] for energy critical wave problems, see also Donninger and Krieger [3]. All these results point out that the critical topology is not enough by itself to classify the flow near the ground state.

3. **Refined ansatz and formal derivation of the dynamics in $\mathcal{A}$**

We first look formally for a solution of (1) of the form

$$u(t,x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(t)}\left(t, \frac{x-x(t)}{\lambda(t)}\right), \tag{28}$$

where Q_b is close to Q for b small. In the case of the (gKdV) equation, a first order approximation of Q_b in b happens to be enough:

$$Q_b = Q + bP,$$

where $P(x)$ is a function to be determined (this is contrast with the (NLS) equation - see [27]).

Let us use the following notation, for a function $f = f(x)$:

$$\begin{aligned} Lf &= -f'' + f - 5Q^4 f, \\ \Lambda f &= \frac{1}{2} f + y f', \\ f^{(\lambda,x)}(t,x) &= \frac{1}{\lambda^{\frac{1}{2}}(t)} f\left(\frac{x-x(t)}{\lambda(t)}\right). \end{aligned}$$

We note the L^2 scalar product: $(f,g) = \int f(x)g(x)dx$.

From (28), we get

$$u_t = -\frac{\lambda_t}{\lambda}(\Lambda Q_b)^{(\lambda,x)} - \frac{x_t}{\lambda}(Q_b')^{(\lambda,x)} + b_t P^{(\lambda,x)},$$

and thus, changing variables, $u(t,x)$ is solution of (1) if

$$-\lambda^2\lambda_t \Lambda Q_b + \left(Q_b'' - \lambda^2 x_t Q_b + Q_b^5\right)' + \lambda^3 b_t P = 0.$$

We fix

$$\lambda^2 x_t = 1 \quad \text{and} \quad -\lambda^2\lambda_t = b,$$

and we expand at order one in b, using $Q'' - Q + Q^5 = 0$, to obtain

$$b\Lambda Q + b(LP)' + \lambda^3 b_t P + O(b^2) = 0. \tag{29}$$

We further fix (recall that the function P is to be chosen, as the geometrical parameters (b,λ,x))

$$(LP)' = -\Lambda Q,$$

and we claim that with this choice of function P, one obtains from (29)

$$\lambda^3 b_t = -2b^2$$

(we refer the reader to [22] for a detailled computation at order 2 in b). Combining the equations of λ_t and b_t obtained above, one gets

$$\frac{d}{dt}\left(\frac{b}{\lambda^2}\right) = \frac{1}{\lambda^2}\left(b_t - 2\frac{\lambda_t}{\lambda}b\right) = 0,$$

and thus

$$-\lambda_t = \frac{b}{\lambda^2} = \ell_0.$$

Formally, we obtain the three scenarios of Theorem 3 depending on the sign of ℓ_0:

- $\ell_0 > 0$: $\lambda_t = -\ell_0 < 0$ and so blow up happens in finite time T and $\lambda(t) = \ell_0(T-t)$. For example, this case is obtained if $E_0 \leq 0$ (and this explains the blow up rate obtained in Theorem 1).
- $\ell_0 = 0$: $\lambda(t) = C$ and the solution behaves essentially as a soliton.
- $\ell_0 < 0$: $\lambda_t = -\ell_0 > 0$ thus the solution first defocuses and then exits any small neighborhood of the soliton (see Section 5).

Let us now stress the main difficulties to achieve the proof of Theorems 1 and 3:

(1) First, when considering a solution close to the soliton, there is an remainder term, which we have to control and maintain small in some functional space. The control of the remainder term, which is a main point in [22], will be sketched in the next section.
(2) Second, the idealized dynamical system in (b, λ, x) obtained above is perturbed by ε and higher order terms in b.
(3) Third, note that since $\int \Lambda Q \neq 0$, there exists no solution $P \in L^2$ of $(LP') = \Lambda Q$. Therefore, one needs to consider a solution which does not tend to zero at $-\infty$ and then apply a cut-off. The approximate profile Q_b is thus slightly more involved than $Q + bP$.

4. **Control of the remainder term**

A solution $u(t,x)$ of (1) close in H^1 to a soliton is canonically decomposed as

$$u(t,x) = \frac{1}{\lambda^{\frac{1}{2}}(t)}(Q_{b(t)} + \varepsilon)\left(t, \frac{x - x(t)}{\lambda(t)}\right)$$

where the parameters $(b(t), \lambda(t), x(t))$ are uniquely adjusted to obtain orthogonality conditions on ε for all time

$$\int \varepsilon Q = \int \varepsilon \Lambda Q = \int \varepsilon y \Lambda Q = 0. \tag{30}$$

Changing time as follows

$$\frac{ds}{dt} = \frac{1}{\lambda^3},$$

the equation of $u(t,x)$ and Q_b implies

$$\varepsilon_s - (L\varepsilon)_y = \left(\frac{\lambda_s}{\lambda} + b\right)\Lambda Q + \left(\frac{x_s}{\lambda} - 1\right)Q' + \frac{\lambda_s}{\lambda}\Lambda\varepsilon + O\left(b^2 + |b_s| + |\varepsilon|^2\right). \tag{31}$$

The dynamical system in (b, λ, x) can be obtained from this equation and (30), with perturbation terms coming from b^2, b_s and ε. The fundamental point to justify the dynamics of the

parameters is to obtain a uniform control of $\varepsilon(s)$ in some norm.

Neglecting for the moment second order term, we concentrate on a toy model:

$$\widetilde{\varepsilon}_s - (L\widetilde{\varepsilon})_y = \alpha(s)\Lambda Q + \beta(s)Q' \tag{32}$$

for which we claim

Lemma 5. *Let $\widetilde{\varepsilon}(s,y)$ be a solution of* (32) *satisfying the orthogonality conditions* (30)*. Then,*

(1) *Energy conservation at $\widetilde{\varepsilon}$ level:*

$$\forall s, \quad (L\widetilde{\varepsilon}(s), \widetilde{\varepsilon}(s)) = Cte. \tag{33}$$

(2) *Virial estimate*

$$\frac{d}{ds}\int y\widetilde{\varepsilon}^2 = -H(\widetilde{\varepsilon},\widetilde{\varepsilon}), \tag{34}$$

where

$$H(\widetilde{\varepsilon},\widetilde{\varepsilon}) = \int \left(3\widetilde{\varepsilon}_y^2 + \widetilde{\varepsilon}^2 - 5Q^4\widetilde{\varepsilon}^2 + 20yQ'Q^3\widetilde{\varepsilon}^2\right) \geq \mu_0\|\widetilde{\varepsilon}(s)\|_{H^1}^2. \tag{35}$$

(3) *Mixed Virial and energy-monotonicity estimate: for $B \gg 1$, $\mu_1 > 0$,*

$$\frac{d}{ds}\left[\int \widetilde{\varepsilon}_y^2\psi\left(\frac{y}{B}\right) + \widetilde{\varepsilon}^2\varphi\left(\frac{y}{B}\right) - 5Q^4\widetilde{\varepsilon}^2\psi\left(\frac{y}{B}\right)\right] + \mu_1\int\left(\widetilde{\varepsilon}_y^2 + \widetilde{\varepsilon}^2\right)\varphi'\left(\frac{y}{B}\right) \leq 0, \tag{36}$$

$$\int \widetilde{\varepsilon}_y^2\psi\left(\frac{y}{B}\right) + \widetilde{\varepsilon}^2\varphi\left(\frac{y}{B}\right) - 5Q^4\widetilde{\varepsilon}^2\psi\left(\frac{y}{B}\right) \geq \mu_1\int \widetilde{\varepsilon}_y^2\psi\left(\frac{y}{B}\right) + \widetilde{\varepsilon}^2\varphi\left(\frac{y}{B}\right), \tag{37}$$

where the smooth functions φ and ψ satisfy

$$\varphi(y) = \begin{cases} e^y \text{ for } y < -1, \\ 1 + y \text{ for } -\frac{1}{2} < y < \frac{1}{2}, \\ y \text{ for } y > 1, \end{cases} \quad \varphi' \geq 0 \text{ on } \mathbb{R}, \tag{38}$$

$$\psi(y) = \begin{cases} e^{2y} \text{ for } y < -1, \\ 1 \text{ for } y > -\frac{1}{2}, \end{cases} \quad \psi' \geq 0 \text{ on } \mathbb{R}. \tag{39}$$

Proof. Identities (33) and (34) are obtained by direct computations from the equation of ε and classical properties of L (see [45]). The coercivity of $H(\varepsilon,\varepsilon)$ is proved in [17]. The estimate (36) follows from direct computations and estimates, and the use of (35) on a suitable localization of ε. It combines in a sharp way monotonicity arguments first derived in L^2 in [17] (reminiscent of the Kato smoothing effect [8]) and localized type Virial estimates. The coercivity property (37) is a consequence of well-known properties of the operator L, such as (for $\mu > 0$)

$$(\varepsilon, Q) = (\varepsilon, Q') = (\varepsilon, \Lambda Q) = 0 \quad \Rightarrow \quad (L\varepsilon,\varepsilon) \geq \mu\|\varepsilon\|_{H^1}^2,$$

(see [45]), and localisation arguments. □

A main novelty in [22] is the introduction of such a mixed Virial and energy-monotonicity functional for ε. Indeed, it follows from long but direct arguments that a similar estimate holds for ε solution of the full nonlinear equation (31) provided we assume some a priori control on ε and the geometrical parameters.

For $B \gg 1$, define

$$\mathcal{N}(s) = \int \varepsilon_y^2 \psi\left(\frac{y}{B}\right) + \varepsilon^2 \varphi\left(\frac{y}{B}\right),$$

$$\mathcal{F}_i(s) = \int \varepsilon_y^2 \psi\left(\frac{y}{B}\right) + \varepsilon^2(1+\mathcal{J}_i)\varphi\left(\frac{y}{B}\right) - \frac{1}{3}\left((\varepsilon+Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5\right)\psi\left(\frac{y}{B}\right),$$

$$\mathcal{J}_i = (1-J_1)^{-4i} - 1, \quad J_1 = (\varepsilon, \rho_1), \quad \rho_1(y) = \frac{4}{\left(\int Q\right)^2}\int_{-\infty}^{y} \Lambda Q.$$

Proposition 6 ([22]). *Assume that on some interval* $[0, s_0]$,
(H1) smallness:

$$\|\varepsilon(s)\|_{L^2} + |b(s)| + \mathcal{N}(s) \le \kappa^*; \tag{40}$$

(H2) comparison between b and λ:

$$\frac{|b(s)| + \mathcal{N}(s)}{\lambda^2(s)} \le \kappa^*; \tag{41}$$

(H3) L^2 weighted bound on the right:

$$\int_{y>0} y^{10}\varepsilon^2(s,x)dx \le 10\left(1 + \frac{1}{\lambda^{10}(s)}\right). \tag{42}$$

Then the following bounds hold on $[0, s_0]$*: for* $B \gg 1$, $\mu > 0$,
(i) Scaling invariant Lyapounov control:

$$\frac{d}{ds}\mathcal{F}_1 + \mu\int \left(\varepsilon_y^2 + \varepsilon^2\right)\varphi'\left(\frac{y}{B}\right) \lesssim |b|^4. \tag{43}$$

(ii) Scaling weighted H^1 Lyapounov control:

$$\frac{d}{ds}\left\{\frac{\mathcal{F}_2}{\lambda^2}\right\} + \frac{\mu}{\lambda^2}\int \left(\varepsilon_y^2 + \varepsilon^2\right)\varphi'\left(\frac{y}{B}\right) \lesssim \frac{|b|^4}{\lambda^2}. \tag{44}$$

(iii) Pointwise bounds:

$$|J_1| + |J_2| \lesssim \mathcal{N}^{\frac{1}{2}}, \tag{45}$$

$$\mathcal{N} \lesssim \mathcal{F}_j \lesssim \mathcal{N}, \quad j = 1, 2. \tag{46}$$

Time integration of equations of the parameters (b, λ, x) (from equation (31)) with the dispersive bounds of Proposition 6 imply the following control of the flow by the parameter b.

Lemma 7 (Control of the flow by b, [22]). *Under assumptions* (H1)-(H2)-(H3) *of Proposition 6, the following hold:*
(i) Control of the dynamics for b. *For all* $0 \le s_1 \le s_2 < s_0$,

$$\int_{s_1}^{s_2} b^2(s)ds \lesssim \int \left(\varepsilon_y^2 + \varepsilon^2\right)(s_1)\varphi'\left(\frac{y}{B}\right) + |b(s_2)| + |b(s_1)|, \tag{47}$$

$$\left|\frac{b(s_2)}{\lambda^2(s_2)} - \frac{b(s_1)}{\lambda^2(s_1)}\right| \le \frac{C^*}{10}\left[\frac{b^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)}{\lambda^2(s_2)} + \frac{1}{\lambda^2(s_1)}\int \left(\varepsilon_y^2 + \varepsilon^2\right)(s_1)\varphi'\left(\frac{y}{B}\right)\right], \tag{48}$$

for some universal constant $C^* > 0$.
(ii) Control of the scaling dynamics. *Let* $\lambda_0(s) = \lambda(s)(1 - J_1(s))^2$. *Then on* $[0, s_0)$,

$$\left|\frac{(\lambda_0)_s}{\lambda_0} + b\right| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b|\mathcal{N}^{\frac{1}{2}} + |b|^2. \tag{49}$$

(iii) Dispersive bounds. *For all* $0 \le s_1 \le s_2 < s_0$,

$$\mathcal{N}(s_2) + \int_{s_1}^{s_2} \left[\int \left(\varepsilon_y^2 + \varepsilon^2\right)(s)\varphi'\left(\frac{y}{B}\right) + |b|^4(s) \right] ds \lesssim \mathcal{N}(s_1) + (|b^3(s_2)| + |b^3(s_1)|). \quad (50)$$

$$\frac{\mathcal{N}(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2} \left[\int \left(\varepsilon_y^2 + \varepsilon^2\right)(s)\varphi'\left(\frac{y}{B}\right) + |b|^4(s) \right] \frac{ds}{\lambda^2(s)} \lesssim \frac{\mathcal{N}(s_1)}{\lambda^2(s_1)} + \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)} \right]. \quad (51)$$

Note that in the proofs of Theorem 1 and Theorem 3, the rigorous estimates (48) and (49), together with the control of ε from (50) and (51), replace the two idealized equations obtained in Section 3:

$$\frac{d}{ds}\left(\frac{b}{\lambda^2}\right) = 0, \qquad \frac{\lambda_s}{\lambda} + b = 0.$$

The quantity J_1, introduced both in the definition of $\mathcal{F}_i$ and the definition of λ_0 may seem mysterious at this point. In these definitions, the introduction of J_1 provides suitable cancellations of diverging terms. Since $\rho_1 \notin L^2$, it is one more indication that the (gKdV) flow cannot be fully explained in H^1 but requires some decay assumption on the solution.

5. **Minimal mass blow up solution and the (Exit) scenario**

The construction of the minimal mass solution and the determination of the universal behavior of solutions in the (Exit) regime follow a unified compactness strategy.

5.1. **Construction of the minimal mass solution.** The minimal element is obtained as the limit of sequences of *defocusing* solutions. Indeed, we pick a sequence of *well prepared* initial data

$$u_n(0) = Q_{b_n(0)}, \quad b_n(0) = -\frac{1}{n}$$

which by construction have subcritical mass

$$\|u_n(0)\|_{L^2} - \|Q\|_{L^2} \sim \frac{c}{n}.$$

Such solutions are necessarily in the (Exit) regime of Theorem 3 and we denote by t_n^* the corresponding exit time (see (24)). Moreover, we have from [22] (see also the formal discussion of Section 3) a precise description of the flow in the time interval $[0, t_n^*]$, in particular, we know that the solution admits a decomposition

$$u_n(t,x) = \frac{1}{\lambda_n^{\frac{1}{2}}(t)}(Q_{b_n(t)} + \varepsilon_n)\left(t, \frac{x - x_n(t)}{\lambda_n(t)}\right) \quad (52)$$

where to leading order (b_n, λ_n) behave as follows

$$\frac{b_n(t)}{\lambda_n^2(t)} \sim b_n(0) = -\frac{1}{n}, \quad (\lambda_n)_t \sim -b_n(0),$$
$$\lambda_n(t) \sim 1 - b_n(0)t, \quad b_n(t) \sim b_n(0)\lambda_n^2(t). \quad (53)$$

The (Exit) time t_n^* is the one for which the solution moves strictly away from the solitary wave which in our setting is equivalent to

$$b_n(t_n^*) \sim -\alpha^*,$$

independent of n. This allows us to compute t_n^* and show using (53) that the solution defocuses:

$$\lambda_n^2(t_n^*) \sim \frac{b_n(t_n^*)}{b_n(0)} \sim n\alpha^* \quad \text{as} \quad n \to +\infty.$$

Next, we renormalize the flow at t_n^*, considering the solution of (gKdV) defined by

$$v_n(\tau, x) = \lambda_n^{\frac{1}{2}}(t_n^*) u_n(t_\tau, \lambda_n(t_n^*)x + x_n(t_n^*)), \quad t_\tau = t_n^* + \tau\lambda_n^3(t_n^*).$$

From direct computations, v_n admits a decomposition

$$v_n(\tau, x) = \frac{1}{\lambda_{v_n}^{\frac{1}{2}}(\tau)} (Q_{b_{v_n}} + \varepsilon_{v_n}) \left(\tau, \frac{x - x_{v_n(\tau)}}{\lambda_{v_n(\tau)}}\right)$$

with from the symmetries of the flow

$$\lambda_{v_n}(\tau) = \frac{\lambda_n(t_\tau)}{\lambda_n(t_n^*)}, \quad x_{v_n}(\tau) = \frac{x_n(t_\tau) - x_n(t_n)}{\lambda_n(t_n^*)}, \quad b_{v_n}(\tau) = b_n(t_\tau), \quad \varepsilon_{v_n}(\tau) = \varepsilon_n(t_\tau).$$

The renormalized parameters can be computed at main order using (53):

$$\begin{aligned} \lambda_{v_n}(\tau) &\sim \frac{1}{\lambda_n(t_n^*)} \left[1 - b_n(0)(t_n^* + \tau\lambda_n^3(t_n^*))\right] \\ &\sim \frac{1}{\lambda_n(t_n^*)} \left[\lambda_n(t_n^*) - \tau b_n(0)\lambda_n^3(t_n^*)\right] \sim 1 - \tau b_n(t_n^*) \sim 1 + \tau\alpha^*. \end{aligned}$$

Observe that the law of $\lambda_{v_n}(\tau)$ at this order does not depend on n, which is a remarkable property, decisive in our approach. Letting $n \to +\infty$, we extract a weak limit $v_n(0) \rightharpoonup v(0)$ such that the corresponding solution $v(\tau)$ to (gKdV) blows up backwards at some finite time $\tau^* \sim -\frac{1}{\alpha^*}$ with the blow up speed $\lambda_v(\tau) \sim \tau - \tau^*$ i.e (18). Note that the extraction of the weak limit requires sharp controls on the remaining radiation ε_{v_n}. Here an essential use is made of the fact that the set of data $u_n(0)$ is *well prepared* as this induces uniform bounds for $\varepsilon_{v_n}(0) = \varepsilon_{u_n}(t_n^*)$ in H^1 and allow us to use the H^1 weak continuity of the flow in the limiting process. Note also that by the weak convergence $\|v\|_{L^2} \leq \|Q\|_{L^2}$, but since the solution $v(\tau)$ blow up in finite time, it has exactly the minimal mass.

For uniqueness, we refer the reader to [23].

5.2. **Solutions in the (Exit) regime.** Now, we prove the universality of S as attractor in the (Exit) case. For this, we consider a sequence of data $(u_0)_n$ with $\|(u_0)_n\|_{L^2} \to \|Q\|_{L^2}$ as $n \to +\infty$ such that the corresponding solution to (gKdV) is in the (Exit) regime. We write the solution at the (Exit) time in the form (52), renormalize the flow and aim at extracting a weak limit as $n \to +\infty$ as before. The strategy of the proof is similar, except that since the data is not well prepared, no uniform H^1 bound on $v_n(0)$ can be obtained. To get around, we introduce two new tools: (1) a concentration compactness argument on sequences of solutions in the critical L^2 space in the spirit of [9] using the tools developed in [44], [12] for the Airy group, which allows us to extract a non trivial weak limit with suitable dynamical controls; (2) refined *local* H^1 bounds on $v_n(\tau)$ in order to ensure that the L^2 limit actually belongs to H^1. Hence the weak limit is a minimal mass H^1 blow up element, and by the uniqueness statement of Theorem 2, it is S up to the symmetries of the equation. Therefore, we get the final conclusion of Theorem 3.

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