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TIME-PERIODIC OSCILLATION OF CAPILLARY FLUID DROPLET

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TIME-PERIODIC OSCILLATION OF CAPILLARY FLUID DROPLET

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ABSTRACT. In this note, we outline the construction of time-periodic oscillating capillary fluid droplet, based on a working paper of the authors. The droplet is modeled by an incompressible, irrotational Eulerian system with free boundary. The key tool developed in the research work is a coordinate-free paradifferential calculus on the compact Lie group $SU(2)$, together with its zonal version. The Eulerian free boundary system is parilinearized into a quasilinear dispersive PDE. Despite the presence of small denominators, paradifferential reduction allows one to rewrite the system into a fixed point form balancing all loss of regularity. The construction of time-periodic solutions is then based on bifurcation theory with computer assistance.

1. THE MAIN RESULTS

1.1. Eulerian System for a Fluid Droplet. Consider a moving three-dimensional perfect, incompressible, irrotational fluid of normalized density 1 with free boundary satisfying the following assumptions:

- (A1) The fluid occupies a smooth, compact region in $\mathbb{R}^3$, diffeomorphic to the closed Euclidean ball.
- (A2) The fluid is described by Eulerian system with no gravity or any other external force in presence.
- (A3) The air-fluid interface is governed by the Young-Laplace law with normalized surface tension coefficient $\sigma_0 = 1$, and the effect of air flow is neglected.

We are interested in the hydrodynamic system obeying (A1)-(A3). In particular, we aim to construct *periodic-in-time* oscillating solution of the system.

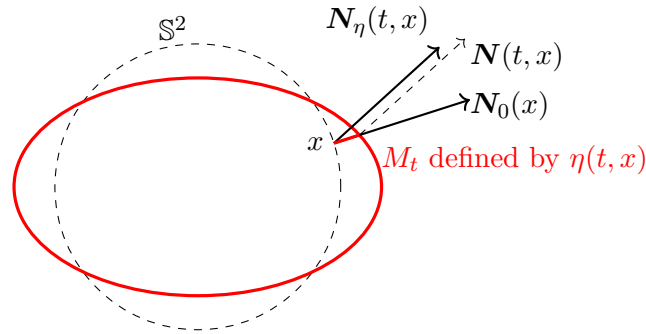


FIGURE 1. The surface defined by a height function.

We denote the fluid region by $\mathcal{O}_t$, and its boundary $M_t := \partial\mathcal{O}_t$, which is given by a height function $\eta(t, x)$. The unit outer normal vector field on M_t is denoted by $\mathbf{N}$, and its pull-back to the standard sphere is denoted by $\mathbf{N}_\eta$. See Figure 1 for the geometry. We write ∇^{Euc} for the flat connection on $\mathbb{R}^3$. The fluid motion is the gradient of some harmonic potential function in $\mathcal{O}_t$. The free boundary

naturally imposes the kinematic condition

$$(1.1) \quad \frac{\partial M_t}{\partial t} \cdot \mathbf{N} = \nabla^{\text{Euc}} \Phi|_{M_t} \cdot \mathbf{N},$$

where $\partial M_t / \partial t$ denotes the velocity vector field on the interface M_t . On the other hand, the boundary behavior of Φ is governed by the Bernolli equation (together with Young-Laplace law):

$$(1.2) \quad \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla^{\text{Euc}} \Phi|^2 \right) \Big|_{M_t} = -H(M_t) + p_e, \quad p_e = \text{constant external pressure}.$$

Here $H(M_t)$ is the (scalar) mean curvature of M_t .

1.2. Zakharov Formulation. We can rewrite the system (1.1)(1.2) as a PDE system for the height function $\eta(t, x)$ and the trace $\phi(t, x)$ of the potential Φ on M_t . Indeed, introducing the *Dirichlet-Neumann operator*

$$(1.3) \quad G[\eta] : \phi \mapsto \frac{\sqrt{(1+\eta)^2 + |\nabla \eta|^2}}{1+\eta} \nabla^{\text{Euc}} \Phi \cdot \mathbf{N}_\eta.$$

The kinetic condition (1.1) becomes $\partial_t \eta = G[\eta] \phi$.

Since we are looking for time-periodic solutions, we may normalize t by the frequency $\omega > 0$ to make the period $\equiv 2\pi$. The system (1.1)(1.2) is transformed into the *capillary droplet equation*, comparable to the classical Zakharov formulation of water waves near a water level:

$$(EQ) \quad \begin{cases} \omega \frac{\partial \eta}{\partial t} = G[\eta] \phi, \\ \omega \frac{\partial \phi}{\partial t} = -\frac{|\nabla \phi|^2}{2(1+\eta)^2} + \frac{((1+\eta)^2 G[\eta] \phi + \nabla \eta \cdot \nabla \phi)^2}{2((1+\eta)^2 + |\nabla \eta|^2)(1+\eta)^2} - H[\eta] + 2. \end{cases}$$

Here ∇ is the standard connection on $\mathbb{S}^2$, and $H[\eta]$ is the mean curvature operator.

The only equilibrium of (EQ) is $(\eta, \phi) = (0, 0)$, which geometrically corresponds to the standard Euclidean sphere. The linearization of (EQ) at the equilibrium reads

$$(1.4) \quad \partial_t \eta = G[0] \phi, \quad \partial_t \phi = (\Delta + 2) \eta.$$

Here Δ is the spherical Laplacian. It is not hard to see that $G[0]$ and $\Delta + 2$ act as the Fourier multiplier ℓ and $-(\ell - 1)(\ell + 2)$ on spherical harmonics of degree ℓ , respectively. Under the complex diagonal unknown u , the linearized system (1.4) becomes a dispersive one:

$$(1.5) \quad \partial_t u = i\Omega(\nabla)u, \quad u = \sqrt{-\Delta_0 - 2\eta} + i\sqrt{G[0]}\phi,$$

where the pseudodifferential operator $\Omega(\nabla)$ is a Fourier multiplier on $\mathbb{S}^2$, given by

$$(1.6) \quad \omega_\ell^{(0)} = \sqrt{\ell(\ell - 1)(\ell + 2)}, \quad \ell \in \mathbb{N}.$$

This shows that (EQ) is a quasilinear dispersive PDE system of order $3/2$.

The system (EQ) is of long-standing interest for hydrophysicists and astronautical engineers. The eigen-frequency formula, or *dispersive relation* (1.6), was deduced in Lord Rayleigh's fundamental work [R⁺79]. Physicists already carried out “weakly nonlinear analysis” [TB83, NB86, LM88], including the frequency-amplitude relation for nonlinear oscillation and initial numerical results. Experimental verification of the formula (1.6) together with the “weakly nonlinear analysis” have

been repeatedly addressed in microgravity hydrodynamical experiments, e.g. [WAL96]. Recent experimental evidence for superfluid helium drop can be found in [BWN⁺23].

1.3. The Main Results. The main result of our paper concerns the existence of time-periodic oscillation modes of the system (EQ). To the authors' knowledge, our theorem provides the first construction of global-in-time special solutions, irreducible to fixed profile in moving frame, of a Hamiltonian *quasilinear dispersive* PDE system defined on a *compact, curved manifold*.

Theorem 1.1. *Suppose $\ell = 6, 7, 9, 11, 13, \dots, 18, 20, 22, \dots, 30$. There exist:*

- A C^2 function of ξ defined near 0 of the form

$$\omega_\ell(\xi) = (1 + \omega_\ell^{(2)}\xi^2 + O(\xi^4))\sqrt{\ell(\ell-1)(\ell+2)}$$

where $\omega_\ell^{(2)} < 0$ is a uniquely determined real number;

- A compact set $K_\ell \subset \mathbb{R}$ with Lebesgue density 1 at 0;
- A C^2 curve in $H_t^1 H_x^s$ parameterized by ξ :

$$\eta(\xi; t, x) = \xi \cos(\omega_\ell(\xi)t)P_\ell + O(\xi^2),$$

$$\phi(\xi; t, x) = \xi \sin(\omega_\ell(\xi)t)P_\ell + O(\xi^2);$$

such that when $\xi \in K_\ell$, the functions $\eta(\xi; t, x), \phi(\xi; t, x)$ form a solution of the Zakharov system (EQ) with period $2\pi/\omega_\ell(\xi)$.

Theorem 1.2. *For the mode $\ell = 2$, there exist:*

- A C^2 function of ξ defined near 0 of the form

$$\omega_2(\xi) = (1 + \omega_2^{(2)}\xi^2 + O(\xi^4))\sqrt{8}, \quad \omega_2^{(2)} = -1.1392750\dots$$

- A compact set $K_2 \subset \mathbb{R}$ with Lebesgue density 1 at 0;
- A C^2 curve in $H_t^1 H_x^s$ parameterized by ξ :

$$\eta(t, x) = \xi \cos(\omega_2(\xi)t)P_2(\cos \theta) + \lambda \xi \cos(3\omega_2(\xi)t)P_4(\cos \theta) + O(\xi^2),$$

$$\begin{aligned} \phi(t, x) = & -\frac{\xi\omega_2(\xi)}{2} \sin(\omega_2(\xi)t)P_2(\cos \theta) \\ & - \frac{3\lambda\xi\omega_2(\xi)}{2} \sin(3\omega_2(\xi)t)P_4(\cos \theta) + O(\xi^2), \end{aligned}$$

with $\lambda = 0.3103295\dots$,

such that when $\xi \in K_2$, the functions $\eta(\xi; t, x), \phi(\xi; t, x)$ form a solution of the Zakharov system (EQ) with period $2\pi/\omega_2(\xi)$.

2. TOOLBOX: PARADIFFERENTIAL CALCULUS ON THE LIE GROUP $SU(2)$

Being an intrinsically curved manifold, the 2-sphere $\mathbb{S}^2$ carries no global coordinate system. This brings technical difficulties for nonlinear analysis defined on $\mathbb{S}^2$. Unlike the toric case, the product of Fourier basis functions is no longer a Fourier basis function. The purpose of this section is to explain how *paraproduct operators*, or more generally, *paradifferential operators*, can be manipulated on the compact Lie group $SU(2)$ in a coordinate-free manner, enabling delicate nonlinear analysis.

The transference to $\mathbb{S}^2$ is then straightforward since $\mathbb{S}^2$ is nothing but the homogeneous space $\mathrm{SU}(2)/\mathrm{U}(1)$. The theory is largely parallel to that on the flat torus.

2.1. Summary of the Toric Case. The notion of *paradifferential operators* was introduced by Bony in [Bon81] for the analysis on $\mathbb{R}^n$. Based on standard Littlewood-Paley decomposition, Bony introduced the *paraproduct decomposition* for products and the notion of *paralinearization* to separate out the least regular part out of a nonlinear expression.

Roughly speaking, the paraproduct operator T_a related to a given function a acts as

$$T_a u := \sum_j S_{j-3} a \cdot \Delta_j u,$$

where $S_{j-3} a$ is the smooth truncation of a up to frequency $\leq 2^{j-3}$ and $\Delta_j u$ is the Littlewood-Paley building block of u of frequency $\simeq 2^j$. Bony found that the operator T_a behave as if it were a usual product operator, in the sense that if $a, b \in C^r$, then the composition $T_a T_b$ is essentially just T_{ab} . Bony further found that for $F \in C^\infty$, a nonlinear composition $F(u)$ can be decomposed as

$$(2.1) \quad F(u) = T_{F'(u)} u + \mathcal{R}_F(u) \in H^s + H^{2s-n/2},$$

so that *the least regular part of $F(u)$ is the paraproduct along $F'(u)$* . This is known as the *paralinearization formula*.

Later developments along this direction include a series of works of Meyer-Coifman (for example [Mey80, CM86]) and also the work of Alinhac [Ali89], who found an operation called *paracomposition* that clarifies change-of-variable for non-smooth functions and coordinates. A more comprehensive introduction can be found in Chapter X of Hörmander's book [Hör97] or Chapter 4 of Métivier's textbook [Mét08]. See also the papers [AS26, AS24].

It turns out that almost every key proposition in the toric paradifferential calculus finds its counterpart on $\mathrm{SU}(2)$. For the purpose of clarity, we record in the following only the results on $\mathrm{SU}(2)$ without proof.

2.2. Littlewood-Paley Decomposition. Throughout this section, the Lie group $\mathrm{SU}(2)$ is equipped with its standard bi-invariant metric. We use $(L_k)_{k=1}^3$ to denote the global orthonormal frame of left-invariant vector fields on $\mathrm{SU}(2)$. We also write the gradient operator $\nabla := (L_1, L_2, L_3)$, and the Laplacian

$$\Delta = \sum_{k=1}^3 L_k^2, \quad \langle \nabla \rangle := (1 - \Delta)^{1/2}.$$

The Laplacian Δ is bi-invariant, so it *commutes with every L_k* , although the L_k do not commute among themselves. This single fact replaces the commutativity of Fourier multipliers and partial derivatives on $\mathbb{T}^d$, and is the reason why the whole theory goes through.

Functions of Δ play the role of Fourier multipliers in the toric case. We first introduce the invariant Littlewood-Paley decomposition. Fix a dyadic partition of unity

$$1 = \sum_{j \geq 0} \psi(2^{-j} \lambda), \quad \psi \in C_0^\infty(\mathbb{R}) \text{ is even and non-negative.}$$

We then set the Littlewood-Paley building block operators and smoothing operators as

$$(2.2) \quad \Psi_j := \psi(2^{-j}\langle \nabla \rangle), \quad S_j := \sum_{i \leq j} \Psi_i,$$

respectively. Each $\Psi_j u$ is a finite linear combination of eigenfunctions of Δ with eigenvalue $\simeq 2^{2j}$, and the integral kernel of Ψ_j is concentrated at scale 2^{-j} with Schwartz-type decay, exactly as on $\mathbb{T}^3$. This is all that is needed to transfer the toric theory of function spaces.

Just as in the toric case, Sobolev and Zygmund spaces are characterized by

$$\|u\|_{H^s}^2 \simeq \sum_{j \geq 0} 2^{2js} \|\Psi_j u\|_{L^2}^2, \quad |u|_{C_*^r} \simeq \sup_{j \geq 0} 2^{jr} |\Psi_j u|_{L^\infty}.$$

The usual embeddings, products and interpolation relations remain valid.

2.3. Paraproducts and Paralinearization. Fix an integer $D \geq 3$. Given a distribution a on $SU(2)$, the *paraproduct operator* T_a is defined by

$$(2.3) \quad T_a u := \sum_{j \geq 0} (S_{j-D} a) \cdot (\Psi_j u), \quad \text{or when } a \text{ has lengthy expression, } \mathbf{Op}^{\text{PM}}(a) := T_a.$$

The idea, due to Bony, is to “freeze” the low frequencies of a against the high frequencies of u : since $S_{j-D} a$ oscillates at frequency $\ll 2^j$, the j ’th summand still has spectrum near 2^j , so T_a acts *almost blockwise* in the Littlewood-Paley decomposition of u . The price is that $T_a u$ differs from au ; the gain is that T_a behaves like a differential operator of order zero even when a is rough, and that the difference $au - T_a u$ is smoother than u when a is.

The basic properties of the paraproduct operator are exactly as in the toric case, and the proof is largely the same due to the common kernel properties (see e.g. [AS26]):

Proposition 2.1. *Let $r > 0$ and $s \in \mathbb{R}$.*

(i) *For $a \in L^\infty$, the operator norm of T_a satisfies*

$$\|T_a; \mathcal{L}(H^s, H^s)\| \lesssim |a|_{L^\infty}.$$

(ii) (Composition, adjoint) *For $a, b \in C^r$, there holds*

$$\begin{aligned} \|T_a T_b - T_{ab}; \mathcal{L}(H^s, H^{s+r})\| &\lesssim_{s,r} |a|_{L^\infty} |b|_{C^r} + |a|_{C^r} |b|_{L^\infty} \\ \|T_a^* - T_{\bar{a}}; \mathcal{L}(H^s, H^{s+r})\| &\lesssim_{s,r} |a|_{C^r}. \end{aligned}$$

(iii) (Bony’s paralinearization) *If $F \in C^\infty$ and $u \in H^s$ with $s > 3/2$, then there holds the following decomposition:*

$$F(u) - T_{F'(u)} u \in H^{2s-3/2}.$$

Item (ii) says that $a \mapsto T_a$ is an algebra homomorphism modulo smoothing operators, i.e. the operator $T_a \circ T_b$ is essentially just T_{ab} . Item (iii) says that a nonlinear function $F(u)$ of u is essentially just the paraproduct of u by the linearization $F'(u)$. This is what allows one to treat nonlinear terms as if they were linear with a smoothing remainder.

A particular case of the parilinearization formula is the *paraproduct decomposition*: if $u, v \in H^s \cap L^\infty$, then there holds

$$uv = T_u v + T_v u + R_{\text{PM}}(u, v) \in H^s + H^s + H^{2s-3/2},$$

where $R_{\text{PM}}(u, v) = \sum_{j,k:|j-k|<3} \Delta_j u \Delta_k v$ is the *high-high interaction* of u, v . In particular, when $s > 3/2$, there holds the following Banach algebra property just as in the toric case:

$$\|uv\|_{H^s} \lesssim_s \|u\|_{H^s} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{H^s}.$$

2.4. Classical Paradifferential Operators and Symbolic Calculus. On the Lie group $\text{SU}(2)$, we can use the invariant frame $\{L_k\}_{k=1}^3$ to represent any classical differential operator. A linear operator of the form

$$A = \sum_{|\beta| \leq m} a_\beta L^\beta$$

is called a *classical differential operator* of order m , where L^β runs over ordered monomials in L_1, L_2, L_3 and a_β are scalar functions. We define the corresponding paradifferential quantization by replacing each coefficient by a paraproduct:

$$(2.4) \quad T_A := \sum_{|\beta| \leq m} T_{a_\beta} L^\beta.$$

In particular, for a vector field $V \cdot \nabla = \sum_{k=1}^3 V^k L_k$, the corresponding paradifferential operator is

$$T_V \cdot \nabla := \sum_{k=1}^3 T_{V^k} L_k.$$

The point of (2.4) is that all the non-commutativity of the frame is absorbed into the differential operator A , which we know how to compute with, while the paradifferential part only involves scalar paraproducts. Since S_j and Ψ_j are functions of Δ , they commute with L_k , and the Leibniz rule survives quantization:

$$L_k T_b = T_{L_k b} + T_b L_k.$$

This yields the symbolic calculus:

Proposition 2.2 (Symbolic calculus). *Let*

$$A = \sum_{|\beta| \leq m} a_\beta L^\beta, \quad B = \sum_{|\beta'| \leq m'} b_{\beta'} L^{\beta'}$$

be classical differential operators of orders m, m' with C^r coefficients, $r > m + m'$. Then the composition $T_A \circ T_B$ satisfies

$$T_A \circ T_B = T_{A \circ B} + \mathcal{R}_{\text{CM}}(A, B),$$

where the remainder is smoothing:

$$\|\mathcal{R}_{\text{CM}}(A, B); \mathcal{L}(H^s, H^{s+r-m-m'})\| \lesssim_{s,r,m,m'} \left(\sum_{\beta} |a_\beta|_{C^r} \right) \left(\sum_{\beta'} |b_{\beta'}|_{C^r} \right).$$

Here $A \circ B$ is the usual composition of differential operators, computed with the Leibniz rule in the non-commutative frame.

In words: classical differential operators and their paradifferential quantizations *enjoy the same algebra* modulo smoothing operators. In particular, commutators are computed as for genuine differential operators,

$$[T_A, T_B] = T_{[A, B]} + (\text{smoothing of order } r - m - m').$$

The proof of Proposition 2.2 is nothing but a consecutive usage of (ii) in Proposition 2.1.

2.5. Commutator with the Fractional Laplacian. Either linear or nonlinear analysis calls for the commutator of fractional Laplacian with products or paraproducts. The major technical difficulty on the group $SU(2)$ is to keep track of *lower order terms*. Indeed, the leading term $T_{\nabla a} \cdot \nabla \langle \nabla \rangle^{m-2}$ is visible in any chart-based calculus, while the sub-leading ones are not. It is necessary to derive a coordinate-free approach to compute these terms, as they play very important role in the analysis of quasilinear PDEs.

Write $\mathbf{Ad}_\Delta B := [\Delta, B]$. By the Leibniz rule,

$$\mathbf{Ad}_\Delta T_a = T_{\Delta a} + 2T_{\nabla a} \cdot \nabla,$$

and inductively $\mathbf{Ad}_\Delta^k T_a$ is a classical paradifferential operator of order k whose coefficients are derivatives of a of order $\leq 2k$. When $m = 2l$ is an even integer, there is an *exact, elementary* formula

$$\langle \nabla \rangle^{2l} T_a = T_a \langle \nabla \rangle^{2l} + \sum_{k=1}^l (-1)^k \binom{l}{k} (\mathbf{Ad}_\Delta^k T_a) \langle \nabla \rangle^{2l-2k}.$$

The following proposition asserts that this binomial expansion *persists for fractional powers*, with generalized binomial coefficients, up to a remainder whose order is limited only by the regularity of the function a :

Proposition 2.3. *Let $m \in \mathbb{R}$, $r \geq 1$, $N = [r]$, and $a \in C^r$. Then*

$$\langle \nabla \rangle^m T_a = T_a \langle \nabla \rangle^m + \sum_{k=1}^N (-1)^k \binom{m/2}{k} (\mathbf{Ad}_\Delta^k T_a) \langle \nabla \rangle^{m-2k} + B_a,$$

where the remainder B_a satisfies

$$\|B_a; \mathcal{L}(H^s, H^{s+r-m})\| \lesssim_{s,r,m} |a|_{C^r}.$$

2.6. Para-transport Equation. The theory of classical linear transport equation corresponding to a smooth time dependent vector field is well-established. It is sometimes necessary to develop its paradifferential counterpart, with a reason to be clarified in the next subsection.

Let $X(t) \in L_t^1 C_x^1$ be a time-dependent vector field. Consider the linear *para-transport equation*

$$(2.5) \quad \partial_t w - T_{X(t)} \cdot \nabla w = f, \quad w(t_0) = h.$$

This is the paradifferential version of the transport equation $\partial_t w - X \cdot \nabla w = f$. However, unlike the classical one, whose flow is only as regular as X , the para-transport equation is well-posed in *every* Sobolev space, because $T_X \cdot \nabla$ is a first order paradifferential operator with an L^2 -skew-adjoint principal part (due to (ii) of Proposition 2.1).

Proposition 2.4. Equation (2.5) defines a flow map (“fundamental solution”) $\mathbf{F}(X; t, t_0)$, bounded on every H^s , $s \in \mathbb{R}$, such that:

(F1) The function $w(t) = \mathbf{F}(X; t, t_0)h$ solves the homogeneous Cauchy problem

$$\partial_t w - T_{X(t)} \cdot \nabla w = 0, \quad w(t_0) = h.$$

There holds the transmission properties

$$\mathbf{F}(X; t, t_1) \mathbf{F}(X; t_1, t_0) = \mathbf{F}(X; t, t_0), \quad \mathbf{F}(X; t_0, t_0) = \text{Id}.$$

(F2) There holds the energy estimate

$$\|\mathbf{F}(X; t, t_0)h\|_{H^s} \leq \exp\left(C_s \int_{t_0}^t |X(\tau)|_{C^1} d\tau\right) \|h\|_{H^s}.$$

(F3) The mapping $X \mapsto \mathbf{F}(X; t, t_0)$ is C^k at the price of losing k Sobolev indices:

$$\|\mathbf{d}_X^k \mathbf{F}(X; t, t_0) \cdot (\delta X)^{\otimes k} h\|_{H^{s-k}} \lesssim_{s,k} |\delta X|_{L_t^1 C_x^1}^k \|h\|_{H^s}.$$

The proof is based on standard energy estimate and is not so different from the toric case; see e.g. [AS24] for the proof in the toric case. A little explanation is needed for (F3): linearizing (2.5) with respect to X , the linearized equation reads

$$\partial_t(\mathbf{d}_X w \cdot \delta X) - T_{X(t)} \cdot \nabla(\mathbf{d}_X w \cdot \delta X) = T_{\delta X} \cdot \nabla w, \quad (\mathbf{d}_X w \cdot \delta X)(t_0) = 0.$$

The source term is one derivative rougher than w , and this causes the loss of one Sobolev index. Higher order linearizations are estimated inductively.

2.7. Paracomposition. It is usually necessary for nonlinear on manifolds to deal with change of coordinates. The classical pull-back

$$\chi_* f := f \circ \chi$$

by a C^r diffeomorphism $\chi : \text{SU}(2) \rightarrow \text{SU}(2)$ maps C^∞ to C^r only. This feature makes it quite useless in nonlinear analysis, where the functions involved cannot be assumed as smooth *in a priori*. It turns out that Alinhac’s *paracomposition operator* χ^* is a substitute: it is bounded on every H^s , and reproduces the algebraic properties of χ_* modulo smoothing errors. On the group $\text{SU}(2)$, we realize it as a time-one para-transport flow, in the spirit of Egorov theorem in classical pseudodifferential analysis.

A diffeomorphism close to the identity is parametrized by an $\mathfrak{su}(2)$ -valued function γ with $|\gamma|_{C^1} < c_0$ via the “normal coordinates”:

$$(2.6) \quad \chi(x) = x \cdot \exp(\gamma(x)).$$

This gives a 1-1 correspondence between C^r diffeomorphisms close to identity and $\mathfrak{su}(2)$ -valued functions close to zero. We then connect χ to the identity through the isotopy

$$\chi_t := x \cdot \exp(\gamma_t(x)), \quad \gamma_t := t e^{-(1-t)\langle \nabla \rangle} \gamma, \quad t \in [0, 1],$$

and let $X_\gamma(t)$ be the vector field transporting it, namely

$$\mathbf{d}\chi_t \cdot X_\gamma(t) = \partial_t \chi_t.$$

The smoothing factor $e^{-(1-t)\langle \nabla \rangle}$ is the essential device: differentiating γ_t in t costs one derivative, but the heat-type kernel regains it after integration in t , so that $X_\gamma \in L_t^1 C_x^{r'}$ for every $r' < r$, making χ_t *almost as regular as* γ . By the chain rule, $(\chi_t)_* f$ solves the classical transport equation

$$(2.7) \quad \partial_t (\chi_t)_* f = X_\gamma \cdot \nabla (\chi_t)_* f,$$

so the pull-back operator χ_* is naturally seen as the classical time-1 flow of the differential operator $X_\gamma \cdot \nabla$.

The corresponding *paracomposition operator* χ^* is then the para-flow corresponding to (2.7).

Definition 2.5. Using the notation in Proposition 2.4, we define $\chi^* := \mathbf{F}(X_\gamma; 1, 0)$, i.e. the time-1 flow of the paradifferential operator $T_{X_\gamma} \cdot \nabla$.

By Proposition 2.4, χ^* and $(\chi^*)^{-1} = \mathbf{F}(X_\gamma; 0, 1)$ are bounded on every H^s , with norms $\leq K_{r,s}(|\gamma|_{C^r})$. This is a huge advantage compared to χ_* .

In parallel to the algebraic relation $T_A \circ T_B = T_{A \circ B} + \text{remainder}$, algebraic properties of χ_* are transferred to χ^* . These include the conjugation of differential operators (i.e. the chain rule) and linearization in γ .

Proposition 2.6 (Conjugation). *Let A be a classical differential operator of order m with C^r coefficients, $r > 2m + N + 1$, and $r' \in (m + N + 1, r - m)$. Then*

$$\chi^* T_A = T_{\chi_* A \chi_*^{-1}} \chi^* + \mathcal{R}_{\text{Con}}(A, \chi),$$

where $\chi_* A \chi_*^{-1}$ is the classical differential operator obtained from A by the change of variables χ (chain rule involving the Jacobian of χ), and for $0 \leq k \leq N$, the remainder $\mathcal{R}_{\text{Con}}(A, \chi)$ is smoothing and differentiable in χ at the price of losing Sobolev indices:

$$\|\mathbf{d}_\gamma^k \mathcal{R}_{\text{Con}}(A, \chi) \cdot (\delta\gamma)^{\otimes k}; \mathcal{L}(H^s, H^{s+r'-m-k-1})\| \leq K_s(|\gamma|_{C^r}) \left(\sum_\beta |a_\beta|_{C^r} \right) |\delta\gamma|_{C^r}^k.$$

Proposition 2.7 (Linearization). *Let $r > N + 2$ and $r' \in (N + 2, r)$. Set $V_{\gamma, \delta\gamma} \cdot \nabla \chi_* f := (\mathbf{d}_\gamma \chi_* \cdot \delta\gamma) f$ to be the linearization of the classical pull-back χ_* . Then*

$$(\mathbf{d}_\gamma \chi^* \cdot \delta\gamma) f = T_{V_{\gamma, \delta\gamma}} \cdot \nabla \chi^* f + \mathcal{R}_{\text{Diff};1}(\chi)(\delta\gamma, f),$$

and for $0 \leq k \leq N$, the remainder $\mathcal{R}_{\text{Diff};1}(\chi)$ is smoothing and differentiable in χ at the price of losing Sobolev indices:

$$\|\mathbf{d}_\gamma^k \mathcal{R}_{\text{Diff};1}(\chi) \cdot (\delta\gamma_1)^{\otimes k}(\delta\gamma, \cdot); \mathcal{L}(H^s, H^{s+r'-2-k})\| \leq K_s(|\gamma|_{C^r}) |\delta\gamma|_{C^r} |\delta\gamma_1|_{C^r}^k.$$

Therefore, up to smoothing errors, χ^* transforms paradifferential operators by the ordinary chain rule. In parallel, the differential of χ^* with respect to the diffeomorphism χ is a first order paradifferential operator, which corresponds to the classical linearization of χ_* .

2.8. Summary of the Toolbox. On the group $\text{SU}(2)$, paradifferential operators T_A are defined via invariant Littlewood-Paley decomposition. The calculus of paradifferential operators obeys the same algebra as classical differential operators as seen in Proposition 2.2. The fractional Laplacian commutes with a paraproduct as in Proposition 2.3. The interaction between the paracomposition χ^* and the paradifferential operator T_A obeys the chain rule, as seen in Proposition 2.6.

3. TOOLBOX: ZONAL PARADIFFERENTIAL CALCULUS ON THE 2-SPHERE

Let us recall that the group $SU(2)$ can be parameterized by the *Euler angles*: the *precession*, *nutation* and *intrinsic rotation* are respectively

$$\varphi \in [0, 2\pi), \quad \theta \in [0, \pi], \quad \psi \in [-2\pi, 2\pi).$$

An element of $SU(2)$ is then factorized as

$$(3.1) \quad \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix}.$$

The parameters $\varphi \in [0, 2\pi)$, $\theta \in (0, \pi)$, $\psi \in [-2\pi, 2\pi)$ are in 1-1 correspondence to elements of $SU(2)$ excluding a lower dimensional closed subset (which corresponds to $\theta = \pm\pi$).

We shall refer to functions on $SU(2)$ that depend only on the angle θ as *zonal*. When projected to $\mathbb{S}^2$ via the Hopf fibration $SU(2) \mapsto SU(2)/U(1)$, such a function becomes a zonal (axi-symmetric) function on $\mathbb{S}^2$ *in the usual sense*. Conversely, any zonal function on $\mathbb{S}^2$ is naturally lifted to $SU(2)$ to a zonal function. Therefore we shall not distinguish these two objects.

Being essentially functions with only one degree of freedom, it turns out that zonal pseudodifferential/paradifferential operators enjoy an algebraic structure parallel to that on the circle $\mathbb{T}^1$. The exact meaning of this assertion will become clear in the sequel.

3.1. Coordinate Singularities. The analysis of zonal functions suffer from the coordinate singularity at the poles $\theta = 0, \pi$. We have to rely on coordinate-free definition of the function norms that are also compatible with the spectral theory. Therefore, we introduce the zonal Sobolev space $H_{\text{Zon}}^s \subset H^s$, consisting of functions of the form

$$f(x) = \sum_{\ell \in \mathbb{N}_0} \widehat{f}(\ell) \tilde{P}_\ell(\cos \theta), \quad \sum_{\ell \in \mathbb{N}_0} |\widehat{f}(\ell)|^2 (1 + \lambda_\ell)^s < +\infty.$$

Here $\tilde{P}_\ell$ are the normalized Legendre polynomials. The space of functions C_{Zon}^r is defined similarly. The Sobolev embedding for zonal functions has the same index as general functions on $\mathbb{S}^2$ due to the coordinate singularity at the poles:

$$H_{\text{Zon}}^s \subset C_{*, \text{Zon}}^{s-1}.$$

To understand how differentiation along “latitude” direction interacts with Fourier multipliers, it is necessary to work with the singular differential operator $\mathcal{C} = -\cot \theta \partial_\theta$ on the group $SU(2)$, which occurs in the latitude part of the Laplacian (restricted to zonal functions):

$$\partial_\theta^2 + \cot \theta \partial_\theta.$$

We name it the *Christoffel correction* on the curved manifold $SU(2)$, since it involves the connection coefficient $\cot \theta$ under the Euler angle coordinate system on $SU(2)$. It turns out that if $s > 2.5$ and $a \in H_{\text{Zon}}^s$, then $[\mathcal{C}, S_j]a$ has spectral support within $\{\sqrt{\lambda} \leq 2^j\}$, and

$$(3.2) \quad \sup_M |[\mathcal{C}, S_j]a| \lesssim_s \|a\|_{H^s} 2^{-j(s-2.5)}.$$

The proof is based on the recursive differential relations of Legendre polynomials, yielding that $\mathcal{C}$ does not expand the spectral support of any function.

3.2. Calculus of Zonal Paradifferential Operators. A classical differential operator on $SU(2)$ is said to be a *zonal* one, if it maps zonal functions to zonal functions. In particular, in terms of Euler angles, we say that a vector field $V \cdot \nabla$ on $SU(2)$ is *zonal*, if it takes the following form:

$$V \cdot \nabla = f(\theta)\partial_\theta,$$

where $f(\theta)$ is a scalar function defined for $\theta \in [0, \pi]$. Note that the angle θ is *not* a globally well-defined function on $SU(2)$, but we require that $f(\theta)\partial_\theta$ should be extendable to the poles $\theta = 0, \pi$ with desired regularity.

It turns out that every regular zonal vector field is a potential one:

Lemma 3.1. *Fix a $r > 1$. Consider a zonal C^r vector field $V \cdot \nabla$ on $SU(2)$. There is a scalar function $v \in C_{\text{Zon}}^{r+1}$, depending linearly on V , such that $\nabla v = V \cdot \nabla$. This function is legitimately referred to as the effective potential of the vector field.*

The proof is based on a standard Poincaré lemma argument on the simply connected manifold $SU(2)$. With the aid of effective potential, we can compute the commutator between a “zonal vector field” type paradifferential operator and the Laplacian Δ :

Proposition 3.2. *Fix $r > 2.5$. Let V be a C^r zonal vector field on $SU(2)$. Then when restricted to zonal functions on $SU(2)$, there holds*

$$[\Delta, T_V \cdot \nabla] = 2\mathbf{Op}^{\text{PM}}(\text{div}V + \mathcal{C}v)\Delta + \mathbf{Op}^{\text{PM}}(\nabla(\text{div}V + 2\mathcal{C}v)) \cdot \nabla + \mathcal{R}(V).$$

Here $\mathcal{C} = -\cot\theta\partial_\theta$, the function $v \in C_{\text{Zon}}^{r+1}$ is the effective potential of V , and for all $s \in \mathbb{R}$, the remainder $\mathcal{R}(V) \in H^{s+r-*}$.

The composition of paradifferential operators corresponding to zonal vector fields also enjoys similar properties:

Proposition 3.3. *Fix $r > 2.5$. Let V, W be C^r zonal vector fields on $M = SU(2)$. Then when restricted to zonal functions, there holds*

$$(T_V \cdot \nabla)(T_W \cdot \nabla) = T_{V \cdot W}\Delta + \mathbf{Op}^{\text{PM}}((\text{div}W)V) \cdot \nabla + 2\mathbf{Op}^{\text{PM}}((\mathcal{C}v)W) \cdot \nabla + \mathcal{R}(V, W),$$

where $v \in C_{\text{Zon}}^{r+1}$ is the effective potential of V , while for all $s \in \mathbb{R}$, the remainder $\mathcal{R}(V, W) \in \mathcal{L}(H_{\text{Zon}}^s, H_{\text{Zon}}^{s+r-*})$ is a smoothing zonal operator.

The proof of these two propositions is nothing but a paradifferential rephrase of some elementary geometric calculation. For example, if a, u are smooth zonal functions, then

$$[\Delta, \nabla a \cdot \nabla]u = 2(\Delta a + \mathcal{C}a)\Delta u + \nabla(\Delta a + 2\mathcal{C}a) \cdot \nabla u.$$

This is comparable with the equality

$$[\partial_x^2, \partial_x a \cdot \partial_x] = 2\partial_x^2 a \cdot \partial_x^2 u + \partial_x^3 a \cdot \partial_x$$

on $\mathbb{T}^1$. Therefore, the principal symbol indicated in Proposition 3.2 is similar to the familiar 1D case, and the difference is only a lower order correction involving $\mathcal{C} = -\cot \theta \partial_\theta$.

3.3. Zonal Paracomposition Operators. The theory of paracomposition operators developed in Subsection 2.7 takes a more explicit form in the zonal case. We say that a diffeomorphism $\chi : \text{SU}(2) \rightarrow \text{SU}(2)$ (not necessarily C^∞) is *zonal*, if in Euler angles, it takes the form $\theta \mapsto \theta + q(\cos \theta) \sin \theta$, where $q : [-1, 1] \rightarrow \mathbb{R}$ is a function of required regularity. In other words, a zonal diffeomorphism *only distorts the nutation angle* θ .

For a zonal vector field $V \cdot \nabla$, the classical chain rule shows that

$$\chi_* V \cdot \nabla \chi_*^{-1} = (J_\chi^{-1} \cdot (V \circ \chi)) \cdot \nabla,$$

where with a little abuse of notation, we write $V \circ \chi$ for the pull-back of the coefficients of V , and J_χ^{-1} is a *scalar* zonal function (involving the Jacobian of χ). Roughly speaking, this shows that *zonal vector fields are closed under conjugation by zonal diffeomorphisms*. By Proposition 2.6, the corresponding paradifferential operator $T_V \cdot \nabla$ satisfies

$$\chi^* T_V \cdot \nabla (\chi^*)^{-1} = \mathbf{Op}^{\text{PM}}(J_\chi^{-1} \cdot (V \circ \chi)) \cdot \nabla + \text{smoothing remainder}.$$

Furthermore, the conjugation of Laplacian Δ under a zonal paracomposition also takes an explicit form. It is, of course, comparable to the classical chain rule on $\mathbb{T}^1$:

$$\chi_* \partial_x^2 (\chi_*)^{-1} = (\chi')^2 \partial_x^2 + \text{lower order terms}.$$

Proposition 3.4. *Fix an integer $N \geq 1$, and let $r > 4 + N + 1$, $r' \in (2 + N + 1, r - 2)$. Let $\chi : \text{SU}(2) \rightarrow \text{SU}(2)$ be a C^r zonal diffeomorphism. Then when restricted to zonal functions, there holds*

$$(3.3) \quad \chi^* \Delta (\chi^*)^{-1} = T_{1+a} \Delta + T_V \cdot \nabla + \text{smoothing remainder}.$$

Here the coefficient $a \in C_{\text{Zon}}^{r-1}$, while $V \cdot \nabla$ is a C^{r-2} zonal vector field. The smoothing remainder gains back $\simeq r$ derivatives on every H^s , and is continuously differentiable in χ at the price of losing derivatives as in Proposition 2.6.

3.4. Conclusion: Cancelable Zonal Operators. We are ready to state a ready-to-use toolbox of calculus specifically designed for zonal paradifferential operators acting on zonal functions.

Let us first recall that on $\mathbb{T}^1$, the most general pseudodifferential operators that one encounters in practice are essentially just linear combinations of

$$(3.4) \quad a_m(x) |D_x|^m, \quad b_m(x) \partial_x |D_x|^{m-1}.$$

We may refer to the first as *product type* and second as *gradient type*. The usual pseudodifferential calculus ensures that these operators are closed under composition (up to smoothing remainder), and also closed under conjugation by smooth diffeomorphisms.

It turns out that on $\text{SU}(2)$, the zonal operators also enjoy similar properties. We introduce the following definition:

Definition 3.5 (Cancelable Zonal Operators). Given indices $N \in \mathbb{N}$, $m \in \mathbb{R}$ and $r > 0$, we consider an operator of the form

$$A = \sum_{k=0}^N a_{m-k} \langle \nabla \rangle^{m-k} + V_{m-k} \cdot \nabla \langle \nabla \rangle^{m-k-1},$$

where the a_{m-k} 's are scalar functions of class C_{Zon}^r , and the V_{m-k} 's are C^r zonal vector fields. The *cancelable zonal paradifferential operator* associated to A , denoted by T_A , is defined as

$$T_A = \sum_{k=0}^N T_{a_{m-k}} \langle \nabla \rangle^{m-k} + T_{V_{m-k}} \cdot \nabla \langle \nabla \rangle^{m-k-1}.$$

The core of our toolbox is then roughly summarized as follows:

- (Zon1) If T_A, T_B are cancelable zonal paradifferential operators, then so is $T_A T_B$ up to smoothing remainders gaining $\simeq r$ derivatives. The symbolic calculus is built upon Proposition 2.3, 3.2, 3.3.
- (Zon2) If T_A is a cancelable zonal paradifferential operator as in Definition 3.5 and χ is a C^r zonal diffeomorphism, then the conjugation $\chi^* T_A (\chi^*)^{-1}$ is, up to smoothing remainders, still a cancelable zonal paradifferential operator. For the principal term $T_{a_m} \langle \nabla \rangle^m$, the conjugation by χ^* reads

$$\mathbf{Op}^{\text{PM}}((1+a)^{m/2} a_m) \langle \nabla \rangle^m + \text{lower order operators},$$

where the function $1+a$ is as in (3.3), explicitly computable involving the Jacobian of χ .

4. SKETCH OF THE PROOF OF THE MAIN RESULTS

In this section, we outline the proof of the main theorems 1.1-1.2. In particular, we notify that despite the presence of small denominators, *no KAM or Nash-Moser iteration is needed in the proof*.

4.1. Dirichlet-Neumann Operator of Axi-symmetric Surface. The Dirichlet-Neumann operator introduced in (1.3) is where the first major technical difficulty arises due to its non-locality and high nonlinearity with respect to η . As shown in (1.4), we have that $G[0]$ acts as the Fourier multiplier ℓ . For a general zonal height function η and zonal trace function ϕ , we have the following:

Theorem 4.1. Fix $\eta \in H_{\text{Zon}}^{s+1/2}$ and $\phi \in H_{\text{Zon}}^s$ with s sufficiently large. The Dirichlet-Neumann operator $G[\eta]$ given by (1.3) admits the following parilinearization formula:

$$(4.1) \quad G[\eta] \phi = T_{\Lambda[\eta]} w - \nabla \cdot (T_{\mathfrak{V}} \eta) - 2T_{(1+\eta)^{-1} \mathfrak{b}} \eta + \mathcal{R}_{\text{DN}}(\eta, \phi),$$

where

- $w = \phi - T_{\mathfrak{b}} \eta$ is Alinhac's good unknown. The zonal function $\mathfrak{b}$ and the zonal vector field $\mathfrak{V}$ are respectively the radial and tangential projection of the velocity field on the boundary:

$$(4.2) \quad \mathfrak{b} = \frac{G[\eta] \phi + (1+\eta)^{-2} \nabla \eta \cdot \nabla \phi}{1 + (1+\eta)^{-2} |\nabla \eta|^2}, \quad \mathfrak{V} = \frac{\nabla \phi - \mathfrak{b} \nabla \eta}{(1+\eta)^2}.$$

- $T_{\Lambda[\eta]}$ is a cancelable operator (see Definition 3.5), associated to the classical operator

$$\Lambda[\eta] = (1+\eta)^{-1} \Omega(\nabla) + \text{cancellable operator of degree 0},$$

where $\Omega(\nabla)$ is as in (1.6).

- The remainder $\mathcal{R}_{\text{DN}}$ is a smooth mapping from $H_{\text{Zon}}^{s+1/2} \times H_{\text{Zon}}^s$ to H_{Zon}^{2s-*} , where the index $2s - *$ differs from $2s$ by a fixed value.

The proof of Theorem 4.1 is essentially the same as in the well-cited references [AM09, ABZ11]; see also [Sha25].

4.2. Parilinearization of the Full System. Theorem 4.1 together with (iii) of Proposition 2.1 yields the following parilinearization formula for the Bernoulli equation:

$$(4.3) \quad -\frac{|\nabla\phi|^2}{2(1+\eta)^2} + \frac{((1+\eta)^2 G[\eta]\phi + \nabla\eta \cdot \nabla\phi)^2}{2((1+\eta)^2 + |\nabla\eta|^2)(1+\eta)^2} \\ = T_b G[\eta]\phi - T_{\mathfrak{W}} \cdot \nabla w + T_{(1+\eta)|\mathfrak{W}|^2 - \mathfrak{W} \cdot \nabla b} \eta \quad \text{mod } H_{\text{Zon}}^{2s-*}.$$

On the other hand, the toolbox of cancellable zonal operators (3.4) gives the following:

$$(4.4) \quad H[\eta] - 2 = T_h \Delta_x \eta + T_{V_{\text{MC}}} \cdot \nabla \eta + T_{r_{\text{MC}}} \eta \quad \text{mod } H_{\text{Zon}}^{2s-*},$$

where h is a smooth function of η and $\nabla\eta$, V_{MC} and r_{MC} are smooth functions of $\eta, \nabla\eta, \text{Hess}\eta$.

Combining (4.1)(4.3)(4.4), introducing a suitable complex diagonal unknown

$$u \simeq (-\Delta - 2)^{1/4} \eta + iw \in H_{\text{Zon}}^s,$$

the capillary droplet system (EQ) then becomes the following form:

$$(4.5) \quad \omega \partial_t u - i(1 + T_a)\Omega(\nabla)u - T_V \cdot \nabla u - T_Y \cdot \nabla \langle \nabla \rangle^{-1/2} u - iT_b \langle \nabla \rangle^{1/2} u \\ = P_1 u + P_2 \bar{u} + \mathcal{R}(u).$$

Here in (4.5), the real coefficients a, b and real zonal vector fields V, Y are smooth functions of $\eta, \nabla\eta, \text{Hess}\eta$, while the operators P_1, P_2 are both cancellable zonal operators of degree 0 (see Definition 3.5). The remainder $\mathcal{R}(u) \in H_{\text{Zon}}^{2s-*}$ is smooth in u .

4.3. Straightening the Coefficients. The toolbox 3.4 enables us to conjugate the diagonalized equation (4.5) by suitable operators to reduce it into *constant coefficient form*:

Proposition 4.2. *A time-periodic function $u \in H^1(\mathbb{T}_t^1; H_{\text{Zon}}^s)$ solves the diagonalized equation (4.5), if and only if it solves the following equation:*

$$(4.6) \quad \mathfrak{L}[\omega; u]u = \mathcal{R}_{\text{Red}}(u).$$

Here in (4.6), the remainder $\mathcal{R}_{\text{Red}}(u) \in H_t^1 H_{\text{Zon}}^{2s-*}$ is smoothing and quadratically small, with suitable (e.g. C^4) regularity in $u \in H_t^1 H_{\text{Zon}}^s$, while $\mathfrak{L}[\omega; u]$ is a Fourier multiplier of the form

$$\mathfrak{L}[\omega; u] = \omega \partial_t - i\mathfrak{a}_{3/2}[\omega; u]\Omega(\nabla) + \left(\mathfrak{a}_{1/2}[\omega; u]\langle \nabla \rangle^{1/2} + \mathfrak{a}_0[\omega; u] + \cdots \right),$$

where the coefficients $\mathfrak{a}_{3/2}[\omega; u], \mathfrak{a}_{1/2}[\omega; u], \cdots$ are all scalar-valued functionals in $(\omega, u) \in \mathbb{R} \times H_t^1 H_{\text{Zon}}^s$.

To obtain (4.6) from (4.5), we first conjugate (4.5) with a sequence of suitable *real linear* operators of the form

$$u \mapsto u + \exp\left(\Omega(\nabla)^{-1} \times \text{cancellable operator of degree } \leq 0\right) \bar{u}$$

to remove the anti-diagonal term $P_2\bar{u}$. Indeed, the conjugation of $(1 + T_a)\Omega(\nabla)$ by this with a suitable choice of the cancellable operator shall remove P_2 .

The next step is to inductively reduce the diagonal terms into constant coefficients. The idea may be illustrated by a model problem on $\mathbb{T}^1$ as follows. Consider the first order operator

$$L = a_1(x)|D_x| + a_0(x), \quad a_1 \in C^\infty(\mathbb{T}^1) \text{ is positive.}$$

Conjugating L with a diffeomorphism χ such that $(a_1 \circ \chi)/\chi'$ is a constant $\mathbf{a}_1$ (which uniquely determines χ), we obtain

$$\chi_* L \chi_*^{-1} = \mathbf{a}_1 |D_x| + a_0 \circ \chi + \text{smoothing operators.}$$

Conjugating this further with $\exp(\mathbf{a}_1^{-1} \int^x a_0 \circ \chi) \operatorname{sgn}(D_x)$, the term $a_0 \circ \chi$ is cancelled out, leaving only its average $\mathbf{a}_0$, producing pseudodifferential operators of order -1 . The process may then continue inductively for N times, finally yielding that L is conjugated to

$$\mathbf{a}_1 |D_x| + \mathbf{a}_0 + \mathbf{a}_{-1} |D_x|^{-1} + \cdots + \mathbf{a}_{-N} |D_x|^{-N} + \text{smoothing remainder.}$$

The core idea of the argument is the closeness of the operator class spanned by (3.4) under composition and diffeomorphisms.

The exactly same idea also applies to the paradifferential operator in (4.5). This is due to the closeness of the *cancellable operators* given in the toolbox (3.4) under composition and zonal paracomposition. For example, by conjugating with a suitable zonal paracomposition operator, the Jacobian shall exactly cancel with the coefficient $1 + T_a$, leaving only a function of t . A reparameterization of time can then cancel out this function of t , leaving only a constant in both t, x . The lower order terms are cancelled by similar arguments as in the $\mathbb{T}^1$ case.

4.4. Para-reduced Equation and Bifurcation Analysis. Having obtained the equivalent formulation (4.6), we are ready to explain the core difficulty in constructing time-periodic solutions and how it can be resolved without using Nash-Moser iteration.

Bifurcation analysis. Obviously, $u \equiv 0$ is a trivial solution of (4.6), and it thus becomes a bifurcation problem with real parameter $\omega \in \mathbb{R}$. The linearized equation at $u = 0$ reads

$$(4.7) \quad [\omega \partial_t - i\Omega(\nabla)] \delta u = f.$$

Equation (4.7) with $f = 0$ has nontrivial solution in $H_t^1 H_{\text{Zon}}^s$ if and only if ω is an eigen-frequency as in (1.6). Suppose now $\omega = \omega_{\ell_0}^{(0)}$ for some $\ell_0 \geq 2$. The kernel of the operator $\omega \partial_t - i\Omega(\nabla)$ is the span of

$$(4.8) \quad \sqrt{(\ell-1)(\ell+2)} \cos(Kt) P_\ell + i\sqrt{\ell} \sin(Kt) P_\ell, \quad \text{where } K^2 = \frac{\ell(\ell-1)(\ell+2)}{\ell_0(\ell_0-1)(\ell_0+2)}.$$

Therefore, the dimension of $\operatorname{Nul}(\mathfrak{L}[\omega; 0])$ equals to the number of non-trivial integer solutions to the Diophantine equation

$$(4.9) \quad K^2 = \frac{\ell(\ell-1)(\ell+2)}{\ell_0(\ell_0-1)(\ell_0+2)}, \quad K \in \mathbb{N}, \ell \geq 2.$$

Since this is an elliptic curve over $\mathbb{Q}$, the dimension is always finite. Counting using standard computational software in algebraic geometry shows that the kernel is 1D if ℓ_0 equals the following numbers:

$$6, 7, 9, 11, 13, 14, 15, 16, 17, 18, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30.$$

If $\ell_0 = 2$, since the system is invariant under reflection with respect to the equator of the sphere, the subspace involving *even* ℓ i.e. even spherical harmonics, is an invariant subspace of the system. When restricted to even spherical harmonics, the kernel is 2-dimensional and spanned by

$$\sqrt{2}\cos(t)P_2 + i\sin(t)P_2, \quad 3\cos(3t)P_4 + i\sqrt{2}\sin(4t)P_4.$$

Para-reduced Equation. On the other hand, the difference $\omega_K - \omega_\ell^{(0)}$ could be very small. The best one could hope is the following Diophantine condition:

$$(4.10) \quad |\omega_K - \omega_\ell^{(0)}| \geq \frac{\gamma}{\ell^\tau}.$$

Therefore, equation (4.7) and thus (4.6) suffers from *loss of regularity due to small denominators*. However, our paradifferential formulation resolves this issue without relying on Nash-Moser type iterations. The argument is roughly as follows.

Given the mode ℓ_0 , when ω is close to the eigen-frequency $\omega_{\ell_0}^{(0)}$, we simply introduce a Fourier multiplier $\mathfrak{A}[\omega; u]$ that inverts $\mathfrak{L}[\omega; u]$ when ω, u satisfy a suitable Diophantine condition. A Whitney type extension argument makes sure that the multiplier $\mathfrak{A}[\omega; u]$ can be defined for *every* ω, u and has C^4 dependence on them. Therefore we are led to the following:

Proposition 4.3. *Suppose the pair $(\omega, u) \in \mathbb{R} \times H_t^1 H_{\text{zon}}^s$ satisfies a suitable Diophantine condition. Then it solves the system (EQ) if and only if it solves the para-reduced equation:*

$$(4.11) \quad \Pi_{\text{Nul}} \mathfrak{L}[\omega; u]u = \Pi_{\text{Nul}} \mathcal{R}_{\text{Red}}(u), \quad \Pi_{\text{Nul}}^\perp u = \mathfrak{A}[\omega; u] \Pi_{\text{Nul}}^\perp \mathcal{R}_{\text{Red}}(u),$$

where Π_{Nul} denotes the orthogonal projection to the kernel.

The key feature is that since $\mathcal{R}_{\text{Red}}(u)$ is smoothing, the loss of regularity caused by $\mathfrak{A}[\omega; u]$ is *compensated* and therefore (4.11) becomes a bifurcation problem in *one single Banach space*.

Having obtained the bifurcation problem (4.11), we are at the place to determine what the solution look like in the (ω, u) -space. The Poincaré-Lindstedt method is used towards the original Zakharov system (EQ) to justify the following assertion:

When $\xi \rightarrow 0$, the solution curve of (4.11) has exactly the Taylor expansion indicated in Theorem 1.1-1.2, depending on which branch it is bifurcating along.

A Python program is then designed to numerically compute the values $\omega_\ell^{(2)}$. Finally, these expansions show that a solution curve of (4.11) is parameterized by ξ in a *non-degenerate* manner. Therefore, a measure theoretic argument shows that the ξ 's making (ω, u) satisfy a suitable Diophantine condition form a set of Lebesgue density 1 as $\xi \rightarrow 0$. This completes the proof.

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