



Séminaire Laurent Schwartz

EDP et applications

Année 2025-2026

Exposé n° III (16 décembre 2025)

Yuchen Mao, Sung-Jin Oh, and Zhongkai Tao

BOGOVSKII-TYPE OPERATORS AND FLEXIBILITY OF
THE EINSTEIN CONSTRAINT EQUATION

<https://doi.org/10.5802/slsedp.191>

© Les auteurs, 2025-2026.



Cet article est mis à disposition selon les termes de la licence

LICENCE INTERNATIONALE D'ATTRIBUTION CREATIVE COMMONS BY 4.0.

<https://creativecommons.org/licenses/by/4.0/>

Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
<http://www.ihes.fr/>

Centre de mathématiques Laurent Schwartz
CMLS, CNRS, École polytechnique,
Institut Polytechnique de Paris
F-91128 PALAISEAU CEDEX
<https://cmls.ip-paris.fr/>



Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2266-0607

BOGOVSKII-TYPE OPERATORS AND FLEXIBILITY OF THE EINSTEIN CONSTRAINT EQUATION

YUCHEN MAO, SUNG-JIN OH, AND ZHONGKAI TAO

ABSTRACT. We give an overview of the method of Bogovskii-type operators and the applications to the conic and obstruction-free gluing problems in general relativity.

1. INTRODUCTION

Let $(\mathcal{M}, \mathbf{g})$ be a Lorentzian manifold of dimension $1+3$ with signature $(-, +, +, +)$. The Einstein vacuum equation reads

$$(1.1) \quad \text{Ric}_{\mathbf{g}} - \frac{1}{2} \mathbf{g} \, \text{tr}_{\mathbf{g}} \text{Ric}_{\mathbf{g}} = 0.$$

Let (M, g) be a spacelike hypersurface of dimension 3 in $\mathcal{M}$, and let k be its second fundamental form. The Gauss and Codazzi equations imply the Einstein constraint equations:

$$(1.2) \quad R_g + (\text{tr}_g k)^2 - |k|_g^2 = 0, \quad \text{div}_g k - \text{d}(\text{tr}_g k) = 0.$$

On the other hand, the fundamental theorem of Choquet-Bruhat [FB52] states that any (sufficiently regular) triple (M, g, k) solving (1.2) corresponds to a geometrically unique spacetime with a metric solving (1.1) (see [CBG69, Sbi16] for the uniqueness).

At each point, (g, k) contains 12 scalar components, whereas (1.2) comprises only 4 scalar equations. The system is therefore very underdetermined.

From this perspective, perhaps the most surprising result for the constraint equation is the *positive mass theorem* of Schoen–Yau [SY79] (see Witten [Wit81] for a different proof and Brendle–Wang [BW26] for a proof in arbitrary dimensions). In the simplest case, it says that if $k = 0$ and g is flat outside a compact set in $\mathbb{R}^3$, then (M, g) is isometric to $\mathbb{R}^3$.

In this note, we will instead focus on the flexibility side of the constraint equation and discuss the following questions:

- (1) Can one build nontrivial vacuum data supported in a prescribed region?
- (2) Can two solutions be glued in a prescribed region? What is the obstruction to glue two solutions?
- (3) How fast may asymptotically flat vacuum data decay?

We will present two results using a new tool: Bogovskii-type operators. The first is a conic gluing result (stated in a special case to glue to the flat data outside a cone).

Theorem 1.1 (Carlotto–Schoen [CS16], Aretakis–Czimek–Rodnianski [ACR25], Mao–Tao [MT22]). *For every cone*

$$C_\theta(\omega) = \{x \in \mathbb{R}^3 : \angle(x, \omega) \leq \theta\}$$

with $0 < \theta < \pi/2$, there exist nontrivial smooth vacuum data on $\mathbb{R}^3$ satisfying

$$(g, k) = (\delta, 0) \quad \text{on } \mathbb{R}^3 \setminus C_\theta(\omega)$$

and the estimates

$$(1.3) \quad |\partial^\ell (g - \delta)(x)| \lesssim_\ell \langle x \rangle^{-1-\ell}, \quad |\partial^\ell k(x)| \lesssim_\ell \langle x \rangle^{-2-\ell}$$

for all $\ell \geq 0$.

This decay rate is sharp because if it decays faster, the ADM energy vanishes, and by the positive mass theorem, the solution must come from a spacelike hypersurface in the Minkowski spacetime. Theorem 1.1 was first proved by Carlotto–Schoen [CS16], except for the decay rate (1.3), which was left as an open problem (see, e.g., [Car21, Open Problem 3.18]). The decay rate (1.3) was obtained by Aretakis–Czimek–Rodnianski

[ACR25] using characteristic gluing. In this note, we present the direct spacelike-gluing proof of Mao–Tao [MT22].

The second is an obstruction-free gluing result on an asymptotically flat annulus. We recall the asymptotically flat condition.

Definition 1.2. Let $(g, k) \in H_{\text{loc}}^s \times H_{\text{loc}}^{s-1}(B_{R_0}^c)$ for $s > 3/2$ and $R_0 \in 2^{\mathbb{N}}$. We say that (g, k) is α -asymptotically flat if, for some $\alpha \in \mathbb{R}$ and $D_0 > 0$, we have

$$(1.4) \quad \|(g - \delta, k)\|_{\dot{H}^s \times \dot{H}^{s-1}(\tilde{A}_r)} \leq D_0 r^{-s+3/2-\alpha} \quad \text{for all } r \in 2^{\mathbb{N}}, r \geq 2R_0,$$

where¹ $\|u\|_{\dot{H}^s(\tilde{A}_R)} := R^{-s+3/2} \|u(Rx)\|_{H^s(\tilde{A}_1)}$.

Theorem 1.3 (Czimek–Rodnianski [CR22], Mao–Oh–Tao [MOT23]). *Let $s > 3/2$ and $\alpha > 1/2$. Let $(g_{\text{in}}, k_{\text{in}}), (g_{\text{out}}, k_{\text{out}}) \in H_{\text{loc}}^s \times H_{\text{loc}}^{s-1}(B_{R_0}^c)$ be α -asymptotically flat pairs solving (1.2). Define*

$$\Delta \mathbf{E} := \mathbf{E}^{\text{ADM}}[(g_{\text{out}}, k_{\text{out}})] - \mathbf{E}^{\text{ADM}}[(g_{\text{in}}, k_{\text{in}})], \quad \Delta \mathbf{P} := \mathbf{P}^{\text{ADM}}[(g_{\text{out}}, k_{\text{out}})] - \mathbf{P}^{\text{ADM}}[(g_{\text{in}}, k_{\text{in}})],$$

and assume that

$$\Delta \mathbf{E} > |\Delta \mathbf{P}|.$$

Then there exists an asymptotically flat pair $(g, k) \in H_{\text{loc}}^s \times H_{\text{loc}}^{s-1}(B_{R_0}^c)$ solving (1.2) and $r \geq R_0$ such that $(g, k) = (g_{\text{in}}, k_{\text{in}})$ on $B_{2r} \cap B_{R_0}^c$ and $(g, k) = (g_{\text{out}}, k_{\text{out}})$ on B_{32r}^c .

Here in the theorem,

$$\mathbf{E}^{\text{ADM}}[(g, k)] = \frac{1}{2} \lim_{r \rightarrow \infty} \int_{\partial B_r} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j \, dS, \quad \mathbf{P}_a^{\text{ADM}}[(g, k)] = \lim_{r \rightarrow \infty} \int_{\partial B_r} (k_{aj} - \delta_{aj} \text{tr}_\delta k) \nu^j \, dS.$$

See §2.5 for more details on the ADM charges.

Theorem 1.3 was first proved by Czimek–Rodnianski [CR22] as a corollary of their obstruction-free characteristic gluing theorem, under the stronger positivity condition $\Delta \mathbf{E} > C|\Delta \mathbf{P}|$ for some $C > 1$. This condition was relaxed to $\Delta \mathbf{E} > |\Delta \mathbf{P}|$ by the authors [MOT23] via an independent proof based on spacelike gluing, which we describe below.

There had been several powerful methods to study the constraint equation; we refer to Carlotto’s survey [Car21] for the history. Briefly speaking, there are the *conformal method* (see Lichnerowicz [Lic44], York [Yor73], Murchadha–York [OY73], Choquet-Bruhat–York [CBY80] and Isenberg [Ise95]) and the *gluing method* (see Fischer–Marsden [FM75], Corvino [Cor00], Corvino–Schoen [CS06] and Chruściel–Delay [CD03]). A common feature of these methods is that the underdetermined system is at some point reduced to a *determined* elliptic system (to wit, the Lichnerowicz equation in the conformal method, and elliptic systems with degenerate coefficients at the boundary in the gluing method). While those methods led to many important developments, the reduction to determined PDE systems loses the information about the underdeterminedness of the system. The method developed in our recent work (with Philip Isett) [MT22, MOT23, IMOT25] exploits the underdetermined nature more directly. See Nützi [Nüt26b, Nüt26a], Chruściel–Cogo–Nützi [CCN25], Giorgi–Shen–Wan [GSW25] and Shen–Wan [SW25, SW26] for related works. We also mention that there are recent advances by Chen–Klainerman [CK25, CK26b, CK26a] that exploit the structure of the Einstein constraint equations in a different way.

Acknowledgements. S.-J.O. was partially supported by a National Science Foundation CAREER Grant under NSF-DMS-1945615, a National Science Foundation Grant under DMS-2452760, and the Miller Research Professorship. Z.T. was partially supported by a grant from the National Research Agency (ANR) under the France 2030 program, reference ANR-25-CFFS-0012.

2. PRELIMINARIES

2.1. Notations. Unless stated otherwise, indices are raised and lowered with the Euclidean metric δ . Repeated indices are summed. The construction of Bogovskii-type operators works in all dimensions $d \geq 1$, while the discussion of constraint equations focuses on $d = 3$.

For a symmetric tensor u , $\text{tr}_\delta u = \delta^{ij} u_{ij}$. Let $B_r(\xi) := \{x \in \mathbb{R}^d : |x - \xi| < r\}$ and $B_r := B_r(0)$. We write

$$\langle x \rangle = (1 + |x|^2)^{1/2}, \quad A_r = B_{2r} \setminus \overline{B_r}, \quad \tilde{A}_r = B_{4r} \setminus \overline{B_{r/2}}.$$

¹When $s \in \mathbb{N}$, observe that $\|u\|_{\dot{H}^s(\tilde{A}_R)} \simeq \|\partial_x^{(s)} u\|_{L^2(\tilde{A}_R)} + R^{-s} \|u\|_{L^2(\tilde{A}_R)}$.

A solid cone with vertex y , axis $\omega \in \mathbb{S}^{d-1}$, and opening angle θ is

$$C_{y,\theta}(\omega) = \{x \in \mathbb{R}^d : \angle(x - y, \omega) \leq \theta\}.$$

When $y = 0$ we write $C_\theta(\omega)$. For $0 < \theta < \pi/2$ this is a convex cone. We use $\widetilde{W}^{s,p}(U)$ for the closure of $C_c^\infty(U)$ under the $W^{s,p}$ topology.

2.2. Linearization at flat data. We recall the linearized constraint equations around the flat data $(\delta, 0)$ on $\mathbb{R}^d$. Write

$$g = \delta + \dot{g}, \quad k = \dot{k}.$$

The linearized equation is

$$(2.1) \quad \partial_i \partial_j \dot{g}^{ij} - \Delta \operatorname{tr}_\delta \dot{g} = 0, \quad \partial^i \dot{k}_{ij} - \partial_j \operatorname{tr}_\delta \dot{k} = 0.$$

We make the following change of variables to make the equations (2.1) in a simpler form:

$$(2.2) \quad h_{ij} = g_{ij} - \delta_{ij} - \delta_{ij} \operatorname{tr}_\delta (g - \delta), \quad \pi_{ij} = k_{ij} - \delta_{ij} \operatorname{tr}_\delta k.$$

Since $\operatorname{tr}_\delta h = -(d-1) \operatorname{tr}_\delta (g - \delta)$ and $\operatorname{tr}_\delta \pi = -(d-1) \operatorname{tr}_\delta k$, the inverse transformation is

$$(2.3) \quad g_{ij} = \delta_{ij} + h_{ij} - \frac{1}{d-1} \delta_{ij} \operatorname{tr}_\delta h, \quad k_{ij} = \pi_{ij} - \frac{1}{d-1} \delta_{ij} \operatorname{tr}_\delta \pi.$$

Thus the linearized constraint operator becomes

$$(2.4) \quad P(h, \pi) = (P_H h, P_M \pi) := (\partial_i \partial_j h^{ij}, \partial_i \pi^{ij}).$$

The two components are called the double divergence and the symmetric divergence.

2.3. The nonlinear equations as a quadratic perturbation. In the variables (h, π) , the constraint equations (1.2) can be written as

$$(2.5) \quad P(h, \pi) = \Phi(h, \pi).$$

Near $(h, \pi) = (0, 0)$, the nonlinear map has the schematic form

$$(2.6) \quad \Phi_H(h, \pi) = M^{(0)}(h)(h, \partial^2 h) + M^{(1)}(h)(\partial h, \partial h) + M^{(2)}(h)(\pi, \pi),$$

$$(2.7) \quad \Phi_M(h, \pi) = N^{(0)}(h)(h, \partial \pi) + N^{(1)}(h)(\partial h, \pi),$$

where $M^{(0)}, M^{(1)}, M^{(2)}, N^{(0)}, N^{(1)}$ are finite linear combinations of contractions with coefficients depending smoothly on h . Since Φ is quadratic,

$$(2.8) \quad \Phi(0, 0) = 0, \quad D\Phi(0, 0) = 0.$$

Consequently, in reasonable function spaces, we expect

$$(2.9) \quad \|\Phi(h, \pi)\|_Y \leq C\|(h, \pi)\|_X^2,$$

$$(2.10) \quad \|\Phi(h_1, \pi_1) - \Phi(h_2, \pi_2)\|_Y \leq C(\|(h_1, \pi_1)\|_X + \|(h_2, \pi_2)\|_X)\|(h_1 - h_2, \pi_1 - \pi_2)\|_X.$$

2.4. Formal adjoints and finite-dimensional obstructions. On a bounded domain, the obstruction to finding compactly supported solutions is given by the kernel of the formal adjoint.

For a scalar test function φ ,

$$\int (\partial_i \partial_j h^{ij}) \varphi \, dx = \int h^{ij} \partial_i \partial_j \varphi \, dx, \quad P_H^* \varphi = \nabla^2 \varphi.$$

On a connected Euclidean domain, $\nabla^2 \varphi = 0$ exactly when φ is affine: $\varphi(x) = a + b_i x^i$. Therefore a compactly supported solution of $P_H h = f$ must satisfy

$$(2.11) \quad \int f \, dx = 0, \quad \int x^a f \, dx = 0, \quad a = 1, \dots, d.$$

For a 1-form ω ,

$$\int (\partial_i \pi^{ij}) \omega_j \, dx = - \int \pi^{ij} \partial_i \omega_j \, dx = - \frac{1}{2} \int \pi^{ij} (\partial_i \omega_j + \partial_j \omega_i) \, dx.$$

Thus

$$(2.12) \quad P_M^* \omega = - \frac{1}{2} (\partial_i \omega_j + \partial_j \omega_i).$$

Its kernel is the space of Euclidean Killing fields. In dimension three, it is spanned by the translations $\mathbf{e}_a = \partial_a$ and rotations $\mathbf{Y}_a = \varepsilon_{aij}x^i\partial_j$ for $a = 1, 2, 3$. Hence a compactly supported solution of $P_M\pi = F$ must satisfy

$$(2.13) \quad \int F \cdot \mathbf{e}_a \, dx = 0, \quad \int F \cdot \mathbf{Y}_a \, dx = 0, \quad a = 1, 2, 3.$$

Together, (2.11) and (2.13) give ten conditions in $d = 3$. These are the linear ancestors of energy, linear momentum, center of mass, and angular momentum.

2.5. The ten surface charges. For data (g, k) on a neighborhood of $\partial B_r \subseteq \mathbb{R}^3$, define

$$(2.14) \quad \mathbf{E}[(g, k); \partial B_r] = \frac{1}{2} \int_{\partial B_r} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j \, dS,$$

$$(2.15) \quad \mathbf{P}_a[(g, k); \partial B_r] = \int_{\partial B_r} (k_{aj} - \delta_{aj} \operatorname{tr}_\delta k) \nu^j \, dS,$$

$$(2.16) \quad \mathbf{C}_a[(g, k); \partial B_r] = \frac{1}{2} \int_{\partial B_r} \left[x^a (\partial_i g_{ij} - \partial_j g_{ii}) - \delta_{ia} (g_{ij} - \delta_{ij}) + \delta_{ja} (g_{ii} - \delta_{ii}) \right] \nu^j \, dS,$$

$$(2.17) \quad \mathbf{J}_a[(g, k); \partial B_r] = \int_{\partial B_r} (k_{ij} - \delta_{ij} \operatorname{tr}_\delta k) Y_a^i \nu^j \, dS.$$

Write

$$\mathbf{Q} = (\mathbf{E}, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3).$$

To avoid traces on rough Sobolev data, one may average the surface charges: for $\eta \in C_c^\infty((1, 2))$ such that $\int \eta = 1$ and $\eta_r(r') = r^{-1}\eta(r^{-1}r')$, define

$$(2.18) \quad \mathbf{Q}[(g, k); A_r] = \int \eta_r(r') \mathbf{Q}[(g, k); \partial B_{r'}] \, dr'.$$

The link between charges and cokernels is given by the following. For $0 < r_0 < r_1$,

$$(2.19) \quad \int_{B_{r_1} \setminus B_{r_0}} \frac{1}{2} \partial_i \partial_j h^{ij} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \, dx = \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g, k); \partial B_{r_1}] - \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g, k); \partial B_{r_0}],$$

$$(2.20) \quad \int_{B_{r_1} \setminus B_{r_0}} \sum_j \partial_i \pi^{ij} \begin{pmatrix} \mathbf{e}_1^j \\ \mathbf{e}_2^j \\ \mathbf{e}_3^j \\ \mathbf{Y}_1^j \\ \mathbf{Y}_2^j \\ \mathbf{Y}_3^j \end{pmatrix} \, dx = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{pmatrix} [(g, k); \partial B_{r_1}] - \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{pmatrix} [(g, k); \partial B_{r_0}],$$

and in terms of averaged charges, if

$$(2.21) \quad \chi_{r_0, r_1}(r) = \int_{-\infty}^r (\eta_{r_0}(r') - \eta_{r_1}(r')) \, dr',$$

we have

$$(2.22) \quad \int \chi_{r_0, r_1}(r) \frac{1}{2} \partial_i \partial_j h^{ij} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \, dx = \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g, k); A_{r_1}] - \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g, k); A_{r_0}],$$

$$(2.23) \quad \int \sum_j \chi_{r_0, r_1}(r) \partial_i \pi^{ij} \begin{pmatrix} \mathbf{e}_1^j \\ \mathbf{e}_2^j \\ \mathbf{e}_3^j \\ \mathbf{Y}_1^j \\ \mathbf{Y}_2^j \\ \mathbf{Y}_3^j \end{pmatrix} \, dx = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{pmatrix} [(g, k); A_{r_1}] - \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{pmatrix} [(g, k); A_{r_0}].$$

Under the invariant rescaling

$$(2.24) \quad g^{(R)}(x) = g(Rx), \quad k^{(R)}(x) = Rk(Rx),$$

the charges transform as

$$(2.25) \quad (\mathbf{E}, \mathbf{P})[g^{(R)}, k^{(R)}; A_1] = R^{-1}(\mathbf{E}, \mathbf{P})[g, k; A_R], \quad (\mathbf{C}, \mathbf{J})[g^{(R)}, k^{(R)}; A_1] = R^{-2}(\mathbf{C}, \mathbf{J})[g, k; A_R].$$

3. BOGOVSKII-TYPE OPERATORS

In [IMOT25], joint with Philip Isett, we develop a general theory of Bogovskii-type operators for underdetermined PDE systems satisfying Definition 3.3 below. In particular, it applies to the Einstein constraint equations (1.2).

3.1. The general operator and its formal cokernel. Let $U \subseteq \mathbb{R}^d$ be open, and let

$$(3.1) \quad \mathcal{P} : C^\infty(U; \mathbb{C}^{s_0}) \longrightarrow C^\infty(U; \mathbb{C}^{r_0}), \quad r_0 \leq s_0,$$

be an $r_0 \times s_0$ matrix of linear differential operators with smooth coefficients. We write $u = (u^K)_{K=1}^{s_0}$ and $f = (f^J)_{J=1}^{r_0}$. Its formal L^2 adjoint is

$$\mathcal{P}^* : C^\infty(U; \mathbb{C}^{r_0}) \longrightarrow C^\infty(U; \mathbb{C}^{s_0}).$$

One should think that $\mathcal{P}$ is underdetermined, while $\mathcal{P}^*$ is overdetermined.

Definition 3.1 (Formal cokernel). The formal cokernel of $\mathcal{P}$ on a connected open set U is

$$(3.2) \quad \ker \mathcal{P}^*(U) = \{\mathbf{Z} \in C^\infty(\overline{U}; \mathbb{C}^{r_0}) : \mathcal{P}^*\mathbf{Z} = 0\}.$$

No boundary condition is imposed on $\mathbf{Z}$.

The formal cokernel comes from compactly supported solvability. If $u \in C_c^\infty(U; \mathbb{C}^{s_0})$ and $\mathcal{P}u = f$, then every $\mathbf{Z} \in \ker \mathcal{P}^*(U)$ satisfies

$$(3.3) \quad \langle \mathbf{Z}, f \rangle = \langle \mathbf{Z}, \mathcal{P}u \rangle = \langle \mathcal{P}^*\mathbf{Z}, u \rangle = 0.$$

Thus compactly supported right inverses can exist only after projecting away the formal cokernel.

This leads us to consider the adjoint problem, namely to recover the value of a function φ from $\mathcal{P}^*\varphi$. For this, we introduce the following notion. This is a flexible generalization of special systems of first-order PDEs utilized by Reshetnyak [Res83, Res94].

Definition 3.2 (Graded augmented system). Let $\mathcal{P}$ be an $r_0 \times s_0$ -matrix-valued differential operator on an open subset $U \subseteq \mathbb{R}^d$, with m_K denoting the order of $(\mathcal{P}^*\varphi)_K$ for each $K \in \{1, \dots, s_0\}$. Given $m'_K \in \mathbb{Z}_{\geq 0}$ for each $K \in \{1, \dots, s_0\}$ and a $\mathbb{C}^{r_0}$ -valued function $(\varphi_J)_{J \in \{1, \dots, r_0\}}$ on U , consider $(\Phi_{\mathbf{A}} = \Phi_{\mathbf{A}}(y))_{\mathbf{A} \in \mathcal{A}}$ (called *augmented variables*) satisfying the following properties:

(Φ-1) $(\Phi_{\mathbf{A}})$ is an **augmentation** of (φ_J) . The index set $\mathcal{A}$ contains $\{1, \dots, r_0\}$; moreover, $\Phi_J = \varphi_J$ for $J \in \{1, \dots, r_0\} \subseteq \mathcal{A}$.

(Φ-2) $(\varphi_J) \mapsto (\Phi_{\mathbf{A}})$ is a **differential operator**. There exist functions $c[\Phi_{\mathbf{A}}]^{(\alpha, J)}$ on U , where α is a multi-index and $J \in \{1, \dots, r_0\}$, such that

$$\Phi_{\mathbf{A}}(y) = c[\Phi_{\mathbf{A}}]^{(\alpha, J)}(y) \partial^\alpha \varphi_J(y),$$

for all $\mathbf{A} \in \mathcal{A}$ and $y \in U$.

(Φ-3) Φ and $\mathcal{P}^*\varphi$ solve a **first-order PDE (augmented PDE system)**. There exist matrix-valued 1-forms $(\mathbf{B}_i)_{\mathbf{A}}^{\mathbf{A}'}$ and $(\mathbf{C}_i)_{\mathbf{A}}^{(\gamma, K)}$ on U , where γ is a multi-index and $K \in \{1, \dots, s_0\}$, such that

$$(3.4) \quad \partial_i \Phi_{\mathbf{A}}(y) = (\mathbf{B}_i)_{\mathbf{A}}^{\mathbf{A}'}(y) \Phi_{\mathbf{A}'}(y) + (\mathbf{C}_i)_{\mathbf{A}}^{(\gamma, K)}(y) \partial^\gamma (\mathcal{P}^*\varphi)_K(y),$$

for all $\mathbf{A} \in \mathcal{A}$, $i \in \{1, \dots, d\}$ and $y \in U$. Furthermore,

$$(\mathbf{C}_i)_{\mathbf{A}}^{(\gamma, K)} = 0 \quad \text{if } |\gamma| > m'_K.$$

(Φ-4) **Graded structure**. Define the *degree* $d_{\mathbf{A}}$ of $\Phi_{\mathbf{A}}$ by

$$d_{\mathbf{A}} := -\max\{|\alpha| : c[\Phi_{\mathbf{A}}]^{(\alpha, J)} \neq 0 \text{ for some } J\}.$$

In particular, φ_J has degree 0, i.e., $d_J = 0$ for $J \in \{1, \dots, r_0\}$. We assume that:

• **Graded structure for the augmented system.**

$$\begin{aligned} (\mathbf{B}_i)_{\mathbf{A}}^{\mathbf{A}'} &= 0 & \text{if } d_{\mathbf{A}} > d_{\mathbf{A}'} + 1, \\ (\mathbf{C}_i)_{\mathbf{A}}^{(\gamma, K)} &= 0 & \text{if } d_{\mathbf{A}} > -m_K - |\gamma| + 1. \end{aligned}$$

We write $N_0 := \max\{d_{\mathbf{A}}\}_{\mathbf{A} \in \mathcal{A}} + 1$ for the maximal degree that occurs in (3.4).

We call a collection $(\mathcal{A}, (\Phi_{\mathbf{A}})_{\mathbf{A} \in \mathcal{A}}, (\mathbf{B}_i)_{\mathbf{A}}^{\mathbf{A}'}, (\mathbf{C}_i)_{\mathbf{A}}^{(\gamma, K)})$ satisfying $(\Phi-1)$ – $(\Phi-4)$ a *(graded) augmented system* for $\mathcal{P}$.

Once we have a graded augmented system, we can integrate the ODE on a family of curves and take average. Under natural geometric assumptions, this leads to a right inverse up to finite dimensional cokernels: $(\mathcal{S}_{\eta})_{\mathcal{J}}^K : \widetilde{W}^{s,p}(U) \rightarrow \widetilde{W}^{s+m_K,p}(U)$ for every $1 < p < \infty$ and $s \in \mathbb{R}$, such that

$$\mathcal{P}\mathcal{S}_{\eta}f = f - \sum_{\mathbf{A}} w_{\mathbf{A}} \langle \mathbf{Z}^{\mathbf{A}}, f \rangle \quad \text{for all } f \in \widetilde{W}^{s,p}(U),$$

where $\mathbf{Z}^{\mathbf{A}}$ is a basis of $\ker \mathcal{P}^*(U)$ and $w_{\mathbf{A}} \in C_c^{\infty}(U)$ is a dual basis of $\mathbf{Z}^{\mathbf{A}}$, and the following support property holds:

$$\text{supp } f \subseteq \overline{U} \implies \text{supp } \mathcal{S}_{\eta}f \subseteq \overline{U}.$$

On flat space, our construction recovers the explicit solution operators for various underdetermined differential operators that have appeared in the literature [Bog79, IO16, OT19].

Instead of stating the general theorem, we will explain this construction by examples below.

3.2. The complex ellipticity condition. The principal symbol of $\mathcal{P}^*$ defines a linear map

$$(3.5) \quad p^*(x, \xi) : \mathbb{C}^{r_0} \longrightarrow \mathbb{C}^{s_0}, \quad \xi \in \mathbb{C}^d.$$

The following definition was considered in the literature [Aro54, Smi70, Kał94, BDG20] from different perspectives.

Definition 3.3 (Complex ellipticity condition). The operator $\mathcal{P}$ satisfies *(CE)* on U if

$$(3.6) \quad p^*(x, \xi) \text{ is injective for every } x \in U \text{ and every } \xi \in \mathbb{C}^d \setminus \{0\}.$$

Equivalently, the principal symbol $p(x, \xi)$ of $\mathcal{P}$ is surjective for every nonzero complex covector.

If we only consider $\xi \in \mathbb{R}^d$, this is the usual ellipticity condition.

It turns out that the complex ellipticity condition implies the existence of a graded augmented system.

Theorem 3.4. *Let U be an open subset of $\mathbb{R}^d$. Suppose that $\mathcal{P}$ has smooth coefficients. Assume that the $s_0 \times r_0$ -matrix-valued principal symbol $p^*(x, \xi)$ of $\mathcal{P}^*$ (where $s_0 \geq r_0$) satisfies*

(CE) $p^(x, \xi)$ is injective for all $x \in U$ and $\xi \in \mathbb{C}^d \setminus \{0\}$.*

Then there exist augmented variables $(\Phi_{\mathbf{A}})_{\mathbf{A} \in \mathcal{A}}$ on U satisfying $(\Phi-1)$ – $(\Phi-4)$ with index set $\mathcal{A} = \{(\alpha, J) : 1 \leq J \leq r_0, |\alpha| \leq N_0 - 1\}$, degree $d_{(\alpha, J)} = -|\alpha|$, and coefficients $\mathbf{B}_i, \mathbf{C}_i \in C^{\infty}(U)$ with quantitative estimates.

For the proof, we recall

Proposition 3.5 (Hilbert's Nullstellensatz). *Suppose $f_1(x_1, \dots, x_d), \dots, f_N(x_1, \dots, x_d) \in \mathbb{C}[x_1, \dots, x_d]$ are polynomials with the set of common zeros*

$$Z = \{(z_1, \dots, z_d) \in \mathbb{C}^d : f_j(z_1, \dots, z_d) = 0, j = 1, 2, \dots, N\}.$$

Then for any polynomial $h(x_1, \dots, x_d) \in \mathbb{C}[x_1, \dots, x_d]$, if h vanishes on Z , then there exist $M > 0$ and polynomials $g_j(x_1, \dots, x_d) \in \mathbb{C}[x_1, \dots, x_d]$, $j = 1, 2, \dots, N$ such that

$$h(x_1, \dots, x_d)^M = \sum_j g_j(x_1, \dots, x_d) f_j(x_1, \dots, x_d).$$

Proof of Theorem 3.4. Step 1: For any $x_0 \in U$, there exist $N_0 > 0$ and polynomial valued $r_0 \times s_0$ matrices $g_{\alpha}(\xi)$ for $|\alpha| = N_0$ such that

$$(3.7) \quad \xi^{\alpha} I_{r_0} = g_{\alpha}(\xi) p^*(x_0, \xi), \quad |\alpha| = N_0.$$

Moreover, $(p^*)_{\mathcal{J}}^K(\xi)$ is homogeneous of degree m_K and $(g_{\alpha})_{\mathcal{J}}^K(\xi)$ is homogeneous of degree $N_0 - m_K$.

The claim follows from applying Hilbert's Nullstellensatz (Proposition 3.5) to $\det(M_j(\xi))$ where $M_j(\xi)$ goes through all the $r_0 \times r_0$ minors of $p^*(x_0, \xi)$. By assumption, the common zeros of $\det(M_j(\xi))$ in $\mathbb{C}^d$ is contained in $\{0\}$. Since ξ_ℓ vanishes at 0, there exist $N_1, \dots, N_d$ and polynomials $h_\ell^j(\xi)$ such that

$$\xi_\ell^{N_\ell} = \sum_j h_\ell^j(\xi) \det(M_j(\xi)).$$

We notice that $\det(M_j(\xi))I_{r_0}$ can be written as the product of its adjugate matrix $\text{adj}(M_j(\xi))$ and itself:

$$\det(M_j(\xi))I_{r_0} = \text{adj}(M_j(\xi))M_j(\xi).$$

Moreover, $M_j(\xi)$ is the product of a constant 0 – 1 matrix and $p^*(x_0, \xi)$. Therefore we conclude that there are polynomial-valued $r_0 \times s_0$ matrices $\tilde{g}^\ell(\xi)$ so that

$$\xi_\ell^{N_\ell} I_{r_0} = \tilde{g}^\ell(\xi) p^*(x_0, \xi).$$

Taking $N_0 > N_1 + \dots + N_d$, we conclude (3.7). We may assume $(g_\alpha)_J^K(\xi)$ is homogeneous of degree $N_0 - m_K$ since terms of other degrees do not contribute.

Step 2: There exist ξ -polynomial-valued $r_0 \times s_0$ matrices $g_\alpha(x, \xi)$ for $|\alpha| = N_0$ such that

$$(3.8) \quad \xi^\alpha I_{r_0} = g_\alpha(x, \xi) p^*(x, \xi), \quad |\alpha| = N_0.$$

The matrix $g_\alpha(x, \xi)$ depends smoothly on the coefficients of $p^*(x, \xi)$ and some smooth cutoff functions. Moreover, $(p^*)_K^J(\xi)$ is homogeneous of degree m_K and $(g_\alpha)_J^K(\xi)$ is homogeneous of degree $N_0 - m_K$.

When $\mathcal{P}$ has constant coefficients, then it suffices to take $g_\alpha(x, \xi) = g_\alpha(\xi)$ (from Step 1). When $\mathcal{P}$ has variable coefficients, it follows from a perturbation argument.

Step 3. We claim that taking $\Phi_{\mathbf{A}} = \partial^\alpha \varphi_J$ to be the jet of φ_J up to order $N_0 - 1$ indexed by $\mathbf{A} = (\alpha, J)$ in the set $\mathcal{A} := \{(\alpha, J) : 1 \leq J \leq r_0, |\alpha| \leq N_0 - 1\}$ gives an augmented system satisfying $(\Phi-1)$ – $(\Phi-4)$. First, $(\Phi-1)$ and $(\Phi-2)$ are obvious from the definition. For $(\Phi-3)$, we take the trivial relation $\partial_i \partial^\alpha \varphi_J(y) = \partial^{\alpha + \mathbf{e}_i} \varphi_J(y)$ to be the equation if $|\alpha| < N_0 - 1$, i.e.,

$$(\mathbf{B}_i)_{(\alpha, J)}^{(\alpha', J')} = \begin{cases} 1 & \text{if } J = J', \alpha' = \alpha + \mathbf{e}_i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad (\mathbf{C}_i)_{(\alpha, J)}^{(\gamma, K)} = 0 \quad \text{for } |\alpha| < N_0 - 1.$$

It remains to check that, for $|\alpha| = N_0 - 1$, there exist $(\mathbf{B}_i)_{(\alpha, J)}^{(\alpha', J')}(y)$ and $(\mathbf{C}_i)_{(\alpha, J)}^{(\gamma, K)}(y)$ such that

$$(3.9) \quad \partial_i \partial^\alpha \varphi_J(y) = (\mathbf{B}_i)_{(\alpha, J)}^{(\alpha', J')}(y) \partial^{\alpha'} \varphi_{J'}(y) + (\mathbf{C}_i)_{(\alpha, J)}^{(\gamma, K)}(y) \partial^\gamma (\mathcal{P}^* \varphi)_K(y).$$

This follows from (3.8) and putting $(\mathcal{P}_{\text{prin}}^* - \mathcal{P}^*)\varphi$ into the first term on the right-hand side. The assumption $(\Phi-4)$ follows by taking $d_{(\alpha, J)} = -|\alpha|$. $\square$

In the constant-coefficient homogeneous case, we have the following equivalence.

Proposition 3.6. *Let U be a connected open subset of $\mathbb{R}^d$, and $\mathcal{P}$ an $r_0 \times s_0$ -matrix-valued operator on U that is homogeneous with constant coefficients. Then the following are equivalent:*

- (1) $p^*(\xi)$ satisfies (CE).
- (2) There exists a graded augmented system $\{\Phi_{\mathbf{A}}\}_{\mathbf{A} \in \mathcal{A}}$ for $\mathcal{P}$ on U .
- (3) $\dim \ker \mathcal{P}^*(U) < +\infty$.

Proof. The implication (1) $\Rightarrow$ (2) is exactly Theorem 3.4. Moreover, (2) $\Rightarrow$ (3) follows from integrating along curves. It remains to prove (3) $\Rightarrow$ (1), or equivalently, its contrapositive. Indeed, if p^* does not satisfy (CE), then there exists a constant vector $\{\varphi_J\}_{J \in \{1, \dots, r_0\}} \neq 0$ and $\xi \in \mathbb{C}^d \setminus \{0\}$ such that $p^*(\xi)\varphi = 0$, which means that $e^{i\lambda \xi x} \varphi \in \ker \mathcal{P}^*$ for every $\lambda \in \mathbb{C}$. Thus $\ker \mathcal{P}^*$ is infinite dimensional. $\square$

3.3. Explicit examples on flat spaces. By averaging over straight line segments $\mathbf{x}(y, y_1, s) = y + s(y_1 - y)$ (straight rays $\mathbf{x}(y, \omega, s) = y + s\omega$, respectively), we obtain the Bogovskii-type solution operator (conic solution operator, respectively).

3.3.1. *Double divergence equation.* We consider the double divergence equation

$$(3.10) \quad P_H h = \partial_j \partial_k h^{jk}, \quad h^{ij} = h^{ji}$$

on $\mathbb{R}^d$ for $d \geq 2$. $P_H^* \varphi = \nabla^2 \varphi$. The graded augmented system is

$$(3.11) \quad \begin{aligned} \nabla_i \varphi &= \omega_i, \\ \nabla_i \omega_j &= (P_H^* \varphi)_{ij}. \end{aligned}$$

(1) Let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \eta = 1$. The Bogovskii-type solution operator for (3.10) is given by

$$(3.12) \quad K_\eta^{ij}(z+y, y) = \left(\int_{|z|}^\infty \eta \left(r \frac{z}{|z|} + y \right) r^{d-1} dr \right) \frac{z^i z^j}{|z|^d}.$$

It satisfies

$$(3.13) \quad P_H \int K_\eta(x, y) f(y) dy = f(x) - \int b_\eta(x, y) f(y) dy,$$

where

$$b_\eta(x, y) = (d+1)\eta(x) + (x-y)^i \partial_i \eta(x).$$

(2) Let $\kappa \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \kappa = 1$. The conic solution operator for (3.10) is given by

$$(3.14) \quad K_\kappa^{ij}(z+y, y) = \frac{z^i z^j}{|z|^d} \kappa \left(\frac{z}{|z|} \right).$$

It satisfies

$$(3.15) \quad P_H \int K_\kappa(x, y) f(y) dy = f(x).$$

3.3.2. *Symmetric divergence equation.* We consider the symmetric divergence equation

$$(3.16) \quad P_M \pi = \partial_j \pi^{jk}, \quad \pi^{ij} = \pi^{ji}$$

on $\mathbb{R}^d$ with $d \geq 2$. $(P_M^* \omega)_{ij} = -\frac{1}{2}(\partial_i \omega_j + \partial_j \omega_i)$. Introduce $\eta_{jk} = \frac{1}{2}(\partial_j \omega_k - \partial_k \omega_j)$. The graded augmented system is

$$(3.17) \quad \begin{aligned} \nabla_i \omega_j &= \eta_{ij} - (P_M^* \omega)_{ij}, \\ \nabla_i \eta_{jk} &= \partial_k (P_M^* \omega)_{ij} - \partial_j (P_M^* \omega)_{ik}. \end{aligned}$$

(1) Let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \eta = 1$. The Bogovskii-type solution operator for (3.16) is given by

$$(3.18) \quad \begin{aligned} (L_\eta)_k^{ij}(z+y, y) &= \frac{1}{2} \left(\int_{|z|}^\infty \eta \left(r \frac{z}{|z|} + y \right) r^{d-1} dr \right) \frac{z^i \delta_k^j + z^j \delta_k^i}{|z|^d} \\ &\quad + \frac{1}{2} \partial_{z^m} \left(\left(\int_{|z|}^\infty \eta \left(r \frac{z}{|z|} + y \right) r^{d-1} dr \right) \frac{z^m (z^i \delta_k^j + z^j \delta_k^i)}{|z|^d} \right) \\ &\quad - \partial_{z^k} \left(\left(\int_{|z|}^\infty \eta \left(r \frac{z}{|z|} + y \right) r^{d-1} dr \right) \frac{z^i z^j}{|z|^d} \right). \end{aligned}$$

It satisfies

$$(3.19) \quad P_M \int (L_\eta)_k^{ij}(x, y) F^k(y) dy = F^j(x) - \int (b_\eta)_k^j(x, y) F^k(y) dy,$$

where

$$(b_\eta)_k^j(x, y) = \frac{d+1}{2} \eta(x) \delta_k^j + \frac{1}{2} (x-y)^\ell \partial_\ell \eta(x) \delta_k^j - \frac{1}{2} (x-y)^j \partial_k \eta(x).$$

(2) Let $\kappa \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \kappa = 1$. The conic solution operator for (3.16) is given by

$$(3.20) \quad (L_\kappa)_k^{ij}(z+y, y) = \frac{1}{2} \frac{z^i \delta_k^j + z^j \delta_k^i}{|z|^d} \kappa \left(\frac{z}{|z|} \right) + \frac{1}{2} \partial_{z^m} \left(\frac{z^m (z^i \delta_k^j + z^j \delta_k^i)}{|z|^d} \kappa \left(\frac{z}{|z|} \right) \right) - \partial_{z^k} \left(\frac{z^i z^j}{|z|^d} \kappa \left(\frac{z}{|z|} \right) \right).$$

It satisfies

$$(3.21) \quad P_M \int (L_\kappa)_k^{ij}(x, y) F^k(y) dy = F^j(x).$$

4. MAPPING PROPERTIES OF SOLUTION OPERATORS

We discuss the mapping properties of the solution operators on weighted Sobolev spaces in §3.3.

4.1. Weighted Sobolev spaces. For an integer $m \geq 0$, define

$$(4.1) \quad \|u\|_{H^{m,0}}^2 := \sum_{|\alpha| \leq m} \|\langle x \rangle^{|\alpha|} \partial^\alpha u\|_{L^2(\mathbb{R}^d)}^2.$$

For $s \in \mathbb{R}$, we define $H^{s,0}$ using complex interpolation and duality. For $\delta \in \mathbb{R}$, set

$$(4.2) \quad H^{s,\delta} = \langle x \rangle^{-\delta} H^{s,0}, \quad \|u\|_{H^{s,\delta}} := \|\langle x \rangle^\delta u\|_{H^{s,0}}.$$

For integer m , the norm is equivalent to

$$(4.3) \quad \sum_{|\alpha| \leq m} \|\langle x \rangle^{\delta+|\alpha|} \partial^\alpha u\|_{L^2}.$$

For an open subset $\Omega \subseteq \mathbb{R}^d$, let $\tilde{H}^{s,\delta}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ under the $H^{s,\delta}$ norm. When Ω is the closure of an open set with Lipschitz boundary, we use the same notation.

In $d = 3$, the weights $\delta = -1/2$ and $\delta = 1/2$ correspond respectively to the decay rate r^{-1} and r^{-2} .

Let $\{\chi_j\}_{j \geq 0}$ be a dyadic partition of unity, with χ_j supported where $r \simeq 2^j$, and let $\varphi_j(x) = 2^j x$. Then we have the dyadic decomposition:

$$(4.4) \quad \|u\|_{H^{s,\delta}}^2 \simeq \sum_{j \geq 0} 2^{2j(\delta+d/2)} \|\varphi_j^*(\chi_j u)\|_{H^s(\mathbb{R}^d)}^2.$$

Using (4.4), we have

(i) If $s > d/2$, then

$$(4.5) \quad \|\langle x \rangle^{d/2+\delta} u\|_{L^\infty} \lesssim \|u\|_{H^{s,\delta}}.$$

(ii) For $s_1 + s_2 > 0$, we have

$$(4.6) \quad \|uv\|_{H^{s_0, \delta_1+\delta_2+d/2}} \lesssim \|u\|_{H^{s_1, \delta_1}} \|v\|_{H^{s_2, \delta_2}}, \quad s_0 = \begin{cases} \min(s_1, s_2), & \max(s_1, s_2) > d/2, \\ s_1 + s_2 - d/2, & \max(s_1, s_2) < d/2. \end{cases}$$

The natural solution space for the constraint equations is

$$(4.7) \quad X^{s,\delta} := H^{s,\delta}(\mathbb{R}^d; \text{Sym}^2) \times H^{s-1, \delta+1}(\mathbb{R}^d; \text{Sym}^2).$$

4.2. Support preservation and the outgoing property. Note that

$$\text{supp}(u * v) \subseteq \text{supp } u + \text{supp } v := \{a + b : a \in \text{supp } u, b \in \text{supp } v\}.$$

Therefore we have

Corollary 4.1. *If $\text{supp}(f, F) \subseteq \Omega$ and Ω is a convex cone containing $\text{supp } \kappa$, then the operator $S_c = (K_\kappa, L_\kappa)$ defined by (3.14) and (3.20) satisfies*

$$\text{supp } S_c(f, F) \subseteq \Omega.$$

The operator is outgoing in the following sense.

Definition 4.2. An integral kernel $K(x, y)$ on a cone Ω is outgoing if there is $C_\Omega > 0$ such that

$$(4.8) \quad K(x, y) \neq 0 \implies |y| \leq C_\Omega(1 + |x|).$$

Similarly, we say K is incoming if there is $C_\Omega > 0$ such that

$$(4.9) \quad K(x, y) \neq 0 \implies |x| \leq C_\Omega(1 + |y|).$$

If an operator is both outgoing and incoming, we say that it is diagonal.

Lemma 4.3. *Let $\Omega = C_\theta(\omega)$ with $0 < \theta < \pi/2$ and $\text{supp } \kappa \subseteq \Omega$. The kernel $(K_\kappa(x - y), L_\kappa(x - y))$ is outgoing on Ω .*

Proof. If $x, y \in \Omega$ and $x - y \in C_\theta(\omega)$, then $(x - y) \cdot \omega \geq 0$. Therefore,

$$|y| \cos \theta \leq y \cdot \omega \leq x \cdot \omega \leq |x|. \quad \square$$

4.3. Weighted estimates for homogeneous kernels. The conic kernels have nice mapping properties between weighted Sobolev spaces.

Theorem 4.4. *Let $K \in C^\infty(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ be homogeneous of degree $k - d$, with $0 < k < d$. If*

$$(4.10) \quad -\frac{d}{2} < \delta < \frac{d}{2} - k,$$

then convolution extends to a bounded map

$$(4.11) \quad K * : H^{s-k, \delta+k} \longrightarrow H^{s, \delta}.$$

If the kernel is outgoing on a cone, the lower bound in (4.10) is not needed for the outgoing part; if it is incoming, the upper bound is not needed for the incoming part.

Proof. Split the Schwartz kernel $K(x - y)$ into three regions:

$$K_{\text{diag}} + K_{\text{out}} + K_{\text{in}},$$

where $|x| \simeq |y|$, $|x| > (1 + c)|y|$, and $|x| < (1 - c)|y|$, respectively.

On the diagonal region, rescale each dyadic annulus to unit size. Homogeneity gives a factor $2^{j(k-d)}$, and standard pseudodifferential estimates show that the localized operator has order $-k$.

On the outgoing region, the off-diagonal kernel is smooth and satisfies

$$|\partial_{x,y}^\alpha K_{\text{out}}(x, y)| \lesssim_\alpha (1 + |x| + |y|)^{k-d-|\alpha|}.$$

This implies that

$$(4.12) \quad \|K_{\text{out}, j, j'}\|_{H^{s-k, \delta+k}(A_{2j'}) \rightarrow H^{N, \delta}(A_{2j})} \lesssim 2^{j(k-d)} 2^{(\delta+d/2)j - (\delta-d/2+k)j'} = 2^{(\delta-d/2+k)(j-j')}.$$

In the outgoing region we have $j > j'$, this sum converges when $\delta < d/2 - k$. On the incoming region we have

$$(4.13) \quad \|K_{\text{in}, j, j'}\|_{H^{s-k, \delta+k}(A_{2j'}) \rightarrow H^{N, \delta}(A_{2j})} \lesssim 2^{j'(k-d)} 2^{(\delta+d/2)j - (\delta-d/2+k)j'} = 2^{(\delta+d/2)(j-j')}, \quad j < j'.$$

The convergence requires $\delta > -d/2$. Combining the three pieces gives (4.11). $\square$

For K_κ we take $k = 2$; for L_κ we take $k = 1$. By the outgoing property Lemma 4.3 we have

Corollary 4.5 (Mapping property of S_c). *Let $d \geq 3$ and $\Omega = C_\theta(\omega)$ for some $0 < \theta < \pi/2$. For every $\delta < \frac{d}{2} - 2$ and $s \in \mathbb{R}$,*

$$(4.14) \quad S_c : \tilde{H}^{s-2, \delta+2}(\Omega; \mathbb{R} \oplus \mathbb{R}^d) \longrightarrow \tilde{H}^{s, \delta}(\Omega; \text{Sym}^2) \times \tilde{H}^{s-1, \delta+1}(\Omega; \text{Sym}^2)$$

is bounded.

The endpoint is sharp: for generic compactly supported f , $K_\kappa * f$ has a leading term of size r^{2-d} , and this power lies just outside $\tilde{H}^{s, d/2-2}(\Omega)$.

4.4. The Bogovskii-type operators on an annulus. The Bogovskii-type operators in (3.12) and (3.18) are designed on star-shaped sets, but it is not difficult to generalize the construction to an annulus. Moreover, the Bogovskii-type operators given in (3.12) and (3.18) are pseudodifferential operators of order -2 and -1 , respectively. Therefore we have

Theorem 4.6 (Bogovskii operators on A_1). *For $P = (P_H, P_M)$, there are linear operators*

$$S_B : C_c^\infty(A_1; \mathbb{R} \oplus \mathbb{R}^d) \rightarrow C_c^\infty(A_1; \text{Sym}^2) \times C_c^\infty(A_1; \text{Sym}^2)$$

with the following properties:

- (i) $PS_B(f, F) = (f, F)$ for (f, F) satisfying (2.11) and (2.13);
- (ii) $S_B : \tilde{H}^{s-2}(A_1) \rightarrow \tilde{H}^s(A_1) \times \tilde{H}^{s-1}(A_1)$;
- (iii) $\text{supp}(f, F) \subseteq A_1$ implies that $\text{supp } S_B(f, F) \subseteq A_1$.

5. CONIC GLUING AND OBSTRUCTION-FREE GLUING

5.1. The Picard iteration.

Proposition 5.1. *Let $P : X \rightarrow Y$ and $S : Y \rightarrow X$ be bounded operators between Banach spaces. Suppose $\Phi : X \rightarrow Y$ is smooth near 0, satisfies (2.8), and obeys (2.9)–(2.10). If $u_0 \in X$ and $f \in Y$ is sufficiently small, then there is a unique u_1 of size $\lesssim \|u_0\|_X^2 + \|f\|_Y$ such that*

$$(5.1) \quad u_1 = S(\Phi(u_0 + u_1) + f).$$

Consequently, if $u_0 \in \ker P$ and $PS = \text{Id}_Y$, then $u := u_0 + u_1$ solves $Pu = \Phi(u) + f$ and

$$\|u - u_0\|_X \lesssim \|u_0\|_X^2 + \|f\|_Y.$$

Proof. Let $\varepsilon = \|u_0\|_X^2 + \|f\|_Y$ and consider the ball $B_{C\varepsilon}(0) \subseteq X$. For u_1 in this ball,

$$\|S(\Phi(u_0 + u_1) + f)\|_X \lesssim \|S\|(\|u_0\|_X^2 + \|f\|_Y + C\varepsilon\|u_0\|_X + (C\varepsilon)^2) \leq C\varepsilon.$$

Moreover,

$$\|S(\Phi(u_0 + u_1) + f) - S(\Phi(u_0 + v_1) + f)\|_X \leq C_1\|S\|(\|u_0\|_X + \varepsilon)\|u_1 - v_1\|_X.$$

For ε small so that $C_1\|S\|(\|u_0\|_X + \varepsilon) < 1/2$, Banach fixed point theorem implies a unique u_1 of norm $\leq C\varepsilon$ satisfying (5.1). Finally, if $u_0 \in \ker P$ and $PS = \text{Id}_Y$, then

$$P(u_0 + u_1) = Pu_1 = PS(\Phi(u_0 + u_1) + f) = \Phi(u_0 + u_1) + f. \quad \square$$

The difficult work is therefore concentrated in two tasks: construct a right inverse S with the correct support, and choose spaces in which S and Φ satisfy the required estimates.

5.2. Solutions localized in a cone. We now show how to construct solutions supported in a cone. Let $\Omega = C_\theta(\omega) \subseteq \mathbb{R}^3$,

$$X^{s,\delta} = \tilde{H}^{s,\delta}(\Omega; \text{Sym}^2) \times \tilde{H}^{s-1,\delta+1}(\Omega; \text{Sym}^2), \quad Y^{s,\delta} = \tilde{H}^{s-2,\delta+2}(\Omega; \mathbb{R} \oplus \mathbb{R}^3).$$

For $s > 3/2$ and $\delta \geq -3/2$, the product estimate (4.6) implies that the nonlinear map (2.6)–(2.7) is smooth

$$(5.2) \quad \Phi : X^{s,\delta} \longrightarrow Y^{s,\delta}$$

near the origin. Combining this with Corollary 4.5, we have the following interval

$$(5.3) \quad -\frac{3}{2} \leq \delta < -\frac{1}{2}.$$

Theorem 5.2. *Let $\Omega = C_\theta(\omega) \subseteq \mathbb{R}^3$ with $0 < \theta < \pi/2$. For $s > 3/2$ and $-3/2 \leq \delta < -1/2$, if $u_0 = (h_0, \pi_0) \in X^{s,\delta}$ is a sufficiently small solution of $Pu_0 = 0$ supported in $C_\theta(\omega)$, then there is a solution $u = (h, \pi)$ to (1.2) supported in $C_\theta(\omega)$ such that*

$$(5.4) \quad \|u - u_0\|_{X^{s,\delta}} \lesssim \|u_0\|_{X^{s,\delta}}^2.$$

If u_0 is smooth, then the solution u is also smooth.

Proof. Set the fixed point problem

$$(5.5) \quad u_1 = S_c \Phi(u_0 + u_1).$$

The mapping statements above and the quadratic estimates for Φ verify the hypotheses of Proposition 5.1. Therefore, there exists a solution $u_1 \in X^{s,\delta}$ solving (5.5) with

$$\|u_1\|_{X^{s,\delta}} \lesssim \|u_0\|_{X^{s,\delta}}^2, \quad P(u_0 + u_1) = \Phi(u_0 + u_1).$$

The local nature of Φ and Corollary 4.1 preserve support. Smoothness follows by differentiating the fixed point equation, commuting derivatives through the diagonal part of S_c , and using that the outgoing pieces are smoothing away from the diagonal. $\square$

In order to prove Theorem 1.1, it remains to estimate the tail (1.3). For this, we take u_0 to be compactly supported in Theorem 5.2.

A compactly supported u_0 does not in general lead to compactly supported nonlinear data. This is because the solution operator S_c generates tails. We note that for $|x| > 2|y| + 1$,

$$(5.6) \quad K_\kappa(x - y) = K_\kappa(x) + O(|x|^{-2}|y|),$$

$$(5.7) \quad L_\kappa(x - y) = L_\kappa(x) + O(|x|^{-3}|y|).$$

Consequently, for (f, F) decaying fast enough,

$$(5.8) \quad K_\kappa * f(x) = K_\kappa(x) \int f(y) dy + O(|x|^{-2}),$$

$$(5.9) \quad L_\kappa * F(x) = L_\kappa(x) \int F(y) dy + O(|x|^{-3}).$$

In the fixed point equation 5.5, the weighted Sobolev estimate 4.5 gives the decay of the nonlinearity $\Phi(u_0 + u_1)$ at the rate $r^{-4+\varepsilon}$. This does not lead to the $O(|x|^{-2})$ and $O(|x|^{-3})$ error since $\int_{\mathbb{R}^3} |y| |\Phi(u_0 + u_1)(y)| dy$ diverges. But this is enough to get the upper bound in (1.3) when $\ell = 0$ by splitting the integral into the region $|x| > 2|y| + 1$ and $|x| \leq 2|y| + 1$.

A careful iteration on derivatives for the fixed point equation (5.5) implies (1.3). This finishes the proof of Theorem 1.1.

5.3. Corvino–Schoen type gluing. Now we study the gluing problem on an annulus, à la Corvino–Schoen [CS06]. The proof we state below is from Mao–Oh–Tao [MOT23].

Definition 5.3 ($\mathcal{Q}$ -admissible initial data sets). Let a bounded open subset $\mathcal{Q} \subseteq \mathbb{R}^{10}$ and $s \in \mathbb{R}$ be given. We say that a 10-parameter family $\{(g_Q, k_Q)\}_{Q \in \mathcal{Q}}$ is a $\mathcal{Q}$ -admissible family of annular initial data sets on $\tilde{A}_r$ with Sobolev regularity s if:

- (1) for each $Q \in \mathcal{Q}$, $(g_Q, k_Q) \in (H^1 \cap C^0) \times L^2(\tilde{A}_r)$ and solves (1.2) in $\tilde{A}_r$;
- (2) for each $Q \in \mathcal{Q}$, $(g_Q, k_Q) \in H^s \times H^{s-1}(\tilde{A}_r)$;
- (3) the map $\mathcal{Q} \rightarrow H^s \times H^{s-1}(\tilde{A}_r)$ defined by $Q \mapsto (g_Q, k_Q)$ is Lipschitz;
- (4) for each $Q \in \mathcal{Q} \subseteq \mathbb{R}^{10}$, we have $Q = \mathbf{Q}[(g_Q, k_Q); A_r]$.

Theorem 5.4 (Gluing up to linear obstructions at unit scale). *For each $s > 3/2$, there exist $\varepsilon_c = \varepsilon_c(s) > 0$ and $M_c = M_c(s) > 0$ such that the following holds. Let $(\mathring{g}, \mathring{k})$ be a solution to (1.2) in $H^s \times H^{s-1}(\tilde{A}_1)$ such that*

$$(5.10) \quad \|(\mathring{g} - \delta, \mathring{k})\|_{H^s \times H^{s-1}(\tilde{A}_1)} \leq \varepsilon.$$

Define $\mathring{Q} \in \mathbb{R}^{10}$ by $\mathring{Q} = \mathbf{Q}[(\mathring{g}, \mathring{k}); A_1]$. Let $\mathcal{Q} \subseteq \mathbb{R}^{10}$ be a bounded open set such that

$$(5.11) \quad \mathring{Q} \in \mathcal{Q}, \quad B_{2M_c\varepsilon^2}(\mathring{Q}) \subseteq \mathcal{Q}.$$

and consider a $\mathcal{Q}$ -admissible family of annular initial data sets $\{(g_Q, k_Q)\}_{Q \in \mathcal{Q}}$ on $\tilde{A}_1$ with Sobolev regularity s such that, for all $Q, Q' \in \mathcal{Q}$,

$$(5.12) \quad \|(g_Q - \delta, k_Q)\|_{H^s \times H^{s-1}(\tilde{A}_1)} \leq \varepsilon,$$

$$(5.13) \quad \|(g_Q - g_{Q'}, k_Q - k_{Q'})\|_{H^s \times H^{s-1}(\tilde{A}_1)} \leq K|Q - Q'|.$$

Then the following holds:

- (1) **Exterior gluing.** *If $\varepsilon < \varepsilon_c$ and $K\varepsilon < \varepsilon_c$, then there exists a solution $(g, k) \in H^s \times H^{s-1}(\tilde{A}_1)$ to (1.2) and $Q \in \mathcal{Q}$ such that $\|(g - \delta, k)\|_{H^s \times H^{s-1}(\tilde{A}_1)} \lesssim \varepsilon$, $|Q - \mathring{Q}| \leq M_c\varepsilon^2$, and*

$$(g, k) = \begin{cases} (\mathring{g}, \mathring{k}) & \text{in } A_{\frac{1}{2}}, \\ (g_Q, k_Q) & \text{in } A_2. \end{cases}$$

(2) **Interior gluing.** If $\varepsilon < \varepsilon_c$ and $K\varepsilon < \varepsilon_c$, then there exists a solution $(g, k) \in H^s \times H^{s-1}(\tilde{A}_1)$ to (1.2) and $Q \in \mathcal{Q}$ such that $\|(g - \delta, k)\|_{H^s \times H^{s-1}(\tilde{A}_1)} \lesssim \varepsilon$, $|Q - \dot{Q}| \leq M_c \varepsilon^2$, and

$$(g, k) = \begin{cases} (g_Q, k_Q) & \text{in } A_{\frac{1}{2}}, \\ (\dot{g}, \dot{k}) & \text{in } A_2. \end{cases}$$

In both cases, the following additional statements hold as well:

- (Lipschitz continuity) The map $(\dot{g}, \dot{k}) \mapsto (g, k, Q)$ is Lipschitz as a map from the subset of $H^s \times H^{s-1}$ restricted by (5.10)–(5.11) into $H^s \times H^{s-1} \times \mathbb{R}^{10}$.
- (Persistence of regularity) If $(\dot{g}, \dot{k}) \in H^{s+m} \times H^{s+m-1}(\tilde{A}_1)$ and the family $\{(g_Q, k_Q)\}_{Q \in \mathcal{Q}}$ is of Sobolev regularity $s + m$ for $m \in \mathbb{N}$, then $(g, k) \in H^{s+m} \times H^{s+m-1}(\tilde{A}_1)$ and

$$\begin{aligned} \|(g - \delta, k)\|_{H^{s+m} \times H^{s+m-1}(\tilde{A}_1)} &\leq C \|(\dot{g} - \delta, \dot{k})\|_{H^{s+m} \times H^{s+m-1}(\tilde{A}_1)} \\ &\quad + C \|(g_Q - \delta, k_Q)\|_{H^{s+m} \times H^{s+m-1}(\tilde{A}_1)}, \end{aligned}$$

where $C = C(s, m, \|(\dot{g} - \delta, \dot{k})\|_{H^{s+1} \times H^s(\tilde{A}_1)}, \|(g_Q - \delta, k_Q)\|_{H^{s+1} \times H^s(\tilde{A}_1)})$.

Proof. Step 1. We begin by forming the first trial for the desired initial data set. Let $\chi(x)$ be a smooth radial function which equals 0 for $|x| < 1$ and 1 for $|x| > 2$. We introduce

$$(g_{in;Q}, k_{in;Q}, g_{out;Q}, k_{out;Q}) = \begin{cases} (\dot{g}, \dot{k}, g_Q, k_Q) & \text{for Statement (1),} \\ (g_Q, k_Q, \dot{g}, \dot{k}) & \text{for Statement (2).} \end{cases}$$

We introduce the first guess for the glued initial data, namely,

$$(5.14) \quad (\bar{h}_Q, \bar{\pi}_Q) = (1 - \chi)(h_{in;Q}, \pi_{in;Q}) + \chi(h_{out;Q}, \pi_{out;Q}).$$

Of course, $(\bar{h}_Q, \bar{\pi}_Q)$ would not solve the constraint equations (1.2); our aim is to find a correction $(\tilde{h}_Q, \tilde{\pi}_Q)$ supported in A_1 and a choice of $Q \in \mathcal{Q}$ such that $(h, \pi) = (\bar{h}_Q + \tilde{h}_Q, \bar{\pi}_Q + \tilde{\pi}_Q)$ solves (1.2).

A quick algebraic computation shows that we want $(\tilde{h}_Q, \tilde{\pi}_Q)$ to satisfy

$$(5.15) \quad \partial_i \partial_j \tilde{h}^{ij}_Q = \tilde{M}_Q, \quad \partial_i \tilde{\pi}^{ij}_Q = \tilde{N}_Q,$$

where $\tilde{M}_Q, \tilde{N}_Q$ are the errors satisfying

$$(\tilde{M}_Q, \tilde{N}_Q) = -(\partial_i \partial_j \bar{h}^{ij}, \partial_i \bar{\pi}^{ij}) + (\tilde{M}_Q^{\text{nonlin}}, \tilde{N}_Q^{\text{nonlin}})$$

and $(\tilde{M}_Q^{\text{nonlin}}, \tilde{N}_Q^{\text{nonlin}})$ are quadratic nonlinearities.

Let S_B be the solution operator defined in Theorem 4.6. For each $Q \in \mathcal{Q}$, we look for $(\tilde{h}_Q, \tilde{\pi}_Q)$ solving the fixed point problems

$$(5.16) \quad (\tilde{h}_Q, \tilde{\pi}_Q) = S_B(\tilde{M}_Q, \tilde{N}_Q).$$

By the first part of Proposition 5.1, there exists such solution $(\tilde{h}_Q, \tilde{\pi}_Q) \in \tilde{H}^s \times \tilde{H}^{s-1}(A_1)$ such that

$$\|(\tilde{h}_Q, \tilde{\pi}_Q)\|_{\tilde{H}^s \times \tilde{H}^{s-1}(A_1)} \lesssim \varepsilon.$$

Step 2: Finding Q . To conclude the proof of existence, it only remains to find $Q \in \mathcal{Q}$ such that

$$(5.17) \quad \int \frac{1}{2} \tilde{M}_Q(1, x_1, x_2, x_3)^\dagger dx = 0,$$

$$(5.18) \quad \int \tilde{N}_Q \cdot (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3)^\dagger dx = 0,$$

Introduce $\chi_{\frac{1}{2},2}$ as in (2.21). By (2.22), the LHS of (5.17) equals

$$\int -\chi_{\frac{1}{2},2}(r) \frac{1}{2} \partial_i \partial_j \bar{h}^{ij}_Q \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} dx + \int_{\tilde{A}_1} \chi_{\frac{1}{2},2}(r) \frac{1}{2} \tilde{M}_Q^{\text{nonlin}} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} dx$$

$$= \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_{\frac{1}{2}}] - \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_2] + \int_{\tilde{A}_1} \chi_{\frac{1}{2},2}(r) \frac{1}{2} \tilde{M}_Q^{\text{nonlin}} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} dx,$$

To compute further the first two terms on the last line, we use (2.22) again:

$$\begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_{\frac{1}{2}}] = \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] - \int_{\tilde{A}_1} \chi_{\frac{1}{2},1}(r) \frac{1}{2} M_{in;Q}^{\text{nonlin}} \begin{pmatrix} 1 \\ \vdots \\ x_3 \end{pmatrix} dx,$$

and similarly for $[(g_{out;Q}, k_{out;Q}); A_2]$. We arrive at

$$(5.19) \quad (\text{LHS of (5.17)}) = \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] - \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_1] + \begin{pmatrix} n[Q]_{\mathbf{E}} \\ \vdots \\ n[Q]_{\mathbf{C}_3} \end{pmatrix}$$

where

$$(5.20) \quad \begin{pmatrix} n[Q]_{\mathbf{E}} \\ \vdots \\ n[Q]_{\mathbf{C}_3} \end{pmatrix} = \int_{\tilde{A}_1} \frac{1}{2} \left(\chi_{\frac{1}{2},2}(r) \tilde{M}_Q^{\text{nonlin}} - \chi_{\frac{1}{2},1}(r) M_{in;Q}^{\text{nonlin}} - \chi_{1,2}(r) M_{out;Q}^{\text{nonlin}} \right) \begin{pmatrix} 1 \\ \vdots \\ x_3 \end{pmatrix} dx.$$

Similarly, we may show that

$$(5.21) \quad (\text{LHS of (5.18)}) = \begin{pmatrix} \mathbf{P}_1 \\ \vdots \\ \mathbf{J}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] - \begin{pmatrix} \mathbf{P}_1 \\ \vdots \\ \mathbf{J}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_1] + \begin{pmatrix} n[Q]_{\mathbf{P}_1} \\ \vdots \\ n[Q]_{\mathbf{J}_3} \end{pmatrix}$$

with

$$(5.22) \quad \begin{pmatrix} n[Q]_{\mathbf{P}_1} \\ \vdots \\ n[Q]_{\mathbf{J}_3} \end{pmatrix} = \int_{\tilde{A}_1} \left(\chi_{\frac{1}{2},2}(r) \tilde{N}_Q^{\text{nonlin}} - \chi_{\frac{1}{2},1}(r) N_{in;Q}^{\text{nonlin}} - \chi_{1,2}(r) N_{out;Q}^{\text{nonlin}} \right) \cdot \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{Y}_3 \end{pmatrix} dx.$$

For the nonlinear contribution $n[Q]$, we note that since $n[Q]$ involves quadratic terms,

$$|n[Q]| \lesssim \varepsilon^2, \quad \|n[\cdot]\|_{Lip} \lesssim K\varepsilon.$$

For the sake of concreteness, we focus on the case of exterior gluing from this point on; the case of interior gluing is similar. Then $\mathbf{Q}[(g_{in;Q}, k_{in;Q}); A_1] = \dot{Q}$, whereas $\mathbf{Q}[(g_{out;Q}, k_{out;Q}); A_1] = Q$. Hence, (5.19)–(5.21) reduce to a fixed point problem

$$(5.23) \quad Q = \dot{Q} + n[Q].$$

Moreover, we have shown that $|n[Q]| \lesssim \varepsilon^2$ and $\|n[Q]\|_{Lip} \lesssim K\varepsilon$. **We take M_c to be the implicit constant in the first inequality** so that $|n[Q]| \leq M_c \varepsilon^2$. By hypothesis, $\text{dist}(\dot{Q}, \partial \mathcal{Q}) > M_c \varepsilon^2$. Hence, for ε_c sufficiently small (recall that $\varepsilon < \varepsilon_c$), the RHS of (5.23) is thus a contraction on $\{Q : |Q - \dot{Q}| \leq M_c \varepsilon^2\} \subseteq \mathcal{Q}$. It follows that a unique $Q \in \mathcal{Q}$ satisfying (5.23) and $|Q - \dot{Q}| \leq M_c \varepsilon^2$ exists by the Banach fixed point theorem. Finally, the Lipschitz continuity follows from the Banach fixed point theorem, and the persistence of regularity follows from differentiating the fixed point equation (5.16). $\square$

5.4. Obstruction-free gluing. At the linear level, a compactly supported annular solution requires equality of all ten charges. The nonlinear constraint equations contain quadratic terms that can create charge. We prove a more quantitative version of Theorem 1.3 using *bump solutions* that generate charges.

Theorem 5.5 (Obstruction-free gluing for almost flat data on annuli). *Given $s > 3/2$ and $\Gamma > 1$, there exist $\varepsilon_o = \varepsilon_o(s, \Gamma) > 0$, $\mu_o = \mu_o(s, \Gamma) > 0$ and $C_o = C_o(s, \Gamma) > 0$ such that the following holds. Let $(g_{in}, k_{in}) \in H^s \times H^{s-1}(A_1)$, $(g_{out}, k_{out}) \in H^s \times H^{s-1}(A_{32})$ be pairs solving (1.2). Define $\Delta Q = (\Delta \mathbf{E}, \dots, \Delta \mathbf{J}_3) \in \mathbb{R}^{10}$ by*

$$(5.24) \quad \Delta Q = \mathbf{Q}[(g_{out}, k_{out}); A_{32}] - \mathbf{Q}[(g_{in}, k_{in}); A_1],$$

and assume that

$$(5.25) \quad \Delta \mathbf{E} > |\Delta \mathbf{P}|,$$

$$(5.26) \quad \frac{\Delta \mathbf{E}}{\sqrt{(\Delta \mathbf{E})^2 - |\Delta \mathbf{P}|^2}} < \Gamma,$$

$$(5.27) \quad \Delta \mathbf{E} < \varepsilon_o^2,$$

$$(5.28) \quad |\Delta \mathbf{C}| + |\Delta \mathbf{J}| < \mu_o \Delta \mathbf{E},$$

$$(5.29) \quad \|(g_{\text{in}} - \delta, k_{\text{in}})\|_{H^s \times H^{s-1}(A_1)}^2 + \|(g_{\text{out}} - \delta, k_{\text{out}})\|_{H^s \times H^{s-1}(A_{32})}^2 < \mu_o \Delta \mathbf{E}.$$

Then there exists $(g, k) \in H^s \times H^{s-1}(B_{64} \setminus \overline{B_1})$ solving (1.2) such that

$$(5.30) \quad (g, k) = (g_{\text{in}}, k_{\text{in}}) \quad \text{in } A_1, \quad (g, k) = (g_{\text{out}}, k_{\text{out}}) \quad \text{in } A_{32},$$

$$(5.31) \quad \|(g - \delta, k)\|_{H^s \times H^{s-1}(B_{64} \setminus \overline{B_1})}^2 < C_o \Delta \mathbf{E}.$$

The condition (5.25) is a reflection of the positive mass theorem. (5.26) is a quantitative version of (5.25). (5.27)–(5.29) are smallness conditions.

Proof sketch of Theorem 5.5. By Theorem 5.4, it suffices to construct a family of solutions $(g_{\Delta Q}, k_{\Delta Q})$ in the annulus A_{16} such that it extends $(g_{\text{in}}, k_{\text{in}})$ and satisfies the conditions in Definition 5.3. For this, we construct bump solutions $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ for $t > 0$, $\ell \in \mathbb{R}^3$ with $|\ell| < 1$ and $\xi \in \mathbb{R}^3$ with the following properties:

- (1) Each bump solution $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ solves (1.2).
- (2) Each bump solution is the sum of a linearized solution of size t , supported in a small ball $B_b(\xi)$, and a nonlinear correction of size t^2 , supported in an outgoing cone.
- (3) The charges are given by

$$(5.32) \quad (\mathbf{E}, \mathbf{P}, \mathbf{C}, \mathbf{J}) = \gamma(\ell) t^2 (1, \ell_i, \xi_i, \varepsilon_i^{jk} \xi_j \ell_k), \quad \gamma(\ell) = (1 - |\ell|^2)^{-\frac{1}{2}}$$

up to small error terms.

We can then use a (carefully chosen) combination of disjoint bump solutions to build a family of solutions $(g_{\Delta Q}, k_{\Delta Q})$ with the desired properties.

The construction of bumps is technical. Instead of going through the details, we give a heuristic explanation of the construction.

Each bump should be thought of as a particle localized in a certain position and moving with certain linear momentum and angular momentum. We start with a localized linear solution, and correct it using the solution operator. This gives the linearized part of size t , and the nonlinear correction of size t^2 . From the formula for the charges (2.14)–(2.17), it is easy to see the charges are of size t^2 .

The key observation is that, through the nonlinear properties of the constraint equation, we can achieve a natural range of charges subject to the size constraint above. To construct the linearized solution, first observe that one can take $k = 0$ and g_t an even function so that $(\mathbf{E}, \mathbf{P}, \mathbf{C}, \mathbf{J}) \simeq (t^2, 0, 0, 0)$; that $\mathbf{E}$ is necessarily positive is a consequence of the positive mass theorem (although it can also be checked directly by computation). Then we apply a Lorentz boost (after solving the linearized Einstein equations) defined by

$$y^\mu = (\Lambda_\ell)^\mu{}_\nu x^\nu + \xi^\mu,$$

where $\Lambda_\ell \in SO^+(1, 3)$ (Lorentz boost by velocity ℓ) and $\xi \in \mathbb{R}^{1+3}$ are defined by

$$(\Lambda_\ell x)^0 = \gamma(\ell)(x^0 - \ell \cdot P_\ell x), \quad (\Lambda_\ell x)^j = \gamma(\ell)(-\ell^j x^0 + (P_\ell x)^j) + (P_\ell^\perp x)^j, \quad \xi^0 = 0, \quad \xi^j = \xi^j,$$

where $P_\ell, P_\ell^\perp : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are orthogonal projections to (ℓ) and $(\ell)^\perp$, respectively. This would give the desired charges in (5.32). Note that we always have $|\mathbf{P}| < \mathbf{E}$ in this construction. This is where the condition (5.25) comes from. $\square$

To obtain Theorem 1.3, we note that after rescaling the asymptotically flat solution satisfying (1.4) using (2.25), we have for $\alpha \in (1/2, 1)$,

$$\begin{aligned} (\mathbf{E}, \mathbf{P})[(g^{(r)}, k^{(r)}), A_R] &= r^{-1}(\mathbf{E}^{ADM}, \mathbf{P}^{ADM})[(g, k)] + O(r^{-2\alpha}), \\ (\mathbf{C}, \mathbf{J})[(g^{(r)}, k^{(r)}), A_R] &= O(r^{-2\alpha}). \end{aligned}$$

Therefore, Theorem 1.3 follows from Theorem 5.5.

References

- [ACR25] S. Aretakis, S. Czimek & I. Rodnianski – The characteristic gluing problem for the Einstein equations and applications, *Duke Math. J.* **174** (2025), no. 2, p. 355–402.
- [Aro54] N. Aronszajn – On coercive integro-differential quadratic forms, in *Conference on Partial Differential Equations, University of Kansas*, vol. 14, 1954, p. 94–106.
- [Bog79] M. E. Bogovskii – Solution of the first boundary value problem for an equation of continuity of an incompressible medium, *Dokl. Akad. Nauk SSSR* **248** (1979), no. 5, p. 1037–1040.
- [BDG20] D. Breit, L. Diening & F. Gmeineder – On the trace operator for functions of bounded A-variation, *Anal. PDE* **13** (2020), no. 2, p. 559–594.
- [BW26] S. Brendle & Y. Wang – A dimension descent scheme for the positive mass theorem in arbitrary dimension, 2026, [arXiv:2604.08473](#).
- [Car21] A. Carlotto – The general relativistic constraint equations, *Living Rev. Relativ.* **24** (2021), p. 170, Id/No 2.
- [CS16] A. Carlotto & R. Schoen – Localizing solutions of the Einstein constraint equations, *Invent. Math.* **205** (2016), no. 3, p. 559–615.
- [CK25] X. Chen & S. Klainerman – Solving the constraint equation for general free data, 2025, [arXiv:2512.22704](#).
- [CK26a] ———, Formation of trapped surfaces from spacelike initial data, 2026, [arXiv:2607.11236](#).
- [CK26b] ———, Forward construction of vacuum initial data with borderline decay, 2026, [arXiv:2606.08716](#).
- [CBG69] Y. Choquet-Bruhat & R. Geroch – Global aspects of the Cauchy problem in general relativity, *Comm. Math. Phys.* **14** (1969), p. 329–335.
- [CBY80] Y. Choquet-Bruhat & J. W. York – The Cauchy problem. General relativity and gravitation, 1980.
- [CCN25] P. T. Chruściel, A. Cogo & A. Nützi – A Bogovskii-type operator for Corvino-Schoen hyperbolic gluing, *Classical Quantum Gravity* **42** (2025), no. 14, p. 12, Id/No 147001.
- [CD03] P. T. Chruściel & E. Delay – On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications, *Mém. Soc. Math. Fr. (N.S.)* (2003), no. 94, p. vi+103.
- [Cor00] J. Corvino – Scalar curvature deformation and a gluing construction for the Einstein constraint equations, *Commun. Math. Phys.* **214** (2000), no. 1, p. 137–189.
- [CS06] J. Corvino & R. M. Schoen – On the asymptotics for the vacuum Einstein constraint equations, *J. Differential Geom.* **73** (2006), no. 2, p. 185–217.
- [CR22] S. Czimek & I. Rodnianski – Obstruction-free gluing for the einstein equations, 2022, [arXiv:2210.09663](#).
- [FM75] A. E. Fischer & J. E. Marsden – Deformations of the scalar curvature, *Duke Math. J.* **42** (1975), p. 519–547.
- [FB52] Y. Fourès-Bruhat – Théorème d’existence pour certains systèmes d’équations aux dérivées partielles non linéaires, *Acta Math.* **88** (1952), p. 141–225.
- [GSW25] E. Giorgi, D. Shen & J. Wan – Cauchy data for multiple collapsing boson stars, 2025, [arXiv:2512.01844](#).
- [Ise95] J. Isenberg – Constant mean curvature solutions of the Einstein constraint equations on closed manifolds, *Classical Quantum Gravity* **12** (1995), no. 9, p. 2249–2274.
- [IMOT25] P. Isett, Y. Mao, S.-J. Oh & Z. Tao – Integral formulas for under/overdetermined differential operators via recovery on curves and the finite-dimensional cokernel condition I: General theory, 2025, [arXiv:2509.04617](#).
- [IO16] P. Isett & S.-J. Oh – On nonperiodic Euler flows with Hölder regularity, *Arch. Ration. Mech. Anal.* **221** (2016), no. 2, p. 725–804.
- [Ka94] A. Kalamajska – Pointwise multiplicative inequalities and Nirenberg type estimates in weighted Sobolev spaces, *Studia Math.* **108** (1994), no. 3, p. 275–290.
- [Lic44] A. Lichnerowicz – L’intégration des équations de la gravitation relativiste et le problème des n corps, *J. Math. Pures Appl. (9)* **23** (1944), p. 37–63.
- [MOT23] Y. Mao, S.-J. Oh & Z. Tao – Initial data gluing in the asymptotically flat regime via solution operators with prescribed support properties, 2023, [arXiv:2308.13031](#).
- [MT22] Y. Mao & Z. Tao – Localized initial data for Einstein equations, 2022, [arXiv:2210.09437](#).
- [Nüt26a] A. Nützi – Solutions of the constraints with controlled decay to Kerr, including Schwartz decay, 2026, [arXiv:2603.23484](#).
- [Nüt26b] A. Nützi – A support preserving homotopy for the de Rham complex with boundary decay estimates, *Pure Appl. Anal.* **8** (2026), no. 2, p. 497–533.
- [OT19] S.-J. Oh & D. Tataru – The hyperbolic Yang-Mills equation for connections in an arbitrary topological class, *Comm. Math. Phys.* **365** (2019), no. 2, p. 685–739.
- [OY73] N. O’Murchadha & J. W. York – Existence and uniqueness of solutions of the Hamiltonian constraint of general relativity on compact manifolds, *J. Math. Phys.* **14** (1973), p. 1551–1557.
- [Res83] Y. G. Reshetnyak – Linear differential operators of finite type, *Siberian Mathematical Journal* **24** (1983), p. 796–808.
- [Res94] ———, *Stability theorems in geometry and analysis*, Mathematics and its Applications, vol. 304, Kluwer Academic Publishers Group, Dordrecht, 1994.
- [Sbi16] J. Sbierski – On the existence of a maximal Cauchy development for the Einstein equations: a dezornification, *Ann. Henri Poincaré* **17** (2016), no. 2, p. 301–329.
- [SY79] R. Schoen & S.-T. Yau – On the proof of the positive mass conjecture in general relativity, *Commun. Math. Phys.* **65** (1979), p. 45–76.
- [SW25] D. Shen & J. Wan – Formation of multiple black holes from Cauchy data, 2025, [arXiv:2506.16117](#).
- [SW26] ———, Vacuum initial data with minimal decay and borderline decay, 2026, [arXiv:2602.01557](#).
- [Smi70] K. T. Smith – Formulas to represent functions by their derivatives, *Math. Ann.* **188** (1970), p. 53–77.

- [Wit81] E. Witten – A new proof of the positive energy theorem., *Commun. Math. Phys.* **80** (1981), p. 381–402.
[Yor73] J. W. York – Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial-value problem of general relativity, *J. Math. Phys.* **14** (1973), p. 456–464.

Department of Mathematics, University of Michigan, Ann Arbor, MI, USA 48109
Email address: yuchnmao@umich.edu

Department of Mathematics, UC Berkeley, Berkeley, CA, USA 94720 and Korea Institute for Advanced Study, Seoul, Korea 02455
Email address: sjoh@math.berkeley.edu

Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France
Email address: ztao@ihes.fr