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GLOBAL IN TIME STABILITY OF STEADY SHOCKS IN NOZZLES

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ABSTRACT. We prove global dynamical stability of steady transonic shock solutions in divergent quasi-one-dimensional nozzles. One of the key improvements compared with previous results is that we assume neither the smallness of the slope of the nozzle nor the weakness of the shock strength. A key ingredient of the proof are the derivation a exponentially decaying energy estimates for a linearized problem.

1. STEADY SHOCKS IN CHANNELS

The inviscid compressible isentropic Euler equations in dimension $d = 1$ are

$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2 + p(\rho))_x = 0,$$

The first expresses conservation of *mass* ρ and the second conservation of *momentum* ρu . The system is strictly hyperbolic when

$$\forall \rho > 0, \quad p(\rho) > 0, \quad p'(\rho) > 0, \quad p''(\rho) \geq 0.$$

The *sound speed* is defined to be

$$c := p_\rho(\rho)^{1/2}.$$

The local speeds are $u \pm c$. When $|u| > c$ the flow is *supersonic* and where $|u| < c$ it is *subsonic*.

We treat 1d channel or nozzle flow with $u > 0$. Transverse averaged 1d nozzle with section $a(x)$ yields the system

$$\rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u, \quad (\rho u)_t + (\rho u^2 + p(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2.$$

This has the conservative form

$$(a(x)\rho)_t + (a(x)\rho u)_x = 0, \quad (a(x)\rho u)_t + (a(x)\rho u^2 + p(\rho))_x = 0.$$

Suppose the channel flat outside $0 \leq x \leq 1$, that is, $a' \in C_0^\infty(]0, 1[)$.

We study the stability of steady transonic shocks. The flow is supersonic to the left and subsonic to the right. The steady hypothesis is crucial. It allows us to truncate the

domain of study to a finite interval. The steady shock stays in such a truncated interval for all time.

In a $1d$ nozzle that shrinks then expands there are often two steady transonic shock solutions with the same end states as indicated in the next figure. As indicated in the figure the shock is expected to be stable where the channel expands and unstable where it shrinks.



The stable shocks are important in a variety of applications. For example in rocket exhausts, which are expanding, steady shocks are a common cause of failure of the exhaust chamber where stable steady shocks meet the boundary.

2. LAX'S GEOMETRIC SHOCK CONDITION

It is classical that smooth initial data lead to smooth solutions whose derivative blows up in finite time and beyond that time one continues solutions as discontinuous weak solutions constrained to satisfy additional constraints at the discontinuity. The constraints are needed since admitting weak solutions leads to nonuniqueness. A selection criterion is needed.

Consider a strictly hyperbolic system of N conservation laws, $u_t + f(u)_x = 0$ with the property that $f'(u)$ has N distinct eigenvalues,

$$\lambda_1(u) < \lambda_2(u) < \dots < \lambda_N(u).$$

A piecewise smooth weak solution jumping across a shock located at $x = s(t) \in C^1$ must satisfy the Rankine-Hugoniot relations,

$$s'[u] = [f(u)], \quad [\cdot] := \text{jump}.$$

The selection criterion of Lax is motivated by a stability computation. Consider a small change in the initial data. Try to determine small perturbations u_{left} and u_{right} plus a perturbed shock location to generate nearby piecewise smooth solution. This yields a boundary value problem defined by partial differential equations on the left and right supplemented by the Rankine-Hugoniot relations.

Denote by K_{left} and K_{right} the number of characteristics curves approaching the shock from the left and right hand sides respectively. A good boundary value problem needs

$$\# \text{ of BC} = \# \text{ of characteristics outgoing from the shock.}$$

The Rankine-Hugoniot relation yields s' plus $N-1$ conditions on the values of perturbation on the left and right. The number of outgoing shocks is the sum of the number outgoing on the left plus outgoing on the right

$$\# \text{ of outgoing shocks} = (N - K_{left}) + (N - K_{right}), .$$

Lax's Geometric Shock Condition asserts that the number of conditions from the jump relation is exactly what is needed,

$$N - 1 = (N - K_{left}) + (N - K_{right}), \quad K_{left} + K_{right} = N + 1.$$

Example 1. *For the 1D isentropic Euler equations, $N = 2$, with $u > 0$ and a steady transonic shock,*

$$u_{left} > c_{left}, \quad u_{right} < c_{right} .$$

To the left, both wave speeds, $u \pm c$ are positive so incoming to the shock. On the right, one is positive and one is negative, so one is incoming,

$$K_{left} + K_{right} = 2 + 1 = 3 = N + 1 .$$

The Lax condition is automatic for steady transonic shocks.

Theorem 1. *In dimension $d = 1$ the Lax Condition is sufficient for local in time stability of shocks.*

I don't know who was the first to prove this. A proof can be found in [11]. The multidimensional case is substantially more subtle.

Theorem 2 (A. Majda, then G. Métivier). *In dimension $d > 1$ the Lax Condition supplemented with a less self evident Kreiss-Lopatinski Condition is sufficient for local in time stability of shocks.*

We study a problem with $d = 1$ so do not need the Kreiss-Lopatinski condition. In contrast to these results we prove global in time stability.

3. MULTIPLE STEADY SHOCKS

Steady flows with transonic shocks occur around airfoils, turbines, jet engine exhausts, and rocket engine exhausts. Including examples of multiple steady shocks with same end states. Some of the steady shocks are not observed because they are unstable for the dynamics. Distinguishing the stable from the unstable is the goal of this paper.

One way to compute steady shocks on the computer is to march in long steps of implicit numerical schemes toward equilibria. One could hope that the unstable ones would not show up in such a method. The analysis of Embid, Goodman, Majda dispells this hope.

Theorem 3 (Embid, Goodman, Majda). *There are examples of multiple steady shocks including dynamically unstable ones that have nontrivial domain of attraction for such numerical schemes.*

For some classic steady and quasi-steady shocks in $2d$ there have been a variety of criteria proposed to detect the stable patterns.

Theorem 4 (Elling). *For a $2d$ planar shock incident on a wedge there are multiple steady or quasisteady solutions. For each stability criterion proposed by physicists Elling has constructed highly nontrivial steady solutions for which the stability criteria gives an incorrect answer.*

4. LIU'S STABILITY THEOREM IN NOZZLES.

In contrast to Elling's $2d$ results, stability of steady shocks in nozzles follows the standard science literature criterion. For the case of the Cauchy problem on $-\infty < x < \infty$ one has the the following important result.

Theorem 5 (Taiping Liu). *In nozzle flow, weak steady transonic shock in nozzles of small slope are unstable if they occur at a compressive part of the nozzle, $a' < 0$, and stable if they are at an expansive part.*

Liu proves this result by constructing BV solutions on \mathbb{R}_x by a Glimm scheme adapted to the nonzero right hand sides in the nozzle equation. Liu proved that in the unstable case arbitrarily small perturbations lead to changes $O(1)$ in the shock position. In the stable case, he proved that for any $\epsilon > 0$ the shock moves no more than ϵ globally in time for sufficiently small perturbations.

If one permits the perturbation to change the values at $\pm\infty$ then one expects that the perturbed solution converges to perturbed steady shock with slightly perturbed location.

5. STRONGER STABILITY

If the states at $\pm\infty$ are fixed and one hopes that as $t \rightarrow \infty$ one approaches a steady state then there are not many candidates. When $a \neq 0$ translational symmetry is broken. The translates of steady shock solutions are not solutions. It is reasonable to expect that the solution converges to the unperturbed steady transonic shock.

Theorem 6 (Xin, Yin). *For bounded spatial domain with appropriate fixed boundary conditions, the stronger stability holds for small shocks in slightly non constant channels provided the channel is expanding at the shock location. The convergence is at an algebraic rate.*

We prove a far reaching generalization.

Theorem 7 (Xie, Xin, Rauch). *The same conclusion without smallness assumptions. The convergence is at an exponential rate.*

The nozzle case leans on techniques used in our earlier work with Luo on steady transonic Euler-Poisson shocks [16].

6. BOUNDARY CONDITIONS AND COMPATIBILITY CONDITIONS

Work on $l \leq x \leq L$ with

$$-\infty < l < 0 < 1 < L < \infty, \quad \text{supp } a' \subset]0, 1[.$$

In the classic text of Courant and Friedrichs [1] it is emphasized that the nature of the boundary conditions required at l, L depends on the solutions in question. The justification of the appropriateness of the boundary conditions depends on the theorems that one can establish. The present research is a contribution to that central theme.

At left hand boundary, take supersonic data. Both characteristics enter so full Cauchy data is needed. Take the values of the steady shock at l . At the right hand boundary, the flow is subsonic. There is one incoming and one outgoing characteristic. One needs one boundary condition. The boundary conditions at the endpoints are,

$$(6.1) \quad (\rho, u)(t, l) = (\bar{\rho}(l), \bar{u}(l)), \quad \rho(t, L) = \bar{\rho}(L).$$

The latter condition is equivalent to prescribing the pressure at the right hand endpoint. That is the boundary condition suggested by Courant and Friedrichs.

Initial data $u_0(x), \rho_0(x)$ for $x \in [l, L]$ are prescribed close to the steady transonic shock $(\bar{u}(x), \bar{\rho}(x))$ connecting these values. Initial condition must satisfy conditions guaranteeing that the initial and the boundary conditions at $x = l, L$ are compatible with there being a $k \geq 15$ times differentiable solution.

Example 2. $(u_0(l), \rho_0(l)) = (\bar{u}(l), \bar{\rho}(l))$ is needed for continuity at $(0, l)$.

7. RANKINE-HUGONIOT CONDITIONS AND SMALLNESS

If s' is eliminated from the Rankine-Hugoniot condition the remaining equation is a necessary condition on initial data for them to support a shock.

Smallness is expressed as follows. The initial wave (u_0, ρ_0) has a shock at \underline{x} . The steady shock is at x_0 , with $|x_0 - \underline{x}| < \varepsilon$. If ϕ_- is the affine map of $[l, x_0]$ to $[l, \bar{x}] := \mathcal{I}_-$ and similarly for ϕ_+ then

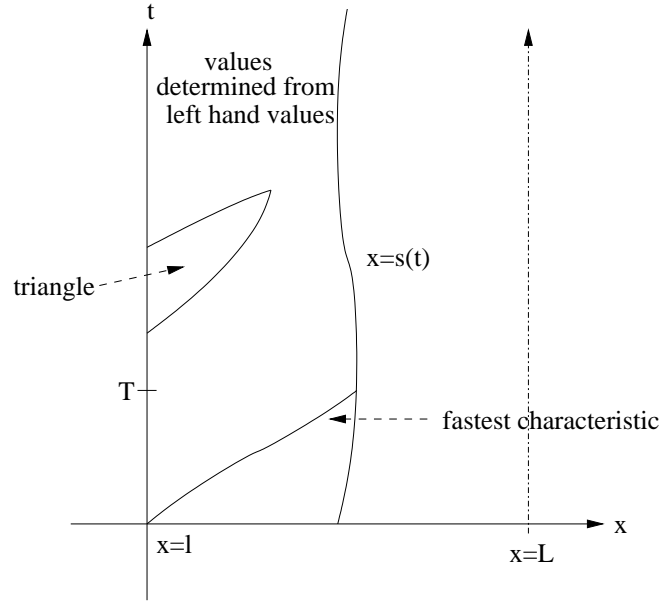
$$\|(\bar{u}(x), \bar{\rho}(x)) - (u_0, \rho_0) \circ \phi_{\pm}^{-1}\|_{H^k(\mathcal{I}_{\pm})} < \varepsilon.$$

The left hand state of a steady transonic shock, $(\bar{u}_-(x), \bar{\rho}_-(x))$ defined for $l \leq x \leq \bar{x}$ extends as a solution or a nonsingular ODE to a strictly larger interval. Similarly $(\bar{u}_+(x), \bar{\rho}_+(x))$.

8. SOLUTION DETERMINED TO THE LEFT OF THE SHOCK FOR $t > T$

For any $\underline{t} > 0$ if the data are close enough there is a piecewise smooth solution with a shock at $x = s(t)$ for $0 \leq t \leq \underline{t}$ and $|s - \bar{x}|$ small. The next proof shows that once $\underline{t} > T$ we know u, ρ on $x < s(t)$.

It is a domain of determinacy argument. Compare (u, ρ) to $(\bar{u}_-(x), \bar{\rho}_-(x))$. They have the same Cauchy data at left. They satisfy the same quasilinear equation to the left. Subtract the equations to find a linear equation for the difference. Fritz John type spacelike deformation sweeps out “triangles” as indicated in the following figure.



9. MAIN RESULT

Thanks to the preceding result, the stability concerns only the shock position and the values of the solution to the right of the shock. Recall that the initial conditions are compatible up to order $k \geq 15$.

Theorem 8. *If $(\bar{\rho}, \bar{u})$ is a steady transonic shock and $a' > 0$ at the shock. Then, there exists an $\varepsilon_0 > 0$ so, if the initial data satisfies smallness with $\varepsilon < \varepsilon_0$, Rankine-Hugoniot, and compatibility, then, the initial boundary value problem has a unique piecewise smooth solution $(\rho, u)(x, t)$ defined for all $t \geq 0$ discontinuous across a single transonic shock at $x = s(t)$.*

For $t > T_0$,

$$(\rho_-, u_-)(t, x) = (\bar{\rho}_-, \bar{u}_-)(x), \quad \text{for } l \leq x < s(t).$$

The solution approaches $(\bar{u}, \bar{\rho})$ exponentially,

$$\|(\rho_+, u_+)(\cdot, t) - (\bar{\rho}_+, \bar{u}_+)(\cdot)\|_{W^{k-7, \infty}(s(t), L)} \leq C \varepsilon e^{-\lambda t},$$

$$\sum_{m=0}^{k-6} |\partial_t^m (s(t) - x_0)| \leq C \varepsilon e^{-\lambda t}.$$

10. OUTLINE OF THE PROOF

- I. Work only for $t > T$ so state to the left is known.
- II. An affine x -transformation $[s(t), L] \mapsto [x_0, L]$ fixes the domain. The function $s(t)$, to be determined, gives the shock location.
- III. Introduce a potential Ψ for the deviation from the steady shock. The potential Ψ satisfies a quasilinear wave equation coupled to an ODE of first order for s with source terms involving $\Psi, \nabla\Psi$.
- IV. Drop terms quadratic or higher in the unknowns. The shock position decouples from the now linear wave equation. Keep the notation Ψ for the unknown of this linearized problem.
- V. *Find an energy dissipation law for the linear wave equation.*
- VI. Prove that solutions of the linear problem decay exponentially.
- VII. Choose $\underline{T} > 0$ so that the linear evolution operator has norm < 1 at time \underline{T} . Conclude that evolution operator has norm < 1 at \underline{T} for nearby coefficients.
- VIII. Use this to control the quasilinear iteration where the equation for the $(n + 1)^{\text{st}}$ has the n^{th} in coefficients.

The next sections sketch key elements from parts **III**, **V** and **VI**.

11. VELOCITY POTENTIAL AND LINEARIZATION.

Recall that $a(x)\rho$ is conserved with flux $a\rho u$. Define a potential for the deviation from the unperturbed state by

$$\Psi_t = a\bar{\rho} - a\rho_+, \quad \Psi_x = a\rho_+ - a\bar{\rho}_+, \quad \Psi(0, L) = 0.$$

The conservation laws guarantee solvability of the equations for Ψ .

The linearization at 0 of the quasilinear wave equation satisfied by $\Psi(t, x), s(t)$ yields a linear wave equation for for small deviations Ψ decoupled from $s(t)$

$$0 = \mathcal{L}_0\Psi := \partial_{tt}\Psi + 2\bar{u}_+\partial_{tx}\Psi + (\bar{u}_+^2 - p'(\bar{\rho}_+))\partial_{xx}\Psi + 2\partial_x\bar{u}_+\partial_t\Psi + B\partial_x\Psi$$

with

$$B := \frac{a'}{a}p'(\bar{\rho}_+) - \partial_x p'(\bar{\rho}_+) + \partial_x \bar{u}_+^2.$$

12. THE DISSIPATION LAW

The linearized boundary value problem is,

$$\begin{aligned} \mathcal{L}_0\Psi &= 0, & x_0 < x < L, & \quad t > 0, \\ \partial_x\Psi &= \frac{2\bar{u}_+}{c^2(\bar{\rho}_+) - \bar{u}_+^2}(x_0)\partial_t\Psi + \frac{a'(x_0)\bar{u}_+u_-}{(c^2(\bar{\rho}_+) - \bar{u}_+^2)a}(x_0)\Psi, & \text{at } x = x_0, \\ \partial_x\Psi &= 0, & \text{at } x = L. \end{aligned}$$

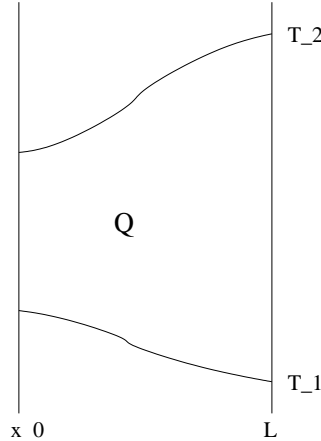
It has variable coefficients depending on the unperturbed shock. The boundary value problem has the strong and well hidden dissipation law,

$$\begin{aligned} E(\Psi, t) + D(\Psi, t) &= E(\Psi, 0), & \text{where,} \\ E &:= \left(\frac{a'\bar{u}_+^2\bar{u}_-\Psi^2}{a} \right) (t, x_0) + \int_{x_0}^L \bar{u}_+ ((\partial_t\Psi)^2 + (p'(\bar{\rho}_+) - \bar{u}_+^2)(\partial_x\Psi)^2) dx, \\ D(\Psi, t) &:= 2 \int_0^t \bar{u}_+^2(x_0)(\partial_t\Psi)^2(\tau, x_0) + \bar{u}_+^2(L)(\partial_t\Psi)^2(\tau, L) d\tau. \end{aligned}$$

The energy E is equivalent to the H^1 norm and is dissipated at both boundaries by an amount $\sim \Psi_t^2$ which, thanks to the boundary conditions, is equivalent up to lower order terms to the boundary H^1 norm.

13. SIDEWAYS ENERGY ARGUMENT

Energy dissipated at right dominates H^1 norm minus the relatively compact L^2 norm on the right.



The sideways energy estimate treating the operator as hyperbolic with time variable x and space variable t as in [21] shows that H^1 norm on right dominates $H^1(Q)$ dominates $E(T_2)$. Therefore

$$\int_{T_1}^{T_2} \Psi_t(t, L)^2 dt \geq C_1 E(T_2)^2 - C_2 \|\Psi(T_1)\|_\gamma^2$$

with $\|\cdot\|_\gamma$ relatively compact compared to the H^1 norm.

14. SPECTRAL ENDGAME

The sideways estimate shows that the time evolution operator by $T_2 - T_1$ has essential spectrum in $|z| \leq (1 + C_1)^{-1/2} < 1$. Since the evolution is dissipative, the spectrum belongs to the disk $|z| \leq 1$. Need to show that there is no eigenvalues on $|z| = 1$. This would imply that the spectral radius is strictly less than 1 and therefore prove exponential decay.

If there were an eigenvalue of modulus 1, the eigenspace would be finite dimensional and invariant. So there would be solutions $e^{i\omega t} \Psi(x)$. To have no dissipation one must have $\Psi_t = 0$ at $x = L$. Since $\Psi(0, L) = 0$ one has $\Psi(t, L) = 0$ for all t . In addition, $\Psi_x(L) = 0$ from the boundary condition. Plugging $e^{i\omega t} \Psi(x)$ into the partial differential equation yields a second order homogenous linear ordinary differential equation for $\Psi(x)$. Therefore $\Psi = 0$ from uniqueness in the Cauchy problem for ordinary differential equations. The absence of eigenvalues of modulus one is proved.

This ends our sketch of the proof.

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