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THE PENROSE INEQUALITY

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KERR BLACK HOLE FORMATION AND A DYNAMICAL APPROACH TO THE PENROSE INEQUALITY

XINLIANG AN AND TAORAN HE

ABSTRACT. In this expository note, we present our recent result on the dynamical formation of Kerr black holes for the $3 + 1$ dimensional Einstein vacuum equations and a newly dynamical approach to the Penrose inequality, based on [6]. Combining the scale-critical short-pulse mechanism with nonlinear Kerr stability, we construct Kerr-black-hole-formation spacetimes with a complete apparent horizon. We prove that this apparent horizon is smooth away from its initial spacetime point, asymptotically null, and converges to the event horizon. In particular, we show that our apparent horizon is achronal, demonstrating a monotonic formula (area increasing law) along it. Building on our analysis, without time symmetric assumption, we then put forward a new mathematical framework and prove both the dynamical Penrose inequality and the space-time Penrose inequality for the perturbations of full subextremal Kerr spacetimes (provided that corresponding Kerr stability result holds), as well as a refinement with angular momentum under axial symmetry. Collectively, this extends Christodoulou’s celebrated trapped surface formation theorem to a black hole formation result without assuming any symmetry.

1. INTRODUCTION

We study the Einstein vacuum equations

$$(1.1) \quad \mathbf{Ric}(\mathbf{g}) = 0$$

on a $3 + 1$ dimensional Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ in the context of gravitational collapse. Here $\mathbf{Ric}(\mathbf{g})$ denotes the Ricci curvatures of the metric $\mathbf{g}$ with signature $(-, +, +, +)$. These equations model the theory of general relativity for the vacuum spacetimes.

One of the central questions concerning the evolution of the Einstein vacuum equations is to understand the dynamical formation of black holes from regular initial data. This problem lies at the intersection of nonlinear wave equations, geometric analysis, and physical perspectives from general relativity and astrophysics. A complete mathematical description of the whole formation process demands detailed mathematical studies of various dynamical stages, i.e., its emergence, evolution, asymptotics and final state. These different stages of black-hole formation are also closely related to Penrose’s geometric perspective on gravitational collapse, which motivated him in 1973 to formulate the Penrose inequality [24].

Meanwhile, the rigorous study of black-hole formation remained to be challenging and has a long history, that Wheeler suggested to Christodoulou in late 1960s. For spherically symmetric Einstein-scalar field system, Christodoulou had a series of celebrated results on this topic in 1990s. A major breakthrough came with Christodoulou’s work [10] in 2008: using the short-pulse method, he proved, *without any symmetry assumption, that a trapped surface (lying inside the black hole region) can form dynamically from regular characteristic initial data.*

The work [7] by An–Luk and the work [2] by An later extend [10] to the scale-critical regime and the emergence of the trapped region and its boundary (apparent horizon) is studied. Our results [6] continues this program by addressing the subsequent evolution, asymptotic behavior and final state of the corresponding gravitational collapse.

A crucial input to this program is the breakthrough on the nonlinear stability of slowly rotating Kerr black holes, established in the series of works of Klainerman–Szeftel [20, 21, 22], together with the companion works of Giorgi–Klainerman–Szeftel [16] and Shen [26]. Their proofs are mainly based on the rather complicated geometric general covariant modulation (GCM) scheme adapted to Kerr, as well as the detailed analysis of the associated coupled wave equations. For

related results on nonlinear Schwarzschild stability, we refer to Klainerman–Szeftel [19] and Dafermos–Holzegel–Rodnianski–Taylor [12].

Against this background, as portrayed in Figure 1, we establish the following global picture: *Arising from admissible characteristic initial data, the Einstein vacuum equations admit black-hole formation solutions: In the process of gravitational collapse, a whole black hole region with its boundary (apparent horizon) emerges from a spacetime point; Subsequently, it grows in evolution, and settles down to a slowly-rotating Kerr black hole. Moreover, the apparent horizon is spacelike in its initial stage, achronal during its evolution, asymptotically null and converging to the event horizon in its final stage. Furthermore, in this process the Penrose inequality is also proved to be true.*

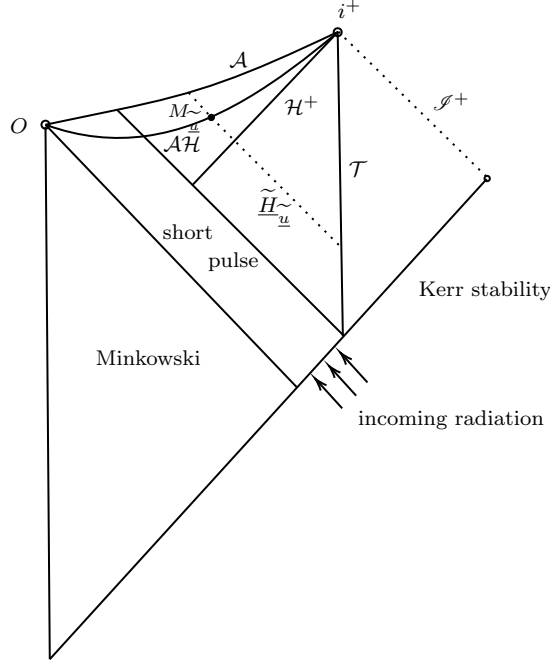


FIGURE 1. Process of Kerr Black Hole Formation

2. MAIN RESULTS

Our first main result concerns the global dynamical formation of Kerr black hole with no symmetry assumption.

Theorem 2.1. *With admissible characteristic initial data, the Einstein vacuum equations (1.1) admit Kerr black hole formation solutions. Each solution processes a complete apparent horizon, originating from an emerging spacetime center point, being spacelike in the short-pulse region, being asymptotically null and converging to the event horizon as the advanced time tends to the timelike infinity.*

The admissibility conditions imposed on the characteristic initial data are open set conditions. Smooth examples of these initial data are given in the appendix of [6] and they allow general dynamical asymptotics, which are used in [22] by Klainerman–Szeftel.

The proof of Lemma 2.1 has two main components. The hyperbolic part connects the scale-critical short-pulse analysis of the first author and Luk [7] with the nonlinear Kerr stability theorem of Klainerman–Szeftel [22], and constructs a suitable Pretorius–Israel type incoming null foliation in the resulting Kerr black hole formation spacetimes. In the elliptic part, we then solve for the marginally outer trapped surface (MOTS) along these incoming null hypersurfaces

via quasi-conformal mappings, and determines the global dynamics and asymptotics of the corresponding apparent horizon. It is worth mentioning that our elliptic arguments described above remain valid for the perturbative spacetimes of subextremal Kerr with full range, as long as the desired Kerr stability holds.

Besides its existence and asymptotic behavior, the apparent horizon enjoys an important causal property, that is the (local) achronality. In [5] we introduced a new concept called the *null comparison principle* to examine this property. Roughly speaking, the null comparison principle says that, relative to a given MOTS on an incoming null hypersurface, trapped surfaces must lie to its interior, whereas untrapped surfaces must lie to its exterior. Based on the detailed geometric information of the MOTS solved as mentioned above, we verify the null comparison principle and hence prove that the resulting apparent horizon is *(locally) achronal for all time*. This further implies our area increasing law of the MOTS. We summarize these properties as

Theorem 2.2. *The apparent horizon $\mathcal{AH} = \cup_{\tilde{u}} M_{\tilde{u}}^{\sim}$ constructed in Lemma 2.1 is either piecewise spacelike or piecewise (outgoing) null. Moreover, the area of MOTS $M_{\tilde{u}}^{\sim}$ is non-decreasing as $\tilde{u}$ grows.*

Compared to the previous method of solving MOTS along the spacelike hypersurface,¹ our elliptic argument here provides more precise geometric information of the MOTS, allowing us to establish the desired area law of the apparent horizon. This monotonicity will be a key ingredient in our proof of the Penrose inequalities below.

Equipped with Lemma 2.1 and Lemma 2.2, we march to fulfill a perspective of Penrose, i.e., the *Penrose inequality* in both Kerr black hole formation spacetimes and perturbed Kerr spacetimes. In 1973, Penrose [24] proposed this inequality to explore the relation between the surface area of the black hole and the total mass of the spacetime.

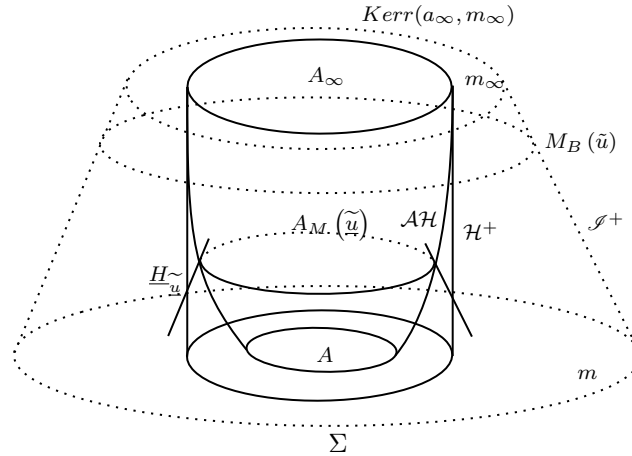


FIGURE 2. A New Approach Toward Penrose Inequality

As depicted in Figure 2, let $A_M(\tilde{u})$ denote the area of the MOTS $M_{\tilde{u}}^{\sim}$, and let $M_B(\tilde{u})$ denote the Bondi mass along future null infinity (with $\tilde{u}$ being an outgoing optical function). The area monotonicity of the apparent horizon, its convergence to the Kerr horizon, and the Bondi mass loss give the string of inequalities

$$(2.1) \quad A_M(\tilde{u}) \leq A_\infty \leq 16\pi m_\infty^2 \leq 16\pi [M_B(\tilde{u})]^2,$$

where m_∞, a_∞ are the final mass, final angular momentum of the dynamical spacetime with $|a_\infty/m_\infty| < 1$ and A_∞ represents the horizon area of limiting Kerr spacetime $Kerr(a_\infty, m_\infty)$:

$$A_\infty = 8\pi m_\infty \left(m_\infty + \sqrt{m_\infty^2 - a_\infty^2} \right).$$

¹See for example [8, 13].

This leads to the following dynamical Penrose inequality.

Theorem 2.3 (Dynamical Penrose Inequality). *The dynamical Penrose inequality holds true in our Kerr black hole formation spacetimes and in Klainerman–Szeftel’s Kerr stability spacetimes. Specifically, letting $A_M(\tilde{u})$ be the area of MOTS $M_{\tilde{u}}$ and defining $M_B(\tilde{u})$ (with $\tilde{u}$ being an outgoing optical function) as the Bondi mass along the future null infinity, we have the following inequality*

$$(2.2) \quad M_B(\tilde{u}) \geq m_\infty \geq \sqrt{\frac{m_\infty(m_\infty + \sqrt{m_\infty^2 - a_\infty^2})}{2}} \geq \sqrt{\frac{A_M(\tilde{u})}{16\pi}}.$$

Here m_∞, a_∞ are the final mass, final angular momentum of the dynamical spacetime with $|a_\infty/m_\infty| \ll 1$.

If assuming nonlinear Kerr stability in full subextremal regime in the sense of [22], we have that (2.2) still holds with $|a_\infty/m_\infty| < 1$.

In the Riemannian setting, when restricting (2.1) to the initial data on Σ , we would expect the spacetime Penrose inequality

$$A \leq 16\pi m^2.$$

Here as in Figure 3, A represents the area of the initial MOTS M_0 along Σ and m denotes the ADM mass of the initial data set (Σ, g, k) . Note that in our constructed spacetimes, we do not impose the time symmetric condition. With the aid of time symmetry (near the MOTS), namely, $k \equiv 0$, the general Riemannian Penrose $A \leq 16\pi m^2$ inequality has been proved by Huisken–Ilmanen [17] via the inverse mean curvature flow and by Bray [9] via the conformal flow.

In [6], as portrayed in Figure 3, with initial data set (Σ, g, k) in perturbative Kerr regime, we evolve Einstein vacuum equations and obtain the same spacetime picture as in Figure 2. Employing the aforementioned strategy, we thus prove the following spacetime Penrose inequality with no time symmetric assumption.

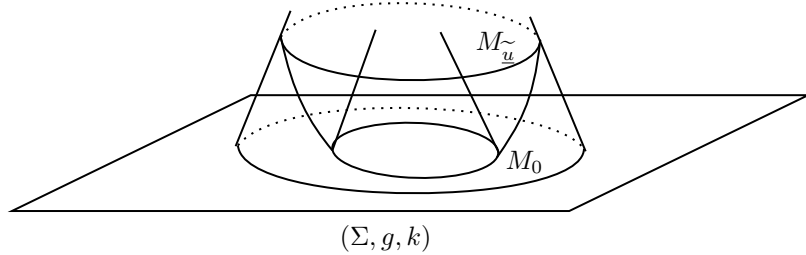


FIGURE 3. Spacetime Penrose Inequality

Theorem 2.4 (Spacetime Penrose Inequality). *Within the perturbative regime of slowly rotating Kerr black holes, assume that we have a spacelike initial data set (Σ, g, k) with a perturbed MOTS M_0 , that is close to the MOTS in the exact Kerr. Let m be the ADM mass of (Σ, g) and denote A to be the area of the MOTS M_0 . Then we have the following Penrose inequality*

$$(2.3) \quad m \geq \sqrt{\frac{A}{16\pi}},$$

with equality if and only if the Gauss curvature of the MOTS M_0 is pointwise equal to $\frac{1}{4m^2}$.

Moreover, assuming the validity of nonlinear Kerr stability for the full subextremal range in the sense of [22], we still have (2.3) with the aforementioned rigidity in the perturbative regime of all subextremal Kerr black holes.

Furthermore, under axial symmetry, conservation of angular momentum gives a refined version of the spacetime Penrose inequality.²

Theorem 2.5 (Penrose Inequality with Angular Momentum). *Let (Σ, g, k) be an axially symmetric initial data set for the Einstein vacuum equations. Assume that (Σ, g, k) is in the perturbative Kerr regime with ADM mass m , Komar angular momentum J and it contains a perturbed MOTS M_0 with area A . Then the following geometric inequality holds*

$$(2.4) \quad \sqrt{\frac{m(m + \sqrt{m^2 - \frac{J^2}{m^2}})}{2}} \geq \sqrt{\frac{A}{16\pi}},$$

with equality if and only if the induced metric on M_0 is isometric to the induced metric on the cross-section along the horizon of a Kerr spacetime with mass m and angular momentum $a = J/m$.

We note that for an exact Kerr spacetime $Kerr(a, m)$ its Komar angular momentum $J = am$. And we can see that inequality (2.4) is sharper than inequality $m \geq \sqrt{A/16\pi}$ in (2.3).

3. KEY POINTS AND NEW INGREDIENTS OF THE PROOF

3.1. Admissible Kerr Black Hole Formation Initial Data. The first step in the hyperbolic part of the proof is to connect two rather different regimes: the scale-critical short-pulse evolution of An–Luk [7], which describes the early stage of gravitational collapse, and the nonlinear Kerr stability theorem (with small angular momentum) of Klainerman–Szeftel [22], which controls the subsequent global evolution. For this purpose, we prescribe admissible characteristic initial data on the outgoing null hypersurface $H_{u_0} = \{u = u_0\}$ up to the spacetime center, as illustrated in Figure 4.

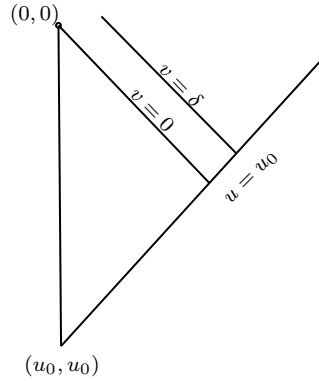


FIGURE 4. Admissible Characteristic Initial Data

We adopt a double-null foliation with outgoing null hypersurfaces H_u and incoming null hypersurfaces $\underline{H}_v$, and denote by $S_{u,v} = H_u \cap \underline{H}_v$ their intersection 2-spheres. The admissible initial data are arranged as follows.

- Along the short-pulse portion $H_{u_0} \cap \{0 \leq v \leq \delta\}$ with $0 < \delta \ll 1$, we prescribe scale-critical short-pulse data that are consistent with [7]. In particular, the outgoing shear $\hat{\chi}$ is large, of size $A^{1/2}/|u_0|$ in L_v^1 , where $A = \delta^{-1} \gg 1$.
- We further impose an isotropy condition on the incoming radiation along $H_{u_0} \cap \{0 \leq v \leq \delta\}$, so that at the end of the pulse, the geometry is sufficiently close to that of a

²We thank Pei-Ken Hung for informing us this interesting problem under axial symmetry.

Schwarzschild metric with mass $m_0 = 1/4$. One of the main conditions encoding this requirement is

$$(3.1) \quad \int_0^\delta |u_0|^2 |\hat{\chi}|^2(u_0, v, \theta^1, \theta^2) dv = 4m_0 + \text{l. o. t.} \quad \text{for all } (\theta^1, \theta^2) \in \mathbb{S}^2.$$

Together with suitably refined initial conditions for $\hat{\chi}$ and the curvature component α on $S_{u_0, \delta}$, this allows us to propagate improved estimates in the incoming direction. As a consequence, the characteristic data on the later incoming hypersurface $\underline{H}_\delta$ are perturbatively close to the corresponding Schwarzschild data.

- After the short pulse, along $H_{u_0} \cap \{v \geq \delta\}$, we continue the data by asymptotically Kerr data satisfying the decay assumptions required in [22]. Typically,

$$|\hat{\chi}| + r|\alpha| \lesssim \frac{\varepsilon_1}{r^{(5/2)+\delta_B}},$$

for suitable small parameters ε_1 and δ_B . Smooth examples of such admissible data are given in [6]. In particular, the late-time characteristic data are not required to agree exactly with Kerr, but can be freely prescribed within the admissible perturbative class.

We then regard the two intersecting null hypersurfaces

$$\{u_0 \leq u \leq -\delta A, v = \delta\}, \quad \{u = u_0, \delta \leq v < \infty\},$$

as the characteristic hypersurfaces from which the subsequent evolution is initiated. To apply the Kerr stability result [22], one further has to reconcile the geometric frameworks used in the two analyses. The short-pulse argument in [7] is formulated in a double-null foliation, whereas [22] employs totally different foliations and frames adapted to perturbed Kerr spacetimes, which are referred to as the principal geodesic (PG) structures. We therefore construct suitable initial data layers near $H_{u_0} \cap \{v \geq \delta\}$ and $\underline{H}_\delta$, and construct the associated PG structures there. Changes of coordinates and frame transformation are also extensively conducted to verify the initial assumptions of [22].

Once these initial conditions are verified, the main theorem of [22] yields a global future development of the admissible characteristic data. The resulting spacetime is ε_0 -close to³ and eventually converges to a nearby slowly rotating Kerr black hole $Kerr(a_\infty, m_\infty)$ with final mass m_∞ and final angular momentum a_∞ satisfying $|a_\infty/m_\infty| \ll 1$.

We mention that a different route to vacuum black hole formation is provided by gluing constructions; see, for instance, Li–Mei [23] and Kehle–Unger [18]. In those approaches, the spacetime is arranged to agree exactly with a (slowly rotating) Kerr black hole in a late-time region, including the event horizon. Our construction instead treats a forward evolution problem: the late-time data are allowed to be genuinely dynamical within the perturbative class, and nonlinear Kerr stability is used to show that the resulting spacetime approaches Kerr only asymptotically. In particular, this gives a family of smooth black hole formation spacetimes with considerable freedom in the incoming radiation at later times.

3.2. Pretorius–Israel Type Coordinate System in Perturbed Kerr Spacetimes. The next step is to construct an incoming null foliation in the perturbative Kerr region that is suitable for the elliptic analysis of MOTSs.

In [22], Klainerman–Szeftel describe the interior region $^{(int)}\mathcal{M}$ using incoming principal geodesic (PG) structures and coordinates $(\underline{u}, r, \theta, \varphi)$ adapted to the Kerr geometry. A distinctive feature of this construction is that $\underline{u}$ is not necessarily an optical function; in particular, its level sets may be slightly timelike. Since our MOTSs are instead solved along incoming null hypersurfaces, we need to construct a genuine incoming optical function in $^{(int)}\mathcal{M}$, together with sufficiently precise information on the associated null hypersurfaces.

³In our proof, we choose $\varepsilon_0 = \varepsilon_1^{\frac{1}{2}}$, where ε_1 measures the size of the perturbation in the initial data layers.

Inspired by the Pretorius–Israel coordinates for exact Kerr [25] and their construction in [11], we proceed as follows.

- The original Pretorius–Israel coordinates are constructed in exact Kerr and rely strongly on the special structure of the Kerr metric, so they cannot be applied directly to the dynamically perturbed Kerr spacetime. We instead construct the incoming optical function $\tilde{u}$ by solving the eikonal equation

$$(3.2) \quad \mathbf{g}^{\mu\nu} \partial_\mu \tilde{u} \partial_\nu \tilde{u} = 0$$

through the method of characteristics. To choose the initial data, we appeal to the incoming optical function

$$\tilde{u}_0 = \underline{u} - \int \frac{r^2 + a_\infty^2}{r^2 - 2m_\infty r + a_\infty^2} dr + r_*(r, \theta)$$

for the exact Kerr spacetime $Kerr(a_\infty, m_\infty)$. We then prescribe $\tilde{u} = \tilde{u}_0$ along $r = r_0$ with $r_0 \approx |u_0| \gg 1$.

- To solve the eikonal equation (3.2), set

$$\underline{h} = \tilde{u} - \tilde{u}_0.$$

For $x = (\underline{u}, r, \theta, \varphi) \in {}^{(int)}\mathcal{M}$ and $p = (p_\underline{u}, p_r, p_\theta, p_\varphi)$, we define

$$F(x, p) := \mathbf{g}^{\alpha\beta} (\partial_\alpha \tilde{u}_0 + p_\alpha) (\partial_\beta \tilde{u}_0 + p_\beta).$$

Then (3.2) is equivalent to

$$F(x, D\underline{h}) = 0, \quad D\underline{h} := (\partial_{\underline{u}} \underline{h}, \partial_r \underline{h}, \partial_\theta \underline{h}, \partial_\varphi \underline{h}).$$

By the classical method of characteristics for first-order nonlinear equations, this is reduced to the Hamiltonian system⁴

$$(3.3) \quad \begin{cases} \frac{d}{ds} p_\alpha = D_{x^\alpha} F(x, p), \\ \frac{d}{ds} x^\alpha = -D_{p_\alpha} F(x, p) = -2\mathbf{g}^{\alpha\beta} (\partial_\beta \tilde{u}_0 + p_\beta). \end{cases}$$

with appropriate initial data for $x(0)$ and $p(0)$ along $r = r_0$.

Once the characteristic system is solved, $\underline{h}$ is recovered along each characteristic curve $x = x(s)$ from

$$\underline{h}(x(s)) = \underline{h}(x(0)) - \int_0^s D_{p_\alpha} F(x(s'), p(s')) p_\alpha(s') ds', \quad D\underline{h}(x(s)) = p(s).$$

- Combining with the hyperbolic estimates from [22] and thanks to the smallness of $p(0)$, we thus solve the characteristic system (3.3) throughout

$${}^{(int)}\mathcal{M} = \{r_{\mathcal{H}} \leq r \leq r_0\}$$

and obtain

$$\tilde{u} = \tilde{u}_0 + O\left(\frac{\varepsilon_0}{\underline{u}^{1+\delta_{dec}}}\right),$$

where ε_0 measures the size of the perturbation and $0 < \delta_{dec} \ll 1$. Thus the level sets $\tilde{H}_{\tilde{u}}$ of $\tilde{u}$ form incoming null hypersurfaces which remain perturbatively close to the corresponding Pretorius–Israel null hypersurfaces of the limiting Kerr spacetime.

We move to construct an incoming geodesic foliation adapted to $\tilde{H}_{\tilde{u}}$. Introduce the Kerr-adapted radial function

$$\tilde{r}(r, \theta) = e^{\kappa r_*(r, \theta)} + r_{+, \infty},$$

⁴Interested readers are referred to Chapter 3 of [14] for more details.

where

$$r_{\pm,\infty} = m_\infty \pm \sqrt{m_\infty^2 - a_\infty^2}, \quad \kappa = \frac{r_{+,\infty} - r_{-,\infty}}{2m_\infty r_{+,\infty}}.$$

We also denote by $\tilde{S}_{\underline{u},\tilde{r}}$ the level sets of $\tilde{r}$ along $\tilde{H}_{\underline{u}}$.

- Setting $\tilde{e}_3 = -\mathbf{D}\tilde{u}$, we propagate angular coordinates $\tilde{\theta}^1, \tilde{\theta}^2$ along $\tilde{e}_3$ and obtain an incoming geodesic coordinate system $(\tilde{u}, \tilde{r}, \tilde{\theta}^1, \tilde{\theta}^2)$ with adapted null frame $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$.
- Writing the MOTS with coordinates

$$M_{\underline{u}} = \{\tilde{r} = R(\tilde{u}, \tilde{\theta}^1, \tilde{\theta}^2)\} =: \tilde{S}_{\underline{u},R} \subset \tilde{H}_{\underline{u}},$$

then schematically its equation takes the form

$$(3.4) \quad \tilde{\Delta}_{\tilde{S}_{\underline{u},R}} R + \tilde{\Gamma} \cdot (\tilde{\nabla} R \cdot \tilde{\nabla} R + \tilde{\nabla} R) + \frac{1}{2} f^{-1} \text{tr} \tilde{\chi} = 0, \quad f \sim -1.$$

Here $\tilde{\Gamma} = \tilde{\partial} \tilde{g}$ denotes the associated Ricci coefficients and

$$\text{tr} \tilde{\chi} = \mathbf{g}(\mathbf{D}_{\tilde{e}_a} \tilde{e}_4, \tilde{e}_a).$$

Thus the leading-order behavior of the outgoing null expansion $\text{tr} \tilde{\chi}$ directly determines the zeroth-order structure of the MOTS equation.

- We first show that

$$\text{tr} \tilde{\chi} = \text{tr} \tilde{\chi}_0 + O\left(\frac{\varepsilon_0}{\tilde{u}^{1+\delta_{dec}}}\right),$$

where $\text{tr} \tilde{\chi}_0$ denotes the corresponding outgoing null expansion in the limiting Kerr space-time $Kerr(a_\infty, m_\infty)$ with respect to $(\tilde{u}_0, \tilde{r})$.

The crucial point is then to identify the precise structure of $\text{tr} \tilde{\chi}_0$. We prove

$$(3.5) \quad \text{tr} \tilde{\chi}_0 = F(\tilde{r}, \theta) (\tilde{r} - r_{+,\infty}), \quad F \sim 1.$$

Consequently,

$$(3.6) \quad \text{tr} \tilde{\chi} = F(\tilde{r}, \theta) (\tilde{r} - r_{+,\infty}) + O\left(\frac{\varepsilon_0}{\tilde{u}^{1+\delta_{dec}}}\right).$$

This is the main geometric output of the construction. It shows that the outgoing expansion vanishes nondegenerately at the limiting Kerr horizon, with a strictly positive first-order coefficient.

- The derivation of (3.5) is a delicate point. A direct computation is not well suited for this purpose, since the explicit expression for $\text{tr} \tilde{\chi}_0$ involves several implicitly defined functions such as $r_*(r, \theta)$, quadratic radicals, and elliptic integrals. Instead, we use an indirect geometric argument:

- We first analyze $\text{tr} \tilde{\chi}_0$ in two regimes: for $\tilde{r} \gg 1$ and for $\tilde{r} < r_{+,\infty}$ sufficiently close to $r_{+,\infty}$. This shows that $\text{tr} \tilde{\chi}_0$ is positive in the former region and negative in the latter.
- After a suitable rescaling of the outgoing null vector to make it geodesic, the Raychaudhuri equation gives a monotonicity property for the corresponding null expansion. The sign information above can therefore be propagated to obtain

$$\text{tr} \tilde{\chi}_0 > 0 \quad \text{for } \tilde{r} > r_{+,\infty}, \quad \text{tr} \tilde{\chi}_0 < 0 \quad \text{for } m_\infty \leq \tilde{r} < r_{+,\infty}.$$

- Finally, writing

$$F(\tilde{r}, \theta) = \frac{\text{tr} \tilde{\chi}_0}{\tilde{r} - r_{+,\infty}},$$

it remains to determine the behavior of F at the horizon. A direct limiting computation gives

$$F(r_{+,\infty}, \theta) = \lim_{\tilde{r} \rightarrow r_{+,\infty}} \frac{\text{tr} \tilde{\chi}_0}{\tilde{r} - r_{+,\infty}} > 0.$$

At the horizon the explicit expression of F simplifies substantially, which makes this limiting calculation much more manageable than estimating F for all $r \geq m$, especially in the range when $|a_\infty/m_\infty|$ is not necessarily small.

The factorization (3.6) is precisely what is needed in the elliptic argument of the next subsection. Since $f \sim -1$, it produces a favorable negative zeroth-order term in the MOTS equation (3.4), which leads first to the C^0 estimate and ultimately to the existence, uniqueness, and asymptotic control of the MOTSs.

Although the black hole formation spacetime obtained from [22] has $|a_\infty/m_\infty| \ll 1$, the construction of the incoming null cones and the analysis of the leading Kerr term (3.5) do not rely on the smallness of a_∞ . They remain valid throughout the subextremal range $|a_\infty| < m_\infty$.

3.3. Existence and Uniqueness of MOTS Along Each Incoming Null Hypersurface.

We now turn to the elliptic part of this note and solve a unique MOTS along each incoming null hypersurface constructed in the preceding subsection. Note that the elliptic argument below does not use the smallness of a_∞ . It therefore applies throughout the full subextremal range $|a_\infty/m_\infty| < 1$, provided the corresponding perturbative Kerr estimates are available.

- Following the deformation procedure in [5], fix an incoming null hypersurface $\widetilde{H}_{\underline{u}}$ and seek the MOTS $M_{\underline{u}}^\sim$ as the graph

$$M_{\underline{u}}^\sim = \{\tilde{r} = R(\tilde{\theta}^1, \tilde{\theta}^2)\} = \widetilde{S}_{\underline{u}, R}^\sim.$$

The MOTS condition $\text{tr}\chi'|_{M_{\underline{u}}^\sim} \equiv 0$ gives the quasilinear elliptic equation⁵

$$(3.7) \quad 0 = L(R, \underline{u}) := \left(\frac{1}{2}f^{-1}\text{tr}\chi'\right)|_{\widetilde{S}_{\underline{u}, R}^\sim} = \widetilde{\Delta}_{\widetilde{S}_{\underline{u}, R}^\sim} R + \widetilde{\Gamma} \cdot (\widetilde{\nabla} R \cdot \widetilde{\nabla} R + \widetilde{\nabla} R) + \frac{1}{2}f^{-1}\text{tr}\tilde{\chi}.$$

Here $f = [\tilde{e}_3(\tilde{r})]^{-1} \sim -1$, $\widetilde{\Gamma}$ represents the Ricci coefficients and $\widetilde{\Delta}$, $\widetilde{\nabla}$ correspond to the Laplace–Beltrami operator and the induced covariant derivative on $\widetilde{S}_{\underline{u}, \tilde{r}}$, respectively.

In [1, 5], where the background geometry is Schwarzschild, a modified Bochner formula was used to derive the required gradient estimate. This approach is no longer effective in the Kerr setting. Indeed, the coefficients entering the first- and second-order gradient terms satisfy only

$$(3.8) \quad |\widetilde{\Gamma}| \lesssim \frac{|a_\infty|}{r^2} + \frac{\varepsilon_0}{\underline{u}^{1+\delta_{dec}}},$$

The first term on the right-hand side is a stationary Kerr contribution and does not decay as $\underline{u} \rightarrow \infty$. Since we do not impose the smallness of a_∞ in the elliptic argument, the corresponding gradient terms cannot be treated simply as small perturbative errors. We therefore replace the modified Bochner argument by the method of quasi-conformal mappings.

- The estimates for the geometry of the incoming null foliation, along with the leading-order formula for $\text{tr}\tilde{\chi}$ obtained in the preceding subsection, allow us to rewrite (3.7) in local coordinates as

$$(3.9) \quad L(R, \underline{u}) = a^{ij}(\omega, R, DR) D_{ij} R + b(\omega, R, DR) + c(\omega, R)(R - r_{+, \infty}) = 0,$$

where ω is a local chart on $\mathbb{S}^2$ and $a^{ij}(\omega, \tilde{r}, p)$, $b(\omega, \tilde{r}, p)$, $c(\omega, \tilde{r})$ are smooth functions satisfying

$$(1) \quad a^{ij}(\omega, \tilde{r}, p) = a^{ji}(\omega, \tilde{r}, p);$$

⁵For notational simplicity, we sometimes suppress the dependence on $\underline{u}$ when there is no danger of confusion.

- (2) there exist constants $\mu \geq \nu > 0$ and $C_0 > c_0 > 0$ independent of $\varepsilon_0, \underline{u}, \tilde{r}, p$, such that

$$(3.10) \quad \nu |\xi|^2 \leq a^{ij}(\omega, \tilde{r}, p) \xi_i \xi_j \leq \mu |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^2,$$

$$(3.11) \quad |b(\omega, \tilde{r}, p)| \lesssim |p|^2 + |p| + \frac{\varepsilon_0}{\underline{u}^{1+\delta_{dec}}}, \quad -C_0 \leq c(\omega, \tilde{r}) \leq -c_0.$$

The estimate (3.11) is precisely where the expansion of $\text{tr} \tilde{\chi}$ from the preceding subsection enters the elliptic argument. The principal term $F(\tilde{r} - r_{+, \infty})$ with $F \sim 1$ gives rise to the strictly negative zeroth-order coefficient c . At the same time, the remainder $O(\frac{\varepsilon_0}{\underline{u}^{1+\delta_{dec}}})$ appears in $b(\omega, \tilde{r}, p)$. Thus the same expansion has two important consequences:

- the negativity of c yields the favorable C^0 estimate and, later, the invertibility of the linearized MOTS operator;
- the decaying remainder $O(\frac{\varepsilon_0}{\underline{u}^{1+\delta_{dec}}})$ gives the quantitative asymptotics of the MOTS as $\underline{u} \rightarrow \infty$.

In this note, we solve the equation of MOTS (3.9) via a Leray–Schauder fixed point argument, instead of the continuity method adopted in [1, 4, 3, 5]. The main idea is as follows.

- The Leray–Schauder fixed point theorem asserts that

Theorem 3.1. *Let $M > 0$ and $\mathcal{B}$ be a Banach space. Suppose that T is a compact mapping from the closed ball*

$$\overline{B}_M := \{x \in \mathcal{B} : \|x\|_{\mathcal{B}} \leq M\}$$

into $\mathcal{B}$. If, for every $x \in \overline{B}_M$ and $\sigma \in [0, 1]$ satisfying $x = \sigma T x$, one has

$$(3.12) \quad \|x\|_{\mathcal{B}} < M,$$

then T has a fixed point.

To apply Lemma 3.1, for any given $S \in C^{1,\alpha}$ with $\alpha \in (0, 1)$, we consider the elliptic equation

$$a^{ij}(\omega, S, DS) D_{ij} R + b(\omega, S, DR) + c(\omega, S)(R - r_{+, \infty}) = 0.$$

In light of the classical theory of elliptic equations and the conditions (3.10)–(3.11), this equation admits a unique solution $R \in C^{2,\alpha}$. We can therefore define a compact map

$$T : C^{1,\alpha} \longrightarrow C^{1,\alpha}, \quad R = T(S).$$

- The condition (3.12) reduces to obtaining a uniform a priori $C^{1,\alpha}$ estimate along the corresponding homotopy,

$$a^{ij}(\omega, R, DR) D_{ij} R + \sigma b(\omega, R, DR) + c(\omega, R)(R - r_{+, \infty}) = 0, \quad \sigma \in [0, 1].$$

In fact, for the estimates below it suffices to treat the case $\sigma = 1$, which is precisely the MOTS equation (3.9).

We then derive the a priori estimates for the MOTS equation according to the scheme

$$C^0 \longrightarrow C^{1,\alpha} \longrightarrow C^{2,\alpha}.$$

The C^0 estimate follows directly from the negativity of $c(\omega, \tilde{r})$ in (3.11) and the maximum principle on $\mathbb{S}^2$. The crucial step is therefore the $C^{1,\alpha}$ estimate. To this end, we employ the new elliptic argument based on quasi-conformal mappings in two dimensions, as inspired by [27].

- Let

$$\alpha := 1 + \frac{\mu}{\nu} - \sqrt{\left(\frac{\mu}{\nu}\right)^2 + \frac{2\mu}{\nu}} \in (0, 1).$$

Given any $S \in C^{1,\alpha}$, we aim to derive a priori $C^{1,\alpha}$ estimates that are independent of S and DS and uniform in $\sigma \in [0, 1]$, for the equation⁶

$$(3.13) \quad \mathcal{L}(S)R + b(\omega, R, DR) - (1 - \sigma)b(\omega, r_{+, \infty}, 0) = 0,$$

where

$$\mathcal{L}(S)R := a^{ij}(\omega, S, DS)D_{ij}R + c(\omega, S)(R - r_{+, \infty}).$$

By (3.11), the zeroth-order term in $b(\omega, \tilde{r}, p)$ is of size $\frac{\varepsilon_0}{\frac{u}{1+\delta_{dec}}}$. Our elliptic argument is in fact robust enough not to rely on the smallness of this term. Accordingly, in what follows we replace $\frac{\varepsilon_0}{\frac{u}{1+\delta_{dec}}}$ in (3.11) by a parameter N , which is allowed to be large.

- Assume first that $\partial_{\tilde{r}}b \leq 0$, and let R_σ solve (3.13). For

$$W = R_{\sigma_1} - R_{\sigma_2} \quad \text{with} \quad 0 \leq \sigma_1 \leq \sigma_2 \leq 1,$$

the maximum principle thus gives

$$(3.14) \quad \|W\|_{C^0} \lesssim |\sigma_1 - \sigma_2|N.$$

Moreover, the structure of b implies schematically

$$|\mathcal{L}(S)W| \lesssim |DR_{\sigma_1}|^2 + |DR_{\sigma_1}| + |DW|^2 + |DW| + N.$$

Using (3.14) and the interpolation inequality

$$\|DW\|_{L^4}^2 \lesssim \|W\|_{C^0} \|W\|_{H^2},$$

we obtain

$$(3.15) \quad \|\mathcal{L}(S)W\|_{L^2} \lesssim \|DR_{\sigma_1}\|_{C^0}^2 + \|DR_{\sigma_1}\|_{C^0} + |\sigma_1 - \sigma_2|N \cdot \|W\|_{H^2} + (|\sigma_1 - \sigma_2|N \cdot \|W\|_{H^2})^{\frac{1}{2}} + N.$$

The key point is to control $\|W\|_{H^2}$ with constants independent of S and DS . The standard elliptic regularity estimate fails as its constant depends on the modulus of continuity of $a^{ij}(\omega, S, DS)$. Instead, in two dimensions the uniform ellipticity implies the Campanato condition

$$\left| \Delta W - \kappa \left(a^{ij} D_{ij} W \right) \right| \leq (1 - \gamma^{-1}) |D^2 W|.$$

Here $\Delta = \sum_{i=1}^2 D_{ii}$. As a result, we obtain

$$\|\Delta W\|_{L^2} \leq (1 - \gamma^{-1}) \|D^2 W\|_{L^2} + \kappa \|a^{ij} D_{ij} W\|_{L^2}.$$

Incorporating with a sharp Miranda–Talenti inequality on $\mathbb{S}^2$, i.e., $\|D^2 W\|_{L^2} \leq \|\Delta W\|_{L^2}$, this yields

$$\|W\|_{H^2} \lesssim \|\mathcal{L}(S)W\|_{L^2} + \|W\|_{L^2},$$

with constants depending only on the uniform ellipticity and boundedness of the coefficients.

Hence, choosing

$$|\sigma_1 - \sigma_2|(N + 1) \ll 1,$$

we may absorb the H^2 term in (3.15). Together with the Sobolev embedding $H^1 \hookrightarrow L^p$ on $\mathbb{S}^2$, for any $2 \leq p < \infty$, we further obtain

$$(3.16) \quad \|\mathcal{L}(S)W\|_{L^p} \lesssim_p \|DR_{\sigma_1}\|_{C^0}^2 + \|DR_{\sigma_1}\|_{C^0} + N.$$

⁶At the end of the argument we take $S = R$.

- For the similar reason, the classical $W^{2,p}$ estimates are not applicable in our setting. To pass from (3.16) to a $C^{1,\alpha}$ estimate without involving any dependence on S, DS , we utilize the quasi-conformal mapping method.

Consider now a linear elliptic equation

$$au_{xx} + 2bu_{xy} + cu_{yy} = f \quad \text{in } \Omega,$$

let λ and Λ be the smaller and larger eigenvalues of the coefficient matrix and set

$$\gamma = \sup_{\Omega} \frac{\Lambda}{\lambda}.$$

For $f \in L^{2\beta}(\Omega)$ with

$$\beta > \frac{1}{1-\alpha},$$

we are able to prove

$$(3.17) \quad \|u\|_{C^{1,\alpha}(\Omega')} \leq C(\gamma, \beta, \Omega, \Omega') \left(\|u\|_{C^0(\Omega)} + \|f/\lambda\|_{L^{2\beta}(\Omega)} \right),$$

with $\Omega' \subset \overline{\Omega'} \subset \Omega$. The constant C depends only on the uniform ellipticity constant γ , rather than on the modulus of continuity for a, b, c . This extends the classical quasi-conformal estimate, where $\beta = \infty$, to large but finite β .

Applying (3.17) to $W = R_{\sigma_1} - R_{\sigma_2}$ hence gives

$$(3.18) \quad \|R_{\sigma_1} - R_{\sigma_2}\|_{C^{1,\alpha}} \lesssim \|DR_{\sigma_1}\|_{C^0}^2 + \|DR_{\sigma_1}\|_{C^0} + N,$$

provided

$$|\sigma_1 - \sigma_2| \leq \min(C(N+1)^{-1}, 1) =: \delta.$$

Starting from $R_0 \equiv r_{+, \infty}$ and dividing $[0, 1]$ into finitely many intervals of size δ , we iterate (3.18) for at most $\lfloor \delta^{-1} \rfloor + 1$ times and obtain, for $R = R_1$,

$$(3.19) \quad \|R - r_{+, \infty}\|_{C^{1,\alpha}} \lesssim \max(N^{2^{C_N}}, N).$$

- It remains to remove the assumption $\partial_{\tilde{r}} b \leq 0$. If R solves (3.9), define

$$\tilde{b}(\omega, \tilde{r}, p) := \frac{b(\omega, R, DR)}{|DR|^2 + |DR| + N} (|p|^2 + |p| + N).$$

Then $\tilde{R} = R$ solves

$$(3.20) \quad \mathcal{L}(R)\tilde{R} + \tilde{b}(\omega, \tilde{R}, D\tilde{R}) = 0,$$

and, by construction,

$$\partial_{\tilde{r}} \tilde{b} = 0.$$

We may therefore apply the preceding argument to the auxiliary elliptic equation (3.20), which yields (3.19) without any monotonicity assumption on b .

In the perturbative Kerr regime,

$$N = \frac{\varepsilon_0}{\tilde{u}^{1+\delta_{dec}}}.$$

Hence (3.19), followed by the standard Schauder estimate, yields

$$(3.21) \quad \|R - r_{+, \infty}\|_{C^{2,\alpha}} \lesssim \frac{\varepsilon_0}{\tilde{u}^{1+\delta_{dec}}}.$$

The Leray–Schauder fixed point theorem then gives the existence of a MOTS along each incoming null hypersurface $\tilde{H}_{\tilde{u}}$.

To prove uniqueness, we consider the linearized MOTS operator at a solution $R = R(\tilde{u})$:

$$\partial_R L(R, \tilde{u})[W] = a^{ij} D_{ij} W + B^i D_i W + \left(\partial_{\tilde{r}} a^{ij} D_{ij} R + \partial_{\tilde{r}} c(R - r_{+, \infty}) + c \right) W.$$

- By virtue of (3.21) and (3.11),

$$\partial_{\tilde{r}} a^{ij} D_{ij} R + \partial_{\tilde{r}} c(R - r_{+, \infty}) + c = c + O\left(\frac{\varepsilon_0}{\tilde{u}^{1+\delta_{dec}}}\right) \leq -\frac{c_0}{2}.$$

Thus $\partial_R L(R, \tilde{u})$ is invertible by the maximum principle. If R_1 and R_2 are two solutions, then

$$0 = L(R_1) - L(R_2) = \left(\int_0^1 \partial_R L(tR_1 + (1-t)R_2) dt \right) [R_1 - R_2],$$

and the same maximum-principle argument gives $R_1 \equiv R_2$.

- The invertibility of $\partial_R L$ at $R = R(\tilde{u})$ comes from both the detailed structure of $\text{tr} \tilde{\chi}_0$ established in Section 3.2 and our desired $C^{1,\alpha}$ estimates via the quasi-conformal method.

Remark 3.1. For MOTSs lying on incoming null hypersurfaces, the coefficient $a^{ij}(\omega, R, DR)$ in (3.9) is in fact independent of DR . The argument above, however, only uses uniform ellipticity of $a^{ij}(\omega, R, DR)$ and therefore allows such a dependence. This is different from the approach in [4, 3], where the structure $a^{ij} = a^{ij}(\omega, R)$ is used essentially in a Moser iteration leading to

$$C^0 \longrightarrow C^{0,\alpha} \longrightarrow C^{1,\alpha}.$$

This additional flexibility also allows the present method to be applied to MOTS equations on slightly timelike or spacelike hypersurfaces, in which we only obtain conditional gradient estimate for R .

3.4. Dynamics of Apparent Horizon. We next study the global dynamics of the apparent horizon

$$\mathcal{AH} := \bigcup_{\tilde{u} > 0} M_{\tilde{u}}, \quad M_{\tilde{u}} = \{\tilde{r} = R(\tilde{u}, \tilde{\theta}^1, \tilde{\theta}^2)\} \cap \tilde{H}_{\tilde{u}}.$$

Its main properties are summarized as follows.

- **Smoothness.** In the short-pulse region, the smoothness of $\mathcal{AH}$ follows directly from [1]. For the later evolution, the invertibility of the linearized MOTS operator established above, together with the implicit function theorem, yields the smooth dependence of R on $\tilde{u}$. Hence $\mathcal{AH}$ is smooth away from its initial spacetime center point.
- **Late-time asymptotics.** Pick the tangential vector X on $\mathcal{AH}$, which is normal to the MOTSs $M_{\tilde{u}}$, as

$$X = e'_4 - \frac{e'_4(\tilde{r} - R)}{e'_3(\tilde{r} - R)} e'_3.$$

Here $\{e'_1, e'_2, e'_3, e'_4\}$ is the null frame associated with the MOTS $M_{\tilde{u}}$, which is given by

$$e'_3 = \tilde{e}_3, \quad e'_a = \tilde{e}_a + f \tilde{e}_a(R) \tilde{e}_3, \quad e'_4 = \tilde{e}_4 + 2f \tilde{e}^a(R) \tilde{e}_a + f^2 |\tilde{\nabla} R|^2 \tilde{e}_3.$$

Using (3.21), together with the analogous estimate for $\partial_{\tilde{u}} R$ obtained by differentiating the MOTS equation

$$L(R(\tilde{u}), \tilde{u}) = 0,$$

we obtain

$$e'_3(\tilde{r} - R) \sim -1, \quad e'_4(\tilde{r} - R) = O\left(\frac{\varepsilon_0}{\tilde{u}^{1+\delta_{dec}}}\right).$$

Consequently,

$$\mathbf{g}(X, X) = \frac{4e'_4(\tilde{r} - R)}{e'_3(\tilde{r} - R)} = O\left(\frac{\varepsilon_0}{\tilde{u}^{1+\delta_{dec}}}\right) \longrightarrow 0 \quad \text{as } \tilde{u} \rightarrow \infty.$$

Thus $\mathcal{AH}$ becomes asymptotically null toward timelike infinity.

Moreover, combining (3.21) with the estimate for the location of the event horizon $\mathcal{H}^+ = \{r = R_{\mathcal{H}^+}(\underline{u}, \tilde{\theta}^1, \tilde{\theta}^2)\}$ from [22], we infer

$$\left| R(\underline{u}, \tilde{\theta}^1, \tilde{\theta}^2) - R_{\mathcal{H}^+}(\underline{u}, \tilde{\theta}^1, \tilde{\theta}^2) \right| \lesssim \frac{\sqrt{\varepsilon_0}}{\underline{u}^{1+\delta_{dec}}}.$$

Hence the apparent horizon $\mathcal{AH}$ also converges to the event horizon as $\underline{u} \rightarrow \infty$.

- **Achronality and the area law.** With the aid of the structure of the linearized MOTS operator and the a priori estimates for R as established in Section 3.3, we verify the null comparison principle introduced in [5] and thus conclude that our constructed $\mathcal{AH}$ is piecewise spacelike or piecewise null.

As a result, the area of MOTS $A_M(\underline{u}) := \text{Area}(M_{\underline{u}})$ is non-decreasing along $\mathcal{AH}$. And we further have

$$\frac{dA_M}{d\underline{u}} = 0 \quad \text{on the null portions of } \mathcal{AH}, \quad \frac{dA_M}{d\underline{u}} > 0 \quad \text{on the spacelike portions of } \mathcal{AH}.$$

This monotonicity will be a key ingredient in the proof of the Penrose inequality.

3.5. Penrose Inequality in Kerr Black Hole Formation Spacetimes and in Perturbative Kerr Spacetimes. We finally explain how the preceding hyperbolic and elliptic constructions combine to give a new approach to the Penrose inequality. The main ingredients are the global apparent horizon constructed above, the monotonicity of the areas of its MOTS sections, the late-time convergence to the final Kerr horizon, and the Bondi mass loss formula at future null infinity.

In our black hole formation spacetimes, we first establish the dynamical version of Penrose inequality, i.e.,

$$(3.22) \quad M_B(\tilde{u}) \geq m_\infty \geq \frac{\sqrt{r_{+, \infty}^2 + a_\infty^2}}{2} \geq \sqrt{\frac{A_M(\tilde{u})}{16\pi}}.$$

Here $\tilde{u}$ is the outgoing optical function and $M_B(\tilde{u})$ represents the Bondi mass along the level set of $\tilde{u}$.

- In light of the a priori estimate (3.21), the MOTS $M_{\underline{u}} = \{\tilde{r} = R(\underline{u}, \tilde{\theta}^1, \tilde{\theta}^2)\} \cap \tilde{H}_{\underline{u}}$ is sufficiently close to the 2-sphere $\tilde{S}_{\underline{u}, r_{+, \infty}}$,⁷ the area of which can be computed directly with hyperbolic estimates that are established in [22] and [6]:

$$(3.23) \quad \text{Area}(\tilde{S}_{\underline{u}, r_{+, \infty}}) = 4\pi(r_{+, \infty}^2 + a_\infty^2) + O\left(\sqrt{\frac{\varepsilon_0}{\underline{u}^{1+\delta_{dec}}}}\right).$$

Note that the induced metric on the MOTS $M_{\underline{u}}$ satisfies

$$g'_{\tilde{\theta}^a \tilde{\theta}^b} = \mathbf{g}(\partial_{\tilde{\theta}^a} + \partial_{\tilde{\theta}^a} R \partial_{\tilde{r}}, \partial_{\tilde{\theta}^b} + \partial_{\tilde{\theta}^b} R \partial_{\tilde{r}}) = \tilde{g}_{\tilde{\theta}^a \tilde{\theta}^b}$$

with $\tilde{g}_{\tilde{\theta}^a \tilde{\theta}^b}$ being the induced metric on $\tilde{S}_{\underline{u}, \tilde{r}}$. Hence, we can compare the areas of $M_{\underline{u}}$ and $\tilde{S}_{\underline{u}, r_{+, \infty}}$ as below

$$\begin{aligned} \left| A_M(\underline{u}) - \text{Area}(\tilde{S}_{\underline{u}, r_{+, \infty}}) \right| &= \left| \int_{\mathbb{S}^2} \left(\sqrt{\det(\tilde{g})} \Big|_{\tilde{S}_{\underline{u}, R}} - \sqrt{\det(\tilde{g})} \Big|_{\tilde{S}_{\underline{u}, r_{+, \infty}}} \right) d\tilde{\theta}^1 d\tilde{\theta}^2 \right| \\ &\lesssim \sup_{r, \tilde{\theta}^1, \tilde{\theta}^2} \left| \partial_{\tilde{r}} \left(\sqrt{\det(\tilde{g})} \right) \right| \sup_{\tilde{\theta}^1, \tilde{\theta}^2} |R - r_{+, \infty}| \lesssim \frac{\varepsilon_0}{\underline{u}^{1+\delta_{dec}}}. \end{aligned}$$

This together with (3.23) gives

$$(3.24) \quad \lim_{\underline{u} \rightarrow \infty} A_M(\underline{u}) = 4\pi(r_{+, \infty}^2 + a_\infty^2).$$

⁷Recall that $r_{+, \infty} = m_\infty + \sqrt{m_\infty^2 - a_\infty^2}$.

- More precisely, we construct a double null foliation in the exterior region $^{(ext)}\mathcal{M}$, with the outgoing optical function $\tilde{u}$ and the incoming optical function $^{(ext)}\underline{u}$ such that $\Sigma \cap ^{(ext)}\mathcal{M} = \{\tilde{u} + ^{(ext)}\underline{u} = 0\}$. Denote $\tilde{S}_{\tilde{u}, \underline{u}}$ to be the intersection of level sets of $\tilde{u}$ and level sets of $^{(ext)}\underline{u}$.
- Define $m_H(\tilde{u}, \underline{u})$ to be the Hawking mass of the 2-sphere $\tilde{S}_{\tilde{u}, \underline{u}}$. Using the hyperbolic estimates in [22], we derive

$$(3.29) \quad \partial_{\tilde{u}} m_H = O(\tilde{r}^{-2}), \quad m_H = m_\infty \left(1 + O(\tilde{r}^{-1}) + O(\langle \tilde{u} \rangle^{-\frac{1}{2} - \delta_{dec}}) \right),$$

where $\tilde{r}$ stands for the area radius of $\tilde{S}_{\tilde{u}, \underline{u}}$, $\langle \tilde{u} \rangle := \sqrt{|u|^2 + 1}$ and $0 < \delta_{dec} \ll 1$.

- In the expansion of m_H in (3.29), letting $\tilde{r} \rightarrow \infty$ first and then setting $\tilde{u} \rightarrow \infty$, we hence deduce

$$M_B(\tilde{u}) = m_\infty \left(1 + O(\langle \tilde{u} \rangle^{-\frac{1}{2} - \delta_{dec}}) \right) \quad \text{and} \quad M_B(\infty) = m_\infty.$$

Here we use the definition of Bondi mass $M_B(\tilde{u}) := \lim_{\underline{u} \rightarrow \infty} m_H(\tilde{u}, \underline{u})$.

- On the other hand, integrating $\partial_{\tilde{u}} m_H$ backwards from ∞ to $\underline{u}$, with (3.29) we obtain

$$|m_H(\tilde{u}, ^{(ext)}\underline{u}) - M_B(\tilde{u})| \lesssim \frac{1}{\tilde{r}}.$$

Observe that along the initial hypersurface Σ , the Hawking mass $m(\tilde{u}, -\tilde{u})$ tends to the ADM mass at spacelike infinity. Hence

$$M_B(-\infty) = m.$$

The Bondi mass loss formula therefore gives

$$(3.30) \quad m = M_B(-\infty) \geq M_B(\infty) = m_\infty.$$

Combining (3.28), (3.26), and (3.30), we obtain

$$A \leq 4\pi(r_{+, \infty}^2 + a_\infty^2) \leq 16\pi m_\infty^2 \leq 16\pi m^2,$$

which is precisely (3.27).

- The same chain of inequalities also reveals the equality case. Suppose

$$A = 16\pi m^2.$$

Then equality must hold at every step above. In particular,

$$m = m_\infty, \quad a_\infty = 0, \quad A_M(\tilde{u}) \equiv A.$$

Thus no area is gained along the apparent horizon. By the area increasing law and the null comparison principle, $\mathcal{AH}$ must therefore be a non-expanding null horizon.

The null structure equations then imply that the relevant geometric quantities along $\mathcal{AH}$ take their Schwarzschild values. In particular,⁸

$$\hat{\chi}' \equiv 0, \quad \rho' = -\frac{1}{4m^2}.$$

Together with the Gauss equation, this shows that the Gauss curvature K_{M_0} of the initial MOTS satisfies

$$K_{M_0} \equiv \frac{1}{4m^2}.$$

Conversely, if $K_{M_0} \equiv \frac{1}{4m^2}$, the Gauss–Bonnet theorem immediately gives

$$A = 16\pi m^2.$$

This proves the rigidity statement.

⁸Here ρ' represents one curvature component.

Under the additional assumption of axial symmetry, the same argument yields a sharper inequality involving angular momentum.

- Let Y be the axial Killing vectorfield and let J denote the corresponding Komar angular momentum. Its conservation under the Einstein vacuum evolution gives

$$(3.31) \quad J = am = a_\infty m_\infty =: J_\infty.$$

Using this identity, the area estimate (3.28) becomes

$$(3.32) \quad \begin{aligned} A &\leq 8\pi m_\infty (m_\infty + \sqrt{m_\infty^2 - a_\infty^2}) = 8\pi (m_\infty^2 + \sqrt{m_\infty^4 - J_\infty^2}) \\ &\leq 8\pi (m^2 + \sqrt{m^4 - J^2}) = 8\pi m \left(m + \sqrt{m^2 - \frac{J^2}{m^2}} \right). \end{aligned}$$

where the last inequality follows from $m \geq m_\infty$ and (3.31). Equivalently,

$$(3.33) \quad \sqrt{\frac{m \left(m + \sqrt{m^2 - \frac{J^2}{m^2}} \right)}{2}} \geq \sqrt{\frac{A}{16\pi}}.$$

- The rigidity argument follows the same pattern. Equality in (3.33) forces

$$m = m_\infty, \quad a = a_\infty, \quad A_M(\tilde{u}) \equiv A.$$

Hence the apparent horizon is again a non-expanding null horizon with vanishing shear. The first variation formula then shows that the induced metric on $M_{\tilde{u}}$ is independent of $\tilde{u}$. Since these induced metrics converge to the metric on a cross-section of the limiting Kerr horizon, M_0 is isometric to the corresponding Kerr horizon cross-section. This gives the rigidity statement for the Penrose inequality with angular momentum.

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